DISTINGUISHING MUTANTS BY KNOT POLYNOMIALS

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ABSTRACT

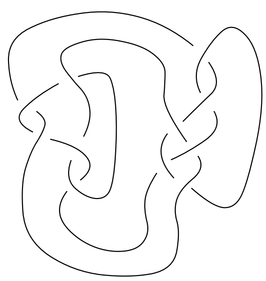
We consider the problem of distinguishing mutant knots using invariants of their satellites. We show, by explicit calculation, that the Homfly polynomial of the 3-parallel (and hence the related quantum invariants) will distinguish some mutant pairs.

Having established a condition on the colouring module which forces a quantum invariant to agree on mutants, we explain several features of the difference between the Homfly polynomials of satellites constructed from mutants using more general patterns. We illustrate this by our calculations; from these we isolate some simple quantum invariants, and a framed Vassiliev invariant of type 11, which distinguish certain mutants, including the Conway and Kinoshita-Teresaka pair.

Keywords: Mutant, Vassiliev invariant, quantum invariant, Homfly polynomial.

1. Introduction.

The two 11-crossing knots with trivial Alexander polynomial found by Conway and Kinoshita-Teresaka are the best-known example of mutant knots. They are shown in figure 1.

$C =$  ,

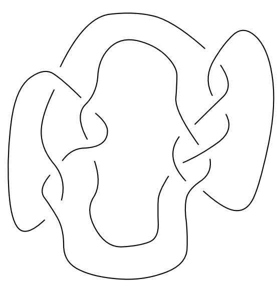
$KT =$  .

Figure 1

The term *mutant* was coined by Conway, and refers to the following general construction.

Suppose that a knot K can be decomposed into two oriented 2-tangles F and G as shown in figure 2.

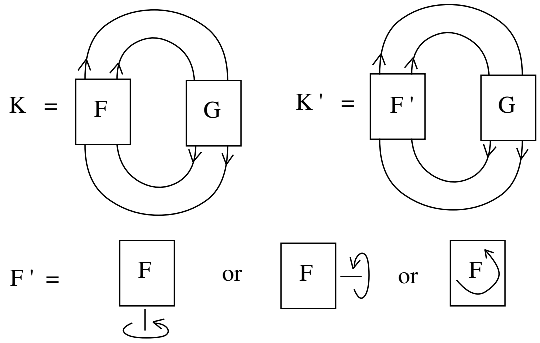


Figure 2

A new knot K' can be formed by replacing the tangle F with the tangle F' given by rotating F through π in one of three ways, reversing its string orientations if necessary. Any of these three knots K' is called a *mutant* of K. It is clear from figure 1 that the knots C and KT are mutants.

It was shown by Conway that mutants always share the same Alexander polynomial, (not necessarily the same as for the unknot). Very shortly after the discovery of the new invariants Lickorish proved, using simple skein theoretic arguments, that mutants must also have identical Homfly and Kauffman polynomials, and hence the same Jones polynomial [4].

Calculations of Morton and Short for a number of examples led to the conjecture [10] that two equally twisted 2-cables of a mutant pair would also share the same Homfly polynomial. This was proved by Lickorish and Lipson [5], also using skein theory. They showed further that the same result holds for reverse-string 2-cables (that is, for 2-cables of two components whose orientations run in opposite directions, giving a 'reverse parallel' satellite). These results were also derived independently by Przytycki [14].

Invariants such as the Homfly polynomial P of any choice of satellite of a knot K may be regarded as an invariant of K itself. These provide a whole range of further invariants, which can be compared for mutants K and K'. When discussing satellites in detail it is assumed that each knot K comes with a chosen framing and that this framing is used in the construction of the satellite $K * Q$ by decorating K with a choice of pattern Q in the annulus. For further discussion of such constructions in connection with knot invariants see [7].

The results above show that $P_{K*Q} = P_{K'*Q}$ for mutants K, K' when the pattern Q is the closure in the annulus of any 2-tangle, and P is the Homfly polynomial. A completely contrasting result occurs when the pattern Q is a closed 3-tangle. We report here an explicit calculation which shows that the Homfly polynomials of the 3-parallels about the Conway knot, C, and its mutant, the Kinoshita-Teresaka knot, KT, are different. From these we isolate a framed Vassiliev invariant of type 11 which distinguishes C from KT, (theorem 2), and we go on in theorem 4 to explain some general features of the difference $P_{K*Q} - P_{K'*Q}$ where the pattern Q is the closure of a 3-braid.

The result on Vassiliev invariants may be compared with recent work by Chmutov, Duzhin and Lando [3] in which they prove that all Vassiliev invariants of type at most eight agree on any mutant pair of knots.

Remark. Morton and Traczyk [12] showed that the Jones polynomial V cannot be used in combination with any choice of satellite to distinguish a mutant pair, K and K'. Thus $V_{K*Q} = V_{K'*Q}$ for any choice of pattern Q, provided that the same framing of K and K' is used.

We turn to the question of distinguishing mutants such as C and KT by means of quantum group invariants. In theorem 5 we give a condition under which a quantum invariant will agree on mutants. The Homfly invariants of satellites of K can be written in terms of quantum invariants of K. We show that theorem 5 gives an alternative proof of the results of Morton-Traczyk and of the results about the Homfly polynomial of the mutants themselves and of their directed 2-cables. It is interesting that theorem 5 does not prove Lickorish and Lipson's result that the Homfly polynomial of *reverse* 2-parallels must agree for mutants; their result then yields a non-trivial consequence for quantum invariants. The simplest example of this is that the $SU(3)_q$ invariant of a knot when coloured by the irreducible module with Young diagram $\begin{smallmatrix}\Box&\Box\\ \Box&\end{smallmatrix}$ will agree on a mutant pair. The calculations quoted above can be used to show, on the other hand, that the $SU(4)_q$ invariant for the module with Young diagram $\begin{smallmatrix}\Box&\Box\\ \Box&\end{smallmatrix}$ does distinguish some mutant pair, namely C and KT, as does the $SU(N)_q$ invariant with the same Young diagram, for every $N \geq 4$.

We have carried out similar calculations for other mutant pairs, and the invariants quoted above again distinguish in very much the same way. We comment later on the algorithms used for the calculations and on the feasibility of extending the calculations to test more complicated cases. It would be interesting to know for example

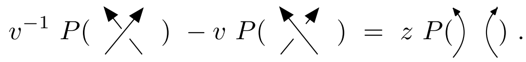
(1) if *any* $SU(3)_q$ invariant can distinguish mutants,

(2) if any Vassiliev invariant of types 9 or 10 can distinguish mutants.

Remark. Calculations made some years ago by Morton and Traczyk showed that the Homfly polynomials P_{C*Q} and P_{KT*Q} were different for the 3-parallel pattern Q. To overcome computational difficulties they fixed the value of one variable and reduced the integer coefficients mod p for some small fixed value of p. The nature of the difference of the two polynomials was thus not easy to appreciate from their calculations. The method of calculation used here has involved a truncation which amounts to retaining only Vassiliev invariants up to a certain type, in this case type 12 is enough. Such a truncation at a fixed type turns out to be very easily implemented in terms of the calculations based on the Morton-Short algorithm for finding Homfly polynomials [11], and it gives a very satisfactory outcome when the difference of the invariants for two mutants is studied.

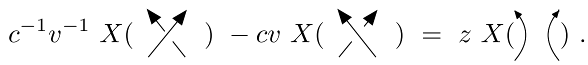
2. Homfly polynomials and Vassiliev invariants.

The Homfly polynomial $P_L(v,z)$ of an oriented link will be taken to satisfy the skein relation



This defines P up to a scalar multiple; we choose the normalisation so that $P=1$ for the unknot.

A framed version, $X_L(v,z) = c^{w(D)} P_L(v,z)$, defined from a diagram D with writhe $w(D)$, having the required framing, can be constructed for any choice of c. The skein relation for X is then



See [7] for a fuller discussion.

In all that follows we shall choose $c = v^{-1}$ so that the skein relation becomes

Set $u = \frac{1-v^2}{z}$. The calculations in [8] show that any negative powers of z in P_L can be accounted for in terms of u. as $v^k Q_k(u,z)$ for some $k \in \mathbf{Z}$ where $Q_k(u,z)$ is a genuine polynomial in u and z, depending on L. The choice of k, which may well be negative, can clearly be altered, since $v^2 = 1 - uz$. When k is as large as possible the degree in u is no more than the braid index of L. Expansion of $v^k = (1-uz)^{k/2}$ as a power series then gives P, and similarly X, as $\sum_{d=0}^{\infty} b_d(u) z^d$.

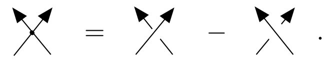
It can be deduced quickly from [13] or [6] that setting $z=0$ in the power series for P_L gives the constant term $b_0(u) = u^{|L|-1}$. The coefficient of z^d is a polynomial in u, which can be shown to have degree at most $d + |L| - 1$ for a link L with $|L|$ components. An alternative direct induction proof can be given, noting that the factor $\delta = \frac{v^{-1}-v}{z} = v^{-1}u$ arising from an extra disjoint unknotted component in a link satisfies $\delta = u + O(z)$ as a power series in z, and the coefficient of z^d in this series has degree $d+1$ in u.

In an adaptation of Bar-Natan's approach to Vassiliev invariants we shall briefly discuss invariants of *framed* links, and define *framed* Vassiliev invariants of type d; all the links studied are assumed to be oriented.

When the framed version X of the Homfly polynomial, with $c = v^{-1}$, is written as a power series in u and z we show that the coefficient $b_d(u)$ of z^d is a framed Vassiliev invariant of type d. This function X gives a convenient way to organise the Vassiliev invariants which arise directly from the Homfly polynomial, including those from the Alexander and Jones polynomials.

We shall work with the set of oriented framed links up to isotopy, or equivalently the set of planar diagrams of links up to regular isotopy, using the convention that the framing curves are chosen to follow the diagrammatic parallels. Suppose that the invariants to be studied take values in Λ, a ring with a 1. Write $\mathcal{L}$ for the space of Λ-linear combinations of oriented framed links. Any Λ-valued invariant V of framed links immediately determines a Λ-linear function $V : \mathcal{L} \to \Lambda$.

From any planar diagram N of a singular link with d nodes we can determine an element $\varphi(N)$ of $\mathcal{L}$ as the alternating sum of 2^d diagrams, by expanding each node of N locally, using the rule



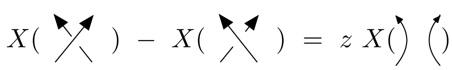
Write $\mathcal{L}_d$ for the linear subspace spanned by all such elements $\varphi(N)$, where N has d nodes. Then

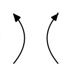
$$\cdots \subset \mathcal{L}_{d+1} \subset \mathcal{L}_d \subset \cdots \subset \mathcal{L}_0 = \mathcal{L}.$$

Definition. An invariant V of framed links is a framed Vassiliev invariant of type d if $V = 0$ on $\mathcal{L}_{d+1}$.

Proposition 1. *The coefficient $b_d(u)$ of z^d in the expansion of the framed Homfly polynomial X_L of an oriented link L as a power series in z and u is a framed Vassiliev invariant of type d.*

Proof: (following Birman and Lin [1]). The invariant X determines a linear function X from $\mathcal{L}$ to the ring of power series in u and z. For any singular link diagram N write $X(N)$ for the power series $X(\varphi(N))$. The relation



shows that if a node  of N is replaced by  to give a diagram N' with one less node then $X(N) = zX(N')$. The power series $X(N)$ thus has a factor of z^d if N has d nodes. Hence the coefficient of z^d in the power series X is a framed Vassiliev invariant of type d, since it is zero for $X(N)$ when N has $d+1$ nodes. □

Remarks. If an invariant V of framed links does not depend on the choice of framing and has type d in the sense above then it is a Vassiliev invariant of type d in the sense of Bar-Natan. Some authors use the term *order* here instead of *type*. The term *degree* d is sometimes used for an invariant of type d but not type $d-1$.

From a framed invariant we can always make an unframed invariant of any knot by evaluating it on the framed knot with framing 0. A framed invariant of Vassiliev type d may conceivably not give rise to an unframed invariant of type d in this way. For the framings on the 2^{d+1} diagrams which come from the expansion of a $d+1$-node singular knot will never all be 0 and so there is no guarantee that the alternating sum of the invariant on these 2^{d+1} knots, when taken with framing 0, will vanish, given that the sum with the diagrammatic framings vanishes. It is true however that framed invariants of finite type can in general be constructed from unframed invariants of finite type, combined with framed invariants of type 1 (which are just writhe and linking numbers). There is discussion of this, for the case of knots, in [3] and [2].

Discussion of satellites of a knot K and the resulting invariants of K are most easily done in terms of a chosen framing of K. Invariants constructed using satellites will thus appear most naturally as framed invariants. Direct comparison with quantum invariants of K, which are also framed invariants, is also most readily done in this context.

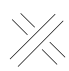
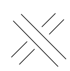
Framed Vassiliev invariants of type d for a satellite link $L = K * Q$ are shown in proposition 3 to be themselves framed Vassiliev invariants of K of type d, for any choice of pattern Q. So in the search for a Vassiliev invariant of type d to distinguish the mutants K and K' we may consider the coefficient of z^d in the expansion of X_{K*Q} for some suitable Q.

Theorem 2. *There exists a framed Vassiliev invariant of type* 11 *which distinguishes the mutants* C *and* KT*, namely the coefficient of* z^{11} *in* X_{K*Q} *where* Q *is the oriented* 3*-parallel pattern.*

This follows from proposition 3, and an explicit calculation of the coefficients for the two knots. Further comments are given after the proof of proposition 3.

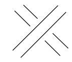
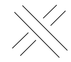
Proposition 3. *Let* V *be a framed Vassiliev invariant of type* d*, and let* Q *be a pattern in the annulus. Then* $V(K*Q)$ *is a framed Vassiliev invariant of the framed knot* K*, of type* d*.*

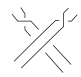
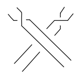
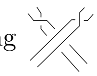
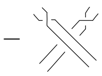
Proof: Write $\mathcal{K} \subset \mathcal{L}$ for the Λ-linear combinations of framed knots, and $\mathcal{K}_d = \mathcal{L}_d \cap \mathcal{K}$. Decoration by Q induces a Λ-linear map $M_Q : \mathcal{K} \to \mathcal{L}$, defined on a framed knot K by $M_Q(K) = K * Q$. We have to show that $V \circ M_Q$ vanishes on $\mathcal{K}_{d+1}$, given that $V(\mathcal{L}_{d+1}) = 0$.

It is enough to show that $M_Q(\mathcal{K}_d) \subset \mathcal{L}_d$ for each d. Let N be any d-node singular knot diagram, and write $\varphi(N) = \sum \pm N_\alpha$ for the alternating sum of 2^d knot diagrams given by expanding each node of N. Then $M_Q(\varphi(N)) = \sum \pm N_\alpha * Q$. Suppose that Q is the closure in the annulus of some n-tangle T. A diagram for $N_\alpha * Q$ is then given by inserting T in the n-strand diagrammatic parallel of N_α, with the strands oriented appropriately. The sum of the diagrams $N_\alpha * Q$, with sign, can then be constructed from the diagram of N by taking the n-strand diagrammatic parallel, away from the nodes, and inserting T, while replacing each node  by the formal difference  $-$  of the two $2n$-tangles.

It is helpful to consider singular tangles and linear combinations of tangles in a similar way to link diagrams, treating

as an expansion of a singular 2-tangle, which can be extended multilinearly to tangles with more nodes. Let T_1 and T_2 be two m-tangles with the same plane projection. It follows readily by induction on the number of crossings at which T_1 and T_2 differ that $T_1 - T_2$ can be written as a linear combination of singular tangles, each with 1 node and the same plane projection as T_1.

In the construction above of $M_Q(\varphi(N))$ we can now replace  −  near each node of N by a linear combination of 1-node tangles. Since N has d nodes this replaces the whole of $M_Q(\varphi(N))$ by a linear combination of d-node link diagrams. Hence $M_Q(\varphi(N)) \in \mathcal{L}_d$. Now $\mathcal{K}_d$ is spanned by elements of the form $\varphi(N)$, and so the result follows. □

Remark. A diagram for the satellite $K^0 * Q$ of K with framing 0 can be drawn from any diagram of K by following the 'framing-compensated' parallel  near  and  near , joined up appropriately. By writing  −  as a linear combination of 1-node tangles a similar argument to proposition 3 shows that the unframed invariant $V(K^0 * Q)$ of K is a Vassiliev invariant of type d when V is an unframed Vassiliev invariant of type d.

Proof of theorem 2: Explicit calculation for the Conway and Kinoshita-Teresaka knots shows that

$$X_{C*Q} - X_{KT*Q} = b_{11}(u) z^{11} + O(z^{12})$$

and that $b_{11}(u) = 6u(u-1)(u+1)(u-2)(u+2)(u-3)^2(u+3)^2$. Thus $b_{11}(u) \neq 0$ as a polynomial in u, and so the coefficient of z^{11} in X_{K*Q} is, by proposition 3, a framed Vassiliev invariant of type 11 which distinguishes C and KT for all u except $u \in \{0, \pm 1, \pm 2, \pm 3\}$. □

Conway's form of the Alexander polynomial is given from Homfly by putting $v = 1$ or equivalently $u = 0$. Now the Alexander polynomial cannot distinguish mutants, even by using satellites, and so $X_{C*Q} = X_{KT*Q}$ when $u = 0$. This explains the root $u = 0$ of b_{11}. The other roots of b_{11} can also be explained by coincidences of invariants on mutants.

Theorem 4. *Let K and K' be mutants and let Q be a pattern for which $X_{K*Q} \neq X_{K'*Q}$. Write*

$$X_{K*Q} - X_{K'*Q} = b_k z^k + O(z^{k+1}).$$

(a) Then b_k has roots $u = 0, \pm 1, \pm 2$.

(b) When Q is the closure of an oriented 3-braid then b_k also has roots $u = \pm 3$.

Proof of theorem 4(a): The root $u = 0$ follows from the coincidence of Alexander polynomials.

The Homfly polynomial of a link L evaluated at $z = \pm(v - v^{-1})$ depends only on the number of components of L, so at this evaluation $X_{K*Q} = X_{K'*Q}$. Putting $z = \pm(v - v^{-1})$ gives $u = \pm v^{-1}$ and hence $u = \pm 1 + O(z)$ as a power series in z. The difference $X_{K*Q} - X_{K'*Q}$ can then be expanded as a power series in z in which the coefficient of the term z^k is $b_k(\pm 1)$. Since this difference is zero the leading coefficient b_k has roots $u = \pm 1$.

The Jones polynomials of $K * Q$ and $K' * Q$ are known to be equal, [12]. The resulting relation between u and z can be used similarly to show that b_k has roots $u = \pm 2$.

The proof of 4(b) is given later. □

It is known that P is either an even or an odd function of z, depending on the number of components of the link. It follows that, as a function of u and z, we have

$$P_L(-u, -z) = (-1)^{|L|-1} P_L(u, z).$$

Hence when we write $X_{K*Q} - X_{K'*Q} = \sum b_d(u) z^d$ we have $b_d(-u) = \pm b_d(u)$, showing that the roots of b_d other than $u = 0$ occur in pairs $\pm\alpha$.

In the next section we use the substitutions in X which give natural unitary quantum invariants. By showing that certain of the quantum invariants agree on mutants we can give a technically more direct proof that b_k has roots $u = \pm 2$. We use Lickorish and Lipson's reverse parallel result to deduce the equality of a further $SU(3)_q$ quantum invariant and hence get the roots $u = \pm 3$ in case (b).

One feature of the leading coefficient b_{11} which remains unexplained is the *repetition* of the roots $u = \pm 3$ as well as the factor 3. Calculations with different choices of Q can avoid the factor 2, but several choices of mutant pairs have so far always given the same leading degree $k = 11$ and the same repeated roots $u = \pm 3$, as well as a factor of 3. It would be interesting to see what roots occur when Q is a closed 4-braid, or when the mutants have a substantially different nature from those chosen; we comment later on the practical constraints which have restricted our calculations.

3. Quantum invariants.

Reshetikhin and Turaev have shown [16] how a finite-dimensional module V_λ over a suitable quantum group can be used to construct an invariant $J(K; V_\lambda)$ of a framed knot K which is a power series in the quantum group parameter h. It can usually be expressed easily in terms of $q = e^h$ or $s = e^{h/2}$.

The construction extends to determine an invariant of framed oriented links when 'coloured' by a choice of module for each component. The invariants are multilinear under direct sums of modules, while a knot K coloured with a tensor product $V \otimes W$ of two modules has the same invariant as the link $K^{(2)}$ made up of two parallel copies of K when coloured by V and W respectively on the two components. Further results allow the quantum invariants of a satellite $K * Q$ when coloured by a module V to be calculated in terms of the quantum invariants of K itself, coloured by summands of the tensor product $V^{\otimes j}$ when the pattern Q is the closure of an oriented j-braid, or (j,j) tangle.

Reshetikhin and Turaev [15,17,16] established a direct connection between the invariants determined by the quantum unitary groups $SU(N)_q$ and the Homfly polynomial invariants. They showed that the invariant $J(L;V_\square)$ for a link coloured by the fundamental N-dimensional $SU(N)_q$-module $V_\square$ equals the invariant X_L, normalised so that $X=1$ for the empty knot, with $c=xv^{-1}$, after substituting $v=s^{-N}=e^{-Nh/2}$, $x=e^{-h/2N}$ and $z=s-s^{-1}=e^{h/2}-e^{-h/2}$. This invariant only differs from the version of the invariant X in the previous section by a factor of $\delta=\frac{v^{-1}-v}{z}=v^{-1}u$, to alter the value at the unknot, and a factor of x^f where f is the chosen framing, to account for the different choice of c.

Some features of the correspondence are discussed in [9]; fuller details can be found in [18]. It is possible to calculate X_{K*Q} with the above substitution for v and z in terms of $SU(N)_q$ invariants of K. For a closed j-braid Q the invariants required have the form $J(K;V_\lambda)$ where V_λ is an irreducible summand of the tensor product $V_\square^{\otimes j}$. Such modules V_λ can be indexed by Young diagrams λ with j cells, of which there are $\Pi(j)$, the number of partitions of j. Thus X_{K*Q} is a linear combination of invariants $J(K;V_\lambda)$, as λ runs through Young diagrams with j cells, and the substitution $v=s^{-N}$, $z=s-s^{-1}$ is made in X.

Conversely each $J(K;V_\lambda)$ for a Young diagram λ with j cells can be expressed as a linear combination of invariants X_{K*Q} where Q runs over an explicit set of $\Pi(j)$ closed j-braids, again substituting $v=s^{-N}$, $z=s-s^{-1}$.

Consequently there are two equivalent tables of invariants of K, one given by X_{K*Q}, with $v=s^{-N}$, $z=s-s^{-1}$, and closed j-braids Q, and the other by $J(K;V_\lambda)$ where V_λ is the $SU(N)_q$-module for a Young diagram with j cells. The invariants with fixed N and j in each table are equivalent in the sense that each invariant from one table is a linear combination (with coefficients independent of K) of invariants from the other table.

The quantum group table has entries only for $N\geq 2$ but the substitutions with $N=0,1$ in the table for X_{K*Q} give evaluations of X yielding the Conway polynomial of $K*Q$, for $N=0$, or a constant depending only on the number of components of $K*Q$ for $N=1$.

We now give conditions on a module V which will ensure that $J(K;V)=J(K';V)$ for a mutant pair of knots K and K'.

Theorem 5. *Let V be a module over a quantum group $\mathcal{G}$, which may be any quantum group which determines knot invariants. Suppose that the module $V\otimes V$ decomposes into a sum of irreducible modules $V\otimes V\cong\sum V_\alpha$ with no repeated summands. Then $J(K;V)=J(K';V)$ for any mutant pair of knots K and K'.*

Proof: Let K be made up of two 2-tangles F and G as in figure 2. In constructing the invariant $J(K;V)$ the tangle F is used to determine a module homomorphism J_F from $V\otimes V$ to itself. It is enough to show that under the conditions of the theorem the tangle F' determines the same module homomorphism, i.e. $J_{F'} = J_F$.

Because no two of the summands in $V \otimes V$ are isomorphic any endomorphism of $V \otimes V$ must preserve the summands (by Schur's lemma), and will act by multiplying the summand V_α by a scalar c_α, say. Any two endomorphisms of $V \otimes V$ will therefore commute.

One of the three possible choices for F' is given by turning F over, so that $F' = \Delta^{-1} F \Delta$ where $\Delta =$  . In this case the endomorphism $J_{F'}$ is the composite $(J_\Delta)^{-1} \circ J_F \circ J_\Delta = J_F$.

The second possibility is where F' is given by reversing the string orientation of F, to give F'', say, and then rotating F'' through π in the plane. Observe first that if the strings in a tangle are reversed then all modules are replaced by their duals [16] so that $J_{F''}$ is the endomorphism of $\sum V_\alpha^*$ in which V_α^* is multiplied by c_α. Now we can draw F'' as in figure 3; this tangle is known to represent the dual of the homomorphism $J_{F'}$. Since the dual of a scalar endomorphism is also scalar it follows that $J_{F'} = J_F$ in this case.

F" =

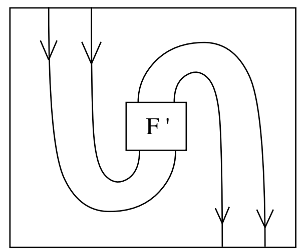


Figure 3

The final possibility is when F' is given from F by composing rotations of the two sorts above, with a reverse of string orientation. The resulting endomorphisms are therefore again the same. □

Among the modules V over $SU(N)_q$ which satisfy the condition of theorem 5 are those modules V_λ whose Young diagram λ consists of a single row or a single column. These include, for all N, the fundamental module $V_\square$, the two modules where λ has two cells and two of the three modules where V_λ has three cells.

This shows directly that none of the quantum invariants in the table for $SU(N)_q$ with $j = 1$ or $j = 2$ can distinguish mutants. By comparing with the table of invariants X_{K*Q} this gives an alternative proof that neither the Homfly polynomial itself nor the Homfly polynomial of 2-cables (oriented in the same sense) can distinguish mutant knots.

When $N = 2$ all irreducible modules satisfy the conditions of theorem 5; translated to the table of invariants X_{K*Q} this proves the result of Morton and Traczyk that the Jones polynomial of satellites can not distinguish mutants.

In order to prove theorem 4(b) we shall use Lickorish and Lipson's reverse parallel result to find certain other quantum invariants which coincide on mutants.

Theorem 6. *Let V be the irreducible $SU(N)_q$-module whose Young diagram λ has $N-1$ cells in the first column and 1 cell in the second column. Then $J(K;V) = J(K';V)$ for any mutant pair K and K'.*

Proof: The module V satisfies

$$V_{\Box} \otimes (V_{\Box})^* \cong V \oplus V_0,$$

where V_0 is the trivial module, since the dual of the fundamental $SU(N)_q$-module $V_{\Box}$ has a single column of $N-1$ cells as its Young diagram. The invariant X_{K*Q} with $v = s^{-N}$, $z = s - s^{-1}$ can be calculated as a quantum invariant by colouring the strings of $K*Q$ with the fundamental $SU(N)_q$-module $V_{\Box}$. When Q is the reverse parallel pattern the direction of the reversed string can be changed provided that the dual module is used on this string. The resulting invariant can then be calculated as $J(K; V_{\Box} \otimes (V_{\Box})^*)$, which is given in turn as $J(K;V) + J(K;V_0)$. It follows from Lickorish and Lipson that $J(K; V_{\Box} \otimes (V_{\Box})^*) = J(K'; V_{\Box} \otimes (V_{\Box})^*)$ and hence that $J(K;V) = J(K';V)$ when K and K' are mutants. $\Box$

When $N = 3$ this shows that the $SU(3)_q$-module V with Young diagram $\begin{smallmatrix}\Box&\Box\\ \Box&\end{smallmatrix}$ does not distinguish mutants. This result does not follow from theorem 5, as $V \otimes V$ has a repeated summand.

Proof of theorem 4(b): Because all three quantum invariants with $N = 3$ and $j = 3$ agree on mutants it follows that all three invariants X_{K*Q} with $N = 3$ and $j = 3$ also agree on mutants. Hence all satellites whose pattern Q is a closed 3-braid will satisfy $X_{K*Q} = X_{K'*Q}$ after substituting $v = s^{-3}$, $z = s - s^{-1}$. Let Q be a closed 3-braid, and suppose that K and K' are mutants. Write $X_{K*Q} - X_{K'*Q} = \sum b_d(u) z^d$, with leading term $b_k(u) z^k$. Rewrite this as power series in h after substituting $s = e^{h/2}$ with $v = s^{-N}$ and $z = s - s^{-1}$. Then $z = h + O(h^2)$ and $u = N + O(h)$ so the leading coefficient in the new series is $b_k(N) h^k$. We have noted earlier that for any choice of pattern Q this series will vanish when $N = 0$, 1 or 2, giving the roots $0, \pm 1, \pm 2$ for b_k. We can now see that when Q is a closed 3-braid the series will also vanish if we put $N = 3$. This proves that b_k also has roots ± 3. $\Box$

In the explicit case of C and KT we know that no integers $N > 3$ are roots of b_{11}. Thus the invariant X_{K*Q} for the 3-parallel Q after substituting $v = s^{-N}$, $z = s - s^{-1}$ will distinguish C and KT for every $N > 3$. Now this invariant is a combination of the three $SU(N)_q$ invariants $J(K; V_\lambda)$ where λ has three cells. Two of these invariants agree on mutants, by theorem 5, so the remaining invariant $J(K; V_\lambda)$ with $\lambda = \begin{smallmatrix}\Box&\Box\\ \Box&\end{smallmatrix}$ distinguishes C from KT for all $N > 3$, and indeed it provides a Vassiliev invariant of type 11 which will distinguish them.

In figure 4 we show on the two equivalent tables of invariants the places where differences in value can occur for mutants. An entry of 0 means that the invariants agree on mutants, $\times$ means that they are known to differ for some mutants and ? means that no evidence is available. The decorations Q in the table for satellites are the closures of the braids shown at the head of each column.

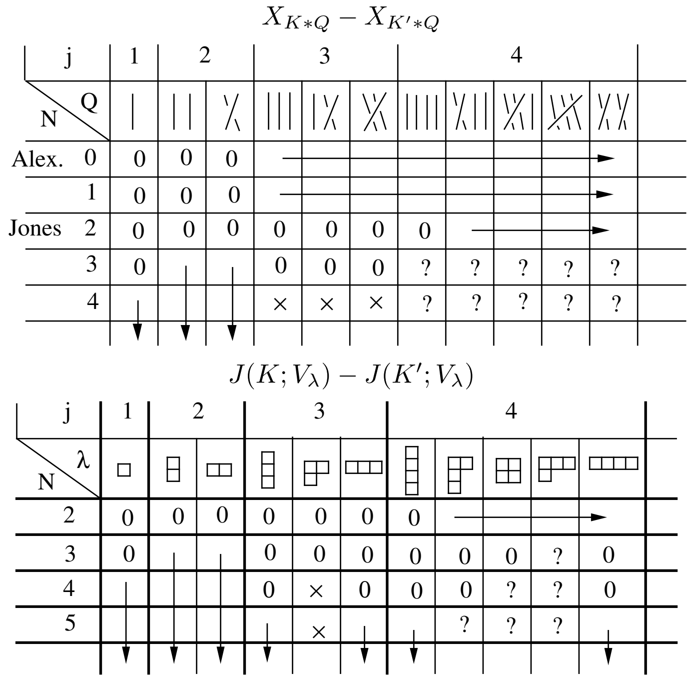


Figure 4

4. Methods of calculation.

The calculations of X_{K*Q} were based on modifications of the algorithm of Morton and Short, [11]. In this algorithm an n-braid β forming part of a diagram for L is written in the Hecke algebra $H_n(z)$ as a linear combination of $n!$ basic braids, with coefficients which are integer polynomials in z. To calculate the invariant X_L with $c = v$ the original braid β is replaced by this combination of basic braids. The invariants calculated from each of the basic braids in turn are combined, with the same choice of coefficients, to give X_L. Thus if only the terms in X_L of degree $\leq d$ in z are needed the computations within $H_n(z)$ may be truncated at degree d without loss.

In the calculations reported in theorem 2 we truncated all polynomials at degree 12 in z, saving a considerable amount of space in the computer calculations. A greater saving was made by calculating X_{C*Q} in two parts, based on the two constituent 2-tangles F and G of the knot C. Because C and KT have braid index 4 the best possible presentation for $C*Q$ is as a closed 12-braid. Now direct calculations in the Hecke algebra $H_{12}(z)$ are impractical, since there are $12! \approx 5\times 10^8$ basis elements, requiring working storage for this number of polynomials. The 2-tangles F and G for C each have the form of a 3-braid in which one string has been closed off. The 3-parallel $C*Q$ can be built out of two 6-tangles, namely the 3-parallels of F and G. Each of these 6-tangles has the form of a 9-braid with three strings closed off, because of the nature of F and G. This enables the contribution of each 6-tangle to be calculated first from an element in $H_9(z)$, which after the closure of the three strings yields a linear combination (with coefficients depending on u) of four elements of $H_6(z)$. Combining the contributions of the two 6-tangles can be done quite easily, and gives X_{C*Q} without having to deal with any algebra larger than $H_9(z)$. The number of basis elements here is $9! \approx 360,000$, which has proved possible to handle without special storage allocation on the local Sun mainframe, given that the polynomials to be stored were truncated at degree 12.

In comparing a pair of mutants K and K' we modified the Morton-Short program to calculate the contribution of the 3-parallel of F in $H_6(z)$ and then subtract the contribution of F', which can be deduced without recalculation, before combining with that of the 3-parallel of G. This provides a direct calculation of the difference $X_{K*Q} - X_{K'*Q}$ at a prescribed level of truncation.

The general oriented 2-tangle F can be presented as shown in figure 5 as the partial closure of a k-braid for some k, in which all but two strings are closed off.

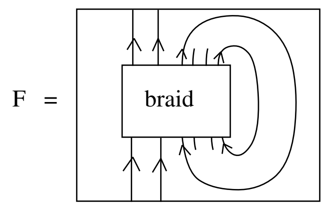


Figure 5

We made a number of calculations for other mutant pairs besides C and KT in which F and G are again partial closures of 3-braids, using a closed 3-braid as the pattern Q. As noted earlier, the difference in each case appeared very similar, with a leading term of degree 11 which was always the same up to a constant factor. Computational problems will clearly arise if we want to use partial closures of k-braids with $k > 3$ for F and G, or if the pattern Q has more than three strings, as in either case we will have to handle braids with at least 12 strings.

The most hopeful route for further computation may be to use the quantum invariants. The simplest unknown case is the $SU(3)_q$ invariant for $\lambda = \begin{smallmatrix}\square&\square&\square\\ \square&&\end{smallmatrix}$, which involves a module of dimension 15, and hence endomorphisms of a module of dimension 15^3 among the contributions of the tangles F and G for the case of C and KT. It would also be interesting to look further at the $SU(4)_q$ invariant with $\lambda = \begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}$ for mutants based on partially closed 4-braids, to see if there were any cases in which a different type Vassiliev invariant was needed to distinguish them.

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