Mutant knots with symmetry

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Abstract

Mutant knots, in the sense of Conway, are known to share the same Homfly polynomial. Their 2-string satellites also share the same Homfly polynomial, but in general their $m$-string satellites can have different Homfly polynomials for $m > 2$. We show that, under conditions of extra symmetry on the constituent 2-tangles, the directed $m$-string satellites of mutants share the same Homfly polynomial for $m < 6$ in general, and for all choices of $m$ when the satellite is based on a cable knot pattern.

We give examples of mutants with extra symmetry whose Homfly polynomials of some 6-string satellites are different, by comparing their quantum $sl(3)$ invariants.

1 Introduction

This paper has been inspired by recent observations of Ochiai and Jun Murakami about the Homfly skein theory of $m$-parallels of certain symmetrical 2-tangles. In [8] Ochiai remarks that the 3-parallels of the tangle $AB$ in figure 1 and its mirror image $BA = AB$ are equal in the Homfly skein of 6-tangles, in other words, in the Hecke algebra $H_6$, [1].

Figure 1:

As a consequence, the 3-parallels of any mutant pair of knots given by composing the 2-tangles $AB$ and $BA$ with any other 2-tangle $C$ and then closing will share the same Homfly polynomial.

This is in contrast with the known fact that 3-parallels of mutant knots in general can have different Homfly polynomials, [7, 4].

There is interest in the extent to which the Homfly polynomial of $m$-parallels or other $m$-string satellites can distinguish mutants which are closures of $ABC$ and $BAC$ with $A$ and $B$ as above. Ochiai has found that the 4-parallels of $AB$ and $BA$ are different in the skein $H_8$. 
The purpose of this paper is to show that if $A$ and $B$ are any two oriented 2-tangles with symmetry

$$A = \begin{array}{c}
    \begin{array}{c}
      A \\
      \uparrow \\
    \end{array} \\
  \end{array}, \quad B = \begin{array}{c}
    \begin{array}{c}
      B \\
      \uparrow \\
    \end{array} \\
  \end{array}$$

then the $m$-parallels, and indeed any directed $m$-string satellite, of knots $\hat{ABC}$ and $\hat{BAC}$ shown in figure 2 share the same Homfly polynomial for $m < 6$.

In contrast there exist examples of $A$, $B$ and $C$, including Ochiai’s case with

$$A = \begin{array}{c}
    \begin{array}{c}
      A \\
      \uparrow \\
    \end{array} \\
  \end{array}, \quad B = \begin{array}{c}
    \begin{array}{c}
      B \\
      \uparrow \\
    \end{array} \\
  \end{array},$$

for which the Homfly polynomials of the 6-fold parallel are different.

As an unexpected extension of the main result we show that the Homfly polynomial of a genuine connected cable, based on the $(m, n)$ torus knot pattern, with $m$ and $n$ coprime, for any number of strings, $m$, will not distinguish mutants with symmetry above, although a more general connected satellite pattern can do so.

The examples which exhibit differences for the directly oriented 6-parallel can also be used to show that the 4-parallels with two pairs of reverse strands have distinct Homfly polynomials.

The proofs are based on the relation of the Homfly satellite invariants to quantum $sl(N)$ invariants, and the techniques are an extension of work with Cromwell [4] and with H. Ryder [6]. The eventual calculations that exhibit the difference of invariants in the specific example depend on the 27 dimensional irreducible module over $sl(3)$ corresponding to the partition 4, 2, and some Maple calculations following similar lines to those in [6].
2 Shared invariants of mutants

The term mutant was coined by Conway, and refers to the following general construction.

Suppose that a knot $K$ can be decomposed into two oriented 2-tangles $F$ and $G$

$$K = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}$$

A new knot $K'$ can be formed by replacing the tangle $F$ with the tangle $F' = \tau_i(F)$ given by rotating $F$ through $\pi$ in one of three ways,

$$\tau_1(F) = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}, \quad \tau_2(F) = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}, \quad \tau_3(F) = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array},$$

reversing its string orientations if necessary. Any of the three knots

$$K' = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}$$

is called a mutant of $K$.

The two 11-crossing knots, $C$ and $KT$, with trivial Alexander polynomial found by Conway and Kinoshita-Teresaka are the best-known example of mutant knots.

$$C = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \quad KT = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}$$
2.1 Satellites

A satellite of $K$ is determined by choosing a diagram $Q$ in the standard annulus, and then drawing $Q$ on the annular neighbourhood of $K$ determined by the framing, to give the satellite knot $K \ast Q$. We refer to this construction as decorating $K$ with the pattern $Q$, as shown in figure 3.

For fixed $Q$ the Homfly polynomial $P(K \ast Q)$ of the satellite is an invariant of the framed knot $K$. The invariants $P(K \ast Q)$ as $Q$ varies make up the Homfly satellite invariants of $K$. We use the alternate notation $P(K; Q)$ in place of $P(K \ast Q)$ when we want to emphasise the dependence on $K$.

The general symmetry result compares the invariants of two knots $K$ and $K'$ made up of 2-tangles $A$, $B$ and $C$, by interchanging $A$ and $B$ as in figure 2.

**Theorem 1.** Suppose that $A$ and $B$ are both symmetric under the half-twist $\tau_3$, so that

\[
A = \begin{array}{c} 1 \\ S \end{array}, \quad B = \begin{array}{c} 1 \\ S \end{array}
\]

Let $K$ and $K'$ be knots which are the closure of $ABC$ and $BAC$ respectively for any tangle $C$, as in figure 2. Then $P(K \ast Q) = P(K' \ast Q)$ for every closed braid pattern $Q$ on $m < 6$ strings.

**Remark 1.** Our proof will apply equally to the case where $Q$ is the closure of a directly oriented $m$-tangle with $m < 6$.

In order to prove the theorem we must rewrite the Homfly satellite invariants in terms of quantum $sl(N)$ invariants, so we now give a brief summary of the relations between these invariants, originally established by Wenzl. Further details can be found in [1] and the thesis of Lukac, [3], including details of variant Homfly skeins with a framing correction factor, $x$. These are isomorphic to the skeins used here but the parameter allows a careful adjustment of the quadratic skein relation to agree directly with the natural relation arising from use of the quantum groups $sl(N)$. 

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Figure 3: Satellite construction
2.2 Homfly skeins

For a surface $F$ with some designated input and output boundary points the (linear) Homfly skein of $F$ is defined as linear combinations of oriented diagrams in $F$, up to Reidemeister moves II and III, modulo the skein relations

1. $\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{skein1}
\end{array}
\end{align*}
= (s - s^{-1}),$

2. $\begin{align*}
\begin{array}{c}
\includegraphics[width=0.15\textwidth]{skein2}
\end{array}
\end{align*}
= v^{-1}.$

It is an immediate consequence that

$\begin{align*}
\begin{array}{c}
\includegraphics[width=0.15\textwidth]{skein3}
\end{array}
\end{align*}
= \delta,
\end{align*}$

where $\delta = \frac{v^{-1} - v}{s - s^{-1}} \in \Lambda$. The coefficient ring $\Lambda$ is taken as $\mathbb{Z}[v^{\pm 1}, s^{\pm 1}]$, with denominators $s^r - s^{-r}, r \geq 1$.

The skein of the annulus is denoted by $\mathcal{C}$. It becomes a commutative algebra with a product induced by placing one annulus outside another.

The skein of the rectangle with $m$ inputs at the top and $m$ outputs at the bottom is denoted by $H_m$. We define a product in $H_m$ by stacking one rectangle above the other, obtaining the Hecke algebra $H_m(z)$, when $z = s - s^{-1}$ and the coefficients are extended to $\Lambda$. The Hecke algebra $H_m$ can also be regarded as the group algebra of Artin’s braid group $B_m$ generated by the elementary braids $\sigma_i$, $i = 1, \ldots, m - 1$, modulo the further quadratic relation $\sigma_i^2 = z \sigma_i + 1$.

The closure map from $H_m$ to $\mathcal{C}$ is the $\Lambda$-linear map induced by mapping a tangle $T$ to its closure $\hat{T}$ in the annulus (see figure 4). We refer to a diagram $Q = \hat{T}$ as a directly oriented pattern.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{closure_map}
\caption{The closure map}
\end{figure}

The image of this map is denoted by $\mathcal{C}_m$, which has a useful interpretation as the space of symmetric polynomials of degree $m$ in variables $x_1, \ldots, x_N$ for large enough $N$. Moreover, the submodule $\mathcal{C}_+ \subset \mathcal{C}$ spanned by the union $\bigcup_{m \geq 0} \mathcal{C}_m$ is a subalgebra of $\mathcal{C}$ isomorphic to the algebra of the symmetric functions.
2.3 Quantum invariants

A quantum group $\mathcal{G}$ is an algebra over a formal power series ring $\mathbb{Q}[[h]]$, typically a deformed version of a classical Lie algebra. We write $q = e^h, s = e^{h/2}$ when working in $sl(N)_q$. A finite dimensional module over $\mathcal{G}$ is a linear space on which $\mathcal{G}$ acts.

Crucially, $\mathcal{G}$ has a coproduct $\Delta$ which ensures that the tensor product $V \otimes W$ of two modules is also a module. It also has a universal $R$-matrix (in a completion of $\mathcal{G} \otimes \mathcal{G}$) which determines a well-behaved module isomorphism

$$R_{VW} : V \otimes W \to W \otimes V.$$ 

This has a diagrammatic view indicating its use in converting coloured tangles to module homomorphisms.

$$\begin{tikzpicture}
    \draw (0,0) -- (0,1);
    \draw (1,0) -- (1,1);
    \draw (0,0) -- (1,1);
    \draw (0,1) -- (1,0);
    \node at (0.5,0.5) {$R_{VW}$};
\end{tikzpicture}$$

A braid $\beta$ on $m$ strings with permutation $\pi \in S_m$ and a colouring of the strings by modules $V_1, \ldots, V_m$ leads to a module homomorphism

$$J_\beta : V_1 \otimes \cdots \otimes V_m \to V_{\pi(1)} \otimes \cdots \otimes V_{\pi(m)}$$

using $R_{V_iV_j}^{h_+1}$ at each elementary braid crossing. The homomorphism $J_\beta$ depends only on the braid $\beta$ itself, not its decomposition into crossings, by the Yang-Baxter relation for the universal $R$-matrix.

When $V_i = V$ for all $i$, we get a module homomorphism $J_\beta : W \to W$, where $W = V^{\otimes m}$. Equally, a directed $m$-tangle $T$ determines an endomorphism $J_T$ of $W = V^{\otimes m}$. Now any $sl(N)$ module $W$ decomposes as a direct sum $\bigoplus \langle W_\mu \otimes V^{(N)}_\mu \rangle$, where $W_\mu$ is the linear subspace consisting of the highest weight vectors of type $\mu$ associated to the module $V^{(N)}_\mu$. Highest weight subspaces of each type are preserved by module homomorphisms, and so $J_T$ determines (and is determined by) the restrictions $J_T(\mu) : W_\mu \to W_\mu$ for each $\mu$.

If a knot $K$ is decorated by a pattern $Q$ which is the closure of an $m$-tangle $T$ then its quantum invariant $J(K \ast Q; V)$ can be found from the endomorphism $J_T$ of $W = V^{\otimes m}$ in terms of the quantum invariants of $K$ and the highest weight maps $J_T(\mu) : W_\mu \to W_\mu$ by the formula

$$J(K \ast Q; V) = \sum c_\mu J(K; V^{(N)}_\mu)$$

with $c_\mu = \text{tr} J_T(\mu)$. This formula follows from lemma II.4.4 in Turaev’s book [11]. Here $\mu$ runs over partitions with at most $N$ parts when we are working with $sl(N)$, and we set $c_\mu = 0$ when $W$ has no highest weight vectors of type $\mu$.

**Proof of theorem 1.** Take $V = V^{(N)}$ as the fundamental module of dimension $N$ for $sl(N)$. Then the only highest weight types $\mu$ which occur in equation (1)
are partitions of \( m \) with at most \( N \) rows. Because \( J(K \ast Q; V^{(N)}) = P(K \ast Q) \) when \( v = s^{-N} \) we can show that \( P(K \ast Q) = P(K' \ast Q) \) by showing that \( J(K \ast Q; V^{(N)}) = J(K' \ast Q; V^{(N)}) \) for all \( N \). By equation 1 it is then enough to show that \( J(K; V^{(N)}_\mu) = J(K' ; V^{(N)}_\mu) \) for all \( N \) and all partitions \( \mu \vdash m \).

Now each tangle \( A \) and \( B \) determines an endomorphism \( J_A, J_B \) of \( V_\mu \otimes V_\nu \). If \( J_A \) and \( J_B \) commute then \( J(K; V_\mu) = J(K' ; V_\mu) \). The endomorphisms \( J_A \) and \( J_B \) are determined by their restriction \( J_A(\nu), J_B(\nu) \) to the highest weight subspaces \( W_\nu \) in the decomposition \( V_\mu \otimes V_\mu = \sum W_\nu \otimes V_\nu \), so it is enough to show that \( J_A(\nu) \) and \( J_B(\nu) \) commute where \( V_\nu \) is a summand of \( V_\mu \otimes V_\mu \). This is certainly the case for all \( \nu \) where \( W_\nu \) is 1-dimensional, which includes the case of single row or column partitions \( \mu \). [4].

As a special case of the work of Rosso and Jones, [9, 5], we know that the endomorphism of \( V_\mu \otimes V_\nu \) for the full twist \( \Delta^2 \) on two strings acts as a scalar \( e^{f(\nu)} \) on each highest weight space \( W_\nu \), while the half twist \( \Delta \), represented by the \( R \)-matrix \( R_{V_\mu V_\nu} \), operates on \( W_\nu \) with two eigenvalues \( \pm e^{\frac{1}{2} f(\nu)} \).

The positive and negative eigenspaces correspond to the classical decomposition of the Schur function \((s_\mu)^2\) into symmetric and skew-symmetric parts, \( h_2(s_\mu) \) and \( e_2(s_\mu) \), and the dimension of each eigenspace of \( W_\nu \) is the multiplicity of \( s_\nu \) in \( h_2(s_\mu) \) and \( e_2(s_\mu) \) respectively.

Now \( A = \tau_3(A) \), so that \( A\Delta = \Delta A \). Hence the endomorphism \( J_A \), and similarly \( J_B \), preserves the positive and negative eigenspaces of each \( W_\nu \). If these eigenspaces have dimension 1 or 0 then \( J_A \) and \( J_B \) will commute on \( W_\nu \).

The theorem is then established by checking that no \( s_\nu \) occurs in \( h_2(s_\mu) \) or \( e_2(s_\mu) \) with multiplicity \( > 1 \) for any \( \mu \) with \( |\mu| \leq 5 \). The decomposition of all of these can be quickly confirmed using the Maple program SF of Stembridge [10].

\[ \square \]

**Corollary 2.** Examples include \( k \)-pretzel knots \( K(a_1, \ldots, a_k) \) with odd \( a_i \).

Here the numbers \( a_i \) can be permuted without changing the Homfly polynomial of any satellite with \( \leq 5 \)-strings.

## 3 Satellites with different Homfly polynomials

A further check with the program SF when \( |\mu| = 6 \) shows that there are just three partitions, \( \mu = 4, 2 \), its conjugate \( \mu = 2, 2, 1, 1 \) and \( \mu = 3, 2, 1 \) whose symmetric square \( h_2(s_\mu) \) contains summands with multiplicity \( > 1 \), as does the exterior squares of \( \mu = 3, 2, 1 \). Explicitly \( h_2(s_{4,2}) = s_{8,4} + s_{8,2,2} + s_{8,4,1} + s_{7,3,2} + s_{7,3,1,1} + s_{8,6,6} + s_{8,6,5,1} + 2 s_{8,6,4,2} + s_{8,6,3,2,1} + s_{8,6,2,2,2} + s_{8,5,1,1} + s_{8,5,4,3} + s_{8,5,4,2,1} + s_{8,5,3,3,1} + s_{8,4,4,4} + s_{8,4,4,2,2} \). This means that, although \( m \)-string satellites of \( K \)
and $K'$ must share the Homfly polynomial when $m \leq 5$, it is possible for the Homfly polynomials of some 6-string satellites to differ.

We give an example now where this does indeed happen.

**Theorem 3.** Let $K$ and $K'$ be the pretzel knots $K = K(1,3,3,-3,-3)$ and $K' = K(1,3,-3,3,-3)$.

The 6-fold parallels $K \ast Q$ and $K' \ast Q$, where $Q$ is the closure of the identity braid on 6 strings, have different Homfly polynomials.

**Proof.** Write $K$ and $K'$ as the closure of the products $\Delta ABAB$ and $\Delta BAAB$ respectively, where

\[
A = \begin{array}{c}
\begin{array}{c}
\text{\Arrowtriangleleft}
\end{array}
\end{array}
\quad B = \begin{array}{c}
\begin{array}{c}
\text{\Arrowtriangleupright}
\end{array}
\end{array}
\]

are the partially closed 3-braids shown, and $\Delta$ is the positive half-twist. We show that $P(K \ast Q) \neq P(K' \ast Q)$ when $v = s^{-3}$. These values are given by the $sl(3)$ quantum invariants $J(K \ast Q; V^{(3)})$ and $J(K' \ast Q; V^{(3)})$, where $V^{(3)}$ is the fundamental 3-dimensional module for $sl(3)$. Since $Q$ is the closure of the identity braid on 6 strings it induces the identity endomorphism on the module $(V^{(3)}) \otimes 6$. This module decomposes as $\bigoplus W_\mu \otimes V^{(3)}_\mu$ where $\mu$ runs through partitions of 6 with at most 3 rows. The trace of the identity on $W_\mu$ is just $d_\mu = \dim W_\mu$, giving

\[
J(K \ast Q; V^{(3)}) = \sum d_\mu J(K; V^{(3)}_\mu).
\]

The only partition $\mu$ in this range for which the exterior or symmetric square contains highest weight vectors of multiplicity $> 1$ is the partition $\mu = 4,2$, since the partition $\mu = 2,2,1,1$ has 4 rows and the repeated factors for $\mu = 3,2,1$ occur for partitions with more than 3 rows. Now $J_A(\mu)J_B(\mu) = J_B(\mu)J_A(\mu)$ for all other $\mu$ since $A$ and $B$ are symmetric up to altering the framing on both strings, while maintaining the writhe. Then

\[
P(K \ast Q) - P(K' \ast Q) = d_\mu (J(K; V^{(3)}_\mu) - J(K'; V^{(3)}_\mu))
\]

when $v = s^{-3}$ and $\mu = 4,2$. Since $d_\mu \neq 0$ it is enough to show that $J(K; V^{(3)}_\mu) \neq J(K'; V^{(3)}_\mu)$. The module $V^{(3)}_\mu$ has dimension 27.

We now work in the quantum group $sl(3)$ and drop the superscript (3) from the irreducible modules.
Decompose the module $V_\mu \otimes V_\mu$ as $\sum W_\nu \otimes V_\nu$ and compare the endomorphisms given by the tangles $T = ABAB\Delta$ and $T' = BAAB\Delta$.

In this case just one of the invariant subspaces of highest weight vectors has dimension $> 1$. It can be shown that the corresponding $2 \times 2$ matrices $A_\mu$ and $B_\mu$ arising from the two mirror-image tangles $A$ and $B$ with 3 crossings satisfy $\text{tr}(A_\mu B_\mu A_\mu B_\mu - A_\mu A_\mu B_\mu B_\mu) \neq 0$, which results in a difference in their $sl(3)$ invariants $J(K; V_\lambda)$.

None of the other 6-cell invariants differ on the two knots. Consequently the 6-parallels have different $sl(3)$ invariants. The $sl(3)$ invariant of the 6-parallels of the two pretzel knots coloured with the fundamental module, and thus their Homfly polynomials, are then different. \square

3.1 Use of the quantum group $sl(3)_q$

The calculation of the $2 \times 2$ matrices $A_\nu$ and $B_\nu$ giving the effect of the two tangles on the highest weight vectors where there is a 2-dimensional highest weight subspace of the symmetric part of the module depends on finding the explicit action of the quantum group on the 27-dimensional module $V_\mu^{(3)}$ with $\mu = 4, 2$ and its tensor square, as well as the homomorphism representing its $R$-matrix. I used the linear algebra packages in Maple to handle the matrix working and subsequent polynomial factorisation, following fairly closely the techniques developed with H. Ryder in the paper [6].

In the interests of reproducibility I give an account of the methods used, and some of the checks applied during the calculations, to test against known properties.

We start from a presentation of the quantum group $sl(3)_q$ as an algebra with six generators, $X_1^\pm$, $X_2^\pm$, $H_1$, $H_2$, and a description of the comultiplication and antipode.

Let $M$ be any finite-dimensional left module over $sl(3)_q$. The action of any one of these six generators $Y$ will determine a linear endomorphism $Y_M$ of $M$. We build up explicit matrices for these endomorphisms on a selection of low-dimensional modules, using the comultiplication to deal with the tensor product of two known modules, and the antipode to construct the action on the linear dual of a known module. We must eventually determine the matrices $Y_M$ for our module $M = V_\mu$ and find the $729 \times 729$ $R$-matrix, $R_{MM}$ which represents the endomorphism of $M \otimes M$ needed for crossings.

We follow Kassel in the basic description of the quantum group from using generators $H_1$ and $H_2$ for the Cartan sub-algebra, but with generators $X_i^\pm$ in place of $X_i$ and $Y_i$. We use the notation $K_i = \exp(hH_i/4)$, and set $a = \exp(h/4)$, $s = \exp(h/2) = a^2$ and $q = \exp(h) = s^2$, unlike Kassel. The generators satisfy the commutation relations

$$[H_i, H_j] = 0, \ [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm, \ [X_i^+, X_i^-] = (K_i^2 - K_i^{-2})/(s - s^{-1}),$$

where $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is the Cartan matrix for $SU(3)$ (and also the Serre
relations of degree 3 between $X_1^\pm$ and $X_2^\pm$).

Comultiplication is given by

$$\Delta(H_i) = H_i \otimes I + I \otimes H_i,$$

(so $\Delta(K_i) = K_i \otimes K_i$),

$$\Delta(X_i^\pm) = X_i^\pm \otimes K_i + K_i^{-1} \otimes X_i^\pm,$$

and the antipode $S$ by $S(X_i^\pm) = -s^{i \pm 1}X_i^\pm$, $S(H_i) = -H_i$, $S(K_i) = K_i^{-1}$.

The fundamental 3-dimensional module, which we denote by $E$, has a basis

in which the quantum group generators are represented by the matrices $Y_E$ as listed here.

$X_1^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $X_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$X_1^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $X_2^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

For calculations we keep track of the elements $K_i$ rather than $H_i$, represented by

$K_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}$

for the module $E$.

We can then write down the elements $Y_{EE}$ for the actions of the generators $Y$ on the module $E \otimes E$, from the comultiplication formulae. The $R$-matrix $R_{EE}$ can be given, up to a scalar, by the prescription

$$R_{EE}(e_i \otimes e_j) = e_j \otimes e_i, \text{ if } i > j,$$

$$= s e_i \otimes e_i, \text{ if } i = j,$$

$$= e_j \otimes e_i + (s - s^{-1}) e_i \otimes e_j, \text{ if } i < j,$$

for basis elements $\{e_i\}$ of $E$.

The linear dual $M^*$ of a module $M$ becomes a module when the action of a generator $Y$ on $f \in M^*$ is defined by $<Y_M, f, v> = <f, S(Y_M)v>$, for $v \in M$. For the dual module $F = E^*$ we then have matrices for $Y_F$, relative to the dual basis, as follows.

$X_1^+ = \begin{pmatrix} 0 & 0 & 0 \\ -s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $X_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -s & 0 \end{pmatrix}$

$X_1^- = \begin{pmatrix} 0 & -s^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $X_2^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -s^{-1} \\ 0 & 0 & 0 \end{pmatrix}$
The most reliable way to work out the $R$-matrices $R_{EF}, R_{FE}$ and $R_{FF}$ is to combine $R_{EE}$ with module homomorphisms $\cup_{EF}, \cup_{FE}, \cap_{EF}$ and $\cap_{FE}$ between the modules $E \otimes F$, $F \otimes E$ and the trivial 1-dimensional module, $I$, on which $X_i^+$ acts as zero and $K_i$ as the identity. The matrices are determined up to a scalar by such considerations; a choice for one dictates the rest.

Once these matrices have been found they can be combined with the matrix $R_{EE}^{-1}$ to construct the $R$-matrices $R_{EF}, R_{FE}, R_{FF}$, using the diagram shown below, for example, to determine $R_{EF}$. This gives

$$R_{EF} = (1_F \otimes 1_E \otimes \cap_{EF}) \circ (1_F \otimes R_{EE}^{-1} \otimes 1_F) \circ (\cup_{FE} \otimes 1_E \otimes 1_F).$$

The module structure of $M = V_\square$ can be found by identifying $M$ as a 27-dimensional submodule of $V_\square \otimes V_\square$, while the two 6-dimensional modules $V_\square$ and $V_\square$ are themselves submodules of $E \otimes E$ and $F \otimes F$ respectively.

We know, by the Pieri formula, that there is a direct sum decomposition of $V_\square \otimes V_\square$ as $M \oplus N$, where $M = V_\square$, and $N$ is the sum of the 8-dimensional module $V_\square$ and the 1-dimensional trivial module.

We first identify the module $V_\square$ as a submodule of $E \otimes E$, knowing that $E \otimes E$ is isomorphic to $V_\square \otimes F$. The full twist element on the two strings both coloured by $E$ is represented by $R_{EE}^2$ which acts on $E \otimes E$ as a scalar on each of the two irreducible submodules $V_\square$ and $F$.

Use Maple to find bases for the two eigenspaces of $R_{EE}^2$. Then we can identify $V_\square$ with the 6-dimensional one, and write $P$ and $Q$ for the $9 \times 6$ and $9 \times 3$ matrices whose columns are these bases. The partitioned matrix $(P | Q)$ is invertible, and its inverse, found by Maple, can be written as $R S$, where $R$ is a $6 \times 9$ matrix with $RP = I_6$ and $RQ = 0$.

Regard $P = \text{inj} M_1 EE$ as the matrix representing the inclusion of the module $V_\square$ into $E \otimes E$. Then $R = \text{proj} EEM_1$ is the matrix, in the same basis, of the projection from $E \otimes E$ to $V_\square$. For $M_1 = V_\square$ the module generators $Y_{M_1}$ are given by $Y_{M_1} = R Y_{EE} P$, giving the explicit action of the quantum group on $V_\square$.

We perform a similar calculation on $F \otimes F$ to identify the module $M_2 = V_\square$ and the matrices $\text{inj} M_2 FF$ and $\text{proj} FFM_2$, giving the action of the quantum
We use inclusion and projection further to find the four $6^2 \times 6^2$ $R$-matrices $R_{M_1, M_2}$. For example, to construct $R_{M_1, M_2} : M_1 \otimes M_2 \to M_2 \otimes M_1$, first map $M_1 \otimes M_2$ to $E \otimes E \otimes F \otimes F$ by $\text{inj}M_1EE \otimes \text{inj}M_2FF$. Then construct the $R$-matrix crossing two strings, one coloured by $X$, with $\times$ vectors in the highest weight space this determines a $2 \times 2$ $W$-space. Explicitly, choose a basis $\{ \epsilon_i \}$ of $M$ and write

$$F_A(v \otimes T_M(e_i)) = \sum_j f_{ij}(v) \otimes e_j$$

with $f_{ij}(v) \in M \otimes M$. Then $J_A(v) = \sum_i f_{ii}(v)$. Applied to each of the two vectors in the highest weight space this determines a $2 \times 2$ matrix $A_\nu$ representing
the restriction of $J_A$ to this subspace. Similarly $B_v$ is found using the mirror image braid $\sigma_2^{-1}\sigma_1\sigma_2^{-1}$.

We know that $R_{MM}$ acts as a scalar on the 2-dimensional space so $J(K; V_v) - J(K'; V_v)$ is a non-zero scalar multiple of $\text{tr}(A_vB_vA_vB_v - B_vA_vA_vB_v)$.

This difference is $2(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)(q^8 + q^6 + 1)^2(q^4 - q^2 + 1)^2(q^4 + q^3 + q^2 + q + 1)^3(q^2 + q + 1)^4(q^2 - q + 1)^4(q + 1)^{10}(q - 1)^{18}$, up to a power of $q = s^2$ and the quantum dimension of $V_v$.

## 3.3 Further examples of difference

Using the same matrices $A_\nu$ and $B_\nu$ it is possible to find further pretzel knot examples based on sequences of the tangles $A$ and $B$ where the 6-parallels have different Homfly polynomial, such as the knots $K(3,3,3,-3,-3)$ and $K(3,3,-3,3,-3)$. The difference here is the same as for the first example multiplied by the factor $2q^{23} - q^{31} - 3q^{30} + 5q^{29} + 3q^{28} - 10q^{27} + q^{26} + 14q^{25} - 6q^{24} - 19q^{23} + 21q^{22} + 20q^{21} - 46q^{20} + 2q^{19} + 61q^{18} - 48q^{17} - 35q^{16} + 83q^{15} - 27q^{14} + 66q^{13} + 72q^{12} + 3q^{11} - 57q^{10} + 10q^9 - 33q^7 + 16q^6 + 7q^5 - 12q^4 + 7q^3 - 4q + 2$. The same calculations guarantee that satellites based on any closed 6-tangle $Q = \tilde{T}$ will have different Homfly polynomial, provided that the trace $c_\mu$ of the endomorphism $J_{\tilde{T}}$ on the highest weight space $W_\mu$ of $V_{\otimes 6}$ is non-zero, where $\mu$ is the partition 4,2. This will be the case for most, but not all, patterns $Q$, and certainly will be the case for many satellites which are knots rather than links.

The calculations in section 3.2 also show that the 4-parallels of the two pretzel knots $K(1,3,3,-3,-3)$ and $K(1,3,-3,3,-3)$ with two strings oriented in one direction and two in the opposite direction will have different Homfly polynomials, by using the decomposition of the corresponding $sl(3)_q$ module $W = V \otimes V \otimes V \otimes V$ into a sum of irreducible $sl(3)_q$ modules. The only module to figure in this decomposition with any multiplicity in its symmetric or exterior square is again $V$. The calculations above, using the fact that Homfly with $v = s^{-3}$ can be calculated by colouring strings with reverse orientation by the dual module $V^*$ to the fundamental module, and that this is $V^*$ for $sl(3)_q$.

## 4 Cable patterns

By way of contrast, if the pattern $Q$ is a cable on any number of strings then $K \ast Q$ and $K' \ast Q$ share the same Homfly polynomial, where $K$ and $K'$ have the same symmetry as in theorem 1.

**Theorem 4.** Suppose that $A$ and $B$ are both symmetric under the half-twist $\tau_3$, so that

$$A = \begin{array}{c} \text{A} \\ \downarrow \end{array}, \quad B = \begin{array}{c} \text{B} \\ \downarrow \end{array}$$

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Let $K$ and $K'$ be knots which are the closure of $ABC$ and $BAC$ respectively for any tangle $C$, as in figure 2. Then $P(K \ast Q) = P(K' \ast Q)$ for every $(m,n)$ cable pattern $Q$ where $m$ and $n$ are coprime.

**Proof.** As in the proof of theorem 1 we show that $J(K \ast Q; V^{(N)}) = J(K' \ast Q; V^{(N)})$ for all $N$. By equation 1 it is then enough to show that $J(K; V^{(N)}_\mu) = J(K'; V^{(N)}_\mu)$ for all $N$ and all partitions $\mu \vdash m$ for which the coefficient $c_\mu \neq 0$. The coefficients $c_\mu$ depend on the pattern $Q$ and arise as the trace of the endomorphism $J_T$ when restricted to the highest weight space $W_\mu \subset V^\otimes m$, where $Q$ is the closure of the $m$-braid $T = (\sigma_1 \sigma_2 \cdots \sigma_{m-1})^n$.

It is shown in [9], (see also [5]), that for any such cable $Q$ the only non-zero coefficients $c_\mu$ occur when the partition $\mu$ is a hook, if $m$ and $n$ are coprime. It is then enough to show that $J(K; V^{(N)}_\mu) = J(K'; V^{(N)}_\mu)$ for all hook partitions $\mu$.

Using the same argument as in theorem 1 it remains to check that no Schur function $s_\nu$ occurs with multiplicity $> 1$ in the decomposition of either the symmetric or exterior squares, $h_2(s_\mu)$ or $e_2(s_\mu)$, for any hook partition $\mu$. This fact has been established by Carbonara, Remmel and Yang in theorem 3 of [2], and so the proof is complete.

**Remark 3.** Theorem 4 highlights the importance of a precise terminology for different types of satellite. The term cable is sometimes used to mean any satellite, while there is a clear distinction here between the behaviour of cables and of parallels or other satellites, which is not primarily a matter of the number of components of the satellite.

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**References**


[2] JO Carbonara, JB Remmel and M Yang. Exact formulas for the plethysm $s_2[s_{(1^n)}]$ and $s_{1^2}[s_{(1^n)}]$. Technical report, Mathematical Sciences Institute, Cornell University, 1992.


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