LaTeX Exercise Sheet 3

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1 Packages

Here are some properties of a pseudo-Anosov homeomorphism $f\colon M \to M$:

(pA 1) The stable and unstable foliations of f are unique up to multiplication of the measures by positive constants.

(pA 2) $h(f) = \log \lambda$.

(pA 3) If M has genus g, and n_p is the number of singular points and boundary components at which the invariant foliations of f have p prongs, then

$$\sum_{p=1}^{\infty} n_p(2-p) = 4(1-g).$$

(pA 4) The periodic points of f are dense in M.

$$\int_C f(z)\,dz = 2\pi i \sum_{\text{poles } z} R_z.$$

$$\begin{array}{ccc} H^p(X) \times H_{p+q}(X, \widehat{A}) & \xrightarrow{\cap} & H_q(X, \widehat{A}) \\ {\scriptstyle \mathrm{id}\times}\Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial} \\ H^p(X) \times H_{p+q-1}(\widehat{A}) & \xrightarrow{\cap} & H_{q-1}(\widehat{A}) \end{array}$$

2 Graphics

Figure 1 is my first example.

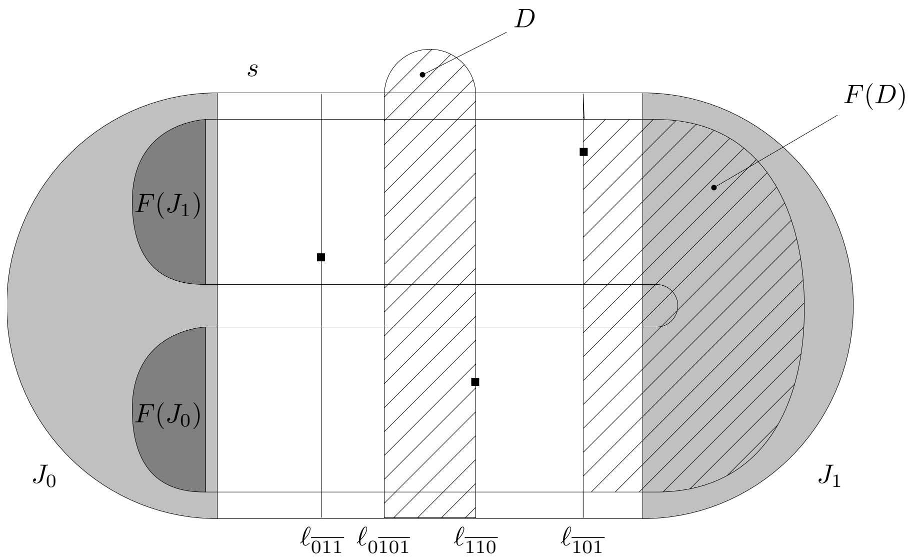

Figure 1: A 1-pruning disk in the horseshoe

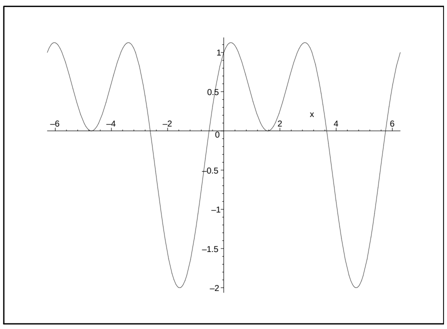

Figure 2: The graph of $y = \sin\, x + \cos\, 2x$

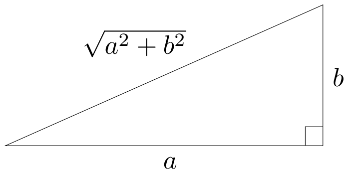

Figure 3: Pythagoras's theorem

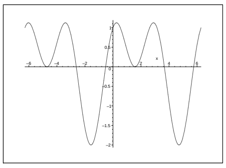

Figure 4: $\sin x + \cos 2x$

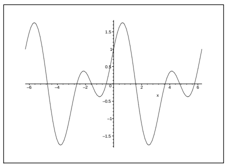

Figure 5: $\sin 2x + \cos x$

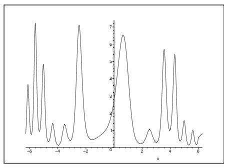

Figure 6: $e^{\sin 2x + \cos x^2}$

3 The End

Theorem 3.1 (Artin, 1925) *The group $\pi_1 B_{0,n}$ admits a presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ and defining relations:*

$$\begin{array}{rcll} \sigma_i \sigma_j & = & \sigma_j \sigma_i & \qquad |\, i - j \,| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1} & \qquad 1 \leq i \leq n-2. \end{array}$$

Proof. (The proof given here is due to Fadell and Van Buskirk, 1962). Let B_n be the abstract group with the presentation of Theorem 3.1. Until we have established the isomorphism between B_n and $\pi_1 B_{0,n}$ we will use the symbols $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n-1}$ for elements of $\pi_1 B_{0,n}$, with $\imath \colon B_n \to \pi_1 B_{0,n}$ defined by $\imath(\sigma_i) = \tilde{\sigma}_i$ for $1 \leq i \leq n-1$. The elements $\tilde{\sigma}_i$ have already been defined pictorially: we now give an equivalent definition which is more precise. Recall the covering projection $p \colon F_{0,n} \to B_{0,n}$. Choose the point $p\left((1,0), \ldots, (n,0)\right) = \tilde{z}^0$ as base point for the group $\pi_1 B_{0,n}$. Lift loops based at $\tilde{z}^0$ in $B_{0,n}$ to paths in $F_{0,n}$ with initial point $((1,0), \ldots, (n,0))$. Then the generator $\tilde{\sigma}_i \in \pi_1 B_{0,n}$ is represented by the path $\mathcal{F}(t)$ in $F_{0,n}$ given by

$$\mathcal{F}(t) = \left((1,0), \ldots, (i-1,0), \mathcal{F}_i(t), \mathcal{F}_{i+1}(t), (i+2,0), \ldots, (n,0)\right),$$

where $\mathcal{F}_i(t) = (i+t, -\sqrt{t-t^2})$ and $\mathcal{F}_{i+1}(t) = (i+1-t, \sqrt{t-t^2})$. That is, $\mathcal{F}(t)$ is constant on all but the ith and $i+1$st strings, and interchanges those two in a nice way.

The proof of Theorem 3.1 will be by induction on n, and will exploit the relationship already developed between $\pi_1 B_{0,n}$ and $\pi_1 F_{0,n}$. Let

$$\tilde{\nu} \colon \pi_1 \left(B_{0,n}, \tilde{z}^0\right) \to \Sigma_n$$

be defined as follows: Let $\tilde{\alpha} \in \pi_1 B_{0,n}$ be represented by a loop

$$\tilde{g} \colon \ (I, \{0,1\}) \to \left(B_{0,n}, \tilde{z}^0\right)$$

and let $g = (g_1, \ldots, g_n) \colon (I, \{0\}) \to (F_{0,n}, z^0)$ be the unique lift of $\tilde{g}$. Define

$$\tilde{\nu}(\alpha) = \begin{pmatrix} g_1(0), \ldots, g_n(0) \\ g_1(1), \ldots, g_n(1) \end{pmatrix} \in \Sigma_n.$$

The kernel of the homomorphism $\tilde{\nu}$ is the pure braid group $\pi_1 F_{0,n}$. Corresponding to the homomorphism $\tilde{\nu}$ is the homomorphism

$$\nu \colon B_n \to \Sigma_n$$

from the abstract group B_n to the symmetric group Σ_n on n letters defined by

$$\nu(\sigma_i) = (i, i+1) \qquad 1 \leq i \leq n-1.$$

Let $P_n = \ker \nu$. The proof can be completed as a consequence of the following lemma:

Lemma 3.2 *The homomorphism* $\imath \colon B_n \to \pi_1 B_{0,n}$ *is an isomorphism onto if* $\imath|_{P_n}$ *is an isomorphism onto* $\pi_1 F_{0,n}$.

Proof. *(Lemma 3.2).* The homomorphism ν is clearly surjective, since the transpositions $\{\nu(\sigma_i)\}$ generate Σ_n. Hence we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_n & \longrightarrow & B_n & \xrightarrow{\nu} & \Sigma_n & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \imath_n = \imath|_{P_n}} & & \downarrow{\scriptstyle \imath} & & \downarrow{\scriptstyle 1} & & \downarrow \\ 1 & \longrightarrow & \pi_1 F_{0,n} & \longrightarrow & \pi_1 B_{0,n} & \xrightarrow[\tilde{\nu}]{} & \Sigma_n & \longrightarrow & 1 \end{array}$$

with exact rows. Applying the Five Lemma, we obtain the desired result. This completes the proof of Lemma 3.2. ∎

Theorem 3.1 now follows. ∎