Mutant knots

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March 16, 2014

Abstract

Mutants provide pairs of knots with many common properties. The study of invariants which can distinguish them has stimulated an interest in their use as a test-bed for dependence among knot invariants. This article is a survey of the behaviour of a range of invariants, both recent and classical, which have been used in studying mutants and some of their restrictions and generalisations.

1 History

Remarkably little of John Conway's published work is on knot theory, considering his substantial influence on it. He had a really good feel for the geometry, particularly the diagrammatic representations, and a knack for extracting and codifying significant information. He was responsible for the terms *tangle*, *skein* and *mutant*, which have been widely used since his knot theory work dating from around 1960. Many of his ideas at that time were treated almost as a hobby and communicated to others either over coffee or in talks or seminars, only coming to be written in published form on a sporadic basis.

His substantial paper [9] is quoted widely as his source of the terms, and the comparison of his and the Kinoshita-Teresaka 11-crossing mutant pair of knots, shown in figure 1. While Conway certainly talks of tangles in [9], and uses methods that clearly belong with linear skein theory and mutants, there is no mention at all of mutants, in those words or any other, in the text. Undoubtedly though he is the instigator of these terms and the paper gives one of the few tangible references to his work on knots.

In [9] Conway gives a table of 11 crossing knots, where he reckons to be confident of differences among them, although without explicit invariants in all cases to be certain of this.

The two 11-crossing knots, C and KT, found by Conway and Kinoshita-Teresaka are probably the best-known example of inequivalent mutant knots. Conway's knot is given in his table of 11-crossing knots [9], while KT appears in [19] as one of a family of knots with trivial Alexander polynomial. These two knots are shown in figure 1.

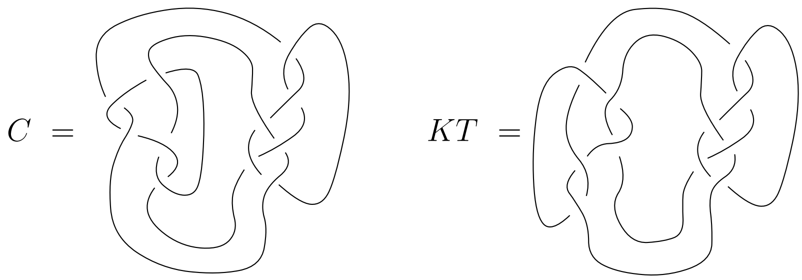


Figure 1: The Conway and Kinoshita-Teresaka mutant pair

The first proof that the two knots in figure 1 are inequivalent was, I believe, given by Riley [39].

Perko [37] tidied up the tables up to 11 crossings, and used double cover techniques in places to distinguish pairs of knots. These methods, however, would not be enough to distinguish a mutant pair, by theorem 3.

Gabai [15] used foliations in showing that C has genus 3 while KT has genus 2. The genus of a knot had until then been a difficult invariant to determine exactly. Gabai's work gave a much wider range of certainty, while in principle the extension of the Alexander polynomial via Heegaard Floer homology gives an exact calculation of the genus.

A recent systematic attempt to document mutant pairs among knots up to 18 crossings has been undertaken by Stoimenow [42]. He also gives comments on the history and techniques available for distinguishing mutants, and the practical limitations for calculations.

Disclaimer

While I have tried to find and credit historical work on mutants I have come across considerable difficulties in even identifying the initial sources of some of the terms, such as *Conway sphere*. I have realised that much of the work has been either in the realm of 'folk-lore', or implicit in places where the authors have not felt it necessary to point up results arising from the general methods being discussed. Certainly one of the benefits of having to read older papers is the realisation of what can be deduced from an understanding of the ideas that underlie the work in question.

I would not want this article to be taken as providing a reliable historical account, and I apologise for any omissions, both in material and in attribution, that I suspect will be found in it.

2 Definitions

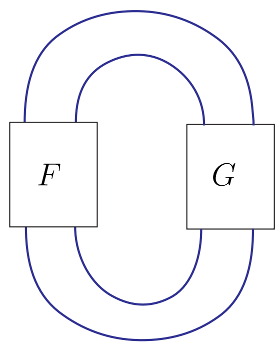
The most commonly used description of mutation is combinatorial, arising directly from Conway's definition of a tangle.

In his setting a tangle is a part of a knot diagram consisting of two arcs contained in a circular region which meet the boundary circle in four diagonally placed points.

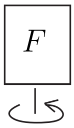
In line with current terminology I shall refer to this as a 2-*tangle*. In general a 2-tangle may contain closed curves as well as the two arcs, but since in this article we will only be considering knots there will not be any occasion to look at 2-tangles with additional curves.

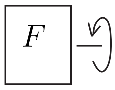
I shall also adjust the diagrams so that the containing region is a rectangle rather than a circle, with two boundary points of the arcs at the top and two at the bottom.

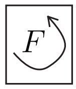
Suppose then that a knot K can be decomposed into two 2-tangles F and G

$K \quad =$


A new knot K' can be formed by replacing the tangle F with the tangle $F' = \rho_i(F)$ given by rotating F through π in one of three ways,

$\rho_1(F) =$ 

, $\rho_2(F) =$ 

, $\rho_3(F) =$ 

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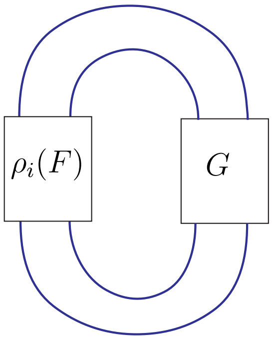
Any of the three knots

$K' =$ 

is called a *mutant* of K.

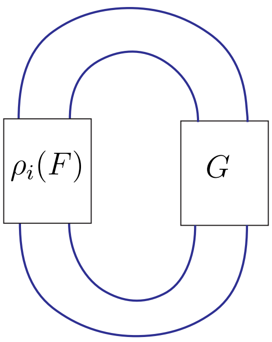
Remark. In my recent paper on mutants with symmetry [33] I have used the notation τ_3, τ_1, τ_2 respectively for these three rotations ρ_1, ρ_2, ρ_3.

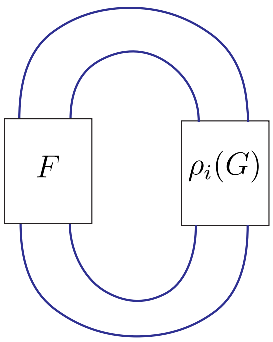
2.1 Equivalence of mutants

As is implicit in the comments above, a knot which can be decomposed into two 2-tangles may well turn out to be equivalent to one or more of the three resulting mutants.

The simplest way in which this may happen is if the tangles F and one or more of $\rho_i(F)$ are equivalent, in other words if F is symmetric under one of the three rotations.

Since the knot K' can be redrawn as

$K' =$ 

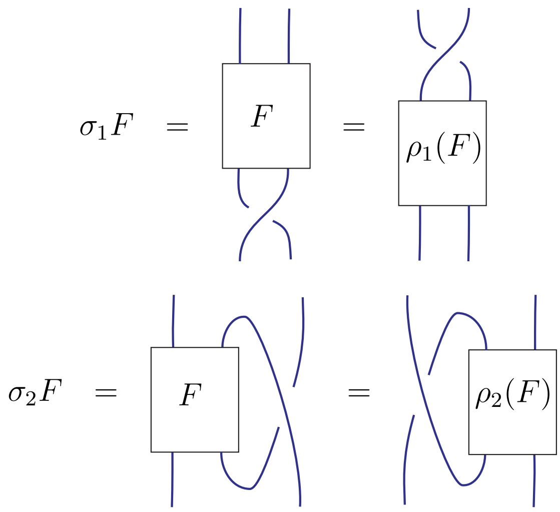
$=$ 

we will equally find that mutants are equivalent if the other tangle G has rotational symmetry. Indeed, if F is symmetric under one of the rotations and G is symmetric under a *different* rotation then all three mutants will be equivalent.

If one of the tangles has all three rotational symmetries then again all the resulting mutants will be equivalent. This is of course the case when the tangle F consists simply of two non-crossing arcs. It is also true where F is a *rational tangle*, in Conway's sense.

Rational tangles arise in Conway's description from nicely arranged consequences of his notation where certain tangles appear in 1-1 correspondence with rational numbers using a continued fraction decomposition.

The 3-string braid group B_3 operates on the set of 2-tangles by braiding the strings coming from three of the boundary points. We can describe the tangles $\sigma_1 F$ and $\sigma_2 F$ by the diagrams



Conway's rational tangles are those which can be generated from a simple tangle without crossings by repeated operation of $\sigma_1^{\pm 1}$ and $\sigma_2^{\pm 1}$.

Theorem 1. *If a 2-tangle F is symmetric under all three rotations then so are the tangles $\sigma_1 F$ and $\sigma_2 F$.*

Proof. After applying each rotation to the tangles $\sigma_1 F$ we can observe that $\rho_1(\sigma_1 F) = \sigma_1 \rho_1(F)$, $\rho_2(\sigma_1 F) = \sigma_1 \rho_3(F)$ and $\rho_3(\sigma_1 F) = \sigma_1 \rho_2(F)$. Similar relations hold for the rotations of $\sigma_2 F$. ■

Corollary 2. *Rational tangles are symmetric under all three π-rotations ρ_i.*

Remark. Stoimenow makes use of this fact in his searches for mutant pairs among knots up to 18 crossings [42], as he is able to exclude decompositions in which one of the tangles is rational. By corollary 2 any mutation of such a decomposition does not produce a different knot.

2.2 A three-dimensional view

There is a natural way of looking at mutants in a three-dimensional context, involving embedded 2-spheres in S^3 meeting a knot transversely in four points. Ruberman [41] adopted the current term *Conway sphere* for such an embedded sphere.

The two 3-balls which lie on either side of a Conway sphere then correspond to a decomposition of the knot into two 2-tangles, although there will be a choice involved in representing each of these by a diagram. In effect the diagram will be determined up to the action of the braid group B_3 on the punctures on the sphere.

If the knot in S^3 is regarded as an orbifold with cone angle π along the knot then the Conway spheres play a natural role in the theory of orbifold decompositions, which is mirrored by their torus covers in the 2-fold cyclic cover of S^3 branched over the knot. Bonahon and Siebenmann [6] prove a uniqueness result for orbifold decompositions in a general setting, Their results apply in this case with suitable Conway spheres providing the counterpart to the tori in the Jaco-Shalen decomposition of the covering manifold.

The following result is noted by Viro [44], who uses the term *twin* rather than mutant.

Theorem 3. *The double covers of S^3 branched over mutant knots are homeomorphic.*

Proof. The double cover of a knot and its mutant by ρ_i are constructed from the double covers of the two constituent tangles by gluing along the torus covering the Conway sphere. The two double covers then differ by the homeomorphism of the torus which covers ρ_i. For each i this homeomorphism is isotopic to the identity. ■

In his paper analysing the behaviour of Conway spheres in knots Lickorish [23] uses the term *untangled* to denote a tangle which is homeomorphic to the trivial tangle, that is, where there is a homeomorphism of the 3-ball carrying the two arcs inside the Conway sphere to a pair of parallel unknotted arcs.

Such tangles are exactly the rational tangles of Conway. (Viro notes that any π-rotated untangled tangle is isotopic to the original tangle by an isotopy which fixes the boundary sphere.)

It is then easy to give a quick proof of the following result (see also the proof of Rolfsen [40]).

Theorem 4. *The only mutant of the unknot is the unknot.*

Proof. Suppose that we have a Conway sphere meeting the unknot in four points. The fundamental group of the four-punctured sphere is free on 3 generators. This cannot inject into both fundamental groups of the tangles on the two sides, otherwise it would inject into the fundamental group of the knot complement. Hence, by Dehn's lemma, there is a non-trivial closed curve on the Conway sphere which bounds a disc disjoint from the arcs in one of the tangles. This disc must separate the two arcs in the ball. Each arc must be unknotted, as it is then a connected summand of the unknot. Hence this tangle is untangled in the sense of Lickorish. It is then a rational tangle and is symmetric under all three rotations, and so the mutants are all equivalent. ■

We have seen here that certain tangle decompositions will only give rise to equivalent mutants. When analysing a tangle decomposition a first check should then be made on the possible symmetries of the constituent tangles. On the other hand, when we suspect that two mutants may not be equivalent, there remains the question of showing that they are indeed different.

In the next section I shall give a selection of classical methods, both geometric and algebraic, which have been been used to distinguish between mutant pairs, and some early limitations which were noted. In the following sections I shall give some of the known invariants which all mutants must share, and further conditions under which a greater range of invariants are shared. In this way the use of mutants contributes a means of looking at possible relations between new and existing invariants, in terms of the extent to which they may agree on various classes of mutants.

3 Classical ways to distinguish mutants

Here the term 'classical' refers to techniques that were in use up to the discovery of the Jones polynomial in 1984. I have loosely separated the methods used under the general headings of algebraic and geometric.

3.1 Algebraic

The simplest method of distinguishing knots is by means of the Alexander polynomial, which can be calculated readily from a Seifert matrix for the knot.

Viro [44] uses comparable Seifert matrices for mutant knots to observe that their Alexander polynomials are the same, along with the homology groups and forms of linking coefficients in branched covers, Minkowski units and signatures, all of which can be found from a Seifert matrix. The Seifert matrices constructed by Viro are either identical or have the form

$$S = \begin{pmatrix} A & C & 0 \\ C^T & a & D^T \\ 0 & D & B \end{pmatrix}, S' = \begin{pmatrix} A & C & 0 \\ C^T & a & D^T \\ 0 & D & B^T \end{pmatrix},$$

where A and B are square matrices, C and D are column matrices.

Viro proves that the Whitehead doubles of the connected sums $K\#K$ and $K\#K^r$ are inequivalent mutants when K is a knot which is inequivalent to its reverse K^r. This follows since equivalence of the Whitehead doubles implies equivalence of the connected sums, and decomposition of connected sums is unique.

He notes also that the whole series of knots with trivial Alexander polynomial described by Kinoshita and Teresaka in [19] all have obvious mutants although he does not give a systematic way of ensuring that all of these pairs are inequivalent.

In the absence of information from the Alexander polynomial, the most basic algebraic way to show that two knots are inequivalent is to compare more directly their groups, in other words the fundamental groups of their complements.

Apart from questions of mirror images and orientation, two inequivalent mutants will have non-isomorphic groups, by the general results of Gordon and Luecke. There is then a good chance of detecting a difference by comparing homomorphisms of their groups into suitable finite groups.

Early distinctions among knots were made by this method by Riley [39], who separated the Conway and Kinoshita-Teresaka 11-crossing knots by means of homomorphisms from the knot group to $PSL(2,7)$. Riley treats this group as a subgroup of the symmetric group S_7, and considers representations in which meridians are mapped to 7-cycles. Calculation of the homology groups of the resulting 7-fold coverings branched over the knot demonstrates a difference between the two knots.

The use of homomorphisms of knot groups to finite groups is known generally as *knot colouring.* These methods include the classical 3-colouring and n-colouring, where the finite group used is the dihedral group D_n. These techniques were much used in their original form by Fox, who extended the methods in [14]. The techniques are essentially those developed, notably by Fenn and Rourke [13], under the current term of *quandle.* It is immediate that n-colouring will not distinguish mutant pairs, since the existence of an n-colouring depends on the Alexander polynomial, and Alexander polynomials are shared by mutants.

3.2 Geometric

Geometric methods available for distinguishing knots include comparison of related 3-dimensional manifolds covering S^3 and branched over the knot in different ways.

3.2.1 Covers

The simplest of such constructions is the double cover. Viro gives a nice summary of the behaviour of mutants, referred to as *twins* in [44]. He proves in Theorem 3 that the double covers of S^3 branched over two mutant knots are homeomorphic. So something more elaborate is needed to distinguish mutants by this type of argument.

More complicated covers can be related to homomorphisms from the knot group to a finite group, as in Riley's arguments in [39], which proved to be effective in distinguishing between the 11-crossing knots C and KT.

3.2.2 Genus

An early geometric invariant of a knot is its *genus*, which is the least genus among orientable surfaces spanning the knot in S^3. The simplicity of its definition has made it a popular invariant, but it is not easy to calculate in general. It can be bounded below in terms of the Alexander polynomial, but in many cases this bound is not exact. Since the Alexander polynomial agrees on mutants this bound will not be helpful in distinguishing mutants.

It is a surprise that the genus of mutants can differ. Gabai developed techniques for calculating the genus, based on the use of foliations, which gave an early distinction between C and KT. He showed in [15] that C has genus 3 while KT has genus 2.

It is only much more recently that the Heegaard-Floer homology of a knot has provided an exact calculation of the genus in all cases.

3.2.3 Diagrammatic invariants

Although the genus can be different for mutants, it is not known whether invariants such as the crossing number, the braid index or the arc index, can ever differ on mutants.

The best hope for settling any of these questions would be to give a direct argument that mutants must have the same braid index (defined as the least number n of strings needed to present the knot as a closed n-braid). Attempts to show that two mutants have different braid index run up against the difficulty that one of the best ways to find a lower bound for the braid index relies on the use of knot polynomials and many of these are shared by mutants.

3.2.4 Hyperbolic geometry

Bonahon and Siebenmann, along with others from Orsay, analysed the structure of classes of knots with the goal of extending and systematising Conway's constructions. Their original work was contained in an influential series of notes, which never itself formed a complete publication, although much is available in their ongoing draft monograph [7]. In the course of their work they made much use of Conway sphere-based decompositions. This culminated in an extensive analysis of knots from the point of view of orbifolds, where the knot formed a subset with cone angle π. These early geometric observations for knots were used in [6] to formulate an orbifold decomposition theorem which is a counterpart to the Jaco-Shalen-Johannson decomposition for 3-manifolds.

Recent work on this, and related bibliographies can be found in work of Boileau et al [5] and Paoluzzi [36], for example.

With the advent of Thurston's work on hyperbolic and other geometric structures on 3-manfolds there followed a more systematic view of knot complements from a geometric point of view.

In particular the default position for a knot complement in the absence of certain special features turns out to be that there is a complete hyperbolic structure of finite volume on the complement. The volume is an invariant of the knot. However Ruberman [41] shows that if a knot K is hyperbolic, then any mutant is also hyperbolic, and they have the same hyperbolic volume.

Weeks [45] developed the amazingly powerful program SnapPea to calculate details of hyperbolic manifolds, including the volume and other details making up an invariant 'canonical structure'. While the volume on its own is not enough, it is possible to use the canonical structures to distinguish inequivalent mutants.

3.2.5 Symmetry

In the same paper [41] Ruberman remarks that the work of Bonahon and Siebenmann [6, 7] ensures that if two mutant knots are equivalent then there must be some rotational symmetries in the constituent 2-tangles. In principle then mutants can be distinguished by showing that the tangles have no suitable symmetry. Subsequently Ruberman and Cochran [8] were able to find a means of ruling out symmetry in some tangles, and apply it to give examples of inequivalent mutants.

4 Polynomials and quantum invariants

The enormous range of invariants which followed the discovery of the Jones polynomial and its generalisations from 1984 onwards has made available many further theoretical and practical ways of comparing knots. Besides using the new invariants to compare mutants it has also proved fruitful to regard mutants and their refinements as a tool for analysing possible dependence among invariants.

Very shortly after the discovery of the new invariants Lickorish proved, using simple skein theoretic arguments, that mutants must also have identical Homfly and Kauffman polynomials, and hence the same Jones polynomial. A good account of this can be found in his survey article [24].

Calculations of Morton and Short for a number of examples led to the conjecture [31] that two equally twisted 2-cables of a mutant pair would also share the same Homfly polynomial. This was proved by Lickorish and Lipson [25], also using skein theory. They showed further that the same result holds for reverse-string 2-cables (that is, for 2-cables of two components whose orientations run in opposite directions, giving a 'reverse parallel' satellite).

This holds equivalently for equally twisted Whitehead doubles. These results were also derived independently by Przytycki [38].

Although the Homfly polynomials of doubles or 2-cables were found not to distinguish mutants it was already clear that invariants of more complicated satellites of knots could provide extra information in general.

4.1 Homfly invariants

In 1984 V.F.R.Jones constructed a new invariant of oriented links $V_L(t) \in \mathbf{Z}[t^{\pm\frac{1}{2}}]$, which turned out to have the property that

$$t^{-1}V_{L_+} - tV_{L_-} = (\sqrt{t} - 1/\sqrt{t})V_{L_0} \tag{1}$$

for links $L_\pm$ and L_0 related as in the Conway polynomial relation. This was quickly extended to a 2-variable invariant $P_L(v, z) \in \mathbf{Z}[v^{\pm1}, z^{\pm1}]$, with the property that

$$v^{-1}P_{L_+} - vP_{L_-} = zP_{L_0}. \tag{2}$$

The name 'Homfly polynomial' has come to be attached to P, being the initial letters of six of the eight people involved in this further development. The name is sometimes extended to the more unwieldy 'HOMFLYPT', to make reference to all eight. The polynomial P contains both the Conway/Alexander polynomial, and Jones' invariant, and can be shown to contain more information in general than both of these taken together. It satisfies the equations

$$\begin{aligned} P(1, z) &= \nabla(z) \\ P(1, s - s^{-1}) &= \Delta(s^2) \\ P(s^2, s - s^{-1}) &= V(s^2) \\ P(s, s - s^{-1}) &= \pm 1 \end{aligned}$$

The skein relation (2) can readily be shown to determine P and V once its value on the trivial knot is given. It has been usual to take $P = 1$ on the trivial knot, although in some recent applications a different normalisation can be more appropriate.

Given the existence of V and P we can then make some calculations. For example, the unlink with two components has

$$\begin{aligned} P &= \frac{v^{-1} - v}{z}, \\ V(s^2) &= -(s + s^{-1}), \end{aligned}$$

while the Hopf link with linking number +1 has

$$\begin{aligned} P &= vz + (v^{-1} - v)v^2 z^{-1}, \\ V(s^2) &= s^3 - s - (s + s^{-1})s^4 = -s(1 + s^4). \end{aligned}$$

The Hopf link with linking number −1 has

$$\begin{aligned} P &= -v^{-1}z + (v^{-1} - v)v^{-2} z^{-1}, \\ V(s^2) &= -s^{-1}(1 + s^{-4}). \end{aligned}$$

This illustrates the general feature that for the mirror image $\overline{L}$ of a link L, (where the signs of all crossings are changed), we have $P_{\overline{L}}(v, z) = P_L(v^{-1}, -z)$ and so $V_{\overline{L}}(s^2) = V_L(s^{-2})$. It is thus quite possible to use V in many cases to distinguish a knot from its mirror-image, while there will be no difference in their Conway polynomials. It is worth noting that although there are still knots which cannot be distinguished from each other by P in spite of being inequivalent, no non-trivial knot has so far been found for which $P = 1$, or even $V = 1$.

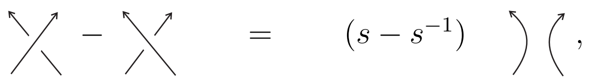
4.1.1 Framed versions

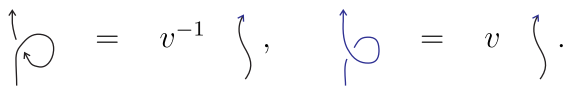
The original Homfly polynomial is invariant under *all* Reidemeister moves, but there is a convenient version which is an invariant of a framed oriented link. A more extended discussion of the exact choice of framing normalisations can be found elsewhere, [2, 27, 26].

In its most adaptable form, $P_L(v, s)$, the framed invariant lies in the ring

$$\Lambda = \mathbf{Z}[v^{\pm 1}, s^{\pm 1}, (s^r - s^{-r})^{-1}], r > 0.$$

Its defining characteristics are the two local skein relations.

1. 

2. 

These relate the invariants of links whose diagrams differ only locally as shown.

They are enough to allow its recursive calculation from simpler diagrams in terms of the value for the unknot.

4.2 Satellite invariants

Invariants such as the Homfly polynomial P of any choice of satellite of a knot K may be regarded as invariants of K itself. These provide a whole range of satellite invariants, which can be compared for mutants K and K'.

4.2.1 Framed links

Framed links are made from pieces of ribbon rather than rope, so that each component has a preferred annulus neighbourhood.

Combinatorially they can be modelled by diagrams in S^2 up to the Reidemeister moves R_{II} and R_{III}, excluding R_I, by use of the 'blackboard framing' convention. The ribbons are determined by taking parallel curves on the diagram. Reidemeister moves R_{II} and R_{III} on a diagram give rise to isotopic ribbons. Any apparent twists in a ribbon can be flattened out using R_I.

Oriented link diagrams D have a *writhe* $w(D)$ which is the sum of the signs of all crossings. This is unchanged by moves R_{II} and R_{III}.

The unframed version of the Homfly polynomial for an oriented link L, invariant under *all* Reidemeister moves, is given from this framed version by $v^{w(D)}P_L(v,s)$ where D is a diagram for the framed link.

Remark. For a framed knot the writhe is sometimes called its 'self-linking number', which is independent of the orientation of the diagram. Generally a framing of a link is determined by a choice of writhe for each component.

4.2.2 Satellites

A *satellite* of a framed knot K is determined by choosing a diagram Q in the standard annulus, and then drawing Q on the annular neighbourhood of K determined by the framing, to give the satellite knot $K*Q$. We refer to this construction as *decorating K with the pattern Q* (see figure 2).

Morton and Traczyk [32] showed that the Jones polynomial V cannot be used in combination with any choice of satellite to distinguish a mutant pair, K and K'. Thus $V_{K*Q} = V_{K'*Q}$ for any choice of pattern Q, provided that the same framing of K and K' is used.

4.2.3 A parameter space for Homfly satellite invariants

The local nature of the Homfly skein relations allows us to make a useful simplification in studying Homfly satellite invariants P_{K*Q} as the pattern Q

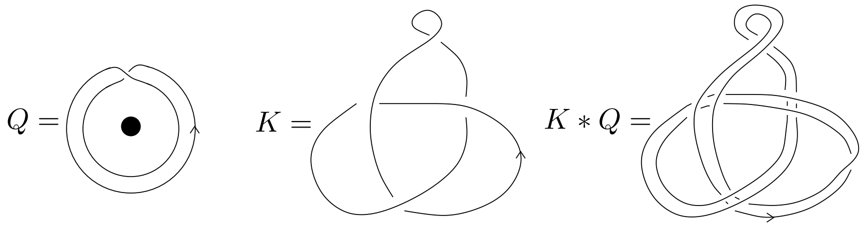
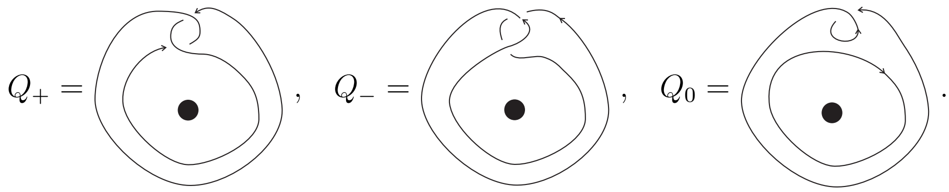


Figure 2: Satellite construction

varies.

Compare for example three patterns $Q_{\pm}$ and Q_0.

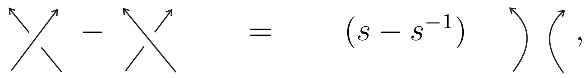


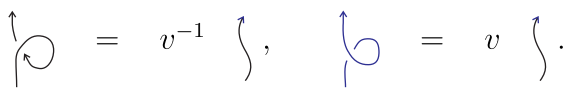
The framed Homfly invariants of $K * Q_{\pm}$ and $K * Q_0$ then satisfy

$$P_{K*Q_+} - P_{K*Q_-} = (s - s^{-1}) P_{K*Q_0}.$$

Since $K * Q_-$ is the unknot for any K, this relates the invariants of the Whitehead double $K * Q_+$ of K and those of its reverse parallel.

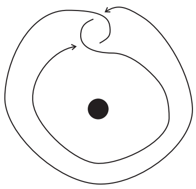
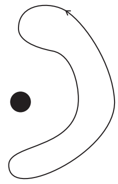
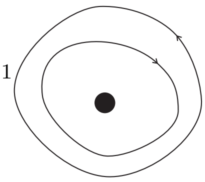
More generally, consider the linear space $\mathcal{C}$ of Λ-linear combinations of diagrams in the annulus (up to R_{II}, R_{III}) and impose the local relations

1. 

2. 

Decorating K by an element $\sum a_i Q_i$ of the linear space $\mathcal{C}$, which is known as the *framed Homfly skein of the annulus*, gives a well-defined Homfly invariant $\sum a_i P_{K*Q_i}$ since the skein relations are respected when the Homfly polynomials of the satellites are compared.

We can summarise our calculation above by saying that in the skein $\mathcal{C}$ we have

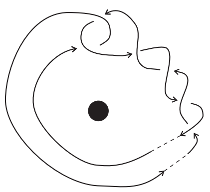
 $=$  $+ (s - s^{-1})v^{-1}$ ,

and hence

$$P_{K*Q_+} = P_{\text{unknot}} + (s - s^{-1})vP_{\text{reverse parallel}}.$$

The space $\mathcal{C}$ then gives a more effective parameter space for satellite invariants, as we only need to know the pattern as an element of $\mathcal{C}$.

For example, any of the twist patterns



is a linear combination of the reverse parallel and the trivial pattern, so the Homfly polynomial of any twisted double can be found from the reverse parallel.

The linear subspaces $\mathcal{C}_m$ for $m > 0$ within $\mathcal{C}$ spanned by the closure of oriented m-tangles with all m ends of arcs directed in the same way are finite dimensional. The space $\mathcal{C}_m$ has a basis consisting of elements Q_λ, one for each partition λ of m. These basis elements Q_λ play an important role in relating Homfly satellite invariants to unitary quantum group invariants,as mentioned in the next section. For a more detailed account of their definition, and interpretation, see the recent article by Morton and Manchon [28], or earlier work of Aiston and Lukac [1, 2, 26].

4.2.4 Homfly satellite invariants on mutants

The results of Lickorish and Lipson [25] show that $P_{K*Q} = P_{K'*Q}$ for mutants K, K' when the pattern Q is the closure in the annulus of any 2-tangle, for example the twist pattern above, and P is the Homfly polynomial.

A contrasting result occurs when the pattern Q is a closed 3-tangle. Homfly invariants of 3-parallels were realised at an early stage to give possibilities for distinguishing mutants.

Calculations made in 1986 by Morton and Traczyk showed that the Homfly polynomials P_{C*Q} and P_{KT*Q} are different for the 3-parallel pattern Q. Since the computing facilities available were limited they fixed the value of one variable and reduced the integer coefficients mod p for some small fixed value of p. Although they were able to establish that the two polynomials were different it was not easy to appreciate the extent and nature of the difference from their calculations. Jun Murakami [34] also made calculations based on 3-parallels of other mutant pairs, and gave necessary conditions for Homfly-based satellite invariants to distinguish mutants. These involve identification of dimension 1 subspaces in representation theory.

Subsequent more sophisticated calculations by Cromwell and Morton [27] give much more detail. Their method of calculation involves a truncation which amounts to retaining only Vassiliev invariants up to a certain type, in this case type 12 is enough. Such a truncation at a fixed type is very easily implemented in terms of the calculations based on the Morton-Short algorithm for finding Homfly polynomials [31], and it gives a very satisfactory outcome when the difference of the invariants for two mutants is studied.

4.2.5 Kauffman polynomial

The 2-variable Kauffman polynomial, discovered shortly after the appearance of the Homfly polynomial, also has the property that it does not distinguish mutants. Nor does it distinguish the 2-parallels of mutants.

Rather less work has been done on establishing which Kauffman satellite invariants can distinguish mutants. Like the Homfly polynomial its satellite invariants are closely related to quantum group invariants [46]. More recent calculations have been made by Stoimenow [42] who has shown that the Kauffman polynomial of the 3-parallel can distinguish some mutants with symmetry, in contrast to the corresponding Homfly polynomial.

5 Unitary quantum group invariants

Following closely after the discovery of the Homfly and Kauffman polynomial invariants came the work of Reshetikhin and Turaev on the development of knot invariants based on quantum groups.

Quantum groups give rise to 1-parameter invariants $J(K;W)$ of an oriented framed knot K depending on a choice of finite dimensional module W over the quantum group, following constructions of Turaev and others [43, 46]. This choice is referred to as *colouring* K by W, and can be extended for a link to allow a choice of colour for each component.

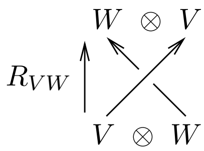
5.1 Basic constructions of quantum invariants

A quantum group $\mathcal{G}$ is an algebra over a formal power series ring $\mathbf{Q}[[h]]$, typically a deformed version of a classical Lie algebra. A finite dimensional module over $\mathcal{G}$ is a linear space on which $\mathcal{G}$ acts.

Crucially, $\mathcal{G}$ has a coproduct Δ which ensures that the tensor product $V \otimes W$ of two modules is also a module. It also has a *universal R-matrix* (in a completion of $\mathcal{G}\otimes\mathcal{G}$) which determines a well-behaved module isomorphism

$$R_{VW} : V \otimes W \to W \otimes V.$$

This has a diagrammatic view indicating its use in converting coloured tangles to module homomorphisms.



A braid β on m strings with permutation $\pi \in S_m$ and a colouring of the strings by modules $V_1, \ldots, V_m$ leads to a module homomorphism

$$J_\beta : V_1 \otimes \cdots \otimes V_m \to V_{\pi(1)} \otimes \cdots \otimes V_{\pi(m)}$$

using $R^{\pm 1}_{V_i,V_j}$ at each elementary braid crossing. The homomorphism J_β depends *only on the braid* β itself, not its decomposition into crossings, by the Yang-Baxter relation for the universal R-matrix.

When $V_i = V$ for all i we get a module homomorphism $J_\beta : W \to W$, where $W = V^{\otimes m}$. Now any module W decomposes as a direct sum $\bigoplus (W_\mu \otimes V^{(N)}_\mu)$, where $W_\mu \subset W$ is a linear subspace consisting of the *highest weight vectors* of type μ associated to the module $V^{(N)}_\mu$. Highest weight subspaces of each type are preserved by module homomorphisms, and so J_β determines (and is determined by) the restrictions $J_\beta(\mu) : W_\mu \to W_\mu$ for each μ, where μ runs over partitions with at most N parts.

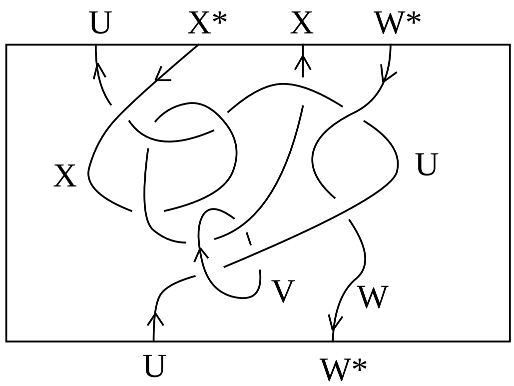
If a knot (or one component of a link) K is decorated by a pattern T which is the closure of an m-braid β, then its quantum invariant $J(K * T; V)$ can be found from the endomorphism J_β of $W = V^{\otimes m}$ in terms of the quantum invariants of K and the restriction maps $J_\beta(\mu) : W_\mu \to W_\mu$ by the formula

$$J(K * T; V) = \sum c_\mu J(K; V_\mu^{(N)}) \tag{3}$$

with $c_\mu = \operatorname{tr} J_\beta(\mu)$. This formula follows from lemma II.4.4 in [43]. We set $c_\mu = 0$ when W has no highest weight vectors of type μ.

More generally the methods of Reshetikhin and Turaev allow the quantum groups $\mathcal{G} = sl(N)_q$ to be used to represent oriented tangles whose components are coloured by $\mathcal{G}$-modules as $\mathcal{G}$-module homomorphisms. One additional feature is needed, namely the use of the dual module V^* defined by means of the antipode in $\mathcal{G}$, (an antiautomorphism of $\mathcal{G}$ which is part of its structure as a Hopf algebra). When the components of the tangle are coloured by modules the tangle itself is represented by a homomorphism from the tensor product of the modules which colour the strings at the bottom to the tensor product of the modules which colour the strings at the top, provided that the string orientations are inwards at the bottom and outwards at the top. The dual module V^* comes into play in place of V when an arc of the tangle coloured by V has an output at the bottom or an input at the top.

For example, the $(4, 2)$-tangle below, when coloured as shown, is represented by a homomorphism $U \otimes W^* \to U \otimes X^* \otimes X \otimes W^*$.



It is possible to build up the definition so that consistently coloured tangles are represented by the appropriate composite homomorphisms, starting from a definition of the homomorphisms for the elementary oriented tangles. Two cases, depending on the orientation, must be considered for both the local maximum and the local minimum, and a little care is needed here to ensure consistency. The final result is a definition of a homomorphism which is invariant when the coloured tangle is altered by R_{II} and R_{III}. When applied to an oriented k-component link diagram L regarded as an oriented $(0,0)$-tangle it gives an element $J(L; V_1, \ldots, V_k) \in \Lambda = \mathbf{Q}[[h]]$ for each colouring of the components of L by $\mathcal{G}$-modules, which is an invariant of the framed oriented link L.

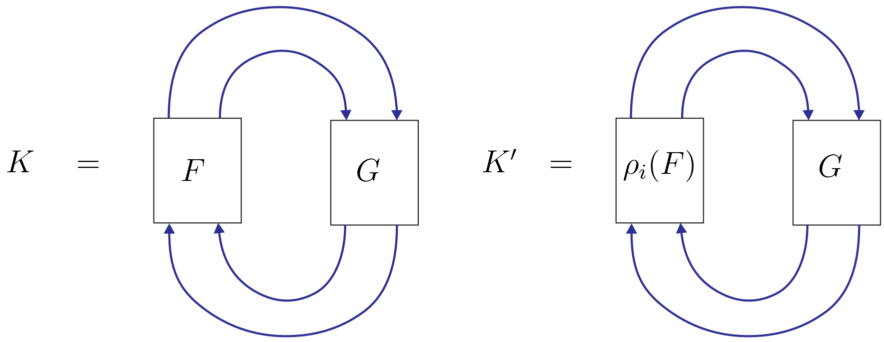
The construction is simplified in the case of $sl(2)_q$ by the fact that all modules are isomorphic to their dual, and so orientation of the strings plays no role.

5.2 Quantum invariants of mutant knots

We turn to the question of distinguishing mutants such as C and KT by means of quantum group invariants, especially those which use the unitary quantum groups $sl(N)_q$. These are closely related to the Homfly satellite invariants of a knot, and can provide complementary insights into their behaviour.

In the case of mutant knots K, K' the basic quantum invariants are the 1-parameter invariants $J(K; V_\lambda)$ and $J(K'; V_\lambda)$ where V_λ is an irreducible module over the quantum group.

The decomposition of oriented knots K and K' into 2-tangles



means that there are endomorphisms of $V_\lambda \otimes V_\lambda$ determined by the tangles F and G and the quantum invariant $J(K; V_\lambda)$ is a suitably weighted trace of their composite. Where F is replaced by $\rho_1(F)$ the endomorphism is replaced by the conjugate with the R-matrix for V_λ. In [27] Morton and Cromwell give conditions on V_λ which ensure that the endomorphism for F will commute with the R-matrix, so that $J(K;V_\lambda) = J(K';V_\lambda)$, when K' is the mutant constructed using the rotation ρ_1.

Remark. This is the case known as the positive mutant in 8.2.1. They also give conditions which ensure equality for quantum invariants of the other mutants.

5.3 Unitary quantum invariants and Homfly invariants

When dealing with $sl(N)_q$ for any fixed natural number N it is usual to write $q = e^h$. Where the framed knot K is coloured by a finite dimensional module W over the unitary quantum group $sl(N)_q$ its invariant $J(K;W)$ depends on the variable h as a Laurent polynomial in one variable $s = e^{h/2} = \sqrt{q}$, up to an overall fractional power of q.

The invariant J is linear under direct sums of modules and all the modules over $sl(N)_q$ are semi-simple, so we can restrict our attention to the irreducible modules $V_\lambda^{(N)}$. For $sl(N)_q$ these are indexed by partitions λ with at most N parts, without distinguishing two partitions which differ in some initial columns with N cells each.

There is a close relation between Homfly satellite invariants and unitary quantum invariants of K. To help in our comparison of these invariants we write $P(K;Q)$ for P_{K*Q} and more generally

$$P(L;Q_1,Q_2,\ldots,Q_k)$$

for the Homfly polynomial of a link L when its components are decorated by $Q_1,\ldots,Q_k$ respectively.

Theorem 5 (Comparison theorem)**.**

1. *The $sl(N)_q$ invariant for the irreducible module $V_\lambda^{(N)}$ is the Homfly invariant for the knot decorated by Q_λ with $v = s^{-N}$, suitably normalised as in [26]. Explicitly,*

$$P(K;Q_\lambda)|_{v=s^{-N}} = x^{k|\lambda|^2} J(K;V_\lambda^{(N)})$$

   where k is the writhe of K, and $x = s^{1/N}$.

2. *Each invariant $P(K;Q)|_{v=s^{-N}}$ is a linear combination of quantum invariants $\sum c_\alpha J(K;W_\alpha)$.*

3. Each $J(K;W)$ is a linear combination of Homfly invariants

$$\sum d_j P(K;Q_j)|_{v=s^{-N}}.$$

Remark.

- In the special case when $N = 2$ we can interpret quantum invariants of K in terms of Kauffman bracket satellite invariants, using the skein of the annulus based on the Kauffman bracket relations. This simpler skein is a quotient of the algebra $\mathcal{C}$. More generally the $sl(N)_q$ invariants depend only on a quotient of the algebra $\mathcal{C}$ for each N.

- The quantum group invariants based on $sl(3)_q$ also admit a combinatorial simplification due to Kuperberg to allow an easier diagrammatic calculation of them. At the same time the quantum group itself is straightforward enough to make it possible to work directly with some of the smaller dimensional modules, [29, 33].

- The 2-variable invariant $P(K;Q)$ can be recovered from the specialisations $P(K;Q)|_{v=s^{-N}}$ for sufficiently many N.

- If the pattern Q is a closed braid on m strings then we only need use partitions $\lambda \vdash m$, since $\mathcal{C}_m$ is spanned by $\{Q_\lambda\}_{\lambda\vdash m}$. Conversely, to realise $J(K;V_\lambda^{(N)})$ with $\lambda \vdash m$ we can use closed m-braid patterns.

The basic condition on the quantum group module V_λ in [27] is that when the module $V_\lambda \otimes V_\lambda$ is decomposed as a direct sum of irreducible modules there should be no repeated summands, up to isomorphism. In this case *any* two endomorphisms of $V_\lambda \otimes V_\lambda$ will commute.

Since this is the case for all irreducible $sl(2)_q$ modules V_λ, it gives an alternative proof of the results of Morton-Traczyk about the Jones polynomial of satellites of mutants.

It is also the case for the fundamental irreducible $sl(N)_q$ module with Young diagram $\square$, which, taken together for all N, determine the Homfly polynomial, and for the irreducible $sl(N)_q$ modules with Young diagrams $\square\square$ and $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$, which determine the Homfly polynomial of the directed 2-cables.

It is interesting that this condition does not establish Lickorish and Lipson's result that the Homfly polynomial of *reverse* 2-parallels must agree for mutants; their result then yields a non-trivial consequence for quantum invariants. The simplest example of this is that the $sl(3)_q$ invariant of a knot when coloured by the irreducible module with Young diagram $\yng(2,1)$ will agree on a mutant pair. I suspect that this invariant is at the heart of Stoimenow's use of the Whitehead double in showing that a pair of knots are not mutants [42]. He gives a pair of knots whose Homfly polynomials of their 2-parallels, and of the knots themselves, agree, and proves that the knots are not mutants because the Homfly polynomials of their Whitehead doubles are different.

The calculations of Cromwell and Morton [27] about the Homfly polynomials of 3-parallels show, on the other hand, that the $sl(4)_q$ invariant for the module with Young diagram $\yng(2,1)$ does distinguish some mutant pair, namely C and KT, as does the $sl(N)_q$ invariant with Young diagram $\yng(2,1)$, for every $N \geq 4$.

6 Vassiliev invariants

The invariants, known variously as *finite type* invariants or *Vassiliev* invariants, developed by Vassiliev and Gusarov in the late 80s, can be related readily to polynomial and quantum group invariants, originally by Birman and Lin [3]. They provide a rather transverse view of a whole collection of these invariants, and their behaviour on mutants has been a matter of continuing interest.

Chmutov, Duzhin and Lando [12] prove that all Vassiliev invariants of degree at most 8 agree on any mutant pair of knots.

This is extended to Vassiliev invariants up to degree 10 by Jun Murakami [35], where he also confirms the degree 11 invariant used by Morton and Cromwell in [27] which can be used to distinguish the knots C and KT.

Morton and Cromwell expand the difference between the Homfly polynomials of the 3-parallels of the knots C and KT to isolate a framed Vassiliev invariant of type 11 which distinguishes these two mutants, and go on to explain some features of the difference $P_{K*Q} - P_{K'*Q}$ for general K, K', where the pattern Q is the closure of a 3-braid.

Further results about Vassiliev invariants on extended and restricted classes of mutants are noted in section 8.

7 Further invariants

Among the homology invariants which have been developed in the past 10 years the Heegaard-Floer homology can certainly distinguish some mutants, since it is able to calculate the genus of the knot.

On the other hand Bloom [4] shows that odd Khovanov homology for knots is unchanged by mutation. As a corollary he notes that Khovanov homology over $\mathbf{Z}_2$ is also mutation invariant. Homfly Khovanov homology is shown by Jaeger [17] to be invariant under *positive* mutation, as defined in 8.2.1.

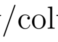
Kim and Livingston [20] show that the 4-ball genus of a knot can be changed by mutation, but the algebraic concordance class is invariant under mutation.

7.1 Behaviour on mutants

I have gathered together here a summary of the results noted about the behaviour of a selection of invariants on mutants. Where the invariants are known to be the same on mutant knots (shared) I give a reference to a proof, not necessarily the original one. Where there are mutants on which the invariant is known to differ I give a reference to an example.

Invariant	**Shared**	**Differs**
Alexander Polynomial	[44]	
Signature	[44]	
Genus		[15]
n-colouring	By Alexander	
Homomorphism to $PSL(2,7)$		[39]
Double branched cover	[44]	
Hyperbolic volume, where available	[41]	
Crossing number	< 16, [42]	Unknown if ≥ 16
Braid index	Unknown	
Arc index	Unknown	
Jones polynomial	[23]	
Homfly polynomial	[23]	
Kauffman polynomial	[23]	
Jones satellites	[32]	
Homfly 2-satellites	[25, 38]	
Kauffman 2-satellites	[25]	
Homfly 3-parallel		[27, 34]
Kauffman 3-parallel		[42]
Vassiliev degree ≤ 8	[12]	
Vassiliev degree ≤ 10	[35]	
Vassiliev degree 11		[27, 35]
Heegaard Floer homology		By genus
Odd Khovanov homology	[4]	
4-ball genus		[18, 20]
Algebraic concordance	[20]	

For unitary quantum groups I give a similar summary. The irreducible module used in colouring is specified by giving its defining Young diagram.

Quantum group	**Colouring**	**Shared**	**Differs**
$sl(2)_q$	Any	[27]	
$sl(N)_q$	row/column	[27]	
$sl(3)_q$		[27]	
$sl(>3)_q$			[27]
$sl(3)_q$			[29]

8 Generalisations and restrictions

8.1 Generalisations

Various generalisations of the original ideas of mutants have been made.

8.1.1 Rotors

An obvious possibility is to decompose a knot by a sphere meeting the knot in $2n$ points with $n > 2$, and then replace one side of the sphere after some transformation. In many cases the resulting knot does not have enough properties in common with the original for this to be worthwhile. However Rolfsen [40] has used the idea of a *rotor*, based on $2n$ intersection points around the equator of a sphere with a rotation of order $2n$ on one side as the transformation. For unoriented knot diagrams this operation preserves the Jones polynomial, although Rolfsen has so far not been able to use the method in his searches for a non-trivial knot with Jones polynomial $V = 1$.

8.1.2 Genus 2 mutants

A more fruitful class of generalised mutants are constructed by finding an embedded genus 2 surface in the knot complement, and regluing the two sides after a suitable degree 2 transformation (a hyperelliptic involution). This construction was used by Ruberman [41] for general 3-manifolds, and by Cooper and Lickorish [10] in the context of knots in S^3.

The construction has a close relation to Conway mutation for knots, which can be realised by applying a sequence of one or two genus 2 mutations. An extensive discussion of genus 2 mutation, and properties which are known to be preserved, is given by Dunfield et al [11]. Further calculations related to genus 2 mutants by Morton and Nathan Ryder appear in [30].

Here are some of the known coincidences and differences.

Invariant	**Shared**	**Differs**
Alexander Polynomial	[10]	
Signature	[10]	
Hyperbolic volume, where available	[41]	
Homfly polynomial		[11]
Kauffman polynomial		[30]
Jones satellites	[11]	
Vassiliev degree 8		[30]
Khovanov homology		[11]

8.2 Restrictions

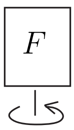
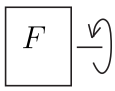
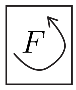
On the other hand certain more tightly defined classes of mutants are known to share more invariants than is true for a general pair of Conway mutants.

Such restricted classes of mutants are consequently more difficult to distinguish, and provide more sensitive test-beds for examining possible dependence among invariants.

8.2.1 Positive mutants

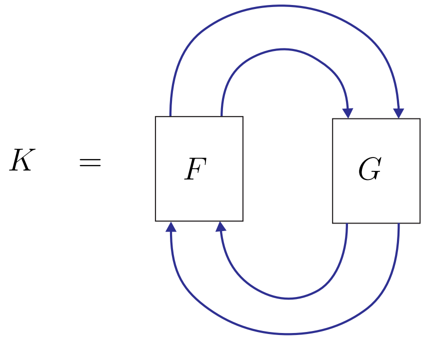
The simplest restriction, which splits mutants into two classes, has led to a number of helpful observations.

Recall that when a knot K can be decomposed into two 2-tangles by a Conway sphere we construct mutants by replacing one of the tangles after one of three possible π-rotations.

$\rho_1(F) =$  , $\rho_2(F) =$  , $\rho_3(F) =$  .

We may orient the original knot K, and we will find that only one of the three rotations respects the string orientation when regluing the 2-tangles.

Assume, as we may do, that the original knot K has been oriented as shown below



Definition. The mutant K' where F is replaced by $\rho_1(F)$ can maintain the orientation of the strings in F, and is called a *positive* mutant of K.

Remark. The classical Conway and Kinoshita-Teresaka pair of knots are positive mutants.

To orient the remaining two mutants where F is replaced by $\rho_2(F)$ or $\rho_3(F)$ the orientations of the strings in one of the two tangles must be reversed. There are then two pairs of positive mutants among the four knots consisting of the original knot K and its three mutants.

Kirk and Livingston [21, 22] adopted the term *positive*, and showed that positive mutants are S-equivalent. This condition on Seifert matrices is necessary, but not sufficient for concordance of the knots. All the same, they prove that concordance is not preserved even for positive mutants.

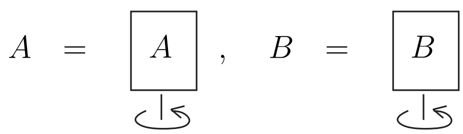
Jaeger [17] shows that the Homfly Khovanov homology is preserved for positive mutants.

8.2.2 Mutants with symmetry

The symmetric mutants discussed by Morton in [33] also have much more restricted properties.

These knots are made up by interchanging three 2-tangles A, B and C as shown in Figure 3.

Assume that the tangles A and B are both be symmetric under the π-rotation ρ_1, so that



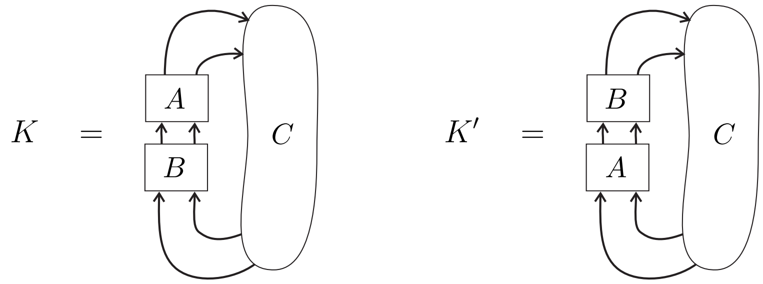


Figure 3: Tangle interchange

and hence A and B both commute with the half-twist.

Under these conditions the two knots K and K' are shown to share the same $sl(N)$ quantum invariants when coloured by the irreducible module V_λ with partition λ of $m = |\lambda|$, for any $m \leq 5$. The proof depends on showing that the module endomorphisms of $V_\lambda \otimes V_\lambda$ determined by A and by B commute, knowing by the symmetry assumption that they commute with the R-matrix.

The relations between Homfly satellite invariants and unitary quantum invariants then ensure that K and K' share the same Homfly satellite invariants for their m-parallels, and indeed any directed m-string satellite, when $m \leq 5$.

In an extension of this result it is shown that the Homfly polynomial of a genuine connected cable, based on the (m, n) torus knot pattern, with m and n coprime, does not distinguish mutants with symmetry K and K' above, for *any* number of strings, m. On the other hand there are examples where the 6-string *parallels* have different Homfly polynomials.

In the course of these calculations [33] the pairs of mutants with symmetry could be distinguished by a Vassiliev invariant of degree 14. Interestingly, calculations of Stoimenow [42] on the 12-crossing Ochiai-Morimura mutants with symmetry 12_{1653} and 12_{1654} using the Kauffman polynomial of the 3-parallel showed that these too were distinguished by a degree 14 Vassiliev invariant. This leads to the speculation that mutants with symmetry may share their Vassiliev invariants of degree ≤ 13.

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