Chapter 3

Fourier Series

3.1 Introduction

Fourier series provide a way of representing *periodic* functions $f : \mathbb{R} \to \mathbb{R}$ as infinite sums of trigonometric functions, in the form

$$f(t) = \frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rt + b_r \sin rt)\,.$$

For example, let $f(t) = t^2$ for $-\pi \leq t \leq \pi$, and extend $f(t)$ periodically to all values of t (Figure 3.1). We shall show that

$$f(t) = \frac{\pi^2}{3} + \sum_{r=1}^{\infty} \frac{4(-1)^r}{r^2} \cos rt.$$

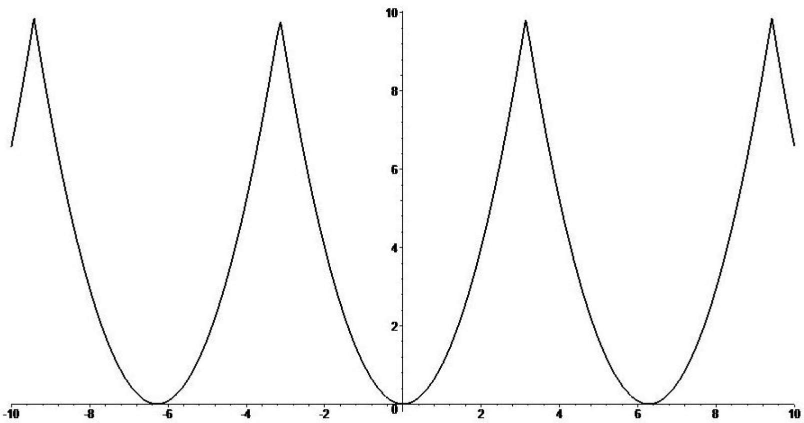

Figure 3.1: The periodic extension of t^2

This is a very important technique in a wide range of applications in physics and engineering, where a periodic signal (e.g. an acoustic wave) can be broken down into its constituent frequencies. (It is because of this that the variable used in Fourier series expansions is traditionally t, representing time, rather than x.) Fourier series also have applications in many branches of mathematics, notably in number theory and the theory of differential equations. Indeed, it was in his attempt to solve the heat equation (a partial differential equation) that Fourier introduced these series in the early 19$^{\text{th}}$ century.

Since trigonometric functions are periodic, they can only be used to represent periodic functions: that is, ones whose graphs repeat themselves over and over. A function is said to be T-periodic if the period of repetition is T: that is, shifting the graph of the function T units to the right leaves it unchanged.

Definitions 3.1 (Periodic function, Periodic extension)
Let $f : \mathbb{R} \to \mathbb{R}$ and $T > 0$. The function f is said to be *T-periodic* if $f(t + T) = f(t)$ for all $t \in \mathbb{R}$.

If $F : [a, b) \to \mathbb{R}$ is a function defined on an interval of length T (i.e. $b - a = T$), its *T-periodic extension* is the unique T-periodic function $f : \mathbb{R} \to \mathbb{R}$ which agrees with F on the interval $[a, b)$. (That is, f is obtained by repeating the graph of F in $[a, b)$ over and over.)

In almost all of this chapter we restrict to the case $T = 2\pi$ (the period of cos and sin), though we will briefly consider functions with other periods in Section 3.6.

A remarkable feature of Fourier series is the wide range of periodic functions f which have Fourier series expansions: even many discontinuous functions can be so represented. This should be contrasted with Maclaurin series expansions

$$f(x) = \sum_{r=0}^{\infty} a_r x^r,$$

which only exist for *analytic* functions $f(x)$. The main theoretical part of this chapter is Section 3.4, in which we discuss what properties a function needs to have in order to be represented by a Fourier series, and exactly what it means for the Fourier series to "represent" the function.

3.2 Revision of trigonometric functions and integration by parts

This section provides a brief review of some trigonometric identities which will be needed, and of the technique of integration by parts which is essential for the calculation of Fourier series expansions.

3.2.1 Trigonometric identities and orthogonality relations

Recall the trigonometric identities:

$$\begin{aligned}
\sin(a+b) &= \sin a\cos b+\cos a\sin b\\
\sin(a-b) &= \sin a\cos b-\cos a\sin b\\
\cos(a+b) &= \cos a\cos b-\sin a\sin b\\
\cos(a-b) &= \cos a\cos b+\sin a\sin b\\
\sin a\sin b &= (\cos(a-b)-\cos(a+b))/2\\
\sin a\cos b &= (\sin(a+b)+\sin(a-b))/2\\
\cos a\cos b &= (\cos(a+b)+\cos(a-b))/2\\
\sin^2 a &= (1-\cos(2a))/2\\
\cos^2 a &= (1+\cos(2a))/2.
\end{aligned}$$

Recall also that, for any integer n,

$$\begin{aligned}
\sin(n\pi) &= 0 \qquad \text{and}\\
\cos(n\pi) &= (-1)^n.
\end{aligned}$$

Of course $(-1)^n$ is equal to $+1$ if n is even, and to -1 if n is odd.

These can be used to prove Theorem 3.1 below, which will be very important in this chapter. We use the following notation:

Definition 3.2 (Kronecker delta)

Given two integers q and r, define δ_{qr}, the *Kronecker delta* by

$$\delta_{qr}=\begin{cases}1 & \text{if } q=r\\ 0 & \text{if } q\neq r\end{cases}$$

Theorem 3.1 (Orthogonality relations) *Let q and r be positive integers. Then*

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi}\cos qt\cos rt\,dt &= \delta_{qr},\\
\frac{1}{\pi}\int_{-\pi}^{\pi}\sin qt\sin rt\,dt &= \delta_{qr},\\
\frac{1}{\pi}\int_{-\pi}^{\pi}\sin qt\cos rt\,dt &= 0,\\
\frac{1}{\pi}\int_{-\pi}^{\pi}1\cdot\cos rt\,dt &= 0,\\
\frac{1}{\pi}\int_{-\pi}^{\pi}1\cdot\sin rt\,dt &= 0, \qquad \textit{and}\\
\frac{1}{\pi}\int_{-\pi}^{\pi}1\cdot 1\,dt &= 2.
\end{aligned}$$

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This result should be interpreted as follows: we regard the functions

$$1,\ \cos t,\ \cos 2t,\ \cos 3t, \ldots,\ \sin t,\ \sin 2t,\ \sin 3t, \ldots$$

as *basis functions* for the space of (suitable) 2π-periodic functions, just as $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are basis vectors for $\mathbb{R}^3$. The equivalent for 2π-periodic functions of the scalar (dot) product of two vectors is

$$< f, g > = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt.$$

Then Theorem 3.1 says that our basis functions are all orthogonal (integrating the product of two different ones gives zero), just as $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are orthogonal in $\mathbb{R}^3$ (the dot product of any two of them is zero).

3.2.2 Integration by parts

Recall the method of *integration by parts.* If u and v are functions of t, then

$$\int u\,\frac{dv}{dt}\;dt = uv - \int \frac{du}{dt}\,v\;dt.$$

We usually use this technique for integration of a product

$$\int f(t)g(t)\,dt,$$

where the derivative of $f(t)$ is "simpler" than $f(t)$, while the integral of $g(t)$ is "no more complicated" than $g(t)$. Then setting $u = f(t)$ and $\frac{dv}{dt} = g(t)$, we can evaluate the original integral provided we can integrate

$$\int \frac{du}{dt}\,v\;dt,$$

which is a "simpler" integral than the one we started with.

Example 3.3 (Integration by parts)
Evaluate

$$\int_{-\pi}^{\pi} t^2\cos rt\,dt \qquad (r \in \mathbb{Z}).$$

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3.3 The Fourier coefficients of a 2π-periodic function

In this section we consider a 2π-periodic function $f(t)$, and *assume* that we can write it in the form

$$f(t) = \frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r\cos rt + b_r\sin rt)$$

(we will see the conditions under which this is possible in Section 3.4 below).

The following theorem provides a recipe for calculating the coefficients a_r and b_r.

Theorem 3.2 (Fourier coefficients theorem) *Suppose that*

$$f(t) = \frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r \cos rt + b_r \sin rt).$$

Then

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt. \\ a_r &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos rt\,dt \qquad (r \geq 1). \\ b_r &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin rt\,dt. \end{aligned}$$

†50

Remark 3.4
In the proof of Theorem 3.2 we assumed that we could swap the order of summation and integration.

Definition 3.5 (Fourier series expansion)
If f is a 2π-periodic function, then we say that

$$\frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r \cos rt + b_r \sin rt)$$

is the *Fourier series expansion of* f, where the coefficients a_r and b_r are as given by Theorem 3.2 (provided the integrals in question exist). We write

$$f(t) \sim \frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r \cos rt + b_r \sin rt).$$

(We use the symbol $\sim$ rather than $=$ unless we are certain that $f(t)$ is actually *equal* to its Fourier series expansion for all $t \in \mathbb{R}$.)

Example 3.6 (Fourier coefficients)
Let f be the 2π-periodic extension of t^2 $(-\pi \leq t < \pi)$. Show that its Fourier series expansion is given by

$$f(t) \sim \frac{\pi^2}{3} + \sum_{r=1}^{\infty}\frac{4(-1)^r}{r^2}\cos rt.$$

Show that, if $\sim$ can be replaced by $=$ (which we will see later that it can), then putting $t = 0$ and $t = \pi$ give

$$\begin{aligned} \frac{\pi^2}{12} &= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \qquad \text{and} \\ \frac{\pi^2}{6} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \end{aligned}$$

respectively. †51

Example 3.7 (Fourier coefficients)
Let f be the 2π-periodic extension of the function $F : [-\pi, \pi) \to \mathbb{R}$ defined by

$$F(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Calculate the Fourier series expansion of f. What is its value when $t = 0$? †52

3.4 The Fourier series theorem

In this section, we investigate the conditions on the periodic function f under which the Fourier series expansion

$$f(t) \sim \frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rt + b_r \sin rt)$$

"makes sense", and the (stronger) conditions under which the $\sim$ can be replaced with $=$. A summary of the results is:

- We will require f to be *piecewise differentiable.* This term is defined precisely below, but roughly speaking it means that the derivative $f'(t)$ exists except at finitely many points in $[-\pi, \pi]$, and at those points it doesn't behave too badly. In particular, this means that f is continuous except at finitely many points in $[-\pi, \pi]$.

- Under these conditions, the Fourier series expansion of f converges, for each fixed value t^* where f is continuous, to $f(t^*)$: that is

$$f(t^*) = \frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rt^* + b_r \sin rt^*) \qquad \text{if } f \text{ is continuous at } t^*.$$

- In particular, if f is a *continuous* function, then the $\sim$ can be replaced with $=$.

- If f is *not* continuous at t^*, then the Fourier series expansion converges to the average of the limits of $f(t)$ as t approaches t^* from above and below:

$$\frac{f(t^*-) + f(t^*+)}{2} = \frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r \cos rt^* + b_r \sin rt^*).$$

For example, if f is the 2π-periodic extension of t $(-\pi \le t < \pi)$ (Figure 3.2), then f is not continuous at $t^* = \pi$, and $f(\pi-) = \pi$, $f(\pi+) = -\pi$. Hence the Fourier series expansion converges to $\frac{f(\pi-)+f(\pi+)}{2} = \frac{\pi-\pi}{2} = 0$ at $t = \pi$.

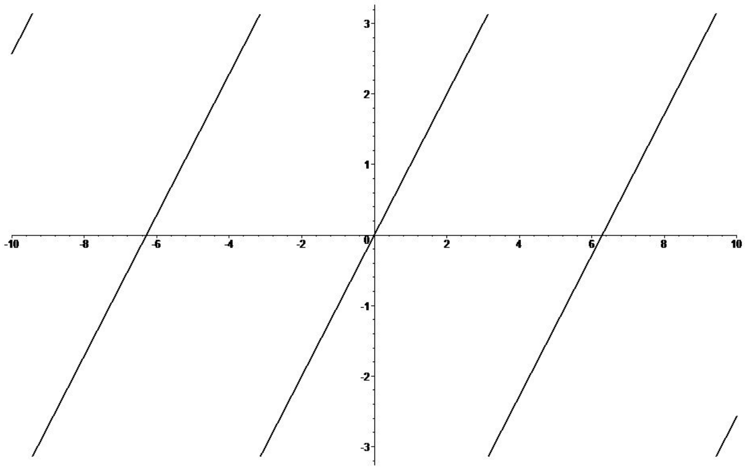

Figure 3.2: The periodic extension of t

- On any interval $[a, b]$ where f is continuous, the Fourier series expansion converges *uniformly* to f. This means that when n is large, the *whole graph* of f is close to the *whole graph* of

$$\frac{a_0}{2} + \sum_{r=1}^{n}(a_r \cos rt + b_r \sin rt)$$

in the range $a \le t \le b$.

- On the other hand, if f is *not* continuous at t^*, then the closer t is to t^*, the larger n needs to be in order for

$$\frac{a_0}{2} + \sum_{r=1}^{n}(a_r \cos rt + b_r \sin rt)$$

to be within some prescribed error $\epsilon > 0$ of $f(t)$. That is, there is no n for which it is within ϵ of $f(t)$ for *all* t.

3.4.1 Piecewise continuous and piecewise differentiable functions

A function $f : [a, b] \to \mathbb{R}$ is piecewise continuous if, roughly speaking, it is made up of a finite number of continuous pieces.

Definition 3.8 (Piecewise continuous)

A function $f : [a, b] \to \mathbb{R}$ is *piecewise continuous* if there are numbers $t_0, t_1, \ldots, t_n$ with $a = t_0 < t_1 < \cdots < t_n = b$, such that f is continuous on each of the intervals (t_i, t_{i+1}), and tends to a finite value at each endpoint of these intervals. That is, the limits

$$\begin{aligned} f(t_i+) &= \lim_{t \searrow t_i} f(t) \quad \text{and} \\ f(t_{i+1}-) &= \lim_{t \nearrow t_{i+1}} f(t) \end{aligned}$$

exist (and are finite).

A function $f : \mathbb{R} \to \mathbb{R}$ is piecewise continuous if it is piecewise continuous on every closed interval $[a, b]$. Thus it can have infinitely many discontinuities, but only finitely many on any finite interval.

Examples 3.9 (Piecewise continuous)

a) The function $f : \mathbb{R} \to \mathbb{R}$ given by the 2π-periodic extension of t $(-\pi \le t < \pi)$ (Figure 3.2) is piecewise continuous. We have $f(\pi-) = \pi$ and $f(\pi+) = -\pi$ (and similarly $f((2n+1)\pi-) = \pi$, $f((2n+1)\pi+) = \pi$ for all integers n).

b) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = \tan t$ (and $f(t) = 0$ if t is an odd multiple of $\pi/2$) is not piecewise continuous. Although it has only finitely many discontinuities on any finite interval, the function "blows up" at these discontinuities, so that (for example) the limits $f(\frac{\pi}{2}-)$ and $f(\frac{\pi}{2}+)$ do not exist.

Similarly, a function $f : [a, b] \to \mathbb{R}$ is piecewise differentiable if it is made up of a finite number of differentiable pieces.

Definition 3.10 (Piecewise differentiable)

A function $f : [a, b] \to \mathbb{R}$ is *piecewise differentiable* if there are numbers $t_0, t_1, \ldots, t_n$ with $a = t_0 < t_1 < \cdots < t_n = b$, such that f is differentiable on each of the intervals (t_i, t_{i+1}), and both $f(t)$ and $f'(t)$ tend to a finite value at each endpoint of these intervals. That is, the limits

$$\begin{aligned} f(t_i+) &= \lim_{t \searrow t_i} f(t), \\ f(t_{i+1}-) &= \lim_{t \nearrow t_{i+1}} f(t), \\ f'(t_i+) &= \lim_{t \searrow t_i} f'(t) \quad \text{and} \\ f'(t_{i+1}-) &= \lim_{t \nearrow t_{i+1}} f'(t) \end{aligned}$$

exist (and are finite).

A function $f : \mathbb{R} \to \mathbb{R}$ is piecewise differentiable if it is piecewise differentiable on every closed interval $[a, b]$.

Notice that a piecewise differentiable function is necessarily piecewise continuous.

Example 3.11 (Piecewise differentiable)
The function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(t) = |t|$ (Figure 3.3) is piecewise differentiable. Although it doesn't have a derivative at $t = 0$, we can take $-\pi = t_0 < t_1 = 0 < t_2 = \pi$, and f is differentiable on $(t_0, t_1) = (-\pi, 0)$, and on $(t_1, t_2) = (0, \pi)$. We have $f'(0-) = -1$ and $f'(0+) = +1$.

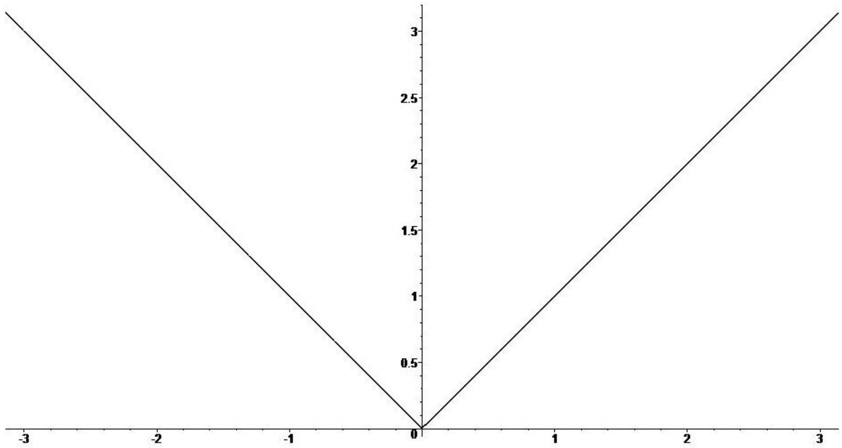

Figure 3.3: $f(t) = |t|$ is piecewise differentiable on $[-\pi, \pi]$

3.4.2 Pointwise and uniform convergence

Recall (Section 1.5) what it means for a series $\sum_{r\geq 1} x_r$ of *numbers* to *converge*: we define partial sums $S_n = \sum_{r=1}^{n} x_r$ for each n, and say that the series converges (to a limit ℓ) if $S_n \to \ell$ as $n \to \infty$.

Similarly, given a Fourier series expansion

$$\frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r \cos rt + b_r \sin rt),$$

we can define partial sums

$$S_n(t) = \frac{a_0}{2} + \sum_{r=1}^{n}(a_r \cos rt + b_r \sin rt),$$

and say that the Fourier series expansion converges to $f(t)$ if $S_n(t)$ converges to $f(t)$ as $n \to \infty$.

However there are two distinct concepts, depending on whether this convergence takes place over the whole range of t *simultaneously*, or only at each fixed value t^* *separately*.

Definitions 3.12 (Pointwise and uniform convergence)
Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of functions, and $f : [a, b] \to \mathbb{R}$.

a) We say that $f_n \to f$ *pointwise* if, for each fixed $t^* \in [a, b]$, the sequence of *numbers* $(f_n(t^*))$ converges to the number $f(t^*)$.

b) We say that $f_n \to f$ *uniformly* if for any $\epsilon > 0$ there is some N such that, if $n \geq N$,

$$|f_n(t) - f(t)| < \epsilon \qquad \textit{for all } t \in [a, b].$$

A good way to think about uniform convergence is this: if $f_n \to f$ uniformly and $\epsilon > 0$, then from some point onwards in the sequence, the graphs of all of the f_n are contained in an ϵ-snake about the graph of f (Figure 3.4).

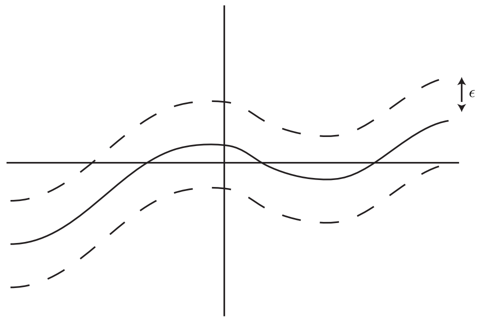

Figure 3.4: When n is large, all of the f_n are in an ϵ-snake about f

Example 3.13 (Limitations of pointwise convergence)
Let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(t) = t^n$, and $f : [0, 1] \to \mathbb{R}$ be given by

$$f(t) = \begin{cases} 0 & \text{if } t < 1, \\ 1 & \text{if } t = 1. \end{cases}$$

Then $f_n \to f$ pointwise. For if we fix any $t^* \in [0, 1]$, then

- If $t^* < 1$, then $f_n(t^*) = (t^*)^n \to 0 = f(t^*)$ as $n \to \infty$.
- If $t^* = 1$, then $f_n(t^*) = 1^n = 1 \to 1 = f(t^*)$ as $n \to \infty$.

Clearly, however, none of the functions f_n looks anything like f. In particular, f isn't even continuous, although all of the f_n are.

These sorts of problems don't arise with uniform convergence, which requires the whole graphs of the f_n to get closer and closer to the whole graph of f. It can be proved, for example, that if $f_n \to f$ uniformly, and all of the f_n are continuous, then f is also continuous.

3.4.3 The Fourier theorem

Theorem 3.3 (Fourier series theorem) *Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and piecewise differentiable. Then*

a) The Fourier series expansion of f converges pointwise to

$$g(t) = \frac{f(t-) + f(t+)}{2}.$$

b) On any interval $[a, b]$ on which f is continuous, the Fourier series expansion of f converges uniformly to f.

Remark 3.14
With regard to a), notice that at any point t^* where f is continuous, $f(t^*-) = f(t^*+) = f(t^*)$, and hence $g(t^*) = f(t^*)$. Thus the Fourier series expansion of f converges to $f(t^*)$.

Example 3.15 (Fourier series theorem)
Calculate the Fourier series expansion of the 2π-periodic extension of t $(-\pi \le t < \pi)$, and check statement a) at $t = \pi$. †53

Figures 3.5 to 3.8 show the graph of $f(t)$, the periodic extension of t, together with the graph of the first 5, 20, 50, and 200 terms of its Fourier series expansion. You can see that away from the discontinuities of f, the Fourier series expansion with 200 terms is indistinguishable from f itself. However, near the discontinuities there is a definite error which does not decrease as we increase the number of terms: what does happen as the number of terms increases is that the region in which this error occurs gets squashed into a smaller and smaller neighbourhood of the discontinuities.

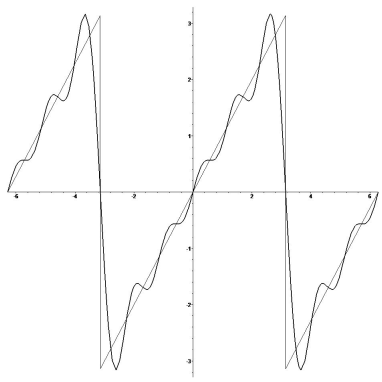

Figure 3.5: 5 terms of the Fourier series expansion of t

3.4.4 Proof of pointwise convergence (non-examinable)

The proof of the Fourier series theorem is hard, and is not part of the examinable material for this module. However, it is also instructive, and in this section the proof

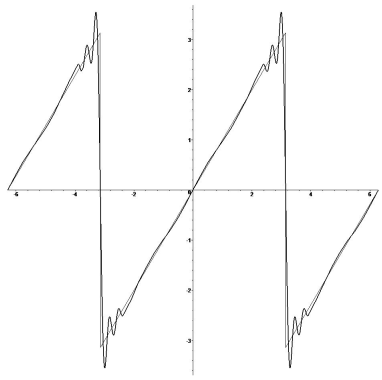

Figure 3.6: 20 terms of the Fourier series expansion of t

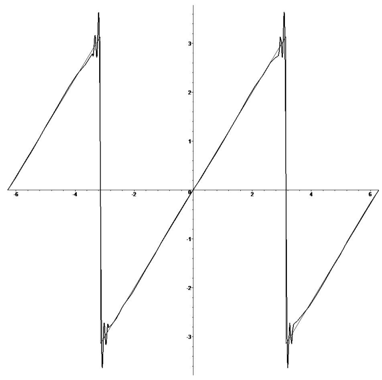

Figure 3.7: 50 terms of the Fourier series expansion of t

of part a) of the theorem, that the Fourier series expansion of a piecewise differentiable periodic function f converges pointwise to $\frac{f(t-)+f(t+)}{2}$, is given.

In fact the proof depends on one result that we will not be able to prove, since it depends on some more advanced properties of integration. It is:

Theorem 3.4 (Riemann-Lebesgue lemma) *Let $g : [a, b] \to \mathbb{R}$ be piecewise continuous. Then*

$$\int_a^b g(u)\sin(\lambda u)\, du \to 0 \qquad \text{as } \lambda \to \infty.$$

You can see intuitively why this result should be true, though. Figure 3.9 shows the graph of a function $g : [0, 1] \to \mathbb{R}$ (with a dashed line), the graph of $-g$ (also with a dashed line), and the graph of $g(u)\sin(100u)$ (with a solid line). You can see that when you integrate $g(u)\sin(100u)$ the areas above and below the axis nearly cancel out, and the result is close to 0. As you increase 100 to larger and larger numbers, the graph oscillates faster and faster, and the areas cancel more and more closely.

Before embarking on the proof of pointwise convergence, let's recap what it is that we're trying to prove. We have a piecewise differentiable 2π-periodic function

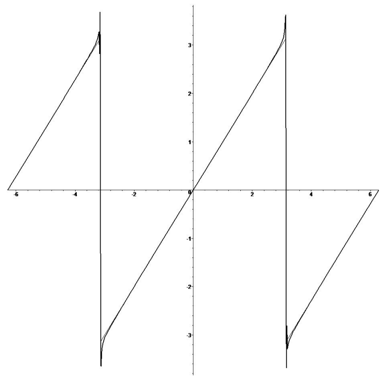

Figure 3.8: 200 terms of the Fourier series expansion of t

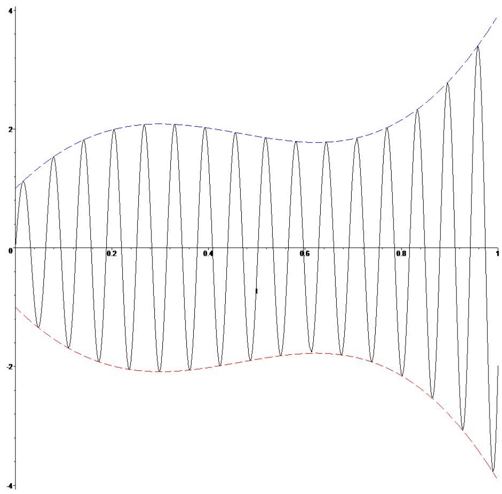

Figure 3.9: Idea of the Riemann-Lebesgue lemma

$f : \mathbb{R} \to \mathbb{R}$, and have defined its Fourier coefficients a_r $(r \geq 0)$ and b_r $(r \geq 1)$ by

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt, \\
a_r &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos rt\,dt \qquad \text{for } r \geq 1, \text{ and} \\
b_r &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin rt\,dt.
\end{aligned}$$

Let $S_n(t)$ be the n^{th} partial sum of the Fourier series expansion of $f(t)$, i.e.

$$S_n(t) = \frac{a_0}{2} + \sum_{r=1}^{n}(a_r \cos rt + b_r \sin rt).$$

We would like to show:

For each fixed value of t, we have

$$S_n(t) \to \frac{f(t+) + f(t-)}{2} \qquad \text{as } n \to \infty.$$

It is also true that the convergence is uniform away from the discontinuities of $f(t)$, but we shall not prove this.

The first step of the proof is to write $S_n(t)$ in terms of another function $\sigma_n(t)$, which we can regard as the nth partial sum of a Fourier series expansion in which all of the a_r are equal to 1, and all of the b_r are zero. That is,

$$\sigma_n(t) = \frac{1}{2} + \sum_{r=1}^{n} \cos rt.$$

Notice that this is a 2π-periodic function; that $\sigma_n(t) = \sigma_n(-t)$ for all t (since $\cos(-rt) = \cos rt$), and that

$$\int_{-\pi}^{\pi} \sigma_n(t) = \pi$$

(since the integral of each $\cos rt$ from $-\pi$ to π is zero).

In fact, we can write down an explicit expression for $\sigma_n(t)$:

Lemma 3.5

$$\sigma_n(t) = \frac{\sin((n+\frac{1}{2})t)}{2\sin\frac{t}{2}}.$$

Proof.

$$\begin{aligned}
\sigma_n(t) &= 1/2 + \cos t + \cos 2t + \cos 3t + \cdots + \cos nt \\
&= \Re(1/2 + e^{it} + e^{2it} + e^{3it} + \cdots + e^{nit}) \quad (\text{since } e^{i\theta} = \cos\theta + i\sin\theta) \\
&= \Re\left(\frac{1}{2} + e^{it}\left(\frac{1-e^{int}}{1-e^{it}}\right)\right) \quad (\text{sum of a geometric progression}) \\
&= \Re\left(\frac{1}{2} + \frac{e^{it}(1-e^{-it})(1-e^{int})}{2-e^{it}-e^{-it}}\right) \quad (\text{multiplying top and bottom by } 1-e^{-it}) \\
&= \frac{1}{2}\Re\left(1 + \frac{(e^{it}-1)(1-e^{int})}{1-\cos t}\right) \\
&= \frac{1}{2}\left(1 + \frac{-1+\cos t+\cos nt-\cos((n+1)t)}{1-\cos t}\right) \\
&= \frac{1}{2}\left(1 + \frac{-2\sin^2\frac{t}{2} + 2\sin\frac{t}{2}\sin((n+\frac{1}{2})t)}{2\sin^2\frac{t}{2}}\right) \\
&= \frac{\sin((n+\frac{1}{2})t)}{2\sin\frac{t}{2}}.
\end{aligned}$$

∎

Now let us see how $S_n(t)$ can be written in terms of $\sigma_n(t)$:

Lemma 3.6 *For each fixed value of t, we have*

$$S_n(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t+u)+f(t-u)}{2}\sigma_n(u)\,du.$$

Proof. Substituting the formulae for the Fourier coefficients into the expression for $S_n(t)$ gives:

$$\begin{aligned}
\pi S_n(t) &= \frac{1}{2}\int_{-\pi}^{\pi} f(u)\,du + \sum_{r=1}^{n}\left(\left(\int_{-\pi}^{\pi} f(u)\cos ru\,du\right)\cos rt + \left(\int_{-\pi}^{\pi} f(u)\sin ru\,du\right)\sin rt\right) \\
&= \int_{-\pi}^{\pi}\left[f(u)\left(\frac{1}{2} + \sum_{r=1}^{n}(\cos ru\cos rt + \sin ru\sin rt)\right)\right] du \\
&= \int_{-\pi}^{\pi} f(u)\left[\frac{1}{2} + \sum_{r=1}^{n}\cos r(t-u)\right] du.
\end{aligned}$$

We now carry out the change of variable $v = t - u$ in this integral, to give

$$\begin{aligned}
\pi S_n(t) &= -\int_{t+\pi}^{t-\pi} f(t-v)\left[\frac{1}{2} + \sum_{r=1}^{n}\cos rv\right] dv \\
&= \int_{t-\pi}^{t+\pi} f(t-v)\sigma_n(v)\,dv \\
&= \int_{-\pi}^{\pi} f(t-v)\sigma_n(v)\,dv
\end{aligned}$$

(since the functions are 2π-periodic, so it doesn't matter which interval of length 2π we integrate over).

Making the change of variable $w = -v$ in this integral gives us

$$\pi S_n(t) = \int_{-\pi}^{\pi} f(t+w)\sigma_n(w)\,dw$$

(since $\sigma_n(w) = \sigma_n(-w)$). Changing the dummy integration variable to u in both of these expressions for $\pi S_n(t)$, adding them, and dividing by 2, gives

$$S_n(t) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(t+u)+f(t-u)}{2}\sigma_n(u)\,du$$

as required. ∎

Now we can substitute in the expression obtained for $\sigma_n(u)$ in Lemma 3.5 to give:

$$S_n(t) = \int_{-\pi}^{\pi}\frac{f(t+u)+f(t-u)}{4\pi\sin(u/2)}\sin\left(\left(n+\frac{1}{2}\right)u\right) du.$$

Recall that our aim is to show that $S_n(t)$ tends to the average of $f(t+)$ and $f(t-)$ as $n \to \infty$. Equivalently, if we subtract this average from $S_n(t)$, we would like to show that

$$\begin{aligned} S_n(t) - \frac{f(t+)+f(t-)}{2} &= S_n(t) - \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t+)+f(t-)}{2}\sigma_n(u)\,du \\ &= \int_{-\pi}^{\pi} \frac{f(t+u)+f(t-u)-f(t+)-f(t-)}{4\pi\sin(u/2)} \sin\left(\left(n+\frac{1}{2}\right)u\right)\,du \end{aligned}$$

tends to 0 as $n \to \infty$.

Lemma 3.7 *For each fixed value of t, the function*

$$g(u) = \frac{f(t+u)+f(t-u)-f(t+)-f(t-)}{4\pi\sin(u/2)}$$

is piecewise continuous on $[-\pi, \pi]$

Proof.

Since $f(t+)$ and $f(t-)$ are just numbers, and $f(u)$ is itself piecewise continuous, it is clear that the numerator is piecewise continuous. The same is therefore true for the whole of $g(u)$ except possibly at points where the denominator vanishes: the only such point in $[-\pi, \pi]$ is $u = 0$. We therefore only have to check that the left and right limits of $g(u)$ exist at $u = 0$. Now

$$\begin{aligned} \lim_{u\to 0+} g(u) &= \lim_{u\to 0+} \frac{f(t+u)-f(t+)+f(t-u)-f(t-)}{4\pi\sin(u/2)} \\ &= \lim_{u\to 0+} \frac{f(t+u)-f(t+)+f(t-u)-f(t-)}{u} \lim_{u\to 0+} \frac{u}{4\pi\sin(u/2)} \\ &= (f'(t+)+f'(t-))/2\pi. \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{u\to 0-} g(u) &= \lim_{u\to 0-} \frac{f(t+u)-f(t-)+f(t-u)-f(t+)}{4\pi\sin(u/2)} \\ &= \lim_{u\to 0-} \frac{f(t+u)-f(t-)+f(t-u)-f(t+)}{u} \lim_{u\to 0-} \frac{u}{4\pi\sin(u/2)} \\ &= (f'(t-)+f'(t+))/2\pi. \end{aligned}$$

∎

This completes the proof of the theorem. For by the Riemann-Lebesgue lemma, it follows that

$$S_n(t) - \frac{f(t+)+f(t-)}{2} = \int_{-\pi}^{\pi} \frac{f(t+u)+f(t-u)-f(t+)-f(t-)}{4\pi\sin(u/2)} \sin\left(\left(n+\frac{1}{2}\right)u\right)\,du$$

tends to zero as $n \to \infty$, so that

$$S_n(t) \to \frac{f(t+)+f(t-)}{2} \qquad \text{as } n \to \infty.$$

3.5 Fourier series of even and odd functions

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *even* if $f(t) = f(-t)$ for all $t \in \mathbb{R}$; and is *odd* if $f(t) = -f(-t)$ for all $t \in \mathbb{R}$.

These definitions reflect symmetries of the graph of f. If f is even, then its graph in $t \leq 0$ can be obtained from the graph in $t \geq 0$ by reflecting in the x-axis; and if f is odd, then its graph in $t \leq 0$ can be obtained from the graph in $t \geq 0$ by rotating through 180 degrees in the origin (see Figure 3.10). (In particular, if f is odd, then $f(0) = 0$.)

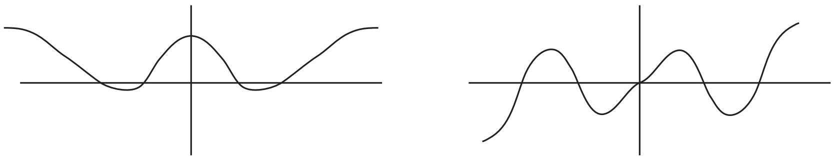

Figure 3.10: Graphs of even and odd functions

Even functions include polynomials with only even powers and $\cos rt$ for all r; odd functions include polynomials with only odd powers and $\sin rt$ for all r.

If f is either even or odd, then the process of computing its Fourier coefficients is simplified.

Theorem 3.8 (Fourier coefficients of even and odd functions) *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic piecewise differentiable function.*

a) If f is even then $b_r = 0$ for all r, and

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt, \\ a_r &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos rt\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\cos rt\,dt \qquad (r \geq 1). \end{aligned}$$

b) If f is odd then $a_r = 0$ for all r, and

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin rt\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\sin rt\,dt.$$

†54

Remark 3.16
The fact that all of the coefficients a_r or b_r vanish clearly reduces the number of calculations which have to be carried out. In some cases, the alternative forms of the non-vanishing coefficients may also be easier to work with.

Example 3.17 (Fourier coefficients of odd function)
Let f be the periodic extension of the function $F : [-\pi, \pi) \to \mathbb{R}$ given by

$$F(t) = \begin{cases} -1 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

Compute the Fourier series expansion of f. †55

Remark 3.18
Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Then $g(t) = \frac{f(t)+f(-t)}{2}$ is an even function, $h(t) = \frac{f(t)-f(-t)}{2}$ is an odd function, and $f(t) = g(t) + h(t)$.

That is, *any function can be written as the sum of an even function and an odd function.* For example, the function e^t is neither even nor odd, but we can write

$$e^t = \frac{e^t + e^{-t}}{2} + \frac{e^t - e^{-t}}{2} = \cosh t + \sinh t,$$

where $\cosh t$ is an even function and $\sinh t$ is an odd function.

3.6 Periodicity other than 2π

If $f : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $T \neq 2\pi$ (that is, if $f(t+T) = f(t)$ for all t), then it can be expanded as a Fourier series in a manner exactly analogous to the 2π-periodic case. The basic T-periodic functions are $\sin\left(\frac{2\pi r}{T}\,t\right)$ and $\cos\left(\frac{2\pi r}{T}\,t\right)$, since

$$\sin\left(\frac{2\pi r}{T}(t+T)\right) = \sin\left(\frac{2\pi r}{T}\,t + 2\pi r\right) = \sin\left(\frac{2\pi r}{T}\,t\right),$$

and similarly for the cosine case.

Theorem 3.9 (Fourier expansion of general periodic function) *Let $f : \mathbb{R} \to \mathbb{R}$ be piecewise differentiable and T-periodic for some $T > 0$. Then f has Fourier expansion*

$$f(t) \sim \frac{a_0}{2} + \sum_{r=1}^{\infty} \left(a_r \cos\left(\frac{2\pi r}{T}\right) t + b_r \sin\left(\frac{2\pi r}{T}\right) t \right),$$

where

$$\begin{aligned} a_0 &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt, \\ a_r &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi r}{T}\right) t\,dt \qquad (r \geq 1), \text{ and} \\ b_r &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi r}{T}\right) t\,dt. \end{aligned}$$

†56

Example 3.19 (Periodicity other than 2π)
Express the logistic map $f : [0,1] \to [0,1]$, given by $f(t) = 4t\,(1-t)$ as a sum of sines; and as a sum of cosines. †57

In the remainder of these notes, we return to the case of 2π-periodicity.

3.7 Integration and differentiation of Fourier series expansions

The techniques described in this section provide further short cuts for computing Fourier series expansions.

3.7.1 Integration of Fourier series expansions

Suppose we want to calculate the Fourier series expansion of (the periodic extension of) t^3. Since this is an odd function the coefficients a_r are all zero, and the coefficients b_r are given by

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} t^3 \sin rt\,dt.$$

Evaluating this integral requires integrating by parts three times, and is a fairly lengthy process.

An easier approach is to integrate the Fourier series expansion

$$t^2 \sim \frac{\pi^2}{3} + \sum_{r=1}^{\infty} \frac{4(-1)^r}{r^2}\cos rt \qquad (-\pi \le t \le \pi)$$

term by term:

$$\int_0^u t^2\,dt \sim \int_0^u \frac{\pi^2}{3}\,dt + \sum_{r=1}^{\infty} \frac{4(-1)^r}{r^2}\int_0^u \cos rt\,dt \qquad (-\pi \le u \le \pi),$$

or

$$\frac{u^3}{3} \sim \frac{\pi^2}{3}u + \sum_{r=1}^{\infty} \frac{4(-1)^r}{r^3}\sin ru \qquad (-\pi \le u \le \pi).$$

Substituting the known Fourier series expansion of u,

$$u \sim \sum_{r=1}^{\infty} \frac{2(-1)^{r+1}}{r}\sin ru \qquad (-\pi \le u \le \pi)$$

gives

$$u^3 \sim \sum_{r=1}^{\infty} \frac{2(-1)^{r+1}\pi^2}{r}\sin ru + \sum_{r=1}^{\infty} \frac{12(-1)^r}{r^3}\sin ru \qquad (-\pi \le u \le \pi),$$

or

$$\begin{aligned} t^3 &\sim \sum_{r=1}^{\infty} \left(\frac{2(-1)^{r+1}\pi^2}{r} + \frac{12(-1)^r}{r^3} \right) \sin rt \\ &= \sum_{r=1}^{\infty} \frac{(-1)^{r+1}(2\pi^2 r^2 - 12)}{r^3} \sin rt \qquad (-\pi \le t \le \pi). \end{aligned}$$

Term by term integration of Fourier series is generally valid: when we integrate $\sin rt$ or $\cos rt$, we bring an extra factor of r into the denominator, which makes the coefficients in the series decrease more rapidly and improves convergence of the series.

Example 3.20 (Integration of Fourier series expansion)
Determine the Fourier series expansion of the periodic extension of t^4, by integrating the expansion of t^3 term by term. †58

3.7.2 Differentiation of Fourier series expansions

Term by term integration of Fourier series expansions is successful because integrating $\sin rt$ and $\cos rt$ brings an extra factor of r into the denominator, which improves convergence. On the other hand, differentiating $\sin rt$ and $\cos rt$ brings an extra factor of r into the denominator, which inhibits convergence. For example, if we try to differentiate the Fourier series expansion of the periodic extension of t,

$$t \sim \sum_{r=1}^{\infty} \frac{2(-1)^{r+1}}{r} \sin rt \qquad (-\pi \le t \le \pi),$$

we get

$$1 \sim \sum_{r=1}^{\infty} 2(-1)^{r+1} \cos rt \qquad (-\pi \le t \le \pi),$$

which is nonsense: the series on the right hand side doesn't converge, since its terms are not decreasing to zero.

On the other hand, differentiating

$$t^2 \sim \frac{\pi^2}{3} + \sum_{r=1}^{\infty} \frac{4(-1)^r}{r^2} \cos rt \qquad (-\pi \le t \le \pi)$$

gives

$$2t \sim \sum_{r=1}^{\infty} \frac{4(-1)^{r+1}}{r} \sin rt \qquad (-\pi \le t \le \pi),$$

or

$$t \sim \sum_{r=1}^{\infty} \frac{2(-1)^{r+1}}{r} \sin rt \qquad (-\pi \le t \le \pi),$$

in agreement with the known Fourier series expansion of t.

The difference between these two cases is that the periodic extension of t^2 is continuous, whereas that of t is not (it is discontinuous, for example, at $t = \pi$).

Theorem 3.10 (Differentiation of Fourier series expansions) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and piecewise differentiable 2π-periodic function with Fourier series expansion*

$$f(t) \sim \frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r \cos rt + b_r \sin rt).$$

Then $f'(t)$ has Fourier series expansion

$$f'(t) \sim \sum_{r=1}^{\infty}(rb_r \cos rt - ra_r \sin rt).$$

†59

Example 3.21 (Differentiation of Fourier series expansion)
Calculate the Fourier series expansion of $|t|$ $(-\pi \leq t < \pi)$. Differentiate it term-by-term. What function is this the Fourier series expansion of? †60

3.8 Complex version of the Fourier series

Recall that $e^{it} = \cos t + i \sin t$, so that $e^{irt} = \cos rt + i \sin rt$.

In this section we first use this fact to show how the Fourier coefficients a_r and b_r can be obtained with a single integration, and then go on to do something more substantial: we will see how it is possible to write the Fourier series expansion of a function f as

$$f(t) \sim \sum_{r=-\infty}^{\infty} c_r e^{irt},$$

where the c_r are complex numbers.

3.8.1 Computing Fourier series expansions using $e^{irt} = \cos rt + i \sin rt$

Theorem 3.11 (Computing Fourier series expansions the complex way) *Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise differentiable 2π-periodic function. For $r \geq 0$, let*

$$A_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)e^{irt}\,dt.$$

Then the Fourier coefficients of f are given by

$$\begin{aligned} a_0 &= A_0, \\ a_r &= \Re(A_r) \qquad (r \geq 1), \text{ and} \\ b_r &= \Im(A_r). \end{aligned}$$

†61

Example 3.22 (Computing Fourier series expansions the complex way)
Show that the Fourier series expansion of the periodic extension of e^t is

$$e^t \sim \frac{\sinh \pi}{\pi}\left(1+\sum_{r=1}^{\infty}\left(\frac{2(-1)^r}{r^2+1}\cos rt+\frac{2(-1)^{r+1}r}{r^2+1}\sin rt\right)\right).$$

†62

3.8.2 The complex Fourier series expansion

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise differentiable periodic function, with Fourier series expansion

$$f(t) \sim \frac{a_0}{2}+\sum_{r=1}^{\infty}(a_r\cos rt+b_r\sin rt).$$

Recall that

$$\begin{aligned}\cos rt &= \frac{e^{irt}+e^{-irt}}{2} \qquad \text{and}\\ \sin rt &= \frac{e^{irt}-e^{-irt}}{2i}=\frac{-i(e^{irt}-e^{-irt})}{2}.\end{aligned}$$

Substituting these into the Fourier series expansion of f gives

$$\begin{aligned}f(t) &\sim \frac{a_0}{2}+\sum_{r=1}^{\infty}\left(a_r\left(\frac{e^{irt}+e^{-irt}}{2}\right)-ib_r\left(\frac{e^{irt}-e^{-irt}}{2}\right)\right)\\ &= \frac{a_0}{2}e^{i0t}+\sum_{r=1}^{\infty}\left(\frac{a_r-ib_r}{2}e^{irt}+\frac{a_r+ib_r}{2}e^{-irt}\right)\\ &= \sum_{r=-\infty}^{\infty}c_re^{irt},\end{aligned}$$

where

$$\begin{aligned}c_0 &= \frac{a_0}{2},\\ c_r &= \frac{a_r-ib_r}{2} \qquad \text{for } r\geq 1, \text{ and}\\ c_{-r} &= \frac{a_r+ib_r}{2} \qquad \text{for } r\geq 1.\end{aligned}$$

Notice that $c_{-r}=c_r^*$, the complex conjugate of c_r.
Inverting these formulae to find the coefficients a_r and b_r in terms of c_r gives

$$\begin{aligned}a_0 &= 2c_0,\\ a_r &= 2\Re(c_r) \qquad (r\geq 1),\ \text{and}\\ b_r &= -2\Im(c_r).\end{aligned}$$

Finally, the coefficients c_r can be calculated directly from f (rather than via a_r and b_r) by

$$c_r = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-irt}\,dt.$$

(Notice that the normalization is achieved by dividing by the *full* length of the range of integration.) To see this, observe that

$$c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i0t}f(t)\,dt,$$

$$\begin{aligned} c_r &= \frac{a_r - ib_r}{2} \qquad \text{(case } r \geq 1\text{)} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(t)\cos rt - if(t)\sin rt\right)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-irt}\,dt, \end{aligned}$$

and

$$\begin{aligned} c_{-r} &= c_r^* \qquad \text{(case } r \geq 1\text{)} \\ &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-irt}\,dt\right)^* \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-i(-r)t}\,dt. \end{aligned}$$

In summary, the complex version of the Fourier series expansion of f is given by

$$\boxed{f(t) \sim \sum_{r=-\infty}^{\infty} c_r e^{irt},}$$

where

$$\boxed{c_r = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-irt}\,dt.}$$

Example 3.23 (Complex Fourier series expansion)
Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic extension of the function $F : [-\pi, \pi) \to \mathbb{R}$ given by

$$F(t) = \begin{cases} 0 & \text{if } \; -\pi \leq t < 0, \\ 1 & \text{if } \; 0 \leq t < \pi. \end{cases}$$

Compute the complex Fourier series expansion of f. †63

3.9 Parseval's theorem

Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both piecewise differentiable 2π-periodic functions, with (complex) Fourier series expansions

$$\begin{aligned} f(t) &\sim \sum_{r=-\infty}^{\infty} c_r e^{irt}, \\ g(t) &\sim \sum_{r=-\infty}^{\infty} d_r e^{irt}. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{r=-\infty}^{\infty} c_r e^{irt}\right) g(t)\,dt \\ &= \sum_{r=-\infty}^{\infty} c_r \, \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{irt}\,dt \\ &= \sum_{r=-\infty}^{\infty} c_r d_{-r} \\ &= \sum_{r=-\infty}^{\infty} c_r d_r^*. \end{aligned}$$

In particular, if $f = g$ then we get:

Theorem 3.12 (Parseval) *Let $f : \mathbb{R} \to \mathbb{R}$ be piecewise differentiable and 2π-periodic, with Fourier series expansion*

$$f(t) \sim \sum_{r=-\infty}^{\infty} c_r e^{irt}.$$

Write

$$P(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)^2\,dt.$$

Then

$$P(f) = \sum_{r=-\infty}^{\infty} |c_r|^2 .$$

We can rewrite this using the real version of the Fourier series expansion as follows:

Theorem 3.13 (Parseval – real version) *Let $f : \mathbb{R} \to \mathbb{R}$ be piecewise differentiable and 2π-periodic, with Fourier series expansion*

$$f(t) \sim \frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r \cos rt + b_r \sin rt).$$

Write

$$P(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)^2 \, dt.$$

Then

$$P(f) = \frac{a_0^2}{4} + \frac{1}{2}\sum_{r=1}^{\infty}\left(a_r^2 + b_r^2\right).$$

†64

Example 3.24 (Parseval's theorem)
Apply Parseval's theorem to the Fourier series expansion of the periodic extension of t^2 to show that

$$\sum_{r=1}^{\infty}\frac{1}{r^4} = \frac{\pi^4}{90}.$$

†65