POLYNOMIALS FROM BRAIDS

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ABSTRACT. An account is given of the construction of the 2-variable polynomial P for an oriented link K by means of the braid groups and their representation in Hecke algebras.

Relations between P and the braid index of K are derived. Reference is also made to other presentations of the algebra, leading to the invariance of specialisations of P under certain alterations of K.

1. INTRODUCTION AND TERMINOLOGY.

The 2-variable polynomial P for an oriented link was developed in the later part of 1984 by a number of authors, [FYHLMO], [PT], following Jones' discovery, [J1], of the new polynomial invariant V through the route of von Neumann algebras and braid groups. In this article I shall give a development of P using a modified form of the method due to Ocneanu and Jones. A wider discussion is given by Jones in his survey [J2]. This article should be regarded as an improved version of [M1], which were notes based on a seminar given in Liverpool in 1984, and whose original results appeared with a completely different proof in [M2]. The approach adopted in [M1] has subsequently proved useful in making explicit computer calculations of P, [MS1], [MS2], particularly because the two variables used can be kept separate for a long time. In this respect it bears a close resemblance to Kauffman's construction of P, starting from Conway's version of the Alexander polynomial.

In the course of the article I shall point out the relation to the variables used by Jones and Ocneanu. The algebra used here can also be recovered naturally from the linear skein approach of Lickorish and Millett, if the existence of P has been already established, see for example [MT]. It is very useful, I believe, to have both approaches in mind when looking at any of the recent family of knot polynomials, as properties can sometimes be anticipated in one framework, but more readily handled in the other.

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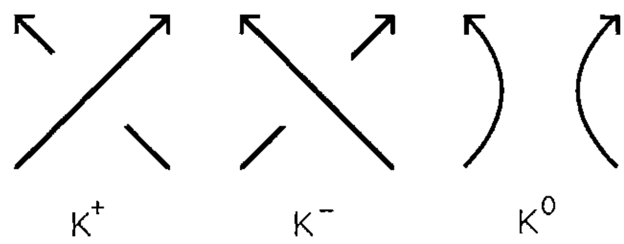
I shall construct P for an oriented link K in the form $P_K(v,z)$, a Laurent polynomial in $Z[v^{\pm 1}, z^{\pm 1}]$. It will satisfy the recurrence relation $v^{-1} P_{K^+} - v P_{K^-} = z P_{K^0}$, where K^+, K^- and K^0 are a triple of links differing only near one crossing as shown in figure 1. With the additional requirement that $P_K = 1$ when K is the unknot this relation can serve as a complete definition of P, as in [LM].

Figure 1



I note the relation here so as to indicate exactly how my choice of variables will relate to this characteristic property of P. It is also worth noting how to change the variables in order to get some useful specialisations and equivalent forms of P.

Alexander polynomial $\Delta_K(t)$;	$v = 1, z = t^{1/2} - t^{-1/2}$.
Conway polynomial $\nabla_K(z)$;	$v = 1, z = z$.
Jones polynomial $V_K(t)$;	$v = 1, z = t^{1/2} - t^{-1/2}$.
Lickorish-Millett polynomial $P_K(\ell,m)$;	$v = (i\ell)^{-1}, z = im$.

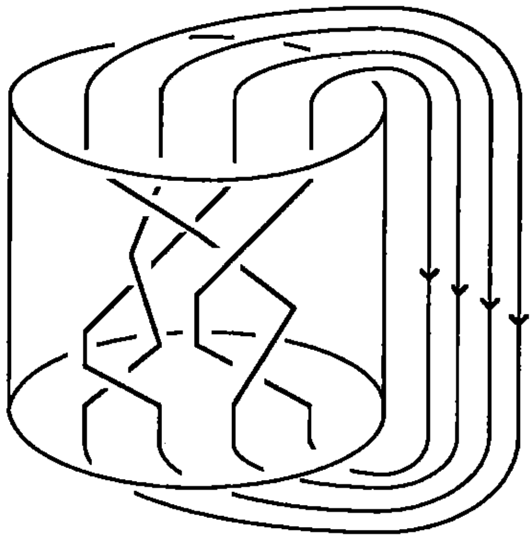
2. BRAIDS.

In constructing an invariant of a knot K we may look for something which clearly depends fairly intrinsically on K, such as the group $G_K = \pi_1(S^3 - K)$, or we may represent K in some way, associate something to the representation, an element of some ring for example, and then show that the result is independent of the choice of representation. There are, for example, invariants which can be described initially in terms of a plane projection of K and then shown to be invariant under the Reidemeister moves, which generate all other projections of K.

Here I shall give a construction of P_K based on closed braid representations of K. A braid on n strings, (β, n), is an embedding of n oriented intervals (strings) in $D^2 \times I$ joining n points $q_i \times \{0\}$, $i = 1,\dots,n$, in the bottom disc $D^2 \times \{0\}$ to the points $q_i \times \{1\}$, possibly permuted, so that the last coordinate increases monotonically on each string. Each intermediate disc $D^2 \times \{t\}$ then meets the strings in exactly n points.

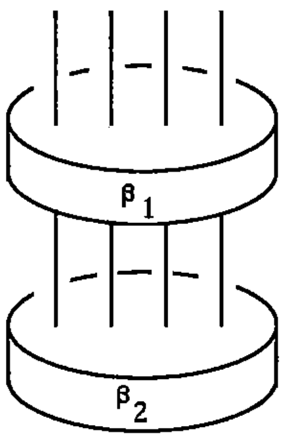
The closure of (β,n) is the link $\hat{\beta}$ formed by joining the top points to the bottom as shown in figure 2. It is given a natural orientation from the increasing direction of the last coordinate within the braid.

Figure 2



Composition of braids on n strings is defined by stacking two copies of $D^2 \times I$, as in figure 3.

Figure 3



The equivalence classes of braids under ambient isotopy of the strings keeping their ends fixed then forms a group under the induced composition, known as Artin's Braid Group, B_n.

Closed braid representations are controlled by two main results:

THEOREM (Alexander)
Every oriented link K can be represented as the closure $\beta\hat{}$ of some braid (β, n).
THEOREM (Markov)
Any two braids whose closures are the same oriented link, up to isotopy, are related by a sequence of moves of two types:

I $(\beta, n) \sim (\alpha^{-1}\beta\alpha, n)$
II $(\beta, n) \sim (\beta\sigma_n^{\pm 1}, n+1)$.

A recent account of these two results is given in [M3].

Both Jones' original polynomial $V_K(t)$ and also $P_K(v,z)$ can be defined by showing how to find $P_\beta(v,z)$ for each braid (β,n) which is unchanged when β is altered by Markov's moves, either of type I (conjugation in B_n) or of type II (changing the number of strings). This was the route originally adopted by Jones in his construction of V_K, and extended by Ocneanu to produce P_K, [J2]. To use it most directly for these purposes I shall construct an algebra (but not a von Neumann algebra) in which we can naturally represent B_n.

3. HECKE ALGEBRA CONSTRUCTION.

The relation of B_n to the symmetric group S_n is exploited here. We may think of S_n as a Coxeter group generated by the transpositions $\tau_i = (i,i+1)$, $1 \le i \le n-1$, with relations

(1) $\tau_i^2 = e$,

(2) $\tau_i \tau_j = \tau_j \tau_i$, $|i-j| > 1$,

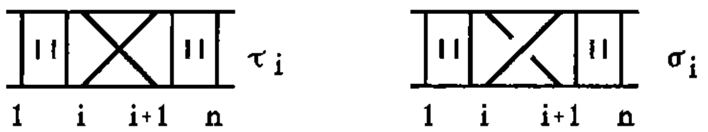
(3) $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$.

The braid group B_n has a corresponding presentation (as the Artin group for S_n), by dropping the relations (1), and taking generators σ_i, $1 \le i \le n-1$, to satisfy relations

(2') $\sigma_i \sigma_j = \sigma_j \sigma_i$, $|i-j| > 1$,

(3') $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

The pictorial representation of τ_i corresponds well with the representation of σ_i as a geometric braid.



The homomorphism $\sigma_i \mapsto \tau_i$ from B_n to S_n then carries a geometric braid to the permutation induced on the end points by the connecting strings.

The Hecke algebra $H_n(z)$ is constructed as an algebra over the ring $Z[z]$ having generators c_i, $1 \le i \le n-1$, which satisfy relations (2) and (3) and a modification of (1). These relations are explicitly

(1") $c_i^2 = z c_i + 1$,

(2") $c_i c_j = c_j c_i$, $|i-j| > 1$,

(3") $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$.

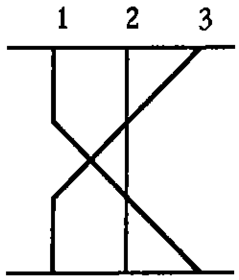
In working with $H_n(z)$, notice that elements are simply linear combinations of monomials in $c_1,\dots,c_{n-1}$ with coefficients in $Z[z]$, where multiplication is controlled by the relations (1")-(3"). If the coefficient ring is specialised by putting $z = 0$ the relations just become those of S_n, with c_i in place of τ_i, and H_n becomes the group algebra $Z[S_n]$. Many structural properties of this algebra are retained by $H_n(z)$, which may be thought of as a perturbation of $Z[S_n]$. This approach is particularly fruitful in the more analytic context where the ring Z is extended to C, and z is viewed as a complex parameter.

4. A LINEAR BASIS FOR $H_n(z)$.

The group algebra $Z[S_n]$ has a Z-linear basis consisting of the permutations, $\{\pi\}, \pi \in S_n$. It is not difficult to construct a corresponding basis $\{\beta_\pi\}, \pi \in S_n$, for $H_n(z)$.

CONSTRUCTION. Write $\pi \in S_n$ as a monomial $\pi(\tau_1,\dots,\tau_{n-1})$ of minimal length, $\ell(\pi)$ say, in the generators $\tau_1,\dots,\tau_{n-1}$ of S_n. This length $\ell(\pi)$ is known as the <u>Bruhat length</u> of π, and can be read from a diagrammatic view of π where points $1,\dots,n$ are joined by lines to the points $\pi(1),\dots,\pi(n)$ respectively so that each pair of lines crosses at most once. Then $\ell(\pi)$ is the total number of crossings of the lines, assuming that only simple crossings occur. The case where $\pi = (1,3)$ is illustrated in figure 4.

Figure 4



Transposition

(1 3)

Any two minimal length monomials for π are related by a sequence of applications of the relations (2) and (3) only.

Now put $b_\pi = \pi(c_1,\dots,c_{n-1}) \in H_n(z)$, using any of the minimal length monomials for π. Since the relations (2") and (3") in $H_n(z)$ exactly correspond with (2) and (3) in S_n this definition depends only on π.
In particular we have $c_i = b_{(i,i+1)}$.

PROPOSITION 1. $H_n(z)$ is linearly generated by $\{b_\pi\}, \pi \in S_n$.

PROOF. Since $H_n(z)$ is generated as an algebra by $c_1,\dots,c_{n-1}$ it is enough to show that $b_\pi c_i$ is a linear combination of $\{b_\pi\}$ for all π, i. Take $\rho = \pi\tau_i$ and write $\pi = \pi(\tau_1,\dots,\tau_{n-1})$ as a minimal length word. Then either the monomial $\pi(\tau_1,\dots,\tau_{n-1})\tau_i$ is a minimal length monomial for ρ, when we have $b_\rho = b_\pi c_i$, or π can itself be written as $\rho(\tau_1,\dots,\tau_{n-1})\tau_i$, where $\rho(\tau_1,\dots,\tau_{n-1})$ is a minimal length

monomial for ρ. In this second case we have $b_\pi = b_\rho c_i$, so
$b_\pi c_i = b_\rho c_i^2 = z b_\rho c_i + b_\rho = z b_\pi + b_\rho$.

In either case b_π is a linear combination of the proposed generators. A further analysis, as in the exercices in [Bo] p53, can be given to show that the elements b_π form a free basis for $H_n(z)$.

5. REPRESENTATIONS OF B_n.

The elements c_i are clearly invertible, with $c_i^{-1} = c_i - z$. We then have a representation $\rho_1 : B_n \longrightarrow H_n(z)$ defined by $\rho_1(\sigma_i) = c_i$.

If we extend the coefficient ring to $Z[z,v^{\pm 1}]$ we may equally give a representation ρ_v by $\rho_v(\sigma_i) = v c_i \in H_n(z,v) = H_n(z) \otimes Z[z,v^{\pm 1}]$. An element $\beta = w(\sigma_1^{\pm 1},\ldots,\sigma_{n-1}^{\pm 1}) \in B_n$ is then represented by

$$\rho_v(\beta) = v^k w(c_1^{\pm 1},\ldots,c_{n-1}^{\pm 1}) = v^k \sum_{\pi \in S_n} a_\pi(z)\, b_\pi,$$

where $a_\pi \in Z[z]$ and $k = c\sim(\beta)$, the exponent sum of the braid β. I shall abuse notation by simply writing β in place of $\rho_v(\beta)$.

6. TRACE FUNCTION.

The other main ingredient in this approach is a linear trace function on H_n. Such a function played a very natural part in Jones' original algebras from the point of view of von Neumann algebras. In this purely algebraic context it proves possible to use the knowledge of the linear structure of H_n and the subalgebra $H_{n-1} \subset H_n$ generated by $c_1,\ldots,c_{n-2}$ to define a (unique) function

$\mathrm{Tr} : H_n(v,z) \longrightarrow Z[z,v^{\pm 1},T]$ with the following properties.

(4) Tr is linear,

(5) $\mathrm{Tr}(ba) = \mathrm{Tr}(ab)$,

(6) $\mathrm{Tr}(1) = 1$,

(7) $\mathrm{Tr}(x c_{n-1}) = T\,\mathrm{Tr}(x)$ for $x \in H_{n-1}$.

A detailed proof of the existence of Tr, for any choice of T, following Ocneanu's construction, is given by Jones in [J2]. An immediate consequence of the definition, since $c_{n-1}^{-1} = c_{n-1} - z$, is that

$$\mathrm{Tr}(x c_{n-1}^{-1}) = \mathrm{Tr}(x c_{n-1}) - \mathrm{Tr}(zx) = (T-z)\,\mathrm{Tr}(x) \quad \text{for any } x \in H_{n-1}.$$

7. DEFINITION OF THE INVARIANT P_K.

Let us now try to define an element P_β for $(\beta,n) \in B_n$ by $P_\beta = k_n \mathrm{Tr}(\beta)$, choosing some normalising factor k_n so that P_β is unaltered when β is changed by Markov moves. If we can do this, then P_β will depend only on $K = \beta\hat{}$.

There is no problem when β is altered by conjugacy in B_n (Markov move I), since $\mathrm{Tr}(ba) = \mathrm{Tr}(ab)$. To ensure invariance of P under Markov move II we

require $k_n \mathrm{Tr}(\beta) = k_{n+1}\mathrm{Tr}(\beta\sigma_n) = k_{n+1}\mathrm{Tr}(\beta\sigma_n^{-1})$ for $\beta \in H_n$.
Now $\qquad \mathrm{Tr}(\beta\sigma_n) = v\mathrm{Tr}(\beta c_n) = vT\,\mathrm{Tr}(\beta)$,
and $\qquad \mathrm{Tr}(\beta\sigma_n^{-1}) = v^{-1}\mathrm{Tr}(\beta c_n^{-1}) = v^{-1}(T-z)\mathrm{Tr}(\beta)$.
We must thus arrange that v, T and z satisfy the equation $vT = v^{-1}(T-z)$.

Formally, we can pass to the quotient of the ring $Z[z,v^{\pm1},T]$ by this relation, or simply imagine that we have chosen $T = z/(1-v^2)$. If we arrange, in addition, that $k_n = vTk_{n+1}$, or equivalently that $k_{n+1} = \delta k_n$, where $\delta = (v^{-1}-v)/z$, then P_β will be invariant under all Markov moves. It is usual to take $k_1 = 1$, so that $k_n = \delta^{n-1}$.

Then $P_\beta = \delta^{n-1}\mathrm{Tr}(\beta)$ depends only on $K = \hat{\beta}$, for any braid (β,n) whose closure is K. For example the unlink with n components can be represented as the closure of $(1,n)$, the identity element of B_n. In this case $P = \delta^{n-1} = z^{-(n-1)}(v^{-1}-v)^{n-1}$.

8. SKEIN RELATION.

It is easy to derive the skein relation $v^{-1}P_{K^+} - vP_{K^-} = zP_{K^0}$ between the invariants for three links K^+, K^- and K^0 which differ only near one crossing as shown in figure 1. This relation is the starting point for other approaches to the definition of P, for example [LM], and a knowledge of P from such an approach can even be used to develop a more geometrically-motivated version of the algebra H_n in the framework of linear skein theory. It is also possible to use Kauffman's polynomial in this manner, to produce and study other algebras of a somewhat similar nature to H_n, [BW].

To establish the relation above arrange a presentation for K^+ as the closure of an n-string braid $\beta\sigma_i$, in which the distinguished crossing appears as σ_i. This can be done with no great difficulty. The links K^- and K^0 can then be represented as the closures of $\beta\sigma_i^{-1}$ and β respectively. Now $c_i - c_i^{-1} = z$ in H_n, so $v^{-1}\sigma_i - v\sigma_i^{-1} = z$ and hence $v^{-1}\beta\sigma_i - v\beta\sigma_i^{-1} = z\beta$ in H_n.
Then $v^{-1}\delta^{n-1}\mathrm{Tr}(\beta\sigma_i) - v\delta^{n-1}\mathrm{Tr}(\beta\sigma_i^{-1}) = z\delta^{n-1}\mathrm{Tr}(\beta)$
giving the skein relation $v^{-1}P_{K^+} - vP_{K^-} = zP_{K^0}$ immediately.

9. P_K AS A POLYNOMIAL.

As described so far, the invariant P_K does not appear to be a polynomial in $Z[v^{\pm1},z^{\pm1}]$, but rather a rational function involving denominators of $1-v^2$. Although this is true of $\mathrm{Tr}(\beta)$, a closer look will show that the factor $k_n = \delta^{n-1}$ is always enough to clear these denominators, so that P_K can be regarded as an element of $Z[v^{\pm1},z^{\pm1}]$. Indeed, with a little more delicacy during the definition it is possible to place P_K sensibly in the ring Λ which is the quotient of $Z[v^{\pm1},z,\delta]$ by the relation $v^{-1}-v = z\delta$.

To establish something of the nature of P, let us first note a result about the image of Tr in $Z[v^{\pm 1},z,T]$.

PROPOSITION 2. $Tr(x) = \sum a_i(z)T^i$, a polynomial of degree $\le n-1$ in T, with coefficients $a_i(z) \in Z[z]$, for $x \in H_n(z)$.

PROOF. It is enough to show this for the generators $b_\pi, \pi \in S_n$. Either $b_\pi \in H_{n-1}$, or it can be written as $b_\pi = x_1 c_{n-1} x_2$ for some $x_1, x_2 \in H_{n-1}$. The result follows by induction on n, directly in the first case, or using the fact that $Tr(x_1 c_{n-1} x_2) = Tr(x_2 x_1 c_{n-1}) = T\,Tr(x_2 x_1)$.

As a consequence, when $\beta = w(\sigma_1^{\pm 1},\ldots,\sigma_{n-1}^{\pm 1})$ we may write $\beta = v^k w(c_1^{\pm 1},\ldots,c_{n-1}^{\pm 1})$, where $k = c\sim(\beta)$ is the exponent sum of β. Then

$$Tr(\beta) = v^k \sum_{i=0}^{n-1} a_i(z)T^i$$

and so $P_\beta = v^k \sum_{i=0}^{n-1} a_i(z)T^i \delta^{n-1} = v^k \sum_{i=0}^{n-1} a_i(z)v^{-i}\delta^{n-i-1}$,

using the relation $T\delta = v^{-1}$. The assignment $\delta = (v^{-1}-v)/z$ may now be made, to arrive unambiguously at a Laurent polynomial in v and z.

10. BRAID INDEX.

From the form of the Laurent polynomial P_K we can demonstrate a lower bound on the braid index of K, the smallest n for which K is the closure of an n-braid.

For if $K = \hat{\beta}$, with $(\beta,n) \in B_n$ then $P_K = P_\beta$ as above. The Laurent degree of P_K in v, $\lambda_v(P_K)$, (the difference between the highest and lowest exponents of v) is the same as that of

$$\sum_{i=0}^{n-1} a_i(z)v^{-i}\delta^{n-i-1} = \sum_{i=0}^{n-1} a_i(z)v^{-i}((v^{-1}-v)/z)^{n-i-1}.$$

Clearly the possible extreme exponents here are $\pm(n-1)$, so $\lambda_v(P_K) \le 2(n-1)$. This gives $\frac{1}{2}\lambda_v(P_K) + 1 \le n$, for any representing braid β, so that $\frac{1}{2}\lambda_v(P_K) + 1$ provides a lower bound for the braid index of K. This bound was noted in [FW] and [M_1].

By way of illustration the case of the knot 9_{42} may be studied, following [MS_1]. Here the Laurent degree of P gives a lower bound of 3 for the braid index, although there is no obvious presentation for 9_{42} as a closed 3-braid. The question of braid index can be resolved here by considering a 2-cable about 9_{42}. If 9_{42} has a 3-braid presentation then any 2-cable will have a 6-braid presentation. Direct calculation of P however for one explicit 2-cable yields a lower bound of 7 for its braid index, forcing 9_{42} to have index at least 4. So even when the bound is not attained for a knot K it can be possible to derive exact information by application to a related knot.

11. IDEMPOTENTS IN H_n.

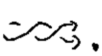
It is interesting to recast H_n in terms of idempotent generators, as it originally appeared in the descriptions of Ocneanu and Jones. In this setting the choice of variables used by Jones shows up in a fairly direct way some results noted recently by Przytycki, [P].

Write the generator c_i of $H_n(z)$ as $c_i = ae_i + bf_i$, where e_i and $f_i = 1-e_i$ are orthogonal idempotents, that is $e_i^2 = e_i$, $f_i^2 = f_i$ and $e_i f_i = 0$. To do this we simply choose a and b as the roots of the equation $x^2 = zx + 1$. Then $ab = -1$ and $a + b = z$. If we write $a = t^{1/2}$ we have $b = -t^{-1/2}$ and $z = t^{1/2} - t^{-1/2}$, with $c_i = t^{1/2} e_i - t^{-1/2} f_i$. The variable t then agrees with Jones' variable t, sometimes written as q, particularly in the context of statistical mechanics operators and the q-state Potts model, where it appears naturally as a fixed integer.

The relation $c_i = (t^{1/2} + t^{-1/2}) e_i - t^{-1/2}$ can be used to recover a presentation of $H_n(z)$ with generators $e_1, \ldots, e_{n-1}$. The defining relations become $e_i^2 = e_i$, $e_i e_j = e_j e_i$, $|i-j| > 1$, and $e_i e_{i+1} e_i - \tau e_i = e_{i+1} e_i e_{i+1} - \tau e_{i+1}$ where $\tau = t/(1+t)^2$. Then z can be defined in terms of τ and vice versa.

As a second variable we may set $\eta = \mathrm{Tr}(e_i)$, and use the fact that $\mathrm{Tr}(c_i) = z/(1-v^2)$ to specify v in terms of t and η. We have $(t^2-v^2)/(1-v^2) = (1+t)\eta$ so that $v^2 = (t-(1+t)\eta)/(1-(1+t)\eta) = (\tau_1 - \eta)/(\tau_2 - \eta)$, where τ_1 and τ_2 are the roots of the polynomial $x^2 - x + \tau$. The parameters τ, η are those used by Ocneanu in the context of subfactors and their index.

The algebra originally used by Jones is the quotient of H_n in this form by the additional relations $e_i e_{i+1} e_i = \tau e_i$. This immediately forces $\mathrm{Tr}(e_i) = \tau$, and hence $\eta = \tau$, requiring $v^2 = t^{\pm 2}$. In his representation of B_n, Jones used $g_i = t^{1/2} c_i = te_i - f_i = (t+1)e_i - 1$ to represent σ_i. In conjunction with the choice of $v = t$ this accounts for the extra normalising factor $t^{1/2 c\sim(\beta)}$ which he needs when calculating V from $\mathrm{Tr}(\beta)$ in this representation, for in the representation described earlier we take $\sigma_i = vc_i = tc_i$ to find V, as compared with $g_i = t^{1/2} c_i$.

The presentation of H_n with idempotent generators give a good viewpoint for Przytycki's results about the way in which P_K can change when the diagram of K is altered by replacing k positive half-twists  with k negative half-twists .

THEOREM (Przytycki). Write K^k and K^{-k} for two links with diagrams related by changing the sense of k half-twists as described above. Let $\phi_k : Z[v^{\pm 1}, z^{\pm 1}] \longrightarrow C[v^{\pm 1}]$ be the homomorphism determined by setting $t^k = 1$, where $z = t^{1/2} - t^{-1/2}$, i.e. put $t^{1/2} = e^{\pi i/k}$ so that $z = 2i \sin \pi/k$. Then $\phi_k(P_{K^k}) = v^{2k} \phi_k(P_{K^{-k}})$.

PROOF. We have $c_i^k = a^k e_i + b^k f_i$ and $c_i^{-k} = a^{-k} e_i + b^{-k} f_i$. Then $c_i^k - c_i^{-k} = (a^k - a^{-k}) e_i + (b^k - b^{-k}) f_i$. Now if we specialise the coefficient ring in H_n to $C[v^{\pm 1}]$ by applying ϕ_k we find that $a^{2k} = b^{2k} = 1$, since $t^k = 1$. Thus $v^{-k}\sigma_i^k - v^k\sigma_i^{-k} = c_i^k - c_i^{-k} = 0$ (modulo ϕ_k) so that $\phi_k(\mathrm{Tr}(v^{-k}\beta\sigma_i^k)) = \phi_k(\mathrm{Tr}(v^k\beta\sigma_i^{-k}))$ for any $\beta \in B_n$. Since $K^{\pm k}$ can be represented by $\beta\sigma_i^{\pm k}$ for some β the theorem follows at once.

Besides the choices of basis of H_n in terms of generators c_i as given above, or the idempotents e_i other choices have been studied in relation to different aspects of H_n. Ocneanu, for example, in one of his presentations of P, [O], uses a basis consisting of braids $\{\alpha_\pi\}$, $\pi \in S_n$, which all close to an unlink, and are conjugate in B_n when the corresponding permutations are conjugate in S_n. The trace of each α_π is then easy to write down, while the description of a product $\alpha_\pi \sigma_i$ in terms of the basis becomes more complicated.

The basis of Kazhdan and Lusztig, [KL], allows the construction of very clean representing matrices corresponding to irreducible representations of S_n and B_n. Its basis elements, like the idempotents e_i, are 'neutral' in the sense that they are unaltered under the change from c_i to c_i^{-1} while the basis $\{b_\pi\}$ is biased towards positive braids. It would be interesting to explore further the possible use of this basis, although it does not appear to be particularly easy to calculate explicitly a single element of B_n in terms of the basis elements, nor to give the trace of the basis elements.

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BIBLIOGRAPHY

[BW] Birman, J.S. and Wenzl, H. "Braids, link polynomials and a new algebra," Preprint, Columbia University, (1986).

[Bo] Bourbaki, N. Groupes et algèbres de Lie, Chapitres 4,5,6. Hermann, Paris, 1968.

[FW] Franks, J. and Williams, R. "Braids and the Jones-Conway polynomial," Preprint, North-Western, (1985).

[FYHLMO] Freyd, P., Yetter, D., Hoste, J., Lickorish, W.B.R., Millett, K.C. and Ocneanu, A. "A new polynomial invariant of knots and links," Bull. Amer. Math. Soc. 12 (1985), 239-246.

[J1] Jones, V.F.R. "A polynomial invariant for knots via von Neumann algebras," Bull. Amer. Math. Soc. 89 (1985), 103-111.

[J2] Jones, V.F.R. "Hecke algebra representations of braid groups and link polynomials," Preprint, Berkeley (1986). To appear in Annals of Math.

[KL] Kazhdan, D. and Lusztig, G. "Representations of Coxeter groups and Hecke algebras," Inv. Math. 53 (1979), 165-184.

[LM] Lickorish, W.B.R. and Millett, K.C. "Topological invariants of knots and links," Preprint (1984). To appear in Topology.

[M1] Morton, H.R. "Closed braid representatives for a link and its 2-variable polynomial," Preprint, Liverpool University (1985).

[M2] Morton, H.R. "Seifert circles and knot polynomials," Math. Proc. Camb. Philos. Soc. 99 (1986), 107-110.

[M3] Morton, H.R. "Threading knot diagrams," Math. Proc. Camb. Philos. Soc. 99 (1986), 247-260.

[MS1] Morton, H.R. and Short, H.B. "The 2-variable polynomial of cable knots," Math. Proc. Camb. Philos. Soc. 101 (1987), 267-278.

[MS2] Morton, H.R. and Short, H.B. "Calculating the 2-variable polynomial of knots presented as closed braids," Preprint, Liverpool University, (1986).

[MT] Morton, H.R. and Traczyk, P. "Knots, skeins and algebras," Preprint, Liverpool University, (1987).

[O] Ocneanu, A. "A polynomial invariant for knots; a combinatorial and algebraic approach," Preprint MSRI (1985).

[P] Przytycki, J. "t_k moves on links," Preprint, Vancouver University, (1986).

[PT] Przytycki, J. and Traczyk, P. "Invariants of links of Conway type," Preprint, Warsaw, (1985).

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