Homfly skeins and the Hopf link

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy by

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June 2001

Abstract

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This thesis exhibits skeins based on the Homfly polynomial and their relations to Schur functions. The closures of skein-theoretic idempotents of the Hecke algebra are shown to be specializations of Schur functions. This result is applied to the calculation of the Homfly polynomial of the decorated Hopf link. A closed formula for these Homfly polynomials is given. Furthermore, the specialization of the variables to roots of unity is considered.

The techniques are skein theory on the one side, and the theory of symmetric functions in the formulation of Schur functions on the other side. Many previously known results have been proved here by only using skein theory and without using knowledge about quantum groups.

Acknowledgements

I am grateful to the DAAD (Deutscher Akademischer Austauschdienst) whose financial support by a HSP-III-grant enabled me to write this thesis. I thank my supervisor Dr. Hugh Morton for sharing his view on skein theory, proposing many of the results that appear in this thesis and for suggesting the necessary techniques. I very much appreciated all comments on my work by members of the department and by visitors.

I would like to express my sincere thanks to Prof. Yongju Bae (Taegu, South Korea) and Dr. Pedro Manchon (Madrid, Spain) for their mathematical encouragement and their friendship. The chapter on quantum invariants has benefited considerably from the careful proofreading by Pedro Manchon.

Während meiner Schulzeit versuchte ich den Grund zu finden, weshalb -1 multipliziert mit -1 wirklich $+1$ ergibt. Da 0 das neutrale Nichts ist, muß -1 etwas Fehlendes sein, das einer Ergänzung bedarf. Die Summe von zwei negativen Zahlen ist immer negativ. Die Multiplikation jedoch scheint die Ordnung von negativ, neutral und positiv zu transzendieren: Das Negative multipliziert mit dem Negativen wird positiv.

Nach meiner Promotion erkannte ich, wie wichtig der Übergang vom Unverständnis zum Wunsch nach Verständnis ist. Die Analogie zwischen der uns umgebenden Welt und der Mathematik ist begrenzt. Vielleicht verläuft diese Grenze bereits durch die Gleichheit von $-1 \cdot -1$ und $+1$.

Whilst at school I tried to understand why -1 multiplied by -1 becomes $+1$. If 0 is the neutral nothing then -1 is something missing which requires a completion. The sum of two negative numbers is always negative. But the multiplication seems to transcend the order of negative, neutral and positive: The negative multiplied by the negative becomes positive.

On completion of my thesis I realized how important the transition is from not understanding to the wish to understand. The analogy between our surrounding world and Mathematics is limited. Perhaps this limit already passes through the equality of $-1 \cdot -1$ and $+1$.

Contents

Introduction

This work exhibits skeins based on the Homfly polynomial and their relations to Schur functions. The results are applied to the calculation of the Homfly polynomial of the decorated Hopf link. Furthermore, the specialization of the variables to roots of unity is considered.

The techniques are skein theory on the one side, and the theory of symmetric functions in the formulation of Schur functions on the other side.

Part of the results are new. For the other results, the approach is new by using skein theory rather than information about quantum groups. This approach has the benefit of generalizing previously known results in many cases.

The first two chapters describe the necessary background of symmetric functions and Homfly skeins. The skeins are based on framed tangles which can be interpreted either as ribbons and annuli with oriented cores or as tangle diagrams with the blackboard framing.

The Homfly skein of the disc with $2n$ boundary points (with suitable 'orientations') becomes an algebra by defining the multiplication as stacking two diagrams one above the other. This algebra H_n is isomorphic to the Hecke algebra which is a deformation of the group algebra of the symmetric group on n letters. Idempotents of H_n indexed by Young diagrams with n cells occur in a natural way as described e.g. in [9], [11], [2], [3] and [25]. I give an account of Blanchet's explicit semi-simple decomposition of H_n and remark in lemma 2.5.6 that the basis elements behave in a nice way under the inclusion $H_n \otimes 1 \subset H_{n+1}$.

The closure of the idempotent of H_n indexed by a Young diagram λ becomes an element Q_λ of the skein of the annulus. It is natural to expect that the map $\lambda \mapsto Q_\lambda$ from the algebra of Young diagrams to the skein of the annulus is an injective algebra homomorphism. This has been proved e.g. in [1], but Aiston's proof used results about quantum groups. The skein-theoretic proof given in chapter 3 was motivated by Kawagoe's ideas in [14]. I interpret the Q_{d_i}'s as the i-th complete symmetric function, and I show that the λ-Schur function in the $\{Q_{d_i}\}$ has the same eigenvalue as Q_λ under some linear map. This eventually leads to the identification of $s_\lambda(\{Q_{d_i}\})$ with Q_λ.

In chapter 4 I describe the Homfly polynomial $\langle\lambda,\mu\rangle$ of the Hopf link with decorations Q_λ and Q_μ on its components. The results are new. The determinantal formula in theorem 4.4.2 for $\langle\lambda,\mu\rangle$ in the case of the substitution $v=s^{-N}$ for some integer $N\geq 2$ was motivated by the results for special cases. It was suggested by a formula in the case $N=2$ by [19], and in the case $N\geq 2$ and s and x substituted by certain roots of unity it was motivated by the formula for the modular transformation matrix S at the end of section 2.3 in [16]. If λ and μ have at most N rows then the formula expresses $\langle\lambda,\mu\rangle$ after the substitution $v=s^{-N}$ as the quotient of two $(N\times N)$-minors of the infinite Vandermonde matrix $V=(s^{2(i-1)(j-1)})_{i,j\geq 1}$. The denominator is the principal minor, and the numerator is given by choosing rows $\lambda_{N-i+1}+i$ and columns $\mu_{N-j+1}+j$ for $i,j=1,\ldots,N$.

In theorems 4.3.4 and 4.3.6 I give a compact formula for the power series $\frac{1}{\langle\lambda\rangle}\sum_{i\geq 0}\langle\lambda,c_i\rangle X^i$, where c_i denotes the column diagram of length i, and $\langle\lambda\rangle=\langle\lambda,\emptyset\rangle$ is the Homfly polynomial of Q_λ. From this, $\langle\lambda,\mu\rangle$ can be calculated directly as a Schur function.

In chapter 5 we substitute the variables x,v and s of the Homfly polynomial by roots of unity, $s^{2(l+N)}=1$, $x^N=s^{-1}$, and $v=s^{-N}$. In this setting, it turns out that the Homfly polynomial of any decorated link does not change when we replace the decoration Q_λ by $Q_{\lambda'}$ whenever $\lambda-\lambda'$ lies in the ideal $I_{N,l}$ of the ring of Young diagrams that is generated by $c_0-c_N, c_{N+1}, c_{N+2},\ldots$ and $d_{l+1},\ldots,d_{l+N-1}$.

This ideal $I_{N,l}$ and the quotient ring $\mathcal{Y}/I_{N,l}$ are considered in chapter 6 which is an algebraic account independent of any skein calculations. It is known that the quotient $\mathcal{Y}/I_{N,l}$ has a basis consisting of the Young diagrams that lie in the $(N-1)\times l$-rectangle. This has been proved by Aiston [1] by using algebraic geometry. I prove the result using only the Littlewood-Richardson rule. The new ingredient here is the algorithm in section 6.4 that produces for any Young diagram λ an element $\overline{\lambda}$ of $\mathcal{Y}$ such that $\lambda-\overline{\lambda}$ lies in $I_{N,l}$, and either $\overline{\lambda}$ is equal to zero, or it is up to a sign a Young diagram in the $(N-1)\times l$-rectangle.

Chapter 7 interprets the results of the previous chapter in a discrete lattice model of $\mathcal{Y}_N=\mathcal{Y}/I_N$, where $I_N\subset I_{N,l}$ is an ideal of $\mathcal{Y}$. The elements of $\mathcal{Y}_N$ that lie in $I_{N,l}$ form a locally finite family of hyperplanes. Quotienting $\mathcal{Y}_N$ by $I_{N,l}$ is the same as quotienting the lattice model by a discrete group of Euclidean isometries generated by the reflections in this family of hyperplanes. The Young diagrams in the $(N-1)\times l$-rectangle correspond to a fundamental simplex next to the origin. The sign appearing in $\overline{\lambda}$ is seen to be the parity of the number of reflections that are needed to bring the lattice point λ to this fundamental simplex.

In chapter 8 I describe the multiplication in $\mathcal{Y}_{N,l}$. In particular, I show that the empty Young diagram appears as a summand of the product of two Young diagrams in the $(N-1)\times l$-diagrams if and only if the two Young diagrams are dual to each other, i.e. they are up to a rotation the complement of each other in a $N\times k$-rectangle for some $k\geq 1$. This result enables us to show in theorem 8.2.9 that the matrix H indexed by Young diagram in the $(N-1)\times l$-rectangle and having the value of the Homfly polynomial of the Hopf link decorated by Q_λ and Q_μ at the position (λ,μ) is quasi-Hermitian, i.e. $H\overline{H}$ is a scalar multiple of the identity matrix after the substitutions $s=x^{-N}$, $v=s^{-N}$, and x by a root of unity of order $2N(l+N)$. Previously known proofs used the knowledge about modular categories, e.g. as in [1] and, more skein-theoretically, in [3].

The Young diagram $\sigma(\lambda)$ derives from λ by adding an initial row of length l and removing then all initial columns of length N. One can show that $\sigma^N(\lambda)=\lambda$. Hence, the cyclic group $\mathbb{Z}_N$ operates on the set of Young diagrams in the $(N-1)\times l$-rectangle. In chapter 9 I give a skein theoretic proof of a result by Kohno and Takata [16] about the Homfly polynomial after the substitution of its variables by roots of unity. The result is that knowledge of the Homfly polynomial of a link L with decorations $Q_\lambda,\dots,Q_\mu$ on its components and the linking matrix of L is sufficient to calculate the Homfly polynomial of L decorated with $Q_{\sigma^{a_1}(\lambda)},\dots,Q_{\sigma^{a_t}(\mu)}$ on its components for any integers a_i.

The second part of chapter 9 explains two approaches that relate the Homfly polynomial of a link L decorated with $Q_\lambda,\dots,Q_\mu$ on its components to the Homfly polynomial of L decorated with $Q_{\lambda^\vee},\dots,Q_{\mu^\vee}$ on its components. Here, $\lambda^\vee$ derives from λ by interchanging rows and columns. Provided one substitutes the variables of the Homfly polynomial by suitable roots of unity, these two Homfly polynomials turn out to be the complex conjugate of each other. One of the two approaches is new, the other approach appeared in [16] in a non-skein-theoretic formulation.

In chapter 10 I classify the ring homomorphisms from $\mathcal{Y}_{N,l}$ to $\mathbb{C}$. In lemma 10.2.1 I characterize these ring homomorphisms by $(N-1)$-tuples of complex numbers called Young-solutions. In section 10.3 I assign to every Young-solution a σ-orbit of Young diagrams in the $(N-1)\times l$-rectangle in a canonical way. It turns out that the number of ring homomorphisms that are assigned the same σ-orbit is equal to the number of Young diagrams in this orbit. The number of ring homomorphisms from $\mathcal{Y}_{N,l}$ to $\mathbb{C}$ is thus equal to the number of Young diagrams in the $(N-1)\times l$-rectangle. At the end of this chapter, I relate Young-solutions and the Homfly polynomials of the decorated Hopf link.

Chapter 11 explains the relation between Homfly polynomials of links decorated by Q_λ's and the $U_h(sl(N))$-invariants. This has been done in [1], but the proof given there had some gaps which are filled here. The results in this chapter are an application of the general theory of quantum-link-invariants as explained e.g. in [22], [15], [12], [13], and [4]. Earlier chapters are independent of this account on quantum groups, thus keeping skein theory and quantum groups on their own grounds.

Chapter 1

Symmetric functions and Young diagrams

1.1 Symmetric functions

This exposition of symmetric functions is based on the first chapter of [17].

We denote by $\mathbb{Z}[x_1, \ldots, x_n]$ the ring of integer polynomials in n variables. It contains the subring Λ_n of symmetric polynomials, i.e. polynomials that are invariant under any permutation of the variables $x_1, \ldots, x_n$. We have

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$$

where Λ_n^k is the $\mathbb{Z}$-submodule of Λ_n that consists of the homogeneous polynomials of degree k, together with the zero polynomial.

For any $m \geq n \geq 0$ we have a ring homomorphism

$$\mathbb{Z}[x_1, \ldots, x_m] \to \mathbb{Z}[x_1, \ldots, x_n]$$

which maps $x_{n+1}, \ldots, x_m$ to zero and keeps any other x_j invariant. This restricts to a module homomorphism $\Lambda_m^k \to \Lambda_n^k$ for any $m \geq n \geq 0$ and $k \geq 0$. These maps are always surjective, and are bijective for $m \geq n \geq k$. We thus define the $\mathbb{Z}$-module Λ^k to be the inverse limit of the $\mathbb{Z}$-modules Λ_n^k,

$$\Lambda^k = \varprojlim_{n} \Lambda_n^k.$$

An element of Λ^k is a sequence $f = (f_n)_{n \geq 0}$, where each $f_n = f_n(x_1, \ldots, x_n)$ is a homogeneous symmetric polynomial of degree k, and $f_m(x_1, \ldots, x_n, 0, \ldots, 0) = f_n(x_1, \ldots, x_n)$ for any $m \geq n \geq 0$. We define the ring Λ of symmetric functions in countably many variables $x_1, x_2, \ldots$ by

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k$$

where the multiplication is componentwise, i.e.

$$(f_0, f_1, \ldots)(g_0, g_1, \ldots) = (f_0 g_0, f_1 g_1, \ldots).$$

The ring Λ is commutative since $\mathbb{Z}[x_1, \ldots, x_n]$ is commutative for any n. The word 'function' is used in the context of the inverse limit, instead of 'polynomial'.

The *r-th elementary symmetric function* e_r is defined by its generating function

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t).$$

The *r-th complete symmetric function* h_r is defined by its generating function

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}.$$

We thus have

$$H(t)E(-t) = 1,$$

or, equivalently,

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \tag{1.1.1}$$

for any $n \geq 1$. We define $e_r = h_r = 0$ for $r < 0$.

Lemma 1.1.1 *Λ is freely generated by $e_1, e_2, \ldots$ as a commutative algebra. It is also freely generated by $h_1, h_2, \ldots$.*

A *pre-partition* is a non-empty (finite or infinite) sequence of non-negative integers

$$\lambda = (\lambda_1, \lambda_2, \ldots)$$

in weakly decreasing order

$$\lambda_1 \geq \lambda_2 \geq \cdots$$

such that only finitely many terms are non-zero. We define an equivalence relation on the set of pre-partitions by saying that two pre-partitions are equivalent if they differ by a (possibly infinite) number of zeros. An equivalence class is called a *partition*. We shall consider partitions but we will mainly use pre-partitions in our arguments. The easy exercise that the statements are correct for partitions is left to the reader in each case.

A *Young diagram* denotes both a partition and a graphical description of this partition. We represent a Young diagram $\lambda = (\lambda_1, \ldots)$ by an array of square cells (of the same size) with λ_i cells in the i-th row, for $i = 1, 2, \ldots$ where we enumerate the rows from top to bottom and the columns from left to right. The j-th cell in the i-th row has the coordinates (i, j). The *content* $\mathrm{cn}(c)$ of the cell $c = (i, j)$ is defined to be $j - i$. The *hook length* $\mathrm{hl}(c)$ of the cell c is defined to be 1 plus the number of cells to the right of c plus the number of cells below c. The number of cells of a Young diagram is denoted by $|\lambda|$. The *length* $l(\lambda)$ is the number of rows of λ, i.e. $\lambda_{l(\lambda)} \neq 0$ and $\lambda_i = 0$ for $i > l(\lambda)$. The *empty Young diagram* $\emptyset$ is the partition (0).

A *standard tableau* of a Young diagram λ is a labelling of the cells of λ by the integers $1, 2, \ldots, |\lambda|$ which is increasing along each row from left to right, and increasing along each column from top to bottom. The number of standard tableaux for a Young diagram λ is denoted by d_λ. We have $d_\lambda \geq 1$ for any Young diagram λ.

We write $\lambda \subset \mu$ for Young diagrams λ and μ if the graphical description of λ is a subset of the graphical description of μ, i.e. if $\lambda_i \leq \mu_i$ for all i. For such Young diagrams, $\mu \backslash \lambda$ denotes the set of cells of μ that do not lie in λ.

Given a Young diagram λ, we define the *transposed* Young diagram $\lambda^\vee$ to be the Young diagram that derives from λ by the reflection in the main diagonal, i.e. the cell (i, j) lies in $\lambda^\vee$ if and only if the cell (j, i) lies in λ. We have $(\lambda^\vee)^\vee = \lambda$ for any Young diagram λ. We have $\mathrm{hl}(i, j) = \lambda_i - i + \lambda_j^\vee - j + 1$.

The single row Young diagram with i cells is denoted by d_i, and the single column Young diagram with i cells is denoted by c_i. We have $d_i^\vee = c_i$ and $c_0 = d_0 = \emptyset$.

We consider a Young diagram λ and an integer $n \geq l(\lambda)$. We define a symmetric polynomial s_λ^n in n variables $x_1, \ldots, x_n$ by

$$s_\lambda^n(x_1, \ldots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}}{\det(x_i^{n-j})_{1 \leq i,j \leq n}}.$$

The numerator and denominator are anti-symmetric, hence the quotient s_λ^n is symmetric in the variables $x_1, \ldots, x_n$. It is a polynomial, indeed. We define $s_\lambda^k = 0$ for $0 \leq k \leq l(\lambda)$. The λ-*Schur function* $s_\lambda = (s_\lambda^n)_{n \geq 0}$ lies in Λ because $s_\lambda^m(x_1, \ldots, x_n, 0, \ldots, 0) = s_\lambda^n(x_1, \ldots, x_n)$ for any $m \geq n$ (which is easily checked).

Lemma 1.1.2 *The set of Schur functions s_λ for all Young diagrams λ is a $\mathbb{Z}$-basis of Λ. The set of Schur functions s_λ such that $|\lambda| = k$ is a $\mathbb{Z}$-basis of Λ^k for any $k \geq 0$.*

The next lemma is sometimes called the Giambelli (or Jacobi-Trudi) formula.

Lemma 1.1.3 *For any $n \geq l(\lambda)$ and $m \geq l(\lambda^\vee)$*

$$\begin{aligned} s_\lambda &= \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n} \\ &= \det(e_{\lambda_i^\vee - i + j})_{1 \leq i,j \leq m}. \end{aligned}$$

This implies that $s_{c_i} = e_i$ and $s_{d_i} = h_i$ for any integer $i \geq 0$.

The following multiplication rule for Schur functions is called *Littlewood-Richardson rule*. A proof is given in chapter I.9 of [17]. The concept of a strict extension as given here is equivalent to Macdonald's description.

Theorem 1.1.4 *For any Young diagrams λ and μ we have*

$$s_\lambda s_\mu = \sum_\nu a^\nu_{\lambda\mu} s_\nu$$

where $a^\nu_{\lambda\mu} = 0$ unless $\lambda \subset \nu$ and $|\lambda| + |\mu| = |\nu|$, in which case $a^\nu_{\lambda\mu}$ denotes the number of strict extensions of λ by μ to ν.

We have to know what a strict extension is.

Let λ, μ and ν be Young diagrams such that $\lambda \subset \nu$ and $|\nu| = |\lambda| + |\mu|$. An *extension* ζ of λ by μ to ν is a labelling of the cells of $\nu\backslash\lambda$ with the integers $1, \ldots, l(\mu)$ such that the label i appears μ_i times, $i = 1, \ldots, l(\mu)$. Furthermore, an extension has to satisfy the following two conditions. First, the labels are strictly increasing downwards along every column of ν. Second, the set of cells $\nu^{(j)}$ which derives from ν by removing all cells with labels greater than or equal to j has to be a Young diagram for any $j = 1, \ldots, l(\mu)$.

An extension ζ determines a word $w(\zeta)$ which is the sequence of labels of ζ read from right to left and top-down.

An extension is called *strict* if for any label i and any prefix (i.e. initial subword) of $w(\zeta)$ the number of occurrences of the label i is not less than the number of occurrences of $i + 1$.

For example, the two extensions of $(3, 1)$ by $(3, 2)$ to $(4, 4, 1)$ in figure 1.1 determine the words $1, 2, 2, 1, 1$ resp. $1, 2, 1, 1, 2$. The second extension is therefore strict whereas the first extension is not strict.

Remark The rows of any extension are weakly increasing when read from left to right. This is because of the condition that $\nu^{(i)}$ is a Young diagram for any label i.

Remark Let $k \geq 1$ be any integer. The number of extensions of $(k, k-1, \ldots, 2, 1)$ by $(k, 1)$ to $(k + 1, k, \ldots, 2, 1)$ is equal to k. Hence, any non-negative integer can occur as a value for $a^\nu_{\lambda\mu}$ for suitable λ, μ and ν.

Remark Instead of checking all prefixes of $w(\zeta)$, one can, equivalently, check all subwords of $w(\zeta)$ that arise as the set of cells that lie above and to the right of some labelled cell of ζ. This alternative definition has been used in [1].

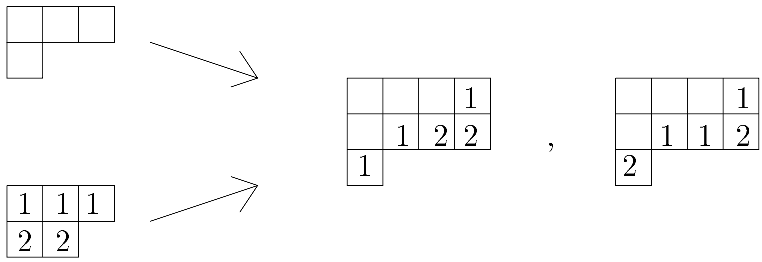


Figure 1.1: The two extensions of $(3,1)$ by $(3,2)$ to $(4,4,1)$.

1.2 The ring of Young diagrams

Definition The *ring of Young diagrams* $\mathcal{Y}$ is the $\mathbb{Z}$-module whose basis is the set of all Young diagrams. The multiplication is defined by

$$\lambda\mu = \sum_{|\nu|=|\lambda|+|\mu|} a^{\nu}_{\lambda\mu}\nu$$

where the coefficients $a^{\nu}_{\lambda\mu}$ are given by the Littlewood-Richardson rule as stated in theorem 1.1.4.

Since the Schur functions s_λ are a linear basis for Λ by lemma 1.1.2, we get a ring isomorphism from Λ to $\mathcal{Y}$ by mapping s_λ to λ. This implies in particular that $\mathcal{Y}$ is a commutative ring. Furthermore, the ring $\mathcal{Y}$ is the free commutative ring generated by all column diagrams $c_1, c_2, \ldots$ which follows from lemma 1.1.1 and the observation the $s_{c_i} = e_i$. Similarly, $\mathcal{Y}$ is the free commutative ring generated by the row diagrams $d_1, d_2, \ldots$. We remark that the empty Young diagram $\emptyset$ is the neutral element for the multiplication. In $\mathcal{Y}$, we define $c_i = d_i = 0$ for integer $i < 0$.

The Giambelli formula from Lemma 1.1.3 becomes

Lemma 1.2.1 *For any $n \geq l(\lambda)$ and $m \geq l(\lambda^\vee)$*

$$\begin{aligned} \lambda &= \det(d_{\lambda_i - i + j})_{1 \leq i,j \leq n} \\ &= \det(c_{\lambda^\vee_i - i + j})_{1 \leq i,j \leq m}. \end{aligned}$$

Equation (1.1.1) now takes the form

$$\sum_{r=0}^{n} (-1)^r c_r d_{n-r} = 0 \tag{1.2.2}$$

for any $n \geq 1$. Equivalently,

$$\left(\sum_{r=0}^{\infty} (-1)^r c_r X^r\right)\left(\sum_{i=0}^{\infty} d_i X^i\right) = 1 \tag{1.2.3}$$

where X is a variable.

Transposing induces a linear map from $\mathcal{Y}$ to $\mathcal{Y}$. This map (called *transposing* as well) is bijective because $(\lambda^\vee)^\vee = \lambda$.

Lemma 1.2.2 *Transposing is a ring automorphism of $\mathcal{Y}$.*

Proof Since $\mathcal{Y}$ is spanned by Young diagrams and generated by column diagrams, it is sufficient to prove that $(\lambda c_i)^\vee = \lambda^\vee (c_i)^\vee$ for any Young diagram λ and any column diagram c_i, $i \geq 1$. We remark that $(c_i)^\vee = d_i$.

The strict extensions of λ by a column c_i of length i are in bijection with the set of Young diagrams that derive from λ by adding i (unnumbered) cells so that at most one cell is added to each row of λ. To turn such a Young diagram into a strict extension, one has to number the added cells with successive numbers $1, 2, \ldots, i$ going the rows downwards.

Similarly, the strict extensions of λ by a row d_i of length i are in bijection with the set of Young diagrams that derive from λ by adding i (unnumbered) cells so that at most one cell is added to each column of λ.

This description of strict extensions is symmetric with respect to columns and rows. Since transposing interchanges columns and rows, it induces a bijection of the strict extensions of λ by c_i and the strict extensions of $\lambda^\vee$ by d_i. ♣

1.3 The ring $\mathcal{Y}_N$

We fix an integer $N \geq 1$.

1.3.1 Definition

We denote by I_N the ideal of $\mathcal{Y}$ generated by the element $c_0 - c_N$ and all column diagrams of length at least $N+1$,

$$I_N = \langle\langle c_0 - c_N, c_{N+1}, c_{N+2}, \ldots \rangle\rangle.$$

We denote

$$\mathcal{Y}_N = \mathcal{Y}/I_N,$$

and we shall denote the image of a Young diagram λ under the quotient map $\mathcal{Y} \to \mathcal{Y}/I_N$ by λ, too.

Definition For a Young diagram $\lambda = (\lambda_1, \ldots, \lambda_N)$ with N rows we define λ' to be the Young diagram derived from λ by removing all columns of length N,

$$\lambda' = (\lambda_1 - \lambda_N, \ldots, \lambda_{N-1} - \lambda_N).$$

Lemma 1.3.1 *For a Young diagram λ with N rows we have $\lambda - \lambda' \in I_N$. If λ has more than N rows then $\lambda \in I_N$.*

Proof For a Young diagram λ we have by the Giambelli formula that

$$\begin{aligned} \lambda &= \det(c_{\lambda_i^\vee - i + j})_{1 \le i,j \le b} \\ &= \begin{vmatrix} c_{\lambda_1^\vee} & c_{\lambda_1^\vee + 1} & \cdots & c_{\lambda_1^\vee + b - 1} \\ c_{\lambda_2^\vee - 1} & c_{\lambda_2^\vee} & \cdots & c_{\lambda_2^\vee + b - 2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\lambda_b^\vee - b + 1} & c_{\lambda_b^\vee - b + 2} & \cdots & c_{\lambda_b^\vee} \end{vmatrix} \end{aligned} \tag{1.3.4}$$

where b denotes the length of $\lambda^\vee$ (which is equal to λ_1). If λ has N rows then the first row of the above determinant reads $c_N, c_{N+1}, \ldots, c_{N+b-1}$. Since $c_N = 1$ and $c_i = 0$ for $i > N$ in $\mathcal{Y}_N$, we can remove the first row and the first column of the determinant without changing its value in $\mathcal{Y}_N$. Hence

$$\lambda = \begin{vmatrix} c_{\lambda_2^\vee} & \cdots & c_{\lambda_2^\vee + b - 2} \\ \vdots & \ddots & \vdots \\ c_{\lambda_b^\vee - b + 2} & \cdots & c_{\lambda_b^\vee} \end{vmatrix}$$

in $\mathcal{Y}_N$. The determinant on the right hand side is the Giambelli formula for the Young diagram that derives from λ by removing the first column (of length N). By applying this argument λ_N-times we get $\lambda = \lambda'$ in $\mathcal{Y}_N$.

If λ has more than N rows, i.e. its length is greater than N, then each entry of the first row of the determinant in equation (1.3.4) is equal to zero in $\mathcal{Y}_N$. Hence $\lambda = 0$ in $\mathcal{Y}_N$. ♣

We define the $\mathbb{Z}$-submodule L_N of $\mathcal{Y}$ to be linearly spanned by all Young diagrams with more than N rows and by the elements $(\lambda - \lambda') \in \mathcal{Y}$ for all Young diagrams λ with N rows,

$$L_N = \langle \lambda - \lambda', \mu \mid \text{ Young diagrams } \lambda \text{ and } \mu \text{ with } l(\lambda) = N \text{ and } l(\mu) \ge N + 1 \rangle.$$

We clearly have

$$\mathcal{Y} = L_N \oplus \langle \lambda \mid \text{ Young diagrams } \lambda \text{ with } l(\lambda) < N \rangle. \tag{1.3.5}$$

Lemma 1.3.2 *L_N is an ideal in $\mathcal{Y}$.*

Proof Since $\mathcal{Y}$ is generated by all the column diagrams $c_1, c_2, \ldots$ it is sufficient to verify that

$$c_i(\lambda - \lambda') \in L_N \text{ for any } i \ge 1 \text{ and any Young diagram } \lambda \text{ with } l(\lambda) = N$$

and

$$c_i\lambda \in L_N \text{ for any } i \geq 1 \text{ and any Young diagram } \lambda \text{ with } l(\lambda) > N.$$

Let $i > N$ and let λ be any Young diagram. Since c_i is a subdiagram of any summand of $c_i\lambda$, we have that $c_i\lambda$ is a linear combination of Young diagrams with more than N rows. Hence $c_i\lambda$ lies in L_N.

Let $i \geq 1$ and let λ be a Young diagram with more than N rows. Then $c_i\lambda$ is a linear combination of Young diagrams with more than N rows since λ is a subdiagram of each summand. Hence $c_i\lambda$ lies in L_N.

Let $1 \leq i \leq N$ and λ be a Young diagram with N rows. We denote by k the number of initial columns of length N of λ. By the multiplication rule for Young diagrams we observe a bijection between the summands of $c_i\lambda$ with N rows and the summands of $c_i\lambda'$ with at most N rows. The bijection being the removal of k initial columns of length N. Hence $c_i(\lambda - \lambda')$ is a linear combination of Young diagrams with more than N rows and terms $(\mu - \zeta)$ where μ and ζ differ by k initial columns of length N. The Young diagrams with more than N rows lie in L_N. The terms $(\mu - \zeta)$ lie in L_N because $\mu' = \zeta'$, hence $\mu - \zeta = (\mu - \mu') - (\zeta - \zeta') \in L_N$. Hence, $c_i(\lambda - \lambda') \in L_N$. ♣

Corollary 1.3.3 *We have $L_N = I_N$. The (images of the) Young diagrams with less than N rows are a basis of $\mathcal{Y}_N$.*

Proof The submodule L_N of $\mathcal{Y}$ is a subset of I_N by lemma 1.3.1. Since L_N is an ideal, we have $L_N = I_N$. The Young diagrams with less than N rows are a basis of $\mathcal{Y}_N$ because of the decomposition of $\mathcal{Y}$ in equation (1.3.5). ♣

1.3.2 Duality

We introduce the concept of duality for Young diagrams with respect to a fixed integer $N \geq 1$. We consider a Young diagram $\lambda = (\lambda_1, \ldots, \lambda_{N-1})$. The complement of λ in the $N \times \lambda_1$-rectangle is not a Young diagram. But after rotating this complement through 180° it becomes a Young diagram denoted by λ^* as depicted in figure 1.2. We denote the dual of the column diagram c_i by c_i^* rather than $(c_i)^*$. We have $c_i^* = c_{N-i}$ for $i = 1, \ldots, N-1$. We have $c_0^* = c_0$.

We have $\lambda_i^* = \lambda_1 - \lambda_{N-i+1}$ for $i = 1, 2, \ldots, N$. It is clear that $(\lambda^*)^* = \lambda$ for any Young diagram λ with less than N rows. Therefore, taking the dual is a permutation of the Young diagrams with less than N rows.

We define

$$\lambda^* = \begin{cases} \lambda^* & \text{if } l(\lambda) \leq N-1, \\ (\lambda')^* & \text{if } l(\lambda) = N, \\ 0 & \text{if } l(\lambda) \geq N+1. \end{cases}$$

Hence, the map $\lambda \mapsto \lambda^*$ induces a linear map $\mathcal{Y}_N \to \mathcal{Y}_N$.

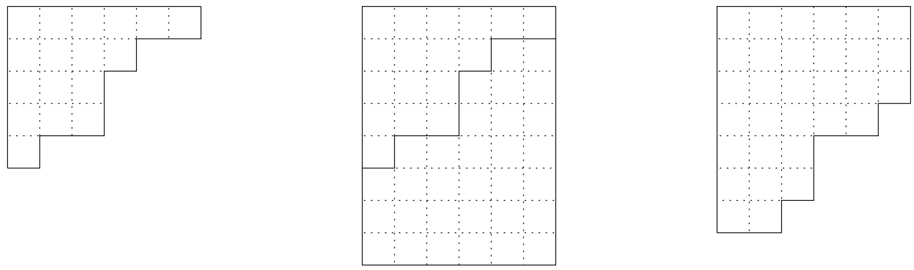


Figure 1.2: The dual of the Young diagram $\lambda = (6,4,3,3,1)$ with respect to $N = 8$ is equal to $\lambda^* = (6,6,6,6,5,3,3,2)$.

Lemma 1.3.4 *The map $\lambda \mapsto \lambda^*$ induces a ring automorphism of $\mathcal{Y}_N$.*

Proof Since $\mathcal{Y}_N$ is spanned by all Young diagrams with less than N rows and generated by the column diagrams $c_1, \ldots, c_{N-1}$, it is sufficient to prove that $(\lambda c_k)^* = \lambda^* c_k^*$ for any Young diagram with less than N rows and any column diagram c_k, $1 \le k \le N-1$.

We have $c_k^* = c_{N-k}$ in $\mathcal{Y}_N$ for any integer k, (with the convention that $c_k = 0$ for $k < 0$), and we thus have to prove that

$$(\lambda c_k)^* = \lambda^* c_{N-k}$$

for any Young diagram λ with at most $(N-1)$ rows, and any integer $k = 1, \ldots, N-1$.

By ε and β we denote variables which are to have values zero or one. The strict extensions of λ by c_k that have at most N rows are all those Young diagrams $(\lambda_1 + \varepsilon_1, \ldots, \lambda_{N-1} + \varepsilon_{N-1}, \varepsilon_N)$ for which $\varepsilon_1 + \ldots + \varepsilon_N = k$.

The strict extensions of λ^* by c_{N-k} that have at most N rows are all those Young diagrams $(\lambda_1^* + \beta_1, \ldots, \lambda_{N-1}^* + \beta_{N-1}, \beta_N)$ for which $\beta_1 + \cdots + \beta_N = N - k$.

Let us consider the sequence of integers

$$(\lambda_1 + \varepsilon_1, \ldots, \lambda_{N-1} + \varepsilon_{N-1}, \varepsilon_N) \tag{1.3.6}$$

for some integers $\varepsilon_1, \ldots, \varepsilon_N$ which are either equal to zero or one, and such that $\varepsilon_1 + \cdots + \varepsilon_N = k$. This is not necessarily a Young diagram. To each such sequence we associate the sequence of integers given by

$$(\lambda_1^* + (1 - \varepsilon_N), \lambda_2^* + (1 - \varepsilon_{N-1}), \ldots, \lambda_{N-1}^* + (1 - \varepsilon_2), (1 - \varepsilon_1)). \tag{1.3.7}$$

We claim that the sequence in equation (1.3.6) is a Young diagram (i.e. weakly decreasing) if and only if the sequence in equation (1.3.7) is a Young diagram.

To see this, we note that the sum of the i-th entry of the first sequence and the $(N-i+1)$-st entry of the second sequence is independent of i for all $i = 1, \ldots, N$ because

$$\begin{aligned}(\lambda_i + \varepsilon_i) + (\lambda^*_{N-i+1} + (1-\varepsilon_i)) &= \lambda_i + \lambda_{N-i+1} + 1 \\ &= \lambda_1 + 1.\end{aligned}$$

Hence, the first sequence is weakly decreasing if and only if the second sequence is weakly decreasing.

Remark that $(1-\varepsilon_N) + \ldots + (1-\varepsilon_1) = N - k$. We thus get a bijection of the strict extensions of λ by c_k and strict extensions of λ^* by c_{N-k}, and associated strict extensions correspond to dual Young diagrams. Therefore, $(\lambda c_k)^* = \lambda^* c_k^*$, and thus $(\lambda\mu)^* = \lambda^*\mu^*$ for any Young diagrams λ and μ. ♣

Chapter 2

Skein theory

2.1 Framed Homfly skeins

Our view is piecewise linear. We denote the interior of a manifold M by $\mathrm{int}(M)$ and the boundary of M by ∂M. We always consider *proper* submanifolds N of a manifold M, i.e. $\mathrm{int}(N) \subset \mathrm{int}(M)$. By an isotopy of a submanifold N of a manifold M we always understand that it is induced by a homeomorphism of M which is isotopic to the identity relative to the boundary ∂M.

Let $\mathcal{F}$ by a surface (with or without boundary) with a fixed orientation. Let $\Theta = \Theta_1 \uplus \Theta_2 = \{\theta_1, \ldots, \theta_k\} \uplus \{\theta_{k+1}, \ldots, \theta_{2k}\}$ be a collection of disjoint oriented arcs in the boundary $\partial(\mathcal{F})$ such that the orientation of each arc θ_i of Θ_1 (resp. Θ_2) agrees (resp. does not agree) with the induced orientation of θ_i by $\mathcal{F}$.

A *ribbon tangle* T *in* $(\mathcal{F} \times (0,1), \Theta)$ is a (possibly empty) collection of pairwise disjoint disks $\{D_1, \ldots, D_k\}$ (also called *ribbons*) and finitely many oriented annuli $\{A_j\}$ in $\mathcal{F} \times (0,1)$ with oriented cores such that

$$
\begin{aligned}
\partial\mathcal{F} \times (0,1) \cap D_i &= \theta_{i_1} \times \left\{\frac{1}{2}\right\} \cup \theta_{i_2} \times \left\{\frac{1}{2}\right\}, \text{ for some } \theta_{i_1} \in \Theta_1, \theta_{i_2} \in \Theta_2, \\
\partial\mathcal{F} \times (0,1) \cap \bigcup_{i=1}^{k} D_i &= \Theta \times \left\{\frac{1}{2}\right\} \\
A_j &\in \mathrm{int}(\mathcal{F}) \times (0,1) \text{ for all } j.
\end{aligned}
$$

We call any arc α in D_i that joins points of θ_{i_1} and θ_{i_2} a *core*. We orient each core α '*from* $\alpha \cap \theta_{i_1}$ *to* $\alpha \cap \theta_{i_2}$'. The set $\partial D_i \backslash (\partial D_i \cap \mathcal{F})$ consists of two cores $\alpha_1 \cup \alpha_2$. The orientations of α_1 and α_2 are induced by different orientations of D_i.

We write $(\mathcal{F} \times (0,1))$ for $(\mathcal{F} \times (0,1), \emptyset)$.

Let A be a commutative ring. We denote by A' the polynomial ring over A in the variables x, x^{-1}, v, v^{-1}, s, s^{-1} and δ, quotiented by the relation $\delta(s - s^{-1}) = v^{-1} - v$.

The *framed Homfly skein* $\mathcal{S}(\mathcal{F},\Theta)$ is the free A'-module over the set of all ribbon tangles in $(\mathcal{F},\Theta)$ quotiented by the following relations

$$T = T' \quad \text{if } T \text{ and } T' \text{ are isotopic ribbon tangles,}$$

and the local relations in figures 2.1 and 2.2.

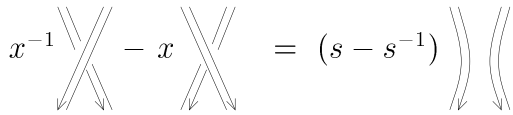


Figure 2.1: Defining relation for $\mathcal{S}(\mathcal{F}\times(0,1),\Theta)$.

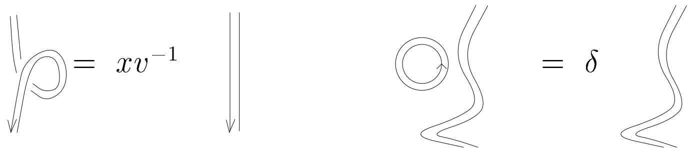


Figure 2.2: More defining relations for $\mathcal{S}(\mathcal{F}\times(0,1),\Theta)$.

We can isotope any ribbon tangle T such that it lies *flat* in $\mathcal{F}\times[\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon]$ for some $\varepsilon>0$ which means that the projection of T to $\mathcal{F}\times\frac{1}{2}$ is an embedding away from finitely many sets $T\cap(N_i\times(0,1))\subset \mathrm{int}(\mathcal{F})\times(0,1)$ each consisting of two local discs of T parallel to a disc N_i in $\mathcal{F}$.

It is straightforward to translate framed Homfly skeins into the language of oriented diagrams by relating 'flat' ribbon tangles with diagrams of arcs and closed curves. To a flat ribbon tangle T in $(\mathcal{F}\times(0,1),\Theta)$ we associate the diagram that is given by the contraction of the ribbons and the annuli to their cores. This is well defined up to isotopy. The cores inherit an orientation from the ribbon tangle.

Each arc of Θ becomes a point under this contraction, and $\Gamma=\Gamma_1\uplus\Gamma_2=\{\gamma_1,\ldots,\gamma_k\}\uplus\{\gamma_{k+1},\ldots,\gamma_{2k}\}$ derives from $\Theta=\Theta_1\uplus\Theta_2$ by making some choice $\gamma_i\in\theta_i$.

Let $\mathcal{F}$ be a surface and $\Gamma=\Gamma_1\uplus\Gamma_2=\{\gamma_1,\ldots,\gamma_k\}\uplus\{\gamma_{k+1},\ldots,\gamma_{2k}\}$ be a set of finitely many points of $\partial\mathcal{F}$. A *diagram in* $(\mathcal{F},\Gamma)$ is a (possibly empty) collection of pairwise disjoint (but we allow crossings) k oriented arcs each joining a point of Γ_1 and Γ_2, and finitely many oriented closed curves in $\mathcal{F}$. The arcs without their endpoints and the closed curves have to lie in $\mathrm{int}(\mathcal{F})$. The arcs are oriented from their intersection with Γ_1 to their intersection with Γ_2. We denote the empty diagram by $\emptyset$.

We call diagrams D_1 and D_2 *regularly isotopic* if they differ by a sequence of moves inside a disc where the moves are the identity on the boundary of this disc. The allowed moves are Reidemeister moves II and III and an orientation preserving homeomorphism of the disc as shown in figure 2.3. We call a sequence of these moves a *regular isotopy.* Note that this has no relation with the usual meaning of isotopy, it is a concept only for diagrams.

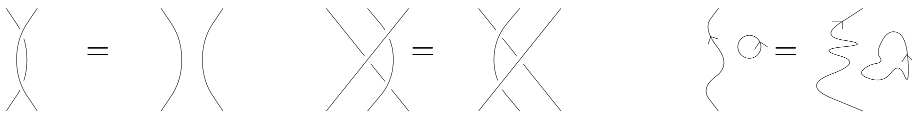


Figure 2.3: Regular isotopy consists of Reidemeister moves II and III, and 'wiggling curves'.

Recall that we denote by A' the polynomial ring in $x, x^{-1}, v, v^{-1}, s, s^{-1}$ and δ, quotiented by the relation $\delta(s - s^{-1}) = v^{-1} - v$.

The *framed Homfly skein* $\mathcal{S}(\mathcal{F}, \Gamma)$ is the free A'-module over the set of all diagrams in $(\mathcal{F}, \Gamma)$ quotiented by the following relations

$$T = T' \quad \text{if } T \text{ and } T' \text{ are regularly isotopic,}$$

and the local relations in figures 2.4 and 2.5.

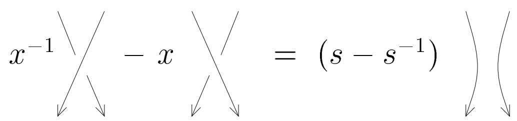


Figure 2.4: Defining relation for $\mathcal{S}(\mathcal{F}, \Gamma)$.

Figure 2.5: More defining relations for $\mathcal{S}(\mathcal{F}, \Gamma)$.

Whenever (here and in the following) the orientation of the cores is not shown then the diagrams represent all possible orientations.

The Whitney-trick is a regular isotopy that relates a straight arc with an arc having two curls. It is depicted in figure 2.6. We can remove one of the curls

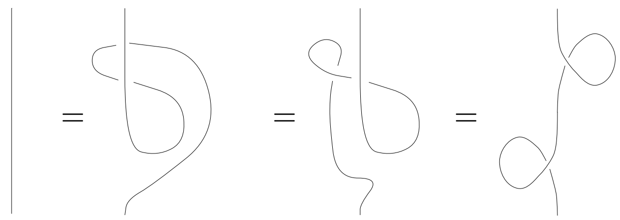
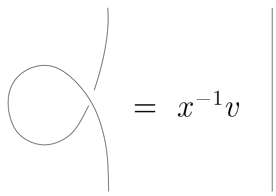


Figure 2.6: The Whitney-trick realizes a cancellation of curls via a regular isotopy.



$$= x^{-1}v$$

Figure 2.7: A derived relation in $\mathcal{S}(\mathcal{F}, \Gamma)$.

at the expense of the additional factor xv^{-1}, and we thus get the local relation depicted in figure 2.7 which is valid in $\mathcal{S}(\mathcal{F}, \Gamma)$.

Let G_1 and G_2 be diagrams in $(\mathcal{F}, \Gamma)$ and denote the associated ribbon tangles in $\mathcal{F} \times (0, 1)$ by T_1 and T_2 (determined up to isotopy). We claim that G_1 and G_2 are equal in $\mathcal{S}(\mathcal{F}, \Gamma)$ if and only if T_1 and T_2 are equal in $\mathcal{S}(\mathcal{F} \times (0, 1), \Theta)$. The only non-trivial part of this claim is that if H_1 and H_2 are isotopic ribbon tangles in $\mathcal{F} \times (0, 1)$ then G_1 and G_2 are equal in $\mathcal{S}(\mathcal{F}, \Gamma)$. If H_1 and H_2 are isotopic then G_1 derives from G_2 by regular isotopy and the local moves shown in figure 2.8 with any orientations on the components. The moves cannot be realized by a regular isotopy in general. (But, e.g. in the sphere S^2 they are regularly isotopic). But the diagrams become equal in $\mathcal{S}(\mathcal{F}, \Gamma)$ since the curls can be removed at the expense of the scalars which cancel. One has to rotate in some instances the diagrams (or oneself) in order to apply the above local skein relations.

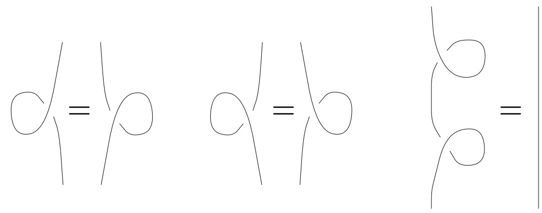


Figure 2.8: Moves in order to handle curls in diagrams.

In what follows we shall be mainly concerned with skeins over the scalars $\mathbb{Z}[x^{\pm 1}, v^{\pm 1}, s^{\pm 1}, \delta]/\langle\langle \delta(s - s^{-1}) = v^{-1} - v \rangle\rangle$. But clearly, a skein makes sense for any extension of this ring. We shall consider as well the subring of the rational functions $Q(x, v, s)$ generated by $x^{\pm 1}, v^{\pm 1}, s^{\pm 1}, \{(s^i - s^{-i})^{-1} \mid i \in I\}$ for some subset $I \subset \mathbb{Z}$. Note that the term $(s^i - s^{-i})^{-1}$ may cause problems when s is substituted by some root of unity.

2.1.1 Useful maps

There are some interesting maps of a Homfly skein to itself without being module homomorphisms.

$$\begin{array}{rcl}
\gamma & : & x \mapsto x,\ v \mapsto v,\ s \mapsto -s^{-1},\ \delta \mapsto \delta \\
\tau & : & x \mapsto -x,\ v \mapsto -v,\ s \mapsto s^{-1},\ \delta \mapsto \delta \\
\rho & : & x \mapsto x^{-1},\ v \mapsto v^{-1},\ s \mapsto s^{-1},\ \delta \mapsto \delta.
\end{array}$$

γ, τ and ρ extend to isomorphisms of the rational functions in x, v, s and δ. They induce isomorphisms of the scalars we are considering since $(s - s^{-1})\delta - (v^{-1} - v)$ is invariant under these maps.

We get maps from the Homfly skein to itself when we leave the diagrams invariant and alter the scalars by γ (resp. τ) because this preserves the skein relations. In the case of ρ, one has to change all crossings of the diagram in order that the skein relations are preserved.

2.2 The Homfly polynomial

Any diagram D in the skein $\mathcal{S}(\mathbb{R}^2)$ of the plane can be transformed via the skein relations to a scalar multiple t of the empty diagram $\emptyset$. An important result states that the $\mathcal{S}(\mathbb{R}^2)$ is isomorphic to its scalars, i.e. the scalar t is well defined. This scalar is denoted as the *Homfly polynomial* $\chi(D)$ of D. The word Homfly is derived from the initial letters of some of the mathematicians who discovered this invariant ([6], [21]).

A *framed link* in $\mathcal{F} \times (0, 1)$ is an oriented link together with a parallel curve to each component, i.e. a longitude in the boundary of a regular neighbourhood of each component. Every oriented link diagram determines a framed link by choosing the blackboard parallel for each component. We shall consider only diagrams of framed links whose blackboard framing gives the framing of the link.

Ribbon tangles in $\mathcal{S}(\mathcal{F} \times (0, 1))$ that consist only of embedded annuli with oriented cores are an equivalent view of framed links, where, for each annulus A, the core of A determines a link component, and a boundary curve of A (it

Figure 2.9: The sign assigned to a crossing.

is irrelevant which one) determines a longitude in a regular neighbourhood N of this link component.

A crossing of an oriented diagram in an oriented surface is given a sign $\varepsilon = \pm 1$ as shown in figure 2.9 where we use the counterclockwise orientation of the local disc. The sum of the signs is denoted as the *writhe* $\mathrm{wr}(D)$ of the diagram D. It is a invariant under regular isotopy.

Let D be a diagram. One gets an invariant χ^{u} that does not involve the variable x by setting $\chi^{\mathrm{u}}(D) = (xv^{-1})^{-\mathrm{wr}(D)}D$.

2.3 The Hecke algebra H_n

We denote by H_n the skein of the disc $[0,1] \times [0,1]$ with the set $\Gamma = \Gamma_1 \uplus \Gamma_2 = \{(\frac{j}{n+1}, 1)\}_{1 \leq j \leq n} \uplus \{(\frac{j}{n+1}, 0)\}_{1 \leq j \leq n}$ and the standard (anti-clockwise) orientation. We call the point $(\frac{j}{n+1}, 1)$ the *j-th point at the top* and $(\frac{j}{n+1}, 0)$ the *j-th point at the bottom*.

The multiplication for diagrams D_1 and D_2 is given by stacking D_1 above D_2. This extends linearly to H_n. The multiplication is associative but not commutative. Every diagram D in H_n determines an element π^D of the symmetric group $\mathcal{S}_n$ on n letters, by saying that the j-th point at the top of the square $[0,1] \times [0,1]$ is joined by an arc of D to the $\pi^D(j)$-th point at the bottom. We have $\pi^{D_1 D_2} = \pi^{D_1}\pi^{D_2}$ in $\mathcal{S}_n$ since we read the product of permutations from left to right.

H_n is known to be isomorphic to the Hecke algebra.

For every permutation $\pi \in \mathcal{S}_n$ there exists a unique braid w_π (called a *positive permutation braid*) such that w_π determines $\pi \in \mathcal{S}_n$, and strings starting at the points i and j at the top with $1 \leq i < j \leq n$ do not cross if $\pi(i) < \pi(j)$, and they cross only once (with the string starting at j overcrossing the string starting at i) if $\pi(i) > \pi(j)$.

It turns out that the set $\{w_\pi \mid \pi \in \mathcal{S}_n\}$ is a basis for H_n (see [20] for a short proof).

The juxtaposition of putting a diagram $D_1 \in H_n$ to the left of $D_2 \in H_m$ induces an inclusion $H_n \otimes H_m \to H_{n+m}$.

2.4 Idempotents in the Hecke algebra

This section describes the interpretation of Gyoja's results [9] by Aiston and Morton [2].

2.4.1 The building blocks a_n and b_n

We denote by $l(\pi)$ the writhe of w_π, which can also be expressed as the minimal number of transpositions to form the permutation π. We define

$$a_n = \sum_{\pi \in \mathcal{S}_n} (x^{-1}s)^{l(\pi)} w_\pi$$

in H_n for any integer $n \geq 0$.

We denote by σ_i the elementary positive braid in which only strings i and $i+1$ cross once positively. The next lemma can be found as Lemma 8 in [2].

Lemma 2.4.1 *We have $\sigma_i a_n = xsa_n$ and $a_n \sigma_i = xsa_n$ for any $1 \leq i \leq n-1$.*

Since H_n is generated as an algebra by the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$, we deduce that a_n lies in the centre of H_n. Even more, for any element h of H_n we have $ha_n = a_n h = \kappa a_n$ for some scalar κ.

In particular, $a_n a_n$ is a scalar multiple α_n of a_n. Lemma 2.4.2 shows that α_n is non-zero. We define $[n] = (s^n - s^{-n})/(s - s^{-1}) = s^{n-1} + s^{n-3} + \cdots + s^{-n+3} + s^{-n+1}$ for any integer $n \geq 0$. We define $[n]! = [n][n-1] \cdots [1]$.

Lemma 2.4.2 *We have $\alpha_n = s^{\frac{(n-1)n}{2}}[n]!$ for any integer $n \geq 1$.*

Proof Using lemma 2.4.1 we get that

$$\begin{aligned} a_n a_n &= a_n \sum_{\pi \in \mathcal{S}_n} (x^{-1}s)^{l(\pi)} w_\pi \\ &= a_n \sum_{\pi \in \mathcal{S}_n} (x^{-1}s)^{l(\pi)} (xs)^{l(\pi)} \\ &= a_n \sum_{\pi \in \mathcal{S}_n} s^{2l(\pi)}. \end{aligned}$$

We can write any permutation π from $\mathcal{S}_n$ uniquely as the product of a permutation τ from $\mathcal{S}_{n-1}$ and the cycle $(\pi(n)\ (\pi(n)+1)\ \ldots\ n)$ of length $n - \pi(n) + 1$. Therefore,

$$\begin{aligned} \sum_{\pi \in \mathcal{S}_n} s^{2l(\pi)} &= \sum_{\tau \in \mathcal{S}_{n-1}} \sum_{i=0}^{n-1} s^{2(l(\tau)+i)} \\ &= \sum_{\tau \in \mathcal{S}_{n-1}} \left(s^{2l(\tau)} \sum_{i=0}^{n-1} s^{2i} \right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{i=0}^{n-1} s^{2i}\right) \sum_{\tau\in\mathcal{S}_{n-1}} s^{2l(\tau)} \\
&= s^{n-1}\frac{s^n - s^{-n}}{s - s^{-1}} \sum_{\tau\in\mathcal{S}_{n-1}} s^{2l(\tau)} \\
&= s^{n-1}[n] \sum_{\tau\in\mathcal{S}_{n-1}} s^{2l(\tau)}.
\end{aligned}
$$

We get by induction that

$$\sum_{\pi\in\mathcal{S}_n} s^{2l(\pi)} = s^{\frac{(n-1)n}{2}}[n]!.$$

♣

The proof of lemma 2.4.2 suggests a decomposition of a_{n+1} given by

$$a_{n+1} = (a_n \otimes 1_1)(1_{n+1} + (x^{-1}s)\sigma_n + (x^{-1}s)^2\sigma_n\sigma_{n-1} + \cdots + (x^{-1}s)^n\sigma_n\sigma_{n-1}\cdots\sigma_1) \tag{2.4.1}$$

This is because we can draw any positive permutation braid w_π on $n+1$ strings in a unique way as the product of a positive permutation braid having a vertical $(n+1)$-st string and the braid $\sigma_n\sigma_{n-1}\cdots\sigma_{\pi(n+1)}$.

We recall from subsection 2.1.1 the isomorphism γ of the rational functions in x, v and s given by $x \mapsto x$, $v \mapsto v$ and $s \mapsto -s^{-1}$. We get a map from H_n to H_n that is the identity on any diagram and behaves on the scalars as γ. We denote this map by γ as well. It satisfies $\gamma(w+y) = \gamma(w)+\gamma(y)$ and $\gamma(wy) = \gamma(w)\gamma(y)$ for any elements w and y of H_n but we remark that it is not an algebra homomorphism since it changes the scalars. We have that γ^2 is equal to the identity. We denote

$$b_n = \gamma(a_n)$$

and $\beta_n = \gamma(\alpha_n)$. We remark that $\gamma(a_n) = \tau(a_n)$, and $\gamma(\alpha_n) = \tau(\alpha_n)$ where τ was defined in subsection 2.1.1.

Lemma 2.4.3 *We have $b_nb_n = \beta_nb_n$ for any integer $n \geq 0$.*

Proof We have $a_na_n = \alpha_na_n$. Applying the map γ we get $\gamma(a_n)\gamma(a_n) = \gamma(\alpha_n)\gamma(a_n)$, hence $b_nb_n = \beta_nb_n$. ♣

We recall the isomorphism ρ of the rational functions in x, v and s given by $x \mapsto x^{-1}$, $v \mapsto v^{-1}$ and $s \mapsto s^{-1}$. We immediately deduce from the skein relations of H_n that the map which reflects any diagram in the plane and that behaves on the scalars as ρ induces a map from H_n to H_n. We denote this map by ρ as well. It satisfies $\rho(w+y) = \rho(w) + \rho(y)$ and $\rho(wy) = \rho(w)\rho(y)$ for any elements w and y of H_n but we remark that it is not an algebra homomorphism. We have that ρ^2 is equal to the identity.

When we consider H_n as an algebra over a subring of the rational functions in x, v and s in which α_n is invertible then $(1/\alpha_n)a_n$ is an idempotent.

Lemma 2.4.4 *We have $\rho(\frac{1}{\alpha_n}a_n) = \frac{1}{\alpha_n}a_n$ for any integer $n \geq 0$.*

Proof We have

$$\sigma_i \rho(a_n) = \rho(\sigma_i^{-1})\rho(a_n) = \rho(\sigma_i^{-1}a_n) = \rho((xs)^{-1}a_n) = xs\rho(a_n)$$

for any $1 \leq i \leq n-1$. Hence,

$$a_n\rho(a_n) = \alpha_n\rho(a_n).$$

Applying the map ρ to this equation we get

$$\rho(a_n)a_n = \rho(\alpha_n)a_n.$$

The element a_n is central in H_n, and therefore $a_n\rho(a_n) = \rho(a_n)a_n$. Therefore, the terms on the right hand sides of the above two equations are equal, i.e. $\alpha_n\rho(a_n) = \rho(\alpha_n)a_n$. We thus get

$$\rho\left(\frac{1}{\alpha_n}a_n\right) = \frac{1}{\alpha_n}a_n.$$

♣

Since ρ and γ commute, we get

Corollary 2.4.5 *We have $\rho\left(\frac{1}{\beta_n}b_n\right) = \frac{1}{\beta_n}b_n$ for any integer $n \geq 0$.*

2.4.2 The quasi-idempotent e_λ

Here, we fix an integer $n \geq 0$ and consider only Young diagrams with n cells. For any Young diagram λ (with n cells) we construct a quasi-idempotent e_λ in H_n in the following way.

We number the cells of any Young diagram μ with the integers $1, 2, \ldots, n$ from left to right and from top to bottom (as reading in a book). The map $(i,j) \in \lambda \to (j,i) \in \lambda^\vee$ determines therefore a permutation π_λ on n letters. We clearly have $\pi_{\lambda^\vee} = \pi_\lambda^{-1}$. We define

$$E_\lambda(a) = a_{\lambda_1} \otimes a_{\lambda_2} \otimes \cdots \otimes a_{\lambda_{l(\lambda)}} \in H_n$$

and

$$E_\mu(b) = b_{\mu_1} \otimes a_{\mu_2} \otimes \cdots \otimes \otimes b_{\mu_{l(\mu)}} \in H_n$$

for any Young diagrams λ and μ. We define

$$e_\lambda = E_\lambda(a) w_{\pi_\lambda} E_{\lambda^\vee}(b) w_{\pi_\lambda}^{-1} \in H_n$$

where $w_{\pi_\lambda}^{-1}$ is the inverse braid of w_{π_λ}. We note that $e_{d_n} = a_n$ and $e_{c_n} = b_n$.

It follows from Lemma 11 in [2] that for any element $T \in H_n$ there exists a scalar t such that

$$E_\lambda(a) T E_{\lambda^\vee}(b) = t E_\lambda(a) w_{\pi_\lambda} E_{\lambda^\vee}(b). \tag{2.4.2}$$

Hence,

$$e_\lambda^2 = \alpha_\lambda e_\lambda$$

for some scalar α_λ. The scalar α_i from lemma 2.4.2 is by definition equal to α_{d_i} (this is a slight abuse of notation). One can also prove that

$$e_\lambda e_\mu = 0 \quad \text{if } \lambda \neq \mu \tag{2.4.3}$$

(of course under the condition that $|\lambda| = |\mu|$).

Remark It might seem more natural to define $e_\lambda = E_\lambda(a) w_{\pi_\lambda} E_{\lambda^\vee}(b)$. The above statements would remain true, but of course with some different scalars α_λ. This is the point. If we define $e_\lambda = E_\lambda(a) w_{\pi_\lambda} E_{\lambda^\vee}(b)$ then $e_\lambda^2 = 0$ unless λ is a single row or column diagram. This is because e_λ^2 contains the factor $E_{\lambda^\vee}(b) E_\lambda(a)$ from which on can extract a factor $a_2 \otimes 1^{n-2}$ from $E_\lambda(a)$ and a factor $b_2 \otimes 1^{n-2}$ from $E_{\lambda^\vee}(b)$ if $\lambda_1 \geq 2$ and $l(\lambda) \geq 2$. One can verify by a direct skein calculation that $b_2 a_2 = 0$ in H_2, and therefore we deduce that $E_{\lambda^\vee}(b) E_\lambda(a) = 0$ in H_n.

Another reason is that the elements $e_\lambda = E_\lambda(a) w_{\pi_\lambda} E_{\lambda^\vee}(b) w_{\pi_\lambda}^{-1}$ specialize to quasi-idempotents of the group algebra $\mathbb{C}[\mathcal{S}_n]$ after the substitutions $x = v = s = \delta = 1$. (One has to consider $\mathbb{C}$ instead of $\mathbb{Z}$.) This is explained in detail in [1].

The elements $e_\lambda \in H_n$ and the scalars α_λ are non-zero. This follows e.g. from their specialization to $\mathbb{C}[\mathcal{S}_n]$. The explicit formula for α_λ is

$$\alpha_\lambda = \prod_{c \in \lambda} s^{\mathrm{cn}(c)} [\mathrm{hl}(c)]. \tag{2.4.4}$$

A proof is given in [26] (see [3] for an exposition).

We define

$$y_\lambda = \frac{1}{\alpha_\lambda} e_\lambda \in H_n$$

which is an idempotent.

The standard closure of a braid (or a tangle) induces a linear map from H_n to the skein of the plane. We give a short skein-theoretic proof that e_λ is non-zero. In fact, we even prove more.

Lemma 2.4.6 *The Homfly polynomial of the closure of e_λ is non-zero for any Young diagram λ.*

Proof We denote the number of cells of λ by n. We specialize $x = v = s = 1$. The scalars are now $\mathbb{Z}[\delta]$ where δ is an indeterminate. We shall consider the Homfly polynomial of the closure of e_λ. The skein relations for $x = v = s = 1$ imply that we can switch any crossings and remove any curls without altering the Homfly polynomial. Therefore, the Homfly polynomial of a diagram in the plane is equal to δ^c where c is the number of components of the diagram. We claim that the closure of e_λ is a linear combination of diagrams with at most n components, and that exactly one diagram occurs with n components (and non-zero coefficient). This implies that the Homfly polynomial of the closure of e_λ is a polynomial in δ of degree n and is thus non-zero. Hence, e_λ is non-zero.

Clearly, the number of components of the closure of an (n,n)-braid is given by the number of cycles in the cycle decomposition of the permutation of $\mathcal{S}_n$ determined by this braid. Since e_λ is a linear combination of (n,n)-braids, all the appearing diagrams in the closure of e_λ have at most n components. It remains to prove that exactly one summand of e_λ determines the identity permutation of $\mathcal{S}_n$.

By simply using distributivity, we can write e_λ as a linear combination of braids,

$$e_\lambda = E_\lambda(a) w_{\pi_\lambda} E_{\lambda^\vee} w_{\pi_\lambda}^{-1} = \sum_\beta t_\beta \beta$$

where

$$\beta = \gamma w_{\pi_\lambda} \gamma' w_{\pi_\lambda}^{-1}$$

for some braids γ and γ' which appear as a summand in $E_\lambda(a)$ resp. $E_{\lambda^\vee}(b)$.

First, we consider strings of β that belong to the same component a_{λ_i} of $E_\lambda(a)$. They do not cross in w_{π_λ} since the r-th string of the component a_{λ_i} is joined to some string of the component $b_{\lambda_r^\vee}$ which appear in order from left to right in $E_{\lambda^\vee}(b)$ and the condition on positive permutation braids ensures that these strings of w_{π_λ} do not cross. Furthermore they do not cross in γ' since they belong to different components $b_{\lambda_j^\vee}$ and $b_{\lambda_k^\vee}$ of $E_{\lambda^\vee}(b)$. Finally, they do not cross in $w_{\pi_\lambda}^{-1}$ since they do not cross in w_{π_λ}.

Similarly, strings of β that belong to the same component $b_{\lambda_j^\vee}$ of $E_{\lambda^\vee}(b)$ do not cross in either γ, w_{π_λ}, or $w_{\pi_\lambda}^{-1}$.

Hence, if two strings of β cross in either γ or γ' then $\beta = \gamma w_{\pi_\lambda} \gamma' w_{\pi_\lambda}^{-1}$ cannot determine the identity permutation. Since every a_{λ_i} and $b_{\lambda_j^\vee}$ contains the identity braid as a summand (with coefficient 1), we see that the identity braid $\beta = \mathrm{id}_n$ is the only summand of the closure of e_λ that has n components. Furthermore, its coefficient is 1 as claimed. ♣

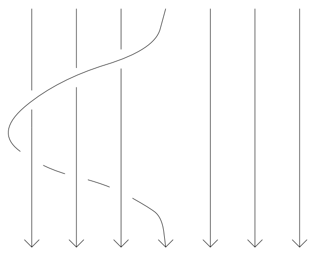


Figure 2.10: The n-string braid $T(j)$ for $n = 7$ and $j = 4$.

By equation 2.4.2 there exists for every central element B of H_n a scalar b such that $Be_\lambda = be_\lambda$. We shall be interested in the tangle $T^{(n)}$ depicted on the left of figure 2.12 which is the identity braid (on n strings) with a simple closed curve encircling it. It is clearly central in H_n.

Lemma 2.4.7 *We have*

$$T^{(n)} e_\lambda = c_\lambda e_\lambda$$

in H_n for any Young diagram λ with n cells. The scalar c_λ is given by

$$c_\lambda = x^{2n} \left(\frac{v^{-1} - v}{s - s^{-1}} + v^{-1} s \sum_{k=1}^{l(\lambda)} (s^{2(\lambda_k - k)} - s^{-2k}) \right).$$

Proof We denote by $T(j)$ the n-string braid $\sigma_j \cdots \sigma_2 \sigma_1 \sigma_1 \sigma_2 \cdots \sigma_j$ as depicted in figure 2.10. By equation (5.1) of the proof of theorem 17 in [2] we have

$$E_\lambda(a) w_{\pi_\lambda} T(j) E_{\lambda^\vee}(b) w_{\pi_\lambda}^{-1} = x^{2(j-1)} s^{2\mathrm{cn}(p(j))} e_\lambda$$

in H_n where $p(j)$ is the cell of λ numbered j in the standard tableau that reads $1, 2, 3, \ldots, n$ from left to right and top to bottom.

We remark that the formula given in [2] differs from this one by a framing factor xv^{-1} because they have used a framing different from the blackboard framing.

The equation in figure 2.11 follows from the skein relation $x^{-1}\sigma_i - x\sigma_i^{-1} = z\,\mathrm{id}$ (where $z = (s - s^{-1})$) which is applied to the upper right crossing. An equivalent relation is depicted in figure 2.12. Inductively, we can therefore write $T^{(n)}$ as the linear combination

$$T^{(n)} = x^{2n} \frac{v^{-1} - v}{s - s^{-1}} \mathrm{id}_n + z x^2 v^{-1} \sum_{j=1}^{n} x^{2(j-1)} T(n - j + 1).$$

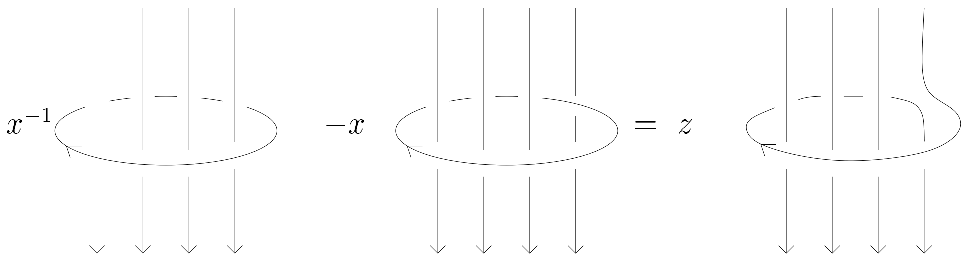


Figure 2.11: A relation in the Hecke algebra H_n.

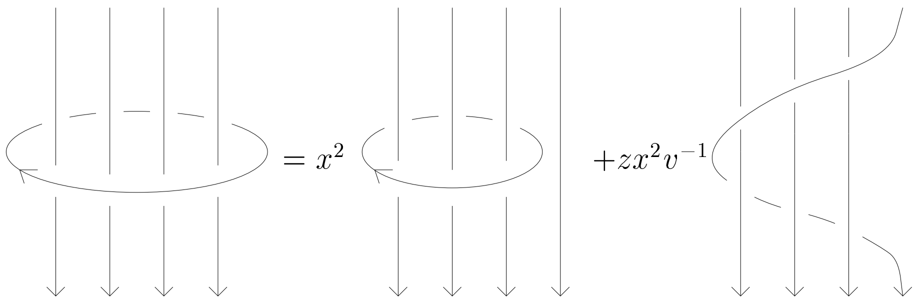


Figure 2.12: An equivalent depiction of the equation in figure 2.11.

Hence, $T^{(n)}e_\lambda = c_\lambda e_\lambda$ with

$$\begin{aligned} c_\lambda &= x^{2n}\frac{v^{-1}-v}{s-s^{-1}} + v^{-1}z\sum_{j=1}^{n} x^{2j}x^{2(n-j)}s^{2\mathrm{cn}(p(j))} \\ &= x^{2n}\left(\frac{v^{-1}-v}{s-s^{-1}} + v^{-1}z\sum_{j=1}^{n} s^{2\mathrm{cn}(p(j))}\right). \end{aligned}$$

Now

$$\begin{aligned} \sum_{j=1}^{n} s^{2\mathrm{cn}(p(j))} &= \sum_{k=1}^{l(\lambda)}\sum_{i=1}^{\lambda_k} s^{2\mathrm{cn}(k,i)} \\ &= \sum_{k=1}^{l(\lambda)}\sum_{i=1}^{\lambda_k} s^{2(i-k)} \\ &= \sum_{k=1}^{l(\lambda)}\sum_{i=0}^{\lambda_k-1} s^{2(i-k+1)} \\ &= \sum_{k=1}^{l(\lambda)} s^{2(-k+1)}\sum_{i=0}^{\lambda_k-1} s^{2i} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{l(\lambda)} s^{2(1-k)} \frac{s^{2\lambda_k}-1}{s^2-1} \\
&= \frac{s}{s-s^{-1}} \sum_{k=1}^{l(\lambda)} (s^{2(\lambda_k-k)} - s^{-2k}).
\end{aligned}
$$

Hence,

$$
c_\lambda = x^{2n} \left(\frac{v^{-1}-v}{s-s^{-1}} + v^{-1}s \sum_{k=1}^{l(\lambda)} (s^{2(\lambda_k-k)} - s^{-2k}) \right).
$$

♣

Lemma 2.4.8 *The scalars c_λ are pairwise different and non-zero for all Young diagrams λ.*

Proof The statement is even true for $x = 1$. We have

$$
c_\lambda = \frac{v^{-1}-v}{s-s^{-1}} + v^{-1}s \sum_{k=1}^{l(\lambda)} (s^{2(\lambda_k-k)} - s^{-2k}).
$$

We can consider c_λ as a Laurent polynomial in v. The coefficient of v in c_λ is $(s-s^{-1})^{-1}$, and therefore c_λ is non-zero. Hence, c_λ is non-zero.

Let λ and η be Young diagrams with $c_\lambda = c_\eta$. Since $s^{2(\lambda_k-k)} - s^{-2k} = 0$ for $k \geq l(\lambda)$, we deduce from $c_\lambda = c_\eta$ that

$$
\sum_{k=1}^{m} (s^{2(\lambda_k-k)} - s^{-2k}) = \sum_{k=1}^{m} (s^{2(\eta_k-k)} - s^{-2k}),
$$

where $m = \max(l(\lambda), l(\eta))$. Hence

$$
\sum_{k=1}^{m} s^{2(\lambda_k-k)} = \sum_{k=1}^{m} s^{2(\eta_k-k)}.
$$

The sequences $(\lambda_k - k)$ and $(\eta_k - k)$, $1 \leq k \leq m$, are strictly decreasing. The above equality implies therefore that $\lambda_k = \eta_k$ for $k = 1, \ldots, m$, hence $\lambda = \eta$. ♣

2.5 Semi-simple decomposition of H_n

This exposition follows the account of Blanchet in [3]. He describes an explicit isomorphism from H_n to a disjoint sum of matrix algebras by generalizing Wenzl's results of [25]. We use the three-dimensional version H_λ of the Hecke algebra as introduced in [2] where the arcs end at the centres of the cells of a Young diagram rather than along a straight line. This model supports the understanding of the construction.

It is helpful but not necessary to know the value of the scalar α_λ from equation (2.4.4). The knowledge of α_λ allows to have a better control of the scalars in lemma 2.5.4.

We fix the index n of H_n throughout this section. Given standard tableaux t and τ of the same Young diagram λ with n cells we construct an element $\alpha_{t\tau}$ of H_n. In this context we say that standard tableaux s and σ are *suitable* for $\alpha_{s\sigma}$ if s and σ belong to the same Young diagram.

We denote by $M_{i\times i}$ the algebra of $(i \times i)$-matrices over the same ring as the ring of scalars for H_n. We recall that d_λ denotes the number of standard tableaux for the Young diagram λ.

To simplify our notation, we denote the Young diagram that underlies a standard tableau t by $\lambda(t)$. We use the notation of the Kronecker-delta $\delta_{\tau s}$ which is defined by $\delta_{ss} = 1$ and $\delta_{\tau s} = 0$ if $\tau \neq s$. We shall prove that $\alpha_{t\tau}\alpha_{s\sigma} = \delta_{\tau s}\alpha_{t\sigma}$ for any (suitable) standard tableaux t, τ, s and σ. This implies that the linear map

$$\bigoplus_{|\lambda|=n} M_{d_\lambda \times d_\lambda} \to H_n$$

mapping the basis element $E_{t\tau}$ (that has all entries equal to zero except the entry 1 at the position (t,τ)) to $\alpha_{t\tau}$ is an algebra homomorphism.

Being careful, we have to consider the scalars of H_n. First of all, we can consider the field $Q(x, v, s)$ of rational functions in x, v and s. But we can restrict the scalars to the subring of the field of rational functions in x, v and s generated by $x^{\pm 1}$, $v^{\pm 1}$, $s^{\pm 1}$, and $(s^i - s^{-i})^{-1}$ for $i = 1, \ldots, n$. This is because the idempotent $y_\lambda = (1/\alpha_\lambda)e_\lambda$ can be defined in this ring since the denominator of α_λ is by equation (2.4.4) a product of terms $(s^j - s^{-j})$ for j being the hook length of some cell of λ. Since λ has n cells in total, it is sufficient to consider $j = 1, \ldots, n$.

We denote by y_λ the three-dimensional version of the idempotent corresponding to λ. Given a Young diagram λ we can remove one of its extreme cells to get a Young diagram μ with one cell less. Given a standard tableau t of λ there is a canonical way to choose an extreme cell by choosing the cell with the highest number in t. We denote the resulting standard tableau by t'. We denote by t^k the k-fold application of this removal of cells.

There is an obvious inclusion of the three-dimensional Hecke algebra H_μ in the Hecke algebra H_λ by adding a straight arc that connects the boundary points based at the removed cell. We denote this inclusion by $g \mapsto g \otimes 1$.

Given a standard tableaux t of a Young diagram λ we define Φ_t in H_λ by

$$\Phi_t = (y_{\lambda(t^{n-1})} \otimes 1_{n-1})(y_{\lambda(t^{n-2})} \otimes 1_{n-2}) \cdots (y_{\lambda(t')} \otimes 1_1)y_{\lambda(t)}$$

where 1_k is the identity braid on k strings. We remark that $y_{t^{n-1}} \otimes 1_{n-1}$ is the identity braid. We define Ω_t in H_λ similarly as

$$\Omega_t = y_\lambda(t)(y_{\lambda(t')} \otimes 1_1) \cdots (y_{\lambda(t^{n-2})} \otimes 1_{n-2})(y_{\lambda(t^{n-1})} \otimes 1_{n-1}).$$

Given standard tableaux t and τ of a Young diagram λ we define an element $\alpha_{t\tau}$ in H_n by

$$\alpha_{t\tau} = F_t \Phi_t \Omega_\tau F_\tau^-$$

where F_t resp. F_τ^- is a tangle that connects upwards resp. downwards the n points arranged along the cells of λ to the n points arranged along a line. We number the points along the line by $1, 2, \ldots, n$ from left to right. The standard tableau t describes a numbering of the upper boundary points of H_λ. We describe in a recursive way the projection of this braid to the plane that contains the upper n points of H_λ. For $i = 1, \ldots, N$ we connect the points numbered $N - i + 1$ by a line that goes only towards right and upwards, that is disjoint to all $i - 1$ previously drawn lines, and that is disjoint to the standard tableaux t^i.

F_τ^- is defined as the mirror image of F_τ.

Lemma 2.5.1 *We have $\alpha_{t\tau}\alpha_{s\sigma} = \delta_{\tau s}\alpha_{t\sigma}$ for any (suitable) standard tableaux t, τ, s and σ.*

Proof We denote by η and μ the Young diagrams given by the standard tableaux τ resp. s. We first consider the case that η and μ are different. We have in the three-dimensional picture of $\alpha_{t\tau}\alpha_{s\sigma}$ a product in which the factor y_η appears in Ω_τ and the factor y_μ appears in Φ_s. Any product containing these factors is equal to zero because of the three-dimensional equivalent of equation (2.4.3).

If the Young diagrams η and μ are equal but τ and s are different then there exists a maximal integer k so that $\lambda(\tau^k) = \lambda(s^k)$ but $\lambda(\tau^{k+1}) \neq \lambda(s^{k+1})$. By the same argument as above we deduce that $\Omega_\tau F_\tau^- F_s \Phi_\sigma$ is equal to zero because we have a product containing $y_{\lambda(\tau^{k+1})}$ and $y_{\lambda(s^{k+1})}$. The other strings do not interfere because of our definition of the connecting braids F_τ^- and F_s.

Finally, if τ and s are equal, we have that

$$\begin{aligned}
\alpha_{t\tau}\alpha_{\tau\sigma} &= F_t \Phi_t \Omega_\tau F_\tau^- F_\tau \Phi_\tau \Omega_\sigma F_\sigma^- \\
&= F_t \Phi_t \Omega_\tau \Phi_\tau \Omega_\sigma F_\sigma^- \\
&= F_t \Phi_t \Omega_\sigma F_\sigma^- \\
&= \alpha_{t\sigma}
\end{aligned}$$

where we used that $\Omega_\tau \Phi_\tau = y_{\lambda(\tau)}$. This is true because $y_\nu(y_\zeta \otimes 1)y_\nu = y_\nu$ in H_ν for any Young diagram ν and subdiagram ζ, $|\nu| = |\zeta| + 1$. We finally show this equality.

First, we note that we can extract a factor $E_{\zeta^\vee}(b) \otimes 1$ from $E_{\nu^\vee}(b)$ at the expense of a scalar β. This is because the quasi-idempotent b_i of H_i satisfies $b_i b_i = \beta_i b_i$ for a non-zero scalar β_i. We thus get

$$E_{\nu^\vee}(b)(E_{\zeta^\vee} \otimes 1) = \beta E_{\nu^\vee}(b)$$

where

$$\beta = \prod_{j=1}^{l(\zeta^\vee)} \beta_{\zeta_j^\vee}.$$

Similarly,

$$(E_\zeta(a) \otimes 1)E_\nu(a) = \alpha E_\nu(a)$$

where

$$\alpha = \prod_{i=1}^{l(\zeta)} \alpha_{\zeta_i}.$$

Second, we have that $E_{\zeta^\vee}(b)E_\zeta(a)$ is a quasi-idempotent of H_λ with the same scalar α_ζ as for $E_\zeta(a)E_{\zeta^\vee}(b)$. This follows from

$$e_\zeta^2 = (E_\zeta(a)E_{\zeta^\vee}(b))^2 = \alpha_\zeta E_\zeta(a)E_\zeta(b) = \alpha_\zeta e_\zeta$$

by reading the involved diagrams from bottom to top which is an anti homomorphism that leaves the a_i and b_j invariant.

Hence,

$$\begin{aligned}
y_\nu(y_\zeta \otimes 1)y_\nu &= \frac{1}{\alpha_\nu^2\alpha_\zeta} E_\nu(a)E_{\nu^\vee}(b)((E_\zeta(a)E_{\zeta^\vee}(b)) \otimes 1)E_\nu(a)E_{\nu^\vee}(b) \\
&= \frac{1}{\alpha_\nu^2\alpha_\zeta\alpha\beta} E_\nu(a)E_{\nu^\vee}(b)(E_{\zeta^\vee}(b) \otimes 1)((E_\zeta(a)E_{\zeta^\vee}(b)) \otimes 1) \\
&\quad \cdot(E_\zeta(a) \otimes 1)E_\nu(a)E_{\nu^\vee}(b) \\
&= \frac{1}{\alpha_\nu^2\alpha_\zeta\alpha\beta} E_\nu(a)E_{\nu^\vee}(b)((E_{\zeta^\vee}(b)E_\zeta(a)) \otimes 1)^2 E_\nu(a)E_{\nu^\vee}(b) \\
&= \frac{1}{\alpha_\nu^2\alpha\beta} E_\nu(a)E_{\nu^\vee}(b)((E_{\zeta^\vee}(b)E_\zeta(a)) \otimes 1)E_\nu(a)E_{\nu^\vee}(b) \\
&= \frac{1}{\alpha_\nu^2} E_\nu(a)E_{\nu^\vee}(b)E_\nu(a)E_{\nu^\vee}(b) \\
&= \frac{1}{\alpha_\nu} E_\nu(a)E_{\nu^\vee}(b) \\
&= y_\nu.
\end{aligned}$$

♣

Lemma 2.5.2 *The closure of $\alpha_{t\tau}$ in the skein of the annulus is equal to zero if t and τ are different tableaux of the same Young diagram. The closure of α_{tt} is equal to the closure of $y_{\lambda(t)}$.*

Proof The closure of $\alpha_{t\tau} = F_t \Phi_t \Omega_t F_t^-$ is equal to the closure of $\Omega_\tau F_\tau^- F_t \Phi_t$ because we can move the factors around in the annulus, i.e. permute them cyclically. By the same argument as in the proof of lemma 2.5.1 we have therefore that the closure of $\alpha_{t\tau}$ is equal to zero if t and τ are different, and the closure of α_{tt} is equal to the closure of $y_{\lambda(t)}$. ♣

Lemma 2.5.3 *The elements $\{\alpha_{t\tau}\}$ of H_n are linearly independent where t and τ range over all suitable standard tableaux of Young diagrams with n cells.*

Proof Assume that

$$\sum_{t,\tau} \kappa_{t\tau} \alpha_{t\tau} = 0$$

for some scalars $\kappa_{t\tau}$. Let s and σ be any suitable Young tableaux. Then multiplication of the above equation by α_{ss} on the left and multiplication by $\alpha_{\sigma s}$ on the right leads to

$$\kappa_{s\sigma} \alpha_{ss} = 0$$

by lemma 2.5.1. In order to deduce that $\kappa_{s\sigma}$ is equal to zero for all suitable Young tableaux s and σ, we have to show that α_{ss} is non-zero in H_n for any standard tableaux s.

As shown in lemma 2.5.2, the closure of α_{ss} in the skein of the annulus is equal to the closure of $y_{\lambda(s)}$ in the skein of the annulus. Even the inclusion of the closure $y_{\lambda(s)}$ in the skein of the plane is non-zero by lemma 2.4.6. Hence, α_{ss} is non-zero in H_n. ♣

Lemma 2.5.4 *The elements $\alpha_{t\tau}$ for any suitable standard tableaux t and τ are a basis for H_n when the scalars are the field of rational functions in x, v and s.*

Proof We recall that d_λ is the number of standard tableaux for the Young diagram λ. The number of elements $\alpha_{t\tau}$ in H_n is therefore given by $\sum_{|\lambda|=n} d_\lambda^2$ which is known to be equal to $n!$ by an argument about the standard decomposition of the group algebra $\mathbb{C}[\mathcal{S}_n]$ into a direct sum of matrix algebras.

Since the elements $\alpha_{t\tau}$ are linearly independent, and the dimension of H_n is $n!$, they form a basis. ♣

In order to define the $\alpha_{t\tau}$ we only need the terms $(s^i - s^{-i})$ to be invertible for all $i \geq 1$. The question is: If r is a subring of the field of rational functions in which all the $(s^i - s^{-i})$ are invertible, are the $\alpha_{t\tau}$ a basis for H_n? They are linearly independent over r, but do they span H_n over r? Blanchet claims in his paper that this already follows from lemma 2.5.1. But it seems that the following additional argument is necessary.

Lemma 2.5.5 *Let k be a field, and r be a subring of k. Let γ be an algebra automorphism of $M_{n\times n}$ over the field k. If γ restricts to an algebra endomorphism $\overline{\gamma}$ of $M_{n\times n}$ over the ring r then $\overline{\gamma}$ is an automorphism of the algebra $M_{n\times n}$ over the ring r.*

Proof We have to show that $\overline{\gamma}^{-1}$ is an algebra endomorphism over the ring r. By the Noether-Skolem-Theorem (see e.g. [10]), we have that the automorphism γ of $M_{n\times n}$ over the field k is the conjugation by some invertible element G of $M_{n\times n}$ whose entries lie in k.

That γ restricts to an endomorphism over the ring r means that GDG^{-1} has entries in r for any $(n\times n)$-matrix D whose entries lie in r. We have to show that the entries of $G^{-1}DG$ lie in r as well since $\overline{\gamma}^{-1}$ is the conjugation with G^{-1}.

We denote by E_{ij} the $(n\times n)$-matrix that differs from the zero-matrix only by the entry (i,j) which is equal to 1. For any $(n\times n)$-matrices A and B we have

$$AE_{ij}B = \begin{pmatrix} (A_{1i}B_{j1}) & (A_{1i}B_{j2}) & \cdots & (A_{1i}B_{jn}) \\ (A_{2i}B_{j1}) & (A_{2i}B_{j2}) & \cdots & (A_{2i}B_{jn}) \\ \vdots & \vdots & & \vdots \\ (A_{ni}B_{j1}) & (A_{ni}B_{j2}) & \cdots & (A_{ni}B_{jn}) \end{pmatrix} = (A_{ki}B_{jl})_{1\le k,l\le n}$$

for any $i,j = 1,\dots,N$. Similarly,

$$BE_{ij}A = (B_{pi}A_{jq})_{1\le p,q\le n}.$$

This means that all the entries of $AE_{ij}B$ for all $1\le i,j\le n$ are a permutation of all the entries of $BE_{ij}A$ for all $1\le i,j\le n$. Hence, if all the entries of $GE_{ij}G^{-1}$ for $1\le i,j\le n$ lie in the ring r then all the entries of $G^{-1}E_{ij}G$ for $1\le i,j\le n$ lie in r. Since the matrices E_{ij} are a linear basis over r, we have that $G^{-1}DG$ has entries in r for any matrix D whose entries lie in r. ♣

We recall that for a standard tableau t we defined t' to be the standard tableau derived from t by deleting the cell with the highest label. Blanchet observes in theorem 1.13 in [3] that

$$y_\lambda \otimes 1 = \sum_{\substack{\lambda\subset\mu \\ |\mu|=|\lambda|+1}} (y_\lambda\otimes 1)y_\mu(y_\lambda\otimes 1).$$

By applying this result to the term $(y_\lambda\otimes 1)$ in the middle of $\alpha_{t\tau}\otimes 1 \in H_{n+1}$, one gets in H_{n+1}

Lemma 2.5.6 *We have*

$$\alpha_{t\tau}\otimes 1 = \sum_{s'=t,\sigma'=\tau} \alpha_{s\sigma}.$$

for any (suitable) tableaux t and τ.

Chapter 3

Closures of idempotents are Schur functions

3.1 Introduction

The Hecke algebra H_n interpreted as the Homfly skein of the disk with $2n$ boundary points with top-down orientation contains idempotents $(1/\alpha_\lambda)e_\lambda$ that are indexed by Young diagrams with n cells. Their closures Q_λ are known to be a basis for the image of H_n under the closure map in the skein of the annulus.

Previous works have shown that the map from the algebra of Young diagrams to the skein of the annulus mapping λ to Q_λ is an algebra isomorphism. But either the proofs used results beyond the scope of skein theory like [1] or they were sketchy and had gaps like [14].

In theorem 3.5.6 we shall give a self contained proof solely based on skein theory. The idea is to consider an element $S_\lambda = \det(Q_{d_{\lambda_i+j-i}})_{1\leq i,j\leq l(\lambda)}$ and to show that it behaves in the same way as Q_λ under the addition of a meridian loop of the annulus. This is sufficient to deduce that $S_\lambda = Q_\lambda$.

The skein of the annulus C' with two boundary points has been considered e.g. in [14], [8] and [18]. The version used here and in [18] enables us to define a commutative multiplication for C' because the boundary points lie on different boundary components of the annulus.

3.2 The skein C of the annulus

The Homfly skein of the annulus shall be denoted by C. We furthermore choose an orientation for the core of the annulus. In all our depictions, the annulus is given the standard anti-clockwise orientation, and the core is oriented anti-clockwise as well.

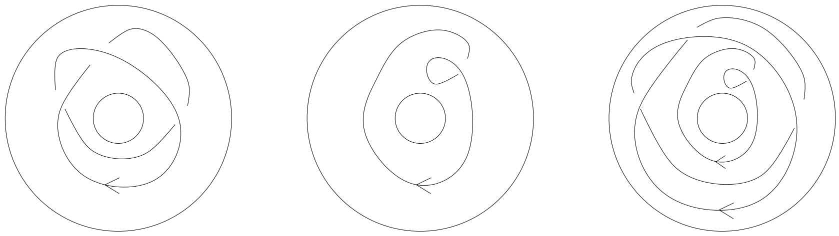


Figure 3.1: The multiplication in the skein of the annulus C.

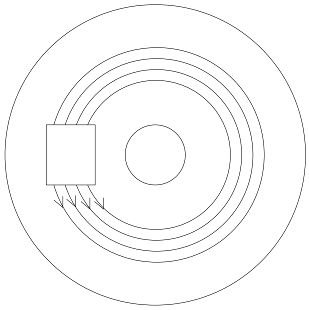


Figure 3.2: The closure map from H_n to C.

Let D_1 and D_2 be two diagrams in the annulus $S^1 \times [0,1]$. We can bring D_1 into $S^1 \times [0,1/2)$, and D_2 into $S^1 \times (1/2,1]$ by a regular isotopy. Then the product of D_1 and D_2 is defined as the diagram $D_1 \cup D_2$. The product is commutative since D_1D_2 and D_2D_1 differ by regular isotopy. The empty diagram is the identity.

The product of D_1 and D_2 is depicted as putting the inward circle of the annulus containing D_1 next to the outward circle of the annulus containing D_2 as shown in figure 3.1.

Figure 3.2 depicts an annulus with a set of n oriented arcs. A disc is removed from the annulus in such a way that we can insert a diagram from H_n such that the orientations of the arcs match. This factors to a map from H_n to C, denoted by $\Delta : D \mapsto \hat{D}$. This is a special case of a *wiring*. We define Q_λ to be the closure of the idempotent y_λ of H_n where n is the number of cells of λ,

$$Q_\lambda = \hat{y}_\lambda \in C.$$

We denote the image of H_n in C of the closing map by C_n. By C_+ we denote the submodule of C spanned by all $C_0, C_1, \ldots$,

$$C_+ = \left\langle \bigcup_{n \geq 0} C_n \right\rangle.$$

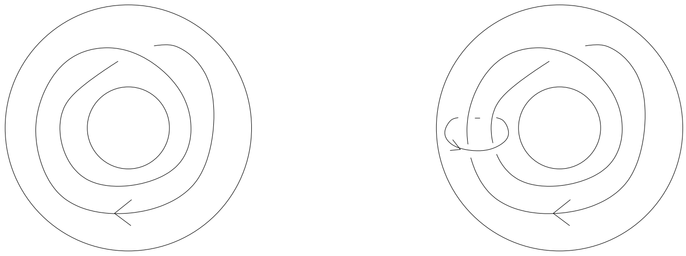


Figure 3.3: Encircling a diagram in the annulus.

We define a linear map Γ from C_+ to C_+ that is the encircling of any diagram in C_+ by a single loop as shown in figure 3.3. Similarly, $\tilde{\Gamma}$ is the same map but with the opposite orientation of the additional loop.

$X_i^+ \in C_i$ is defined as the closure of the braid $\sigma_{i-1}\sigma_{i-2}\ldots\sigma_1$. X_i^- derives from the diagram X_i^+ by reversing the orientation. Any diagram D in the annulus can be written in the skein of the annulus C as a linear combination of totally descending curves. It thus follows that X_i^+ and X_j^- for all integers i and j generate C. In fact, Turaev proved in [23] that they generate C freely as a commutative algebra. We shall prove the weaker result that $X_1^+, X_2^+, \ldots$ generate C_+ freely as a commutative algebra. The *weighted degree* of a monomial $(X_{i_1}^+)^{j_1} \cdots (X_{i_k}^+)^{j_k}$ is defined as $i_1 j_1 + \cdots i_k j_k$.

Lemma 3.2.1 *The dimension of C_k is equal to the number of partitions of k. The elements $X_1^+, X_2^+, \ldots$ are algebraically independent in C_+.*

Proof Inductively one proves that C_k is spanned by the monomials in $\{X_i^+\}$ of weighted degree k for any integer $k \geq 0$. Hence, the dimension of C_k is at most $p(k)$ by which we denote the number of partitions of k. We denote by $C_{\leq n}$ the submodule of C_+ which is spanned by all elements of C_k, $0 \leq k \leq n$. Therefore, the dimension of $C_{\leq n}$ is at most $p_n = p(0) + p(1) + \cdots + p(n)$.

On the other hand, all the closures of e_λ lie in $C_{\leq n}$ provided that the Young diagrams λ have at most n cells. The closures of the e_λ are non-zero by lemma 2.4.6, and they are linearly independent since they have different eigenvalues under the map Γ. Hence, the dimension of $C_{\leq n}$ is at least p_n. Hence, the dimension of $C_{\leq n}$ is equal to p_n. Since every element of $C_{\leq n}$ is a linear combination of monomials in $\{X_i^+\}$ of weighted degree lower than or equal to n, these monomials have to be linearly independent. Since this is true for all $n \geq 0$, we have that all the monomials in $\{X_i^+\}$ are linearly independent. ♣

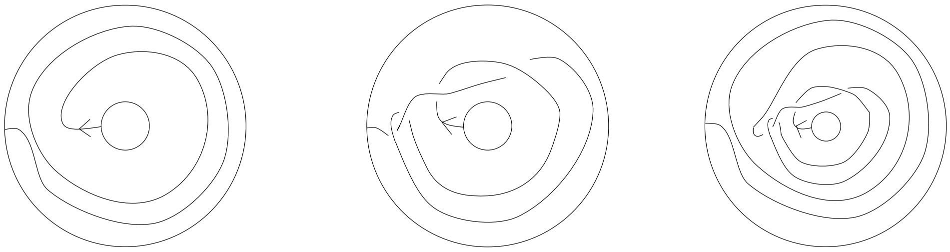


Figure 3.4: The multiplication in C'.

3.3 The variant skein C' of the annulus

We require an orientation of the core of the annulus. The orientation of the annulus induces an orientation on each of its boundary curves. We call c_1 the boundary curve for which this orientation agrees with the orientation of the parallel core. We call c_2 the other boundary component. We pick points $\gamma_1 \in c_1$ and $\gamma_2 \in c_2$. We denote by C' the skein $\mathcal{S}(S^1 \times [0,1], \Gamma = \{\gamma_1\} \uplus \{\gamma_2\})$.

When we embed the annulus in the plane with the standard counter-clockwise orientation and the core oriented counter-clockwise as well, then c_1 is the outer boundary component, and c_2 the inner.

Similarly to C, we turn C' into an algebra. In the standard picture, the inner boundary point of a diagram α comes together with the outer boundary point of a diagram β as shown in Figure 3.4.

The single straight arc e connecting the two marked points is the identity element, as shown in figure 3.5. The commutativity is not immediate but nevertheless turns out to be true as we shall see in lemma 3.3.3 and in the remark following it.

The skein used in [14] has both of its two boundary points on the outer boundary circle of the annulus. Furthermore, they lie at the right. There is a map from C' to this variant skein. First, one turns the annulus over to itself keeping a vertical line fixed. Then one adds the arc from figure 3.6 from below.

We have two operations of C on C'. If α is an element of C and x is an element of C' we define αx as stacking α above x as shown in Figure 3.7. Similarly $x\alpha$ is defined as putting α below x.

We define a closing operation $r \mapsto \hat{r}$ from C' to C which means adding the arc in figure 3.8 from above to a diagram r. In order that this is possible, the annulus for C has to be slightly larger than C'. The framing of the diagram $\hat{r}$ is defined to be its blackboard framing. We remark that this closing operation is not an algebra homomorphism. The linear map from H_n to C given by closing a tangle t is denoted by $t \mapsto \hat{t}$ as well. This should not lead to confusion.

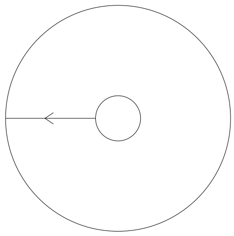


Figure 3.5: The identity e in C'.

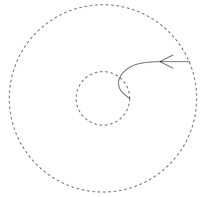


Figure 3.6: A map between different skeins of the annulus after turning the annulus over.

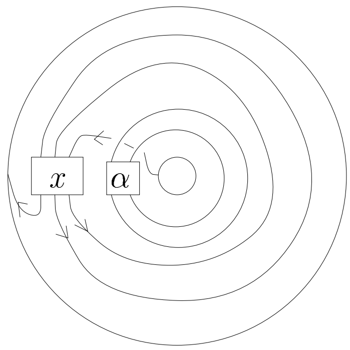


Figure 3.7: Operation of C on C' from the left.

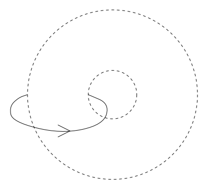


Figure 3.8: The additional arc for the closure.

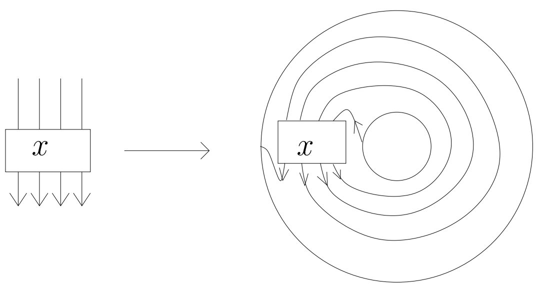


Figure 3.9: Map Δ' from H_n to C'.

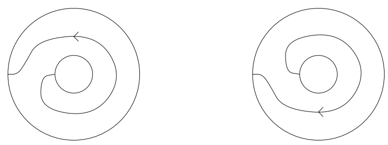


Figure 3.10: The arc a (at the left) and its inverse a^{-1} (at the right).

For any integer $n \geq 1$ we have a linear map $\Delta' : H_n \to C'$ as shown in figure 3.9. We denote the image of H_n under this map by C'_n. We define C'_+ to be the submodule of C' spanned by all $C'_0, C'_1, \ldots$,

$$C'_+ = \left\langle \bigcup_{n \geq 0} C'_n \right\rangle.$$

We shall use the notation $\Delta'_n : H_n \to C'$ if it is necessary to emphasize n.

We shall denote by a the element of C' that is the image of the identity braid 1_2 of H_2 under the map Δ'. It is the arc that joins the two boundary points of C' as shown on the left in figure 3.10.

Lemma 3.3.1 *For any integer $n \geq 1$ we have*

$$C'_n = \left\langle \bigcup_{k=0}^{n-1} C_{n-k-1} a^k \right\rangle.$$

Proof We have $\langle \cup_{k=0}^{n-1} C_{n-k-1} a^k \rangle \subset C'_n$ because

$$\hat{\gamma} a^k = \Delta'_n[(\gamma \otimes \mathrm{id}_{k+1}) \sigma_k \sigma_{k-1} \cdots \sigma_1]$$

for any $\gamma \in H_{n-k-1}$.

Since H_n is spanned by braids, C'_n is spanned by the images under Δ'_n of braid diagrams. We prove for any n-string braid diagram β that $\Delta'_n(\beta) \in \langle \cup_{k=0}^{n-1} C_{n-k-1} a^k \rangle$ by induction on the number of crossings of β.

If β has no crossings then it is the identity braid on n strings, hence $\Delta'_n(\beta)$ is equal to a^{n-1}.

Let β have $r \geq 1$ crossings. Let $\tilde{\beta}$ be another braid diagram on n strings that differs from β by switching some crossings from under- to overcrossings or vice-versa. Then $\beta - \tilde{\beta}$ is in the Hecke algebra a linear combination of diagrams with less than r crossings because of the skein relation. We may assume inductively that the image under Δ'_n of each of those summands lies in $\langle \cup_{k=0}^{n-1} C_{n-k-1} a^k \rangle$. Hence

$$\Delta'_n(\beta) \in \langle \cup_{k=0}^{n-1} C_{n-k-1} a^k \rangle \text{ if and only if } \Delta'_n(\tilde{\beta}) \in \langle \cup_{k=0}^{n-1} C_{n-k-1} a^k \rangle.$$

We change the crossings of β in such a way to a new braid $\tilde{\beta}$ so that the arc c of $\Delta'_n(\tilde{\beta})$ which connects the boundary points is totally descending along its orientation, and c lies below any other component of $\Delta'_n(\tilde{\beta})$. Then c is regularly isotopic to a power of a, say $a^l,\ l \geq 0$.

The other components of $\Delta'_n(\tilde{\beta})$ are the closure of the braid that derives from $\tilde{\beta}$ by deleting the $(l+1)$ strings that belong to c. Hence $\Delta'_n(\tilde{\beta}) \in C_{n-l-1}a^l$, hence $\Delta'_n(\beta) \in \langle \cup_{k=0}^{n-1} C_{n-k-1}a^k \rangle$. ♣

We immediately deduce

Corollary 3.3.2 *C'_+ is a graded commutative subalgebra of C'.*

We have thus proved that C'_+ is linearly spanned as a left-module over C_+ by the powers of a. We can prove even more.

Lemma 3.3.3 *C'_+ is the polynomial algebra in a with the action of C_+ on the left.*

Proof We have to show that the powers of a are linearly independent for coefficients in C_+. So let us assume that

$$c_0 e + c_1 a + c_2 a^2 + \cdots + c_m a^m = 0 \tag{3.3.1}$$

for $m \geq 0$ and coefficients $c_0, c_1, \ldots, c_m$ in C_+. The closure of e is equal to δ times the empty diagram $\emptyset$. The closure g_i of a^i is very similar to X_i^+, and the g_i are algebraically independent in C_+ by essentially the same argument as in the proof of lemma 3.2.1.

Taking the closure transforms the equation (3.3.1) in C'_+ into the following equation in C_+

$$c_0\delta\emptyset + c_1 g_1 + c_2 g_2 + \cdots + c_m g_m = 0.$$

If we first multiply equation (3.3.1) by a^k for some $1 \leq k \leq m$ then we get after taking the closure that

$$c_0 g_k + c_1 g_{k+1} + c_2 g_{k+2} + \cdots + c_m g_{k+m} = 0.$$

We can summarize these $(m+1)$ equations in matrix form as

$$\begin{pmatrix} \delta\emptyset & g_1 & g_2 & \cdots & g_m \\ g_1 & g_2 & g_3 & \cdots & g_{m+1} \\ g_2 & g_3 & g_4 & \cdots & g_{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_m & g_{m+1} & g_{m+2} & \cdots & g_{2m} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

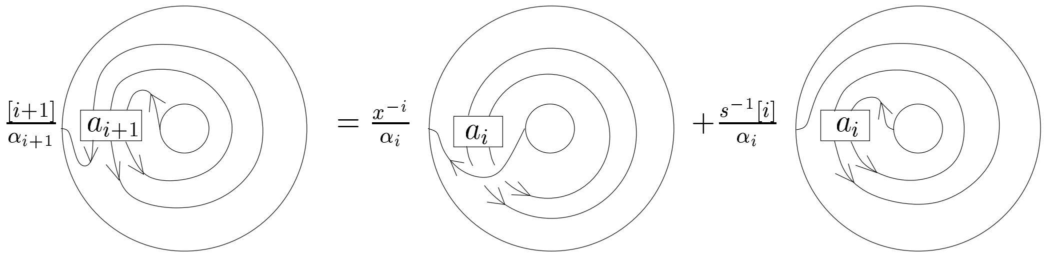


Figure 3.11: Depiction of lemma 3.4.1.

When we express the determinant of the $(m+1) \times (m+1)$-matrix as a sum via the Leibniz rule we see that the monomial $g_2 g_4 \cdots g_{2m}$ appears only once and its coefficient is equal to δ. Since C_+ is freely generated by the empty diagram and $g_1, g_2, \ldots$, the determinant is non-zero. Since C_+ is an integral ring, we can embed it into a field k. Therefore the linear module endomorphism of $C_+^{\oplus n}$ given by the matrix can be extended to a endomorphism of a vector space over the field k. Since the determinant of this vector space endomorphism is equal to the determinant of the module endomorphism, the module endomorphism is finally seen to be injective. Hence $(c_0, c_1, \ldots, c_m) = (0, 0, \ldots, 0)$.

Hence $e, a, a^2, \ldots$ are linearly independent over C_+. ♣

Remark By essentially the same argument, C'_+ as a right-module over C_+ is the polynomial algebra over C_+ in a. Similarly, for either operation of C on C', C' is the Laurent polynomial algebra over C in a.

3.4 Basic skein relations

Let D be an element of the skein of the annulus C. The inclusion of the annulus in the plane induces a (non-injective) linear map from the skein of the annulus C to the skein of the plane $\mathcal{S}(\mathbb{R}^2)$. We denote the Homfly polynomial of the image of D in $\mathcal{S}(\mathbb{R}^2)$ by $\langle D \rangle$. The map $D \mapsto \langle D \rangle$ is an algebra homomorphism.

We define A'_i to be the element $\Delta'(a_i)$ of C'_+, and A_i to be the element $\Delta(a_i)$ of C_+ for any integer $i \geq 0$. We recall that $a_i a_i = \alpha_i a_i$ for some non-zero scalar α_i. We define $h_i = \frac{1}{\alpha_i} A_i$ for any integer $i \geq 0$, and we define $h_i = 0$ for $i < 0$.

The following lemma is depicted in figure 3.11.

Lemma 3.4.1 *We have*

$$\frac{[i+1]}{\alpha_{i+1}} A'_{i+1} = \frac{x^{-i}}{\alpha_i}(eA_i) + \frac{s^{-1}[i]}{\alpha_i} A'_i a$$

in C'_+ for any integer $i \geq 0$.

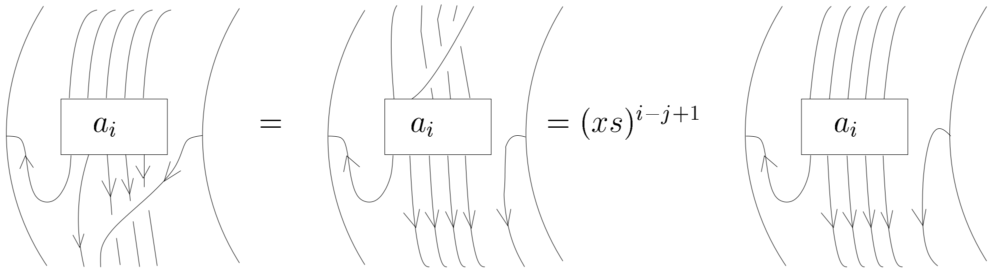


Figure 3.12: Moving crossings around in the annulus.

Proof We have

$$a_{i+1} = (a_i \otimes 1_1)(1_{i+1} + (x^{-1}s)\sigma_i + (x^{-1}s)^2\sigma_i\sigma_{i-1} + \cdots + (x^{-1}s)^i\sigma_i\sigma_{i-1}\cdots\sigma_1)$$

by equation (2.4.1). We consider the term $\Delta'((a_i \otimes 1)\sigma_i\sigma_{i-1}\cdots\sigma_j)$ as depicted in figure 3.12. If $2 \le j \le i$ then we can move the braid $\sigma_i\sigma_{i-1}\cdots\sigma_j$ around the annulus to the top of a_i where the braid is read as $\sigma_{i-1}\sigma_{i-2}\cdots\sigma_{j-1}$ and these $(i-j+1)$ crossings are swallowed by a_i at the expense of the scalar $(xs)^{i-j+1}$. We thus get

$$\Delta'((a_i \otimes 1_1)\sigma_i\sigma_{i-1}\cdots\sigma_j) = (xs)^{i-j+1}\Delta'(a_i)a$$

for $2 \le j \le i$. For $j = 1$ we have the summand $\Delta'((a_i \otimes 1_1)\sigma_i\sigma_{i-1}\cdots\sigma_1)$ which is equal to eA_i. We thus get

$$\begin{aligned}
A'_{i+1} &= \Delta'(a_{i+1}) \\
&= \Delta'(a_i \otimes 1_1) + \sum_{j=1}^{i}(x^{-1}s)^{i-j+1}\Delta'((a_i \otimes 1_1)\sigma_i\sigma_{i-1}\cdots\sigma_j) \\
&= A'_i a + (x^{-1}s)^i(eA_i) + \sum_{j=2}^{i} s^{2(i-j+1)}A'_i a \\
&= (x^{-1}s)^i(eA_i) + A'_i a\sum_{j=2}^{i+1} s^{2(i-j+1)} \\
&= (x^{-1}s)^i(eA_i) + s^{i-1}[i]A'_i a.
\end{aligned}$$

Since $\alpha_{i+1} = \alpha_i s^i[i+1]$ by lemma 2.4.2, we get

$$\frac{[i+1]}{\alpha_{i+1}}A'_{i+1} = \frac{x^{-i}}{\alpha_i}(eA_i) + \frac{s^{-1}[i]}{\alpha_i}A'_i a.$$

♣

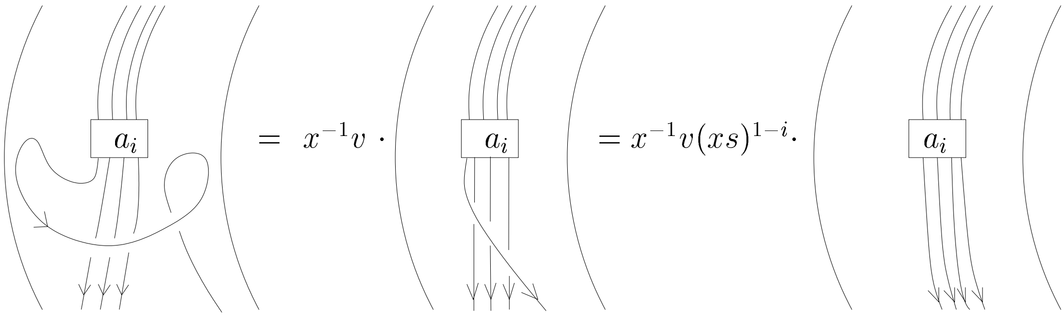


Figure 3.13: The closure of $A'_i a$.

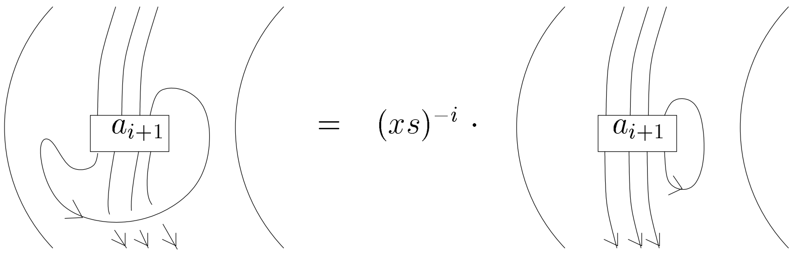


Figure 3.14: The closure of A'_i.

Lemma 3.4.2 *We have*

$$\langle h_{i+1} \rangle = \langle h_i \rangle \frac{v^{-1}s^i - vs^{-i}}{s^i - s^{-i}}$$

for any integer $i \geq 0$.

Proof Using the skein relations in figures 3.13 and 3.14 we deduce from lemma 3.4.1 by taking the closure and Homfly polynomial in $\mathbb{R}^2$ that

$$[i+1](xs)^{-i} \langle h_{i+1} \rangle = x^{-i} \frac{v^{-1} - v}{s - s^{-1}} \langle h_i \rangle + s^{-1}[i]x^{-1}v(xs)^{1-i} \langle h_i \rangle .$$

Hence,

$$\begin{aligned} [i+1] \langle h_{i+1} \rangle &= \langle h_i \rangle \left(s^i \frac{v^{-1} - v}{s - s^{-1}} + [i]v \right) \\ &= \langle h_i \rangle \frac{s^i v^{-1} - s^i v + s^i v - s^{-i} v}{s - s^{-1}} \\ &= \langle h_i \rangle \frac{s^i v^{-1} - s^{-i} v}{s - s^{-1}}. \end{aligned}$$

♣

We define B_i to be the closure of the quasi-idempotent $b_i \in H_i$ in the skein of the annulus.

Corollary 3.4.3 *We have*

$$\left\langle \frac{1}{\beta_{i+1}} B_{i+1} \right\rangle = \left\langle \frac{1}{\beta_i} B_i \right\rangle \frac{vs^i - v^{-1}s^{-i}}{s^i - s^{-i}}$$

for any integer $i \geq 0$.

Proof This follows directly from lemma 3.4.2 by applying the map γ from subsection 2.4.1 which interchanges the quasi-idempotents a_i and b_i. ♣

We define an element

$$t_i = x^i(h_i e) - x^{-i}(e h_i)$$

in C'_+ for any integer i. We remark that $t_i = 0$ for $i \leq 0$.

Lemma 3.4.4 *We have*

$$t_i = (s^{-1} - s)\frac{[i]}{\alpha_i} A'_i a$$

for any integer $i \geq 0$.

Proof We have

$$\frac{[i+1]}{\alpha_{i+1}} A'_{i+1} = \frac{x^{-i}}{\alpha_i}(eA_i) + \frac{s^{-1}[i]}{\alpha_i} A'_i a$$

by lemma 3.4.1. By applying the map ρ from subsection 2.4.1 we get

$$\frac{[i+1]}{\alpha_{i+1}} A'_{i+1} = \frac{x^{i}}{\alpha_i}(A_i e) + \frac{s[i]}{\alpha_i} A'_i a.$$

The right hand sides of the above two equations show that

$$\frac{x^i}{\alpha_i}(A_i e) - \frac{x^{-i}}{\alpha_i}(eA_i) = (s^{-1} - s)\frac{[i]}{\alpha_i} A'_i a$$

for any integer $i \geq 0$. ♣

Corollary 3.4.5 *We have*

$$\hat{t}_i = (s^{1-2i} - s)x^{-i} v h_i$$

for any integer i.

Proof From lemma 3.4.4 and the skein relation in figure 3.13 we deduce that

$$\begin{aligned}\hat{t_i} &= (s^{-1}-s)[i]x^{-1}v(xs)^{1-i}h_i \\ &= (s^{1-2i}-s)x^{-i}vh_i\end{aligned}$$

for any integer $i \geq 0$. This equation holds for negative integers i as well because h_i and t_i are equal to zero for negative i. ♣

Corollary 3.4.6 *We have*

$$(h_ie)^{\wedge} = x^{-2i}\left(\frac{v^{-1}-v}{s-s^{-1}} + v(s^{1-2i}-s)\right)h_i$$

for any integer i.

Proof We have $t_i = x^i(h_ie) - x^{-i}(eh_i)$. Taking the closure we deduce

$$\hat{t_i} = x^i(h_ie)^{\wedge} - x^{-i}\frac{v^{-1}-v}{s^{-1}-s}h_i$$

because the closure of eh_i is equal to h_i with a disjoint loop. By corollary 3.4.5 we immediately get

$$(h_ie)^{\wedge} = x^{-2i}\left(\frac{v^{-1}-v}{s-s^{-1}} + v(s^{1-2i}-s)\right)h_i.$$

♣

Lemma 3.4.7 *We have*

$$t_it_{j+1} - t_jt_{i+1} = (s^2-1)(x^{-i}(eh_i)t_{j+1} - x^{-j}(eh_j)t_{i+1})$$

for any integers i and j.

Proof If either i or j is negative then the lemma is obviously true. Let $i \geq 0$ and $j \geq 0$ from now on. We have

$$\frac{[i+1]}{\alpha_{i+1}}A'_{i+1} = \frac{x^{-i}}{\alpha_i}(eA_i) + \frac{s^{-1}[i]}{\alpha_i}A'_ia$$

by lemma 3.4.1. We multiply both sides by $\frac{[j+1]}{\alpha_{j+1}}A'_{j+1}a$ (on the right) and get

$$\frac{[i+1][j+1]}{\alpha_{i+1}\alpha_{j+1}}A'_{i+1}A'_{j+1}a = \frac{x^{-i}[j+1]}{\alpha_i\alpha_{j+1}}(eA_i)A'_{j+1}a + s^{-1}\frac{[i][j+1]}{\alpha_i\alpha_{j+1}}A'_iaA'_{j+1}a.$$

We multiply both sides by the scalar $(s^{-1}-s)^2$ and use lemma 3.4.4 to get

$$\frac{[i+1][j+1](s^{-1}-s)^2}{\alpha_{i+1}\alpha_{j+1}}A'_{i+1}A'_{j+1}a=(s^{-1}-s)\frac{x^{-i}}{\alpha_i}(eA_i)t_{j+1}+s^{-1}t_it_{j+1}.$$

The left hand side of the above equation is invariant under the interchange of i and j because C'_+ is commutative, and thus the right hand side is invariant under this interchange. Hence,

$$(s^{-1}-s)\frac{x^{-i}}{\alpha_i}(eA_i)t_{j+1}+s^{-1}t_it_{j+1}=(s^{-1}-s)\frac{x^{-j}}{\alpha_j}(eA_j)t_{i+1}+s^{-1}t_jt_{i+1}.$$

Equivalently,

$$t_it_{j+1}-t_jt_{i+1}=(s^2-1)\left(\frac{x^{-i}}{\alpha_i}(eA_i)t_{j+1}-\frac{x^{-j}}{\alpha_j}(eA_j)t_{i+1}\right).$$

♣

3.5 Determinantal calculations

Lemma 3.5.1 *For any integer $r\geq 2$ and integers $i_1,i_2,\dots,i_r$ we have an equality of $(r\times r)$-determinants in C'*

$$\begin{vmatrix} h_{i_1}e & \cdots & h_{i_1+r-2}e & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ h_{i_r}e & \cdots & h_{i_r+r-2}e & t_{i_r+r-1} \end{vmatrix} = s^{2(r-1)} \begin{vmatrix} eh_{i_1} & \cdots & eh_{i_1+r-2} & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ eh_{i_r} & \cdots & eh_{i_r+r-2} & t_{i_r+r-1} \end{vmatrix}$$

when we set $x=1$.

Proof The reason for the substitution $x=1$ is the fact that we can then write Lemma 3.4.7 in determinantal form as

$$\begin{vmatrix} t_i & t_{i+1} \\ t_j & t_{j+1} \end{vmatrix} = (s^2-1)\begin{vmatrix} eh_i & t_{i+1} \\ eh_j & t_{j+1} \end{vmatrix} \tag{3.5.2}$$

for any integers i and j. Using the multilinearity of the determinant together with $t_i=h_ie-eh_i$ we deduce from the above equation that

$$\begin{vmatrix} h_ie & t_{i+1} \\ h_je & t_{j+1} \end{vmatrix} = s^2\begin{vmatrix} eh_i & t_{i+1} \\ eh_j & t_{j+1} \end{vmatrix}, \tag{3.5.3}$$

which is our claim in the case $r=2$.

From equations (3.5.2) and (3.5.3) we deduce that

$$\begin{vmatrix} t_i & t_{i+1} \\ t_j & t_{j+1} \end{vmatrix} = (1-s^{-2}) \begin{vmatrix} h_i e & t_{i+1} \\ h_j e & t_{j+1} \end{vmatrix}. \tag{3.5.4}$$

From now on let $r \geq 3$. We see that

$$\begin{vmatrix} t_{i_1} & t_{i_1+1} & t_{i_1+2} & \cdots & t_{i_1+r-1} \\ \vdots & \vdots & \vdots & & \vdots \\ t_{i_r} & t_{i_r+1} & t_{i_r+2} & \cdots & t_{i_r+r-1} \end{vmatrix} = (1-s^{-2}) \begin{vmatrix} h_{i_1} e & t_{i_1+1} & t_{i_1+2} & \cdots & t_{i_1+r-1} \\ \vdots & \vdots & \vdots & & \vdots \\ h_{i_r} e & t_{i_r+1} & t_{i_r+2} & \cdots & t_{i_r+r-1} \end{vmatrix}$$

by developing the determinant on the left hand side by the first two columns, applying equation (3.5.4) to each summand, and redeveloping the determinant. By doing this successively for the columns 1 and 2, 2 and 3, ..., $(r-1)$ and r, we deduce that

$$\begin{vmatrix} t_{i_1} & \cdots & t_{i_1+r-2} & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ t_{i_r} & \cdots & t_{i_r+r-2} & t_{i_r+r-1} \end{vmatrix} = (1-s^{-2})^{r-1} \begin{vmatrix} h_{i_1} e & \cdots & h_{i_1+r-2} e & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ h_{i_r} e & \cdots & h_{i_r+r-2} e & t_{i_r+r-1} \end{vmatrix}.$$

On the other hand, if we use equation (3.5.2) instead of (3.5.4) in the above argument, we get

$$\begin{vmatrix} t_{i_1} & \cdots & t_{i_1+r-2} & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ t_{i_r} & \cdots & t_{i_r+r-2} & t_{i_r+r-1} \end{vmatrix} = (s^2-1)^{r-1} \begin{vmatrix} e h_{i_1} & \cdots & e h_{i_1+r-2} & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ e h_{i_r} & \cdots & e h_{i_r+r-2} & t_{i_r+r-1} \end{vmatrix}.$$

Hence

$$\begin{vmatrix} h_{i_1} e & \cdots & h_{i_1+r-2} e & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ h_{i_r} e & \cdots & h_{i_r+r-2} e & t_{i_r+r-1} \end{vmatrix} = s^{2(r-1)} \begin{vmatrix} e h_{i_1} & \cdots & e h_{i_1+r-2} & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ e h_{i_r} & \cdots & e h_{i_r+r-2} & t_{i_r+r-1} \end{vmatrix}.$$

♣

We define

$$S_\lambda = \det(h_{\lambda_i+j-i})_{1\leq i,j\leq l(\lambda)} \in C_n$$

where $n = |\lambda|$. We remark that we have proved the following theorem for the case λ equal to a row diagram already in Corollary 3.4.6.

Theorem 3.5.2 *We have $(S_\lambda e)^\wedge = q_\lambda S_\lambda$ in C_+ with the scalar*

$$q_\lambda = \frac{v^{-1}-v}{s-s^{-1}} + vs^{-1} \sum_{k=1}^{l(\lambda)} (s^{2(k-\lambda_k)} - s^{2k})$$

when we set $x = 1$.

Proof We shall set $x = 1$ throughout our calculations. For any elements α and β of the skein of the annulus C we have $(\alpha e) \cdot (\beta e) = (\alpha\beta)e$ in C'. Hence

$$S_\lambda e = \det(h_{\lambda_i+j-i}e)_{1\le i,j\le l(\lambda)}.$$

Similarly

$$eS_\lambda = \det(eh_{\lambda_i+j-i})_{1\le i,j\le l(\lambda)}.$$

We denote $l(\lambda)$ by n from now on. We remark that the closure $(eS_\lambda)^\wedge$ is equal to S_λ and a disjoint circle which can be removed at the expense of the scalar $(v^{-1} - v)/(s - s^{-1})$.

By the multilinearity of the determinant we can write the difference of any two $(n \times n)$-determinants as a telescope sum of n $(n \times n)$-determinants.

$$\begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{vmatrix} - \begin{vmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & & \vdots \\ z_{n1} & \cdots & z_{nn} \end{vmatrix} =$$

$$\sum_{k=1}^{n} \begin{vmatrix} y_{1\,1} & \cdots & y_{1\,k-1} & (y_{1\,k} - z_{1\,k}) & z_{1\,k+1} & \cdots & z_{1\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ y_{n\,1} & \cdots & y_{n\,k-1} & (y_{n\,k} - z_{n\,k}) & z_{n\,k+1} & \cdots & z_{n\,n} \end{vmatrix}.$$

Applying this formula to the determinants for $S_\lambda e$ and eS_λ we get

$$S_\lambda e - eS_\lambda =$$
$$\sum_{k=1}^{n} \begin{vmatrix} h_{\lambda_1}e & \cdots & h_{\lambda_1+k-2}e & t_{\lambda_1+k-1} & eh_{\lambda_1+k} & \cdots & eh_{\lambda_1+n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{\lambda_n+1-n}e & \cdots & h_{\lambda_n+k-1-n}e & t_{\lambda_n+k-n} & eh_{\lambda_n+k+1-n} & \cdots & eh_{\lambda_n} \end{vmatrix}.$$

By lemma 3.5.1 we deduce

$$S_\lambda e - eS_\lambda =$$
$$\sum_{k=1}^{n} s^{2(k-1)} \begin{vmatrix} eh_{\lambda_1} & \cdots & eh_{\lambda_1+k-2} & t_{\lambda_1+k-1} & eh_{\lambda_1+k} & \cdots & eh_{\lambda_1+n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ eh_{\lambda_n+1-n} & \cdots & eh_{\lambda_n+k-1-n} & t_{\lambda_n+k-n} & eh_{\lambda_n+k+1-n} & \cdots & eh_{\lambda_n} \end{vmatrix}.$$

The appearing n determinants are very special because each of them is a sum of terms of the form of a t_i above a product of h_j's. Therefore the closure of each determinant is $\hat{t}_i$ above a product of h_j's. Explicitly,

$$(S_\lambda e)^\wedge - (eS_\lambda)^\wedge =$$
$$\sum_{k=1}^{n} s^{2(k-1)} \begin{vmatrix} h_{\lambda_1} & \cdots & h_{\lambda_1+k-2} & \hat{t}_{\lambda_1+k-1} & h_{\lambda_1+k} & \cdots & h_{\lambda_1+n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{\lambda_n+1-n} & \cdots & h_{\lambda_n+k-1-n} & \hat{t}_{\lambda_n+k-n} & h_{\lambda_n+k+1-n} & \cdots & h_{\lambda_n} \end{vmatrix}.$$

We know by corollary 3.4.5 that $\hat{t}_i$ is a scalar multiple of h_i. Hence

$$(S_\lambda e)^\wedge - (eS_\lambda)^\wedge = \sum_{k=1}^{n} \begin{vmatrix} h_{\lambda_1} & \cdots & h_{\lambda_1+k-2} & \beta_{1\,k} h_{\lambda_1+k-1} & h_{\lambda_1+k} & \cdots & h_{\lambda_1+n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{\lambda_n+1-n} & \cdots & h_{\lambda_n+k-1-n} & \beta_{n\,k} h_{\lambda_n+k-n} & h_{\lambda_n+k+1-n} & \cdots & h_{\lambda_n} \end{vmatrix}$$

where $\beta_{i\,k} = s^{2(k-1)}(s^{1-2(\lambda_i+k-i)} - s)v$. We use the notation $\alpha_i = s^{2i-2\lambda_i-1}v$ and $\gamma_k = -s^{2k-1}v$, hence $\beta_{i\,k} = \alpha_i + \gamma_k$. By the multilinearity of the determinant we get

$$(S_\lambda e)^\wedge - (eS_\lambda)^\wedge = (\gamma_1 + \ldots + \gamma_n)S_\lambda + \sum_{k=1}^{n} \begin{vmatrix} h_{\lambda_1} & \cdots & h_{\lambda_1+k-2} & \alpha_1 h_{\lambda_1+k-1} & h_{\lambda_1+k} & \cdots & h_{\lambda_1+n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{\lambda_n+1-n} & \cdots & h_{\lambda_n+k-1-n} & \alpha_n h_{\lambda_n+k-n} & h_{\lambda_n+k+1-n} & \cdots & h_{\lambda_n} \end{vmatrix}.$$

We bring the sum over the determinants in a more appropriate form via the general formula for variables w_{ij} and α_k,

$$\sum_{k=1}^{n} \begin{vmatrix} w_{1\,1} & \cdots & w_{1\,k-1} & \alpha_1 w_{1\,k} & w_{1\,k+1} & \cdots & w_{1\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ w_{n\,1} & \cdots & w_{n\,k-1} & \alpha_n w_{n\,k} & w_{n\,k+1} & \cdots & w_{n\,n} \end{vmatrix} = (\alpha_1 + \cdots + \alpha_n) \begin{vmatrix} w_{1\,1} & \cdots & w_{1\,n} \\ \vdots & & \vdots \\ w_{n\,1} & \cdots & w_{n\,n} \end{vmatrix}.$$

Applying this formula we get

$$\begin{aligned} (S_\lambda e)^\wedge - (eS_\lambda)^\wedge &= (\gamma_1 + \cdots + \gamma_n)S_\lambda + (\alpha_1 + \cdots + \alpha_n)S_\lambda \\ &= (\beta_{1\,1} + \cdots + \beta_{n\,n})S_\lambda. \end{aligned}$$

Since $(eS_\lambda)^\wedge = (v^{-1} - v)/(s - s^{-1})S_\lambda$, we have $(S_\lambda e)^\wedge = q_\lambda S_\lambda$ with

$$\begin{aligned} q_\lambda &= \frac{v^{-1} - v}{s - s^{-1}} + \beta_{1\,1} + \cdots + \beta_{n\,n} \\ &= \frac{v^{-1} - v}{s - s^{-1}} + vs^{-1} \sum_{k=1}^{n} (s^{2(k-\lambda_k)} - s^{2k}). \end{aligned}$$

♣

We now formulate theorem 3.5.2 for general x.

Theorem 3.5.3 *We have* $(S_\lambda e)^\wedge = q_\lambda S_\lambda$ *in* C_+ *with the scalar*

$$q_\lambda = x^{-2|\lambda|}\frac{v^{-1}-v}{s-s^{-1}} + x^{-2|\lambda|}vs^{-1}\sum_{k=1}^{l(\lambda)}(s^{2(k-\lambda_k)}-s^{2k}).$$

Proof We define two maps from C_+ to C_+. The map Φ is the specialization of x to 1. The map $\overline{\Phi}$ maps every diagram D to $x^{-wr(D)}D$. The maps Φ and $\overline{\Phi}$ are not inverse to each other in general. But, from the definition of the quasi-idempotent $a_i \in H_i$ we see that $\overline{\Phi}\Phi(A_i) = A_i$ for every integer $i \geq 0$. Since the scalar α_i does not involve x, we deduce that $\overline{\Phi}\Phi(P) = P$ for every polynomial in $h_i = \frac{1}{\alpha_i}A_i$. In particular, $\overline{\Phi}\Phi(S_\lambda) = S_\lambda$. Hence $q_\lambda = x^{-2|\lambda|}(q_\lambda|_{x=1})$. ♣

We recall the linear maps Γ and $\tilde{\Gamma}$ from C_+ to C_+ as defined in section 3.2. They encircle a diagram by a single loop with a specified orientation.

Corollary 3.5.4 *We have* $\Gamma(S_\lambda) = q_\lambda S_\lambda$ *and* $\tilde{\Gamma}(S_\lambda) = \tilde{q}_\lambda S_\lambda$ *where*

$$\begin{aligned} q_\lambda &= x^{-2|\lambda|}\frac{v^{-1}-v}{s-s^{-1}} + x^{-2|\lambda|}vs^{-1}\sum_{k=1}^{l(\lambda)}(s^{2(k-\lambda_k)}-s^{2k}), \\ \tilde{q}_\lambda &= x^{2|\lambda|}\frac{v^{-1}-v}{s-s^{-1}} + x^{2|\lambda|}v^{-1}s\sum_{k=1}^{l(\lambda)}(s^{2(\lambda_k-k)}-s^{-2k}) \end{aligned}$$

for any Young diagram λ.

Proof The equality $\Gamma(S_\lambda) = q_\lambda S_\lambda$ is the statement of theorem 3.5.3. We recall the map ρ from subsection 2.4.1. We have $\rho(h_i) = h_i$ by lemma 2.4.4. Hence, $\rho(S_\lambda) = S_\lambda$ because S_λ is a polynomial in the h_i. Hence, $\rho(\Gamma(S_\lambda)) = \tilde{\Gamma}(S_\lambda)$, and thus $\tilde{\Gamma}(S_\lambda) = \rho(q_\lambda)S_\lambda$. ♣

We recall that Q_λ is the element of C_+ which is the closure of the idempotent $(1/\alpha_\lambda)e_\lambda$ of H_n. We thus have to consider some suitable subring of the rational functions in x, v and s as the ring of scalars for the skein modules. We shall describe the structure of the denominators appearing for Q_λ in lemma 3.6.3.

Theorem 3.5.5 S_λ *is equal to* Q_λ *for any Young diagram* λ.

Proof Q_λ is non-zero by lemma 2.4.6. Since the scalars c_λ and $\tilde{q}_\lambda$ from lemma 2.4.7 and corollary 3.5.4 are equal, we have that S_λ and Q_λ are eigenvectors with the same eigenvalue under the map $\tilde{\Gamma}$. Possibly, $S_\lambda = 0$. The set of Q_λ for all Young diagrams λ with n cells is a linear basis for C_n by lemma 3.2.1. Furthermore, the eigenvalues c_λ are pairwise different by lemma 2.4.8.

Hence, we deduce that S_λ is a scalar multiple of Q_λ for any Young diagram λ with n cells. This scalar is a rational function in x, v and s, and it is possibly equal to zero.

We denote the Young diagram consisting of a single cell by $\square$. We have that $S_\square = Q_\square = \hat{a}_1$ is the single core circle in the annulus. Hence, $S_\square^n$ is equal to the closure of the identity braid of H_n. On the other hand, by the multiplication rule for Young diagrams, we have

$$\square^n = \sum_{|\lambda|=n} d_\lambda \lambda$$

where d_λ is the number of standard tableaux of λ. Therefore,

$$S_\square^n = \sum_{|\lambda|=n} d_\lambda S_\lambda.$$

We have the following equality in the skein of the annulus

$$Q_\square^n = \sum_{|\lambda|=n} d_\lambda Q_\lambda.$$

This follows from the results in section 2.5 as we explain now. We have proved that $\sum_s \alpha_{ss} = \mathrm{id}_n \in H_n$ where the sum is over all standard tableaux of Young diagrams with n cells. The closure of any α_{ss} in the annulus is equal to Q_λ when s is a standard tableaux of Q_λ. Finally, the closure of the identity braid of H_n is the n-th power of the core of the annulus which is equal to $Q_\square$.

Since $S_\square = Q_\square$, we deduce from the above two equations that

$$\sum_{|\lambda|=n} d_\lambda Q_\lambda = \sum_{|\lambda|=n} d_\lambda S_\lambda.$$

Since $\{Q_\lambda \mid \lambda \text{ has } n \text{ cells}\}$ is a basis of C_n, and any S_λ lies in C_n, and any S_λ differs from Q_λ by a scalar, we get that $Q_\lambda = S_\lambda$. ♣

Theorem 3.5.6 *The map $\lambda \mapsto Q_\lambda$ is an isomorphism from the algebra of Young diagrams to C_+ provided that any α_λ is invertible in the ring of scalars.*

Proof The ring of Young diagrams $\mathcal{Y}$ is a free Abelian ring generated by the column diagrams $c_1, c_2, \ldots$. This is also true when we consider $\mathcal{Y}$ as an algebra over any subring of the rational functions in x, v and s.

C_+ is commutative, hence there is a unique algebra homomorphism that extends the map $c_i \mapsto Q_{c_i}$. This becomes an algebra homomorphism for any ring of scalars. In order that Q_{c_i} is defined, we need the invertibility of the scalar $(s^i - s^{-i})$.

The Q_λ for all Young diagrams λ are linearly independent. Hence the map $\mathcal{Y} \to C_+$ is injective. It is also surjective because the set of the Q_λ for Young diagrams λ with n cells is a basis for C_n. ♣

3.6 Applications

We shall abbreviate $\langle Q_\lambda \rangle$ by $\langle \lambda \rangle$.

Lemma 3.6.1 *For any Young diagram λ we have*

$$\langle \lambda \rangle = \prod_{x \in \lambda} \frac{v^{-1} s^{\mathrm{cn}(x)} - v s^{-\mathrm{cn}(x)}}{s^{\mathrm{hl}(x)} - s^{-\mathrm{hl}(x)}}.$$

Proof We have by corollary 3.4.3 that

$$\langle c_k \rangle = \prod_{i=1}^{k} \frac{v^{-1} s^{1-i} - v s^{i-1}}{s^i - s^{-i}}. \tag{3.6.5}$$

By exercises I.2.5 and I.3.3 of [17] with $q = s^2,\ a = vs,\ b = v^{-1}s$ we deduce from the above equation that

$$\begin{aligned} \sum_{i \geq 0} \langle c_i \rangle X^i &= \prod_{i \geq 0} \frac{1 + a q^i X}{1 + b q^i X} \\ &= \prod_{i \geq 0} \frac{1 + v s^{2i+1} X}{1 + v^{-1} s^{2i+1} X}, \end{aligned} \tag{3.6.6}$$

and

$$\begin{aligned} s_\lambda &= q^{n(\lambda)} \prod_{x \in \lambda} \frac{a - b q^{\mathrm{cn}(x)}}{1 - q^{\mathrm{hl}(x)}} \\ &= s^{2n(\lambda)} \prod_{x \in \lambda} s^{1 + \mathrm{cn}(x) - \mathrm{hl}(x)} \frac{v^{-1} s^{\mathrm{cn}(x)} - v s^{-\mathrm{cn}(x)}}{s^{\mathrm{hl}(x)} - s^{-\mathrm{hl}(x)}}, \end{aligned}$$

where $n(\lambda) = \sum_{i=1}^{l(\lambda)} (i-1)\lambda_i$. The Schur function s_λ is understood to be expressed as a polynomial in the elementary symmetric functions $e_1, e_2, \ldots$ and then any e_i is replaced by $\langle c_i \rangle$. The isomorphism of Schur functions and Young diagrams implies that $s_\lambda = \langle \lambda \rangle$ because $D \mapsto \langle D \rangle$ induces an algebra homomorphism from C to the scalars. By examples 2 and 3 in section I.1 of [17] we have

$$2n(\lambda) + \sum_{x \in \lambda} (1 + \mathrm{cn}(x) - \mathrm{hl}(x)) = 0.$$

Hence $\langle \lambda \rangle = s_\lambda = \prod_{x \in \lambda} (v^{-1} s^{\mathrm{cn}(x)} - v s^{-\mathrm{cn}(x)}) / (s^{\mathrm{hl}(x)} - s^{-\mathrm{hl}(x)})$. ♣

Let $\mathcal{F}$ be an oriented surface. We recall that a framed link in $\mathcal{F} \times (0,1)$ is an embedded annulus with an oriented core. Let L be a framed link with k components with a fixed numbering. Let $S^1 \times [0,1]$ be an annulus with an oriented core. For diagrams $D_1, \dots, D_k$ in $S^1 \times [0,1]$ we define the *decoration of L with $D_1, \dots, D_k$* as the link

$$(L; D_1, \dots, D_k)$$

which derives from L by replacing each annulus L_i by the annulus with the diagram D_i such that the orientations of the cores match. Each component of each D_i has a small blackboard neighbourhood in the annulus, and this turns the decorated link $(L; D_1, \dots, D_k)$ into a framed link.

The linear extension of decorating satisfies the skein relations, and thus the decoration of a framed link with elements of the skein of the annulus C gives a well defined element of the skein $\mathcal{S}(\mathcal{F} \times (0,1))$.

Lemma 3.6.2 *We have*

$$\begin{aligned}\chi(L; Q_{\lambda^\vee}, \dots, Q_{\mu^\vee}) &= \chi(L; Q_\lambda, \dots, Q_\mu)_{s \mapsto -s^{-1}} \\ &= \chi(L; Q_\lambda, \dots, Q_\mu)_{x \mapsto -x,\ v \mapsto -v,\ s \mapsto s^{-1}}\end{aligned}$$

for any framed link L and any Young diagrams $\lambda, \dots, \eta$.

Proof We recall from subsection 2.4.1 the map γ from H_n to H_n that simply replaces s by $-s^{-1}$. We similarly define γ in other skeins, e.g. in the skein of the annulus or the skein of the plane. γ permutes the idempotents derived from the quasi-idempotents a_n and b_n. Hence $\gamma(Q_{d_n}) = Q_{c_n}$. We have $(\eta\mu)^\vee = \eta^\vee \mu^\vee$ by lemma 1.2.2. Using the ring homomorphism $\mathcal{Y} \to C_+$ from theorem 3.5.6, the fact that $\mathcal{Y}$ is generated by column diagrams, and $\gamma(Q_{d_n}) = Q_{c_n}$, we deduce that

$$\gamma(Q_\lambda) = Q_{\lambda^\vee}$$

for any Young diagram λ. Hence

$$\gamma(L; Q_\lambda, \dots, Q_\eta) = (L; Q_{\lambda^\vee}, \dots, Q_{\eta^\vee})$$

in the skein of the plane $\mathbb{R}^2$.

The second claim follows by repeating the same argument with the map τ from subsections 2.1.1 and 2.4.1 instead of γ. ♣

The hook length $\mathrm{hl}(\lambda)$ of a Young diagram λ is defined as the maximum among the hook lengths of its cells. We have $\mathrm{hl}(\lambda) = \lambda_1 + l(\lambda) - 1$.

Lemma 3.6.3 *The element Q_λ of the skein of the annulus can be written as a linear combination of diagrams $\sum_D t_D D$ where the scalars t_D are fractions whose denominators are products of terms $(s^i - s^{-i})$ for $1 \le i \le \mathrm{hl}(\lambda)$.*

Proof We have
$$Q_\lambda = \det(h_{\lambda_i+j-i})_{1\le i,j\le l(\lambda)}$$
by theorem 3.5.5. We have by definition that $h_k = Q_{d_k} = (1/\alpha_k)\hat{a}_k$, and we know by lemma 2.4.2 that the denominator of α_k is a product of terms $(s^i - s^{-i})$, $1 \le i \le k$. The maximum of the integers $\lambda_i + j - i$ with $1 \le i, j \le l(\lambda)$ is equal to $\lambda_1 + l(\lambda) - 1$ which is the hook length of λ. ♣

Chapter 4

The decorated Hopf link

4.1 The Hopf link

We consider the Hopf link with linking number 1 as depicted in figure 4.1. Let a and b be any elements of the skein of the annulus. We denote by $\langle a, b\rangle$ the Homfly polynomial of the Hopf link with decorations a and b on its components. We have $\langle a, b\rangle = \langle b, a\rangle$, and we abbreviate $\langle Q_\lambda, Q_\mu\rangle$ by $\langle \lambda, \mu\rangle$ for any Young diagrams λ and μ.

The scalars we are looking at are rational functions in x, v and s to ensure that the idempotents $(1/\alpha_\lambda)e_\lambda$ of the Hecke algebra exist.

In order to simplify the calculations of the Homfly polynomial of the decorated Hopf link, we often specialize x to 1. The initial value of the Homfly polynomial may be recovered from this specialized value as described in the next lemma. This is similar to the proof of theorem 3.5.3.

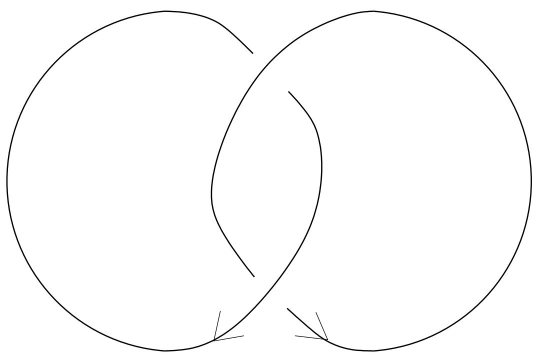


Figure 4.1: The Hopf link.

Lemma 4.1.1 *Let β be an m-braid and γ be an n-braid. Then*

$$\langle x^{-\mathrm{wr}(\beta)}\hat{\beta}, x^{-\mathrm{wr}(\gamma)}\hat{\gamma}\rangle = x^{2nm}\langle\hat{\beta},\hat{\gamma}\rangle_{x=1}.$$

Proof We get a variant of the Homfly polynomial by setting

$$\chi^{\mathrm{u}}(D) = (xv^{-1})^{-\mathrm{wr}(D)}\chi(D)$$

for any link diagram D. This Homfly polynomial χ^{u} satisfies the skein relation $v^{-1}\sigma_1 - v\sigma_1^{-1} = (s - s^{-1})\,\mathrm{id}$ and a disjoint unknot can be removed at the expense of the scalar $(v^{-1} - v)/(s - s^{-1})$. We see that $\chi^{\mathrm{u}}(D)$ does not involve x for any diagram D. Hence,

$$\chi(D) = x^{\mathrm{wr}(D)}\chi(D)_{x=1}$$

for any link diagram D. The writhe of the Hopf link with decorations $\hat{\beta}$ and $\hat{\gamma}$ is equal to $\mathrm{wr}(\beta) + \mathrm{wr}(\gamma) + 2nm$ because the concept of decorations requires the orientation of the braids to be parallel to the orientation of the core of the annulus. Hence

$$\begin{aligned}\langle x^{-\mathrm{wr}(\beta)}\hat{\beta}, x^{-\mathrm{wr}(\gamma)}\hat{\gamma}\rangle &= x^{-(\mathrm{wr}(\beta)+\mathrm{wr}(\gamma))}\langle\hat{\beta},\hat{\gamma}\rangle \\ &= x^{-(\mathrm{wr}(\beta)+\mathrm{wr}(\gamma))}x^{\mathrm{wr}(\beta)+\mathrm{wr}(\gamma)+2nm}\langle\hat{\beta},\hat{\gamma}\rangle_{x=1} \\ &= x^{2nm}\langle\hat{\beta},\hat{\gamma}\rangle_{x=1}.\end{aligned}$$

♣

Corollary 4.1.2 *Let λ and μ be Young diagrams. Then*

$$\langle\lambda,\mu\rangle = x^{2|\lambda|\,|\mu|}\langle\lambda,\mu\rangle_{x=1}.$$

Proof The closures of the quasi-idempotents a_n and b_n of H_n are sums of terms $fx^{-wr(\beta)}\hat{\beta}$ where f is a power of $\pm s$ and β is an n-braid. Since the normalized idempotents differ from the quasi-idempotents by a rational function in s, their closures Q_{d_n} and Q_{c_n} are sums of terms $fx^{-wr(\beta)}\hat{\beta}$ where f is a rational function in s and β is an n-braid. By the relation of Q_λ with Schur functions we can write Q_λ as a homogeneous polynomial in Q_{d_i} (or Q_{c_j}) of degree $|\lambda|$. Hence, Q_λ is a sum of terms $fx^{-wr(\beta)}\hat{\beta}$ where f is a rational function in s and β is a braid on $|\lambda|$ strings. We can now apply lemma 4.1.1 to Q_λ and Q_μ for any Young diagrams λ and μ. We get $\langle\lambda,\mu\rangle = x^{2|\lambda|\,|\mu|}\langle\lambda,\mu\rangle_{x=1}$. ♣

Lemma 4.1.3 *We have*

$$\langle Q_\lambda, b\rangle\langle Q_\lambda, c\rangle = \langle Q_\lambda\rangle\,\langle Q_\lambda, bc\rangle.$$

for any elements b and c of C_+ and any Young diagram λ.

Proof Let λ be any Young diagram and b and c be any elements of C_+. We denote the number of cells of λ by n. The element Q_λ of C_+ is the closure of the idempotent derived from the quasi-idempotent e_λ of the Hecke algebra H_n. The product of e_λ with any central element of H_n is a scalar multiple of e_λ by equation (2.4.2). The identity braid on n strings encircled by a loop decorated with b (resp. c) is obviously a central element of H_n, and we denote it by b' (resp. c'). The closure of $b'c'y_\lambda$ is equal to the Hopf link decorated with Q_λ on one component and bc on the other. The closure of $b'y_\lambda$ (resp. $c'y_\lambda$) is equal to the Hopf link with decorations b (resp. c) and Q_λ.

There exists a scalar t such that $b'e_\lambda = te_\lambda$. By closing the elements on both sides of this equation, we see that $t = \langle Q_\lambda, b\rangle / \langle Q_\lambda\rangle$. We know by lemma 2.4.6 that $\langle Q_\lambda\rangle$ is non-zero. Similarly, $c'e_\lambda = \langle Q_\lambda, c\rangle / \langle Q_\lambda\rangle\, e_\lambda$. Hence,

$$\frac{1}{\alpha_\lambda} b'c'e_\lambda = \frac{\langle Q_\lambda, b\rangle\langle Q_\lambda, c\rangle}{\langle Q_\lambda\rangle^2 \alpha_\lambda} e_\lambda.$$

Taking the closure and Homfly polynomial in the above equation, we get

$$\langle Q_\lambda, bc\rangle = \frac{\langle Q_\lambda, b\rangle\langle Q_\lambda, c\rangle}{\langle Q_\lambda\rangle^2} \langle Q_\lambda\rangle$$

and therefore $\langle Q_\lambda, b\rangle\langle Q_\lambda, c\rangle = \langle Q_\lambda\rangle\, \langle Q_\lambda, bc\rangle$. ♣

We immediately deduce from lemma 4.1.3 that

Corollary 4.1.4 *The linear map $\eta \mapsto \langle\lambda, \eta\rangle / \langle\lambda\rangle$ from the ring of Young diagrams to the ring of rational functions in x, v and s is a ring homomorphism for any Young diagram λ.*

Since any Young diagram η can be written as a polynomial in column diagrams, we only need to know the values of $\langle\lambda, c_i\rangle$ for sufficiently many integer $i \geq 0$ in order to compute $\langle\lambda, \eta\rangle$. Hence, it is useful to define a formal power series

$$E_\lambda(X) = \frac{1}{\langle\lambda\rangle} \sum_{r\geq 0} \langle\lambda, c_r\rangle X^r$$

for any Young diagram λ.

For any formal power series $P(X)$ whose coefficients are rational functions in x, v and s we define $s_\lambda(P(X))$ as first expressing the Schur function s_λ as a polynomial in the elementary symmetric functions $e_0, e_1, \ldots$ and then replacing any e_j by the coefficient of X^j in $P(X)$. We recall that $s_{c_r} = e_r$ for any $r \geq 0$. Note that this is well defined because the elementary symmetric functions are algebraically independent in the ring of symmetric functions.

We state our above considerations in the following lemma.

Lemma 4.1.5 *We have*

$$s_\eta(E_\lambda(X)) = \frac{1}{\langle\lambda\rangle}\langle\lambda,\eta\rangle$$

for any Young diagrams λ and η.

From corollary 4.1.2 we see how to recover $E_\lambda(X)$ from the power series $E_\lambda(X)_{x=1}$ where we substituted x by 1 in every coefficient of the power series. We simply replace X by $x^{2|\lambda|}X$ in $E_\lambda(X)_{x=1}$. Equivalently, we have

$$E_\lambda(x^{-2|\lambda|}X) = E_\lambda(X)_{x=1}.$$

We define

$$H_\lambda(X) = \frac{1}{\langle\lambda\rangle}\sum_{r\geq 0}\langle\lambda, d_r\rangle X^r$$

for any Young diagram λ. The next lemma shows how $E_\lambda(X)$ and $H_\lambda(X)$ are related.

Lemma 4.1.6 *We have*

$$E_\lambda(X)H_\lambda(-X) = 1$$

for any Young diagram λ.

Proof We have by equation (1.2.3) that

$$\left(\sum_{r\geq 0} c_r X^r\right)\left(\sum_{k\geq 0} d_k(-X)^k\right) = 1$$

in the algebra of Young diagrams. By corollary 4.1.4 we have that the map $a \mapsto \langle\lambda, a\rangle / \langle\lambda\rangle$ is an algebra homomorphism from the algebra of Young diagrams to the scalars for any Young diagram λ. Hence

$$\left(\sum_{r\geq 0}\frac{1}{\langle\lambda\rangle}\langle\lambda, c_r\rangle X^r\right)\left(\sum_{k\geq 0}\frac{1}{\langle\lambda\rangle}\langle\lambda, d_k\rangle(-X)^k\right) = 1.$$

♣

The following lemma explains the relation between the power series $E_\lambda(X)$ and the power series $E_{\lambda^\vee}(X)$ for the transposed Young diagram $\lambda^\vee$.

Lemma 4.1.7 *We have*

$$E_{\lambda^\vee}(-X)E_\lambda(X)_{s\mapsto -s^{-1}} = 1$$

for any Young diagram λ.

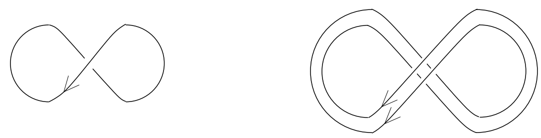


Figure 4.2: The unknot with framing 1 and its 2-parallel.

Proof We have by lemma 3.6.2 that $\langle\lambda\rangle_{s\mapsto -s^{-1}} = \langle\lambda^\vee\rangle$ and in particular we have that $\langle\lambda, c_k\rangle_{s\mapsto -s^{-1}} = \langle\lambda^\vee, d_k\rangle$. Hence

$$\begin{aligned} E_\lambda(X)_{s\mapsto -s^{-1}} &= \frac{1}{\langle\lambda^\vee\rangle}\sum_{k\geq 0}\langle\lambda^\vee, d_k\rangle X^k \\ &= H_{\lambda^\vee}(X) \\ &= E_{\lambda^\vee}^{-1}(-X). \end{aligned}$$

♣

4.2 Hopf link decorated with columns and rows

We now compute $E_{c_k}(X)$ for any integer $k \geq 0$. To do this, we start with a surprisingly simple formula for $\langle c_k, d_j\rangle$.

Lemma 4.2.1 *We have*

$$\langle c_k, d_j\rangle = \langle c_k\rangle \langle d_j\rangle\, x^{2jk}\frac{v^{-1}(s^{2j} - s^{2(j-k)} + s^{-2k}) - v}{v^{-1} - v}$$

for any integers $k \geq 0$ and $j \geq 0$.

Proof We claim that

$$\langle c_k, d_j\rangle = \langle c_k\rangle \langle d_j\rangle\, \frac{v^{-1}(s^{2j} - s^{2(j-k)} + s^{-2k}) - v}{v^{-1} - v}$$

when we set $x = 1$. The lemma then follows from the above claim because $\langle c_k, d_j\rangle = x^{2jk}\langle c_k, d_j\rangle_{x=1}$ by corollary 4.1.2. We shall prove our claim by expressing the Homfly polynomial of a certain decorated link in two different ways and comparing the results. The link in question is the 2-parallel of the unknot with framing 1 as depicted in figure 4.2 decorated with Q_{c_j} on one component and Q_{d_k} on the other component. We denote its Homfly polynomial by R.

Already in the Hecke algebra H_i, the product of the positive curl on i strings and any quasi-idempotent e_λ, $|\lambda| = i$, is a scalar multiple of e_λ. The scalar was calculated using skein theory in theorem 17 in [2] as $f(\lambda) = x^{|\lambda|^2} v^{-|\lambda|} s^{n_\lambda}$ where n_λ is twice the sum of the contents of all cells of λ. Since we specialize x to 1, we have

$$f(\lambda) = v^{-|\lambda|} s^{n_\lambda}.$$

By removing the two curls in figure 4.2 after the decoration we get

$$R = f(c_k) f(d_j) \langle c_k, d_j \rangle. \tag{4.2.1}$$

The other way to calculate R is to consider first the product of Q_{c_k} and Q_{d_j} in the skein of the annulus. R is the Homfly polynomial of the unknot with framing 1 decorated by the product of Q_{c_k} and Q_{d_j}. Since the Q_λ multiply like Schur functions by theorem 3.5.6, we get $Q_{c_k} Q_{d_j} = Q_{\mu_{k,j+1}} + Q_{\mu_{k+1,j}}$ where $\mu_{a,b}$ is the hook Young diagram with $(a+b-1)$ cells of which a are in the first column and b are in the first row. Hence

$$R = f(\mu_{k,j+1}) \langle \mu_{k,j+1} \rangle + f(\mu_{k+1,j}) \langle \mu_{k+1,j} \rangle. \tag{4.2.2}$$

From the above formula for $f(\lambda)$ we deduce

$$f(c_{k+1}) = v^{-1} s^{-2k} f(c_k),\ f(d_{j+1}) = v^{-1} s^{2j} f(d_j),\ f(\mu_{k,j}) = v f(c_k) f(d_j).$$

We have by lemma 3.6.1

$$\begin{aligned}
\langle c_{k+1} \rangle &= \frac{v^{-1} s^{-k} - v s^k}{s^{k+1} - s^{-k-1}} \langle c_k \rangle, \\
\langle d_{j+1} \rangle &= \frac{v^{-1} s^{j} - v s^{-j}}{s^{j+1} - s^{-j-1}} \langle d_j \rangle, \\
\langle \mu_{k,j} \rangle &= \frac{(s^j - s^{-j})(s^k - s^{-k})}{(v^{-1} - v)(s^{k+j-1} - s^{-k-j+1})} \langle c_k \rangle \langle d_j \rangle.
\end{aligned}$$

By these relations we get from equation 4.2.2 that

$$\begin{aligned}
R &= f(\mu_{k,j+1}) \langle \mu_{k,j+1} \rangle + f(\mu_{k+1,j}) \langle \mu_{k+1,j} \rangle \\
&= v f(c_k) v^{-1} s^{2j} f(d_j) \frac{(s^k - s^{-k})(v^{-1} s^j - v s^{-j})}{(v^{-1} - v)(s^{k+j} - s^{-k-j})} \langle c_k \rangle \langle d_j \rangle \\
&\quad + s^{-2k} f(c_k) f(d_j) \frac{(s^j - s^{-j})(v^{-1} s^{-k} - v s^k)}{(v^{-1} - v)(s^{k+j} - s^{-k-j})} \langle c_k \rangle \langle d_j \rangle \\
&= \frac{v^{-1}(s^{2j} - s^{2(j-k)} + s^{-2k}) - v}{v^{-1} - v} f(c_k) f(d_j) \langle c_k \rangle \langle d_j \rangle.
\end{aligned} \tag{4.2.3}$$

Since $f(c_k)$ and $f(d_j)$ are non-zero, we deduce from equations (4.2.1) and (4.2.3) that

$$\langle c_k, d_j \rangle = \langle c_k \rangle \langle d_j \rangle \frac{v^{-1}(s^{2j} - s^{2(j-k)} + s^{-2k}) - v}{v^{-1} - v}$$

when we set $x = 1$. ♣

Corollary 4.2.2 *We have*

$$H_{c_k}(X) = \frac{1 - v^{-1}s^{-2k+1}x^{2k}X}{1 - v^{-1}sx^{2k}X} H_\emptyset(x^{2k}X)$$

for any integer $k \geq 0$.

Proof As usual, it is sufficient to work with the substitution $x = 1$. We have to show that

$$(1 - v^{-1}sX)\frac{1}{\langle c_k\rangle}\sum_{j\geq 0}\langle c_k, d_j\rangle_{x=1}X^j = (1 - v^{-1}s^{-2k+1}X)\sum_{j\geq 0}\langle d_j\rangle X^j.$$

The constant terms of the power series in the above equation are equal to 1. In order that the coefficient of X^j on the left hand side agrees with the coefficient of X^j on the right hand side, we have to show that

$$\frac{1}{\langle c_k\rangle}\langle c_k, d_j\rangle - v^{-1}s\frac{1}{\langle c_k\rangle}\langle c_k, d_{j-1}\rangle = \langle d_j\rangle - v^{-1}s^{-2k+1}\langle d_{j-1}\rangle \tag{4.2.4}$$

after the substitution $x = 1$. By lemma 4.2.1 we can write the left hand side of equation (4.2.4) as

$$\langle d_j\rangle \frac{v^{-1}(s^{2j} - s^{2(j-k)} + s^{-2k}) - v}{v^{-1} - v} - v^{-1}s\langle d_{j-1}\rangle \frac{v^{-1}(s^{2(j-1)} - s^{2(j-1-k)} + s^{-2k}) - v}{v^{-1} - v}.$$

Because

$$\langle d_j\rangle = \langle d_{j-1}\rangle \frac{v^{-1}s^{j-1} - vs^{-j+1}}{s^j - s^{-j}},$$

the left hand side of equation (4.2.4) can be transformed further into

$$\left(\frac{(v^{-1}s^{j-1} - vs^{-j+1})(v^{-1}(s^{2j} - s^{2(j-k)} + s^{-2k}) - v)}{(s^j - s^{-j})(v^{-1} - v)}\right.$$
$$\left. - v^{-1}s\frac{v^{-1}(s^{2(j-1)} - s^{2(j-1-k)} + s^{-2k}) - v}{v^{-1} - v}\right)\langle d_{j-1}\rangle. \tag{4.2.5}$$

The right hand side of equation (4.2.4) is equal to

$$\left(\frac{v^{-1}s^{j-1} - vs^{-j+1}}{s^j - s^{-j}} - v^{-1}s^{-2k+1}\right)\langle d_{j-1}\rangle. \tag{4.2.6}$$

It is straightforward to confirm the equality of the terms in equations (4.2.5) and (4.2.6), and thus equation (4.2.4) is proven. ♣

As an immediate consequence of corollary 4.2.2 and lemma 4.1.6 we get

Corollary 4.2.3 *We have*

$$E_{c_k}(X) = \frac{1 + v^{-1}sx^{2k}X}{1 + v^{-1}s^{-2k+1}x^{2k}X} E_\emptyset(x^{2k}X)$$

for any integer $k \geq 0$.

4.3 Hopf link decorated with any Young diagrams

We shall from now on use symmetric functions as well. On the first sight this seems to be superfluous because the ring of symmetric functions is isomorphic to the ring of Young diagrams via the Schur functions. The crucial bonus of the symmetric functions is that under certain circumstances a ring homomorphism ρ from the symmetric functions to a ring R factors through the symmetric functions in some finitely many variables. A necessary condition for this factorization is that ρ maps the i-th elementary symmetric function e_i to zero for all i large enough. This condition is also sufficient in an appropriate extension of R (if it exists). All one has to do in the case that $\rho(e_i) = 0$ for all $i > i_0$ is to solve the equation $\sum_{i=0}^{i_0} \rho(e_i)t^i = \prod_{i=1}^{i_0}(1 + x_i t)$ for $x_1, \dots, x_{i_0}$ in R where t is a variable.

If we make the substitution $v = s^{-N}$ for some integer $N \geq 0$ then $E_\lambda(X)$ becomes a polynomial in X of degree N. In fact, we shall be able in lemma 4.3.3 to solve the above equation without extending the ring of rational functions in x and s.

In order to calculate the Homfly polynomial of the Hopf link decorated with Q_λ and Q_μ we first have to improve our understanding of Schur functions by proving lemma 4.3.1.

Definition Given a Young diagram λ and elements $r_1, \dots, r_N$ in a commutative ring R, $N \geq l(\lambda)$, we denote by $s_\lambda(r_1, \dots, r_N)$ the element of R that derives from the Schur function s_λ in N variables $x_1, \dots, x_N$ by substituting x_i by r_i for $i = 1, \dots, N$. Equivalently we shall use the notation '$s_\lambda(r_i)$ where $i = 1, \dots, N$'.

Lemma 4.3.1 *Let N be a positive integer, and let λ and μ be Young diagrams with at most N rows. Then*

$$s_\lambda(q^{\mu_i+N-i})s_\mu(q^{N-i}) = s_\mu(q^{\lambda_i+N-i})s_\lambda(q^{N-i})$$

where q is a variable and $i = 1, \dots, N$.

Proof The Schur polynomial s_λ is by definition the quotient of two $(N \times N)$-determinants in variables $x_1, \dots, x_N$,

$$s_\lambda(x_1, \dots, x_N) = \frac{a_{\lambda+\delta}(x_1, \dots, x_N)}{a_\delta(x_1, \dots, x_N)} = \frac{\det(x_i^{\lambda_j+N-j})}{\det(x_i^{N-j})}$$

where $i = 1, \dots, N$ and $j = 1, \dots, N$. We thus get

$$\begin{aligned} s_\lambda(q^{\mu_i+N-i}) &= \frac{a_{\lambda+\delta}(q^{\mu_i+N-i})}{a_\delta(q^{\mu_i+N-i})} \\ &= \frac{\det(q^{(\mu_i+N-i)(\lambda_j+N-j)})}{\det(q^{(\mu_i+N-i)(N-j)})} \end{aligned}$$

where $i = 1, \ldots, N$ and $j = 1, \ldots, N$. Note that the denominator is different from zero. Since the determinant of a matrix is invariant under transposition we can interchange i and j in the determinant of the $N \times N$-matrix in the denominator and get

$$\begin{aligned} s_\lambda(q^{\mu_i+N-i}) &= \frac{\det(q^{(\mu_i+N-i)(\lambda_j+N-j)})}{\det(q^{(N-i)(\mu_j+N-j)})} \\ &= \frac{\det(q^{(\mu_i+N-i)(\lambda_j+N-j)})}{a_{\mu+\delta}(q^{N-i})}. \end{aligned}$$

We thus get

$$s_\lambda(q^{\mu_i+N-i})a_{\mu+\delta}(q^{N-i}) = \det(q^{(\mu_i+N-i)(\lambda_j+N-j)}).$$

Dividing both sides by $a_\delta(q^{N-i})$ we get

$$s_\lambda(q^{\mu_i+N-i})s_\mu(q^{N-i}) = \frac{\det(q^{(\mu_i+N-i)(\lambda_j+N-j)})}{\det(q^{(N-i)(N-j)})}.$$

By interchanging λ and μ we derive

$$s_\mu(q^{\lambda_i+N-i})s_\lambda(q^{N-i}) = \frac{\det(q^{(\lambda_i+N-i)(\mu_j+N-j)})}{\det(q^{(N-i)(N-j)})}.$$

Using the invariance of the determinant under transposition we deduce from the two above equations that

$$s_\lambda(q^{\mu_i+N-i})s_\mu(q^{N-i}) = s_\mu(q^{\lambda_i+N-i})s_\lambda(q^{N-i}).$$

♣

Corollary 4.3.2 *Let N be a positive integer, and let λ and μ be Young diagrams with at most N rows. Then*

$$s_\lambda(\alpha q^{\mu_i+N-i})s_\mu(\alpha q^{N-i}) = s_\mu(\alpha q^{\lambda_i+N-i})s_\lambda(\alpha q^{N-i})$$

where α and q are variables, and $i = 1, \ldots, N$.

Proof The Schur polynomial s_λ is a homogeneous polynomial of degree $|\lambda|$. Hence

$$s_\lambda(\alpha x_1, \ldots, \alpha x_N) = \alpha^{|\lambda|} s_\lambda(x_1, \ldots, x_N).$$

Hence

$$s_\lambda(\alpha q^{\mu_i+N-i})s_\mu(\alpha q^{N-i}) = \alpha^{|\lambda|+|\mu|} s_\lambda(q^{\mu_i+N-i})s_\mu(q^{N-i}), \tag{4.3.7}$$

and

$$s_\mu(\alpha q^{\lambda_i+N-i}) s_\lambda(\alpha q^{N-i}) = \alpha^{|\lambda|+|\mu|} s_\mu(q^{\lambda_i+N-i}) s_\lambda(q^{N-i}). \tag{4.3.8}$$

Lemma 4.3.1 implies that the right hand side of equation (4.3.7) agrees with the right hand side of (4.3.8). Hence the left hand side of equation (4.3.7) agrees with the left hand side of equation (4.3.8) which is our claim. ♣

For $\langle\lambda\rangle$ and $\langle\lambda,\mu\rangle$ we shall denote by an additional subscript N the substitution $v = s^{-N}$ where N is a positive integer, i.e. we write $\langle\lambda\rangle_N$ and $\langle\lambda,\mu\rangle_N$. We denote by $E_\lambda^N(X)$ the substitution $v = s^{-N}$ in $E_\lambda(X)$. Note that $E_\lambda^N(X)$ is only defined if $N \geq l(\lambda)$ in order that $\langle\lambda\rangle_N$ is different from zero.

Lemma 4.3.3 *Let λ be a Young diagram and let $N \geq l(\lambda)$ be an integer. Then*

$$E_\lambda^N(X) = \prod_{i=1}^{N}(1 + s^{N+2\lambda_i-2i+1} x^{2|\lambda|} X).$$

Proof We consider a Young diagram λ and an integer $N \geq l(\lambda)$. An equivalent formulation of our claim is that

$$E_\lambda^N(X)_{x=1} = \prod_{i=1}^{N}(1 + s^{-N+1} q^{\lambda_i+N-i} X)$$

where $q = s^2$. For the rest of the proof we always set $x = 1$ without indicating this substitution by the usual subscript.

By equation (3.6.6) we have

$$E_\emptyset(X) = \sum_{r\geq 0} \langle c_r\rangle X^r = \prod_{k=0}^{\infty} \frac{1 + vs^{2k+1}X}{1 + v^{-1}s^{2k+1}X}.$$

The substitution $v = s^{-N}$ reduces this to the finite product

$$\begin{aligned} E_\emptyset^N(X) &= \prod_{k=0}^{N-1}(1 + s^{-N+2k+1}X) \\ &= \prod_{i=1}^{N}(1 + s^{N-2i+1}X). \end{aligned} \tag{4.3.9}$$

Note that this is our claim in the case $\lambda = \emptyset$.

Let k be an integer, $k \leq N$. By corollary 4.2.3 we have

$$E_{c_k}(X) = \frac{1 + v^{-1}sX}{1 + v^{-1}s^{-2k+1}X} E_\emptyset(X).$$

Substituting $v = s^{-N}$ in the above equation and using equation (4.3.9) we get

$$\begin{aligned} E_{c_k}^N(X) &= \frac{1+s^{N+1}X}{1+s^{N-2k+1}X}\prod_{i=1}^{N}(1+s^{N-2i+1}X) \\ &= \prod_{i=1}^{k}(1+s^{N-2i+3}X)\prod_{i=k+1}^{N}(1+s^{N-2i+1}X) \end{aligned}$$

which is our claim in the case $\lambda = c_k$.

By lemma 4.1.5 we have $s_\lambda(E_{c_r}(X)) = \langle\lambda, c_r\rangle / \langle c_r\rangle$ for any $r \geq 0$. Hence

$$\frac{1}{\langle\lambda\rangle}\langle\lambda, c_r\rangle = \frac{s_{c_r}(E_\emptyset(X))}{s_\lambda(E_\emptyset(X))}s_\lambda(E_{c_r}(X)).$$

Restricting to $0 \leq r \leq N$ and substituting $v = s^{-N}$ we get

$$\begin{aligned} \frac{1}{\langle\lambda\rangle_N}\langle\lambda, c_r\rangle_N &= \frac{s_{c_r}(E_\emptyset^N(X))}{s_\lambda(E_\emptyset^N(X))}s_\lambda(E_{c_r}^N(X)) \\ &= \frac{s_{c_r}(s^{-N+1}q^{N-i})}{s_\lambda(s^{-N+1}q^{N-i})}s_\lambda(s^{-N+1}q^{(c_r)_i+N-i}) \end{aligned}$$

where $q = s^2$, and $i = 1, \ldots, N$. By Corollary 4.3.2 with $\mu = c_r$ and α specialized to s^{-N+1} we deduce from the above equation that

$$\frac{1}{\langle\lambda\rangle_N}\langle\lambda, c_r\rangle_N = s_{c_r}(s^{-N+1}q^{\lambda_i+N-i}).$$

In particular, we deduce from the above equation that $\langle\lambda, c_r\rangle = 0$ for all $r \geq N+1$ because the r-th elementary symmetric function s_{c_r} becomes zero when only N of the infinitely many variables are substituted by non-zero terms. We thus get

$$\begin{aligned} E_\lambda^N(X) &= \sum_{r=0}^{N}\frac{1}{\langle\lambda\rangle_N}\langle\lambda, c_r\rangle_N X^r \\ &= \sum_{r=0}^{N}s_{c_r}(s^{-N+1}q^{\lambda_i+N-i})X^r \\ &= \prod_{i=1}^{N}(1+s^{-N+1}q^{\lambda_i+N-i}X) \end{aligned}$$

because s_{c_r} is the r-th elementary symmetric function. ♣

We now deduce a formula for $E_\lambda(X)$ from the formula for $E_\lambda^N(X)$, $N \geq l(\lambda)$.

Theorem 4.3.4 *We have*

$$E_\lambda(X) = E_\emptyset(x^{2|\lambda|}X) \prod_{j=1}^{l(\lambda)} \frac{1+v^{-1}s^{2\lambda_j-2j+1}x^{2|\lambda|}X}{1+v^{-1}s^{-2j+1}x^{2|\lambda|}X}$$

for any Young diagram λ.

Proof For any integer $N \geq l(\lambda)$ we have that by lemma 4.3.3

$$\begin{aligned} E_\lambda^N(X) &= \prod_{i=1}^{N}(1+s^{N+2\lambda_i-2i+1}x^{2|\lambda|}X) \\ &= \prod_{i=1}^{l(\lambda)}(1+s^{N+2\lambda_i-2i+1}x^{2|\lambda|}X) \prod_{i=l(\lambda)+1}^{N}(1+s^{N-2i+1}x^{2|\lambda|}X). \end{aligned}$$

In particular, for λ equal to the empty Young diagram,

$$E_\emptyset^N(X) = \prod_{i=1}^{N}(1+s^{N-2i+1}X),$$

which we had obtained earlier, too. Combining the above expressions for $E_\lambda^N(X)$ and $E_\emptyset^N(X)$ we get

$$E_\lambda^N(X) = E_\emptyset^N(x^{2|\lambda|}X) \prod_{i=1}^{l(\lambda)} \frac{1+s^{N+2\lambda_i-2i+1}x^{2|\lambda|}X}{1+s^{N-2i+1}x^{2|\lambda|}X}.$$

This means that the power series $E_\lambda(X)$ and

$$E_\emptyset(x^{2|\lambda|}X) \prod_{j=1}^{l(\lambda)} \frac{1+v^{-1}s^{2\lambda_j-2j+1}x^{2|\lambda|}X}{1+v^{-1}s^{-2j+1}x^{2|\lambda|}X}. \tag{4.3.10}$$

are equal for any substitution $v = s^{-N}$ provided that $N \geq l(\lambda)$.

The equality of $E_\lambda(X)$ and the power series in (4.3.10) follows now from the observation that if there exists an integer $n_0 \geq 1$ such that two rational functions $r_1(v,s)$ and $r_2(v,s)$ in v and s are equal for any substitution $v = s^{-n}$, $n \geq n_0$, then $r_1(v,s) = r_2(v,s)$.

Equivalently, let $r(v,s)$ be a rational function in v and s that becomes zero for any substitution $v = s^{-n}$, $n \geq n_0 \geq 1$. In order to show that $r(v,s) = 0$ we write the rational function $r(v,s)$ as the quotient of two polynomials in v and s, say $r(v,s) = p(v,s)/q(v,s)$. Now $p(1,s)$ is a polynomial in s. For any n-th root of unity ζ we have $p(1,\zeta) = 0$ provided that $n \geq n_0$. The only polynomial that has infinitely many roots is the zero polynomial. Hence $p(1,s) = 0$. Hence

$(v-1)$ is a factor of $p(v,s)$, i.e. there exists a polynomial $p_2(v,s)$ such that $p(v,s) = (v-1)p_2(v,s)$. Since $(v-1)$ is different from zero for any substitution $v = s^{-n}$, $n \geq 1$, we have that $p_2(s^{-n}, s) = 0$ for any $n \geq n_0$. Applying the whole argument again we find a polynomial $p_3(v,s)$ such that $p_2(v,s) = (v-1)p_3(v,s)$ and $p_3(s^{-n}, s) = 0$ for any $n \geq n_0$. Applying this argument again and again, we deduce that $(v-1)^k$ is a factor of $p_1(v,s)$ for any $k \geq 1$. Hence $p_1(v,s) = 0$, hence $r(v,s) = 0$.

We have thus proved that

$$E_\lambda(X) = E_\emptyset(x^{2|\lambda|}X) \prod_{j=1}^{l(\lambda)} \frac{1 + v^{-1}s^{2\lambda_j - 2j+1}x^{2|\lambda|}X}{1 + v^{-1}s^{-2j+1}x^{2|\lambda|}X}.$$

♣

By the definition of $E_\lambda(X)$ we have that the coefficient of X in $E_\lambda(X)$ is equal to the scalar $\tilde{q}_\lambda$ from corollary 3.5.4. In fact, we can verify this quickly as follows. We have

$$E_\lambda(X) = E_\emptyset(x^{2|\lambda|}X) \prod_{j=1}^{l(\lambda)} \frac{1 + v^{-1}s^{2\lambda_j - 2j+1}x^{2|\lambda|}X}{1 + v^{-1}s^{-2j+1}x^{2|\lambda|}X}$$

for any Young diagram λ. We have that

$$(1 + aX + \cdots)\frac{1 + bX + \cdots}{1 + cX + \cdots} = 1 + (a + b - c)X + \cdots$$

for any formal power series. We have

$$\begin{aligned}
E_\emptyset(x^{2|\lambda|}X) &= 1 + x^{2|\lambda|}\frac{v^{-1} - v}{s - s^{-1}}X + \cdots, \\
\prod_{j=1}^{l(\lambda)}(1 + v^{-1}s^{2\lambda_j - 2j+1}x^{2|\lambda|}X) &= 1 + \left(v^{-1}x^{2|\lambda|}\sum_{j=1}^{l(\lambda)} s^{2\lambda_j - 2j+1}\right) X + \cdots, \\
\prod_{j=1}^{l(\lambda)}(1 + v^{-1}s^{-2j+1}x^{2|\lambda|}X) &= 1 + \left(v^{-1}x^{2|\lambda|}\sum_{j=1}^{l(\lambda)} s^{-2j+1}\right) X + \cdots.
\end{aligned}$$

Hence, the coefficient of X in $E_\lambda(X)$ is equal to

$$x^{2|\lambda|}\frac{v^{-1} - v}{s - s^{-1}} + v^{-1}x^{2|\lambda|}\sum_{j=1}^{l(\lambda)} s^{2\lambda_j - 2j+1} - v^{-1}x^{2|\lambda|}\sum_{j=1}^{l(\lambda)} s^{-2j+1}$$

which is equal to

$$x^{2|\lambda|}\left(\frac{v^{-1} - v}{s - s^{-1}} + v^{-1}s\sum_{j=1}^{l(\lambda)}\left(s^{2(\lambda_j - j)} - s^{-2j}\right)\right)$$

which is equal to $\tilde{q}_\lambda$ given in corollary 3.5.4.

When we apply theorem 4.3.4 to the case $\lambda = c_k$ and compare the result with corollary 4.2.3 we note a number of cancellations in

$$\prod_{j=1}^{l(\lambda)} \frac{1 + v^{-1}s^{2\lambda_j - 2j+1}x^{2|\lambda|}X}{1 + v^{-1}s^{-2j+1}x^{2|\lambda|}X}.$$

We prove in the next lemma that the number of fractions after cancellations is given by the number of cells in the main diagonal of λ which we denote by $d(\lambda)$.

Lemma 4.3.5 *For any Young diagram λ we have*

$$\prod_{j=1}^{l(\lambda)} \frac{1 + v^{-1}s^{2\lambda_j - 2j+1}x^{2|\lambda|}X}{1 + v^{-1}s^{-2j+1}x^{2|\lambda|}X} = \prod_{i=1}^{d(\lambda)} \frac{1 + v^{-1}s^{2\lambda_i - 2i+1}x^{2|\lambda|}X}{1 + v^{-1}s^{-2\lambda_i^\vee + 2i-1}x^{2|\lambda|}X},$$

and the fractions at the right hand side admit no further cancellations.

Proof With $p = s^{-2}$ and $Y = v^{-1}sx^{2|\lambda|}X$ we have to show that

$$\prod_{j=1}^{l(\lambda)} \frac{1 + p^{j-\lambda_j}Y}{1 + p^jY} = \prod_{i=1}^{d(\lambda)} \frac{1 + p^{i-\lambda_i}Y}{1 + p^{\lambda_i^\vee - i+1}Y}. \tag{4.3.11}$$

Equivalently, we show that

$$\{j - \lambda_j \mid d(\lambda) + 1 \le j \le l(\lambda)\} \cup \{\lambda_i^\vee - i + 1 \mid 1 \le i \le d(\lambda)\}$$

is a decomposition of the set of integers $\{1, 2, \ldots, l(\lambda)\}$.

First, we note that the sequence $(j - \lambda_j)_{j\ge1}$ is strictly increasing and the sequence $(\lambda_i^\vee - i + 1)_{i\ge1}$ is strictly decreasing. This implies that the elements of each of the two sets on its own are pairwise different.

Second, we have $1 \le j - \lambda_j \le l(\lambda)$ for all $j = d(\lambda)+1, \ldots, l(\lambda)$, and we have $1 \le \lambda_i^\vee - i + 1 \le l(\lambda)$ for all $i = 1, \ldots, d(\lambda)$. Hence, it is sufficient to show that the above two sets are disjoint, i.e.

$$\lambda_j - j + \lambda_i^\vee - i + 1 \ne 0 \tag{4.3.12}$$

for all $i = 1, \ldots, d(\lambda)$ and $j = d(\lambda)+1, \ldots, l(\lambda)$.

In fact, equation (4.3.12) is true for all $i \ge 1$ and $j \ge 1$. To see this, we note that if the cell (j, i) lies in the Young diagram λ then equation (4.3.12) denotes the hook length of the cell (j, i) which is greater than zero. On the other hand, if the cell (j, i) does not lie in λ then $\lambda_j < i$ and $\lambda_i^\vee < j$, hence $\lambda_j - j + \lambda_i^\vee - i + 1 \le -1$. Hence, equation (4.3.12) is also true in the case that the cell (j, i) does not lie in λ. We have thus proved equation (4.3.11).

Finally, there are no cancellations in $\prod_{i=1}^{d(\lambda)}(1+p^{i-\lambda_i}Y)/(1+p^{\lambda_i^\vee - i+1}Y)$ because p occurs with non-positive exponents in the numerator, whereas p occurs with positive exponents in the denominator. ♣

The combination of theorem 4.3.4, equation (3.6.6) and lemma 4.3.5 immediately gives the following formula for $E_\lambda(X)$. This form has the benefit that we can make the substitution $v = s^{-n}$ for any integer $n \geq 0$.

Theorem 4.3.6 *We have*

$$E_\lambda(X) = \prod_{k=0}^{\infty} \frac{1 + vs^{2k+1}x^{2|\lambda|}X}{1 + v^{-1}s^{2k+1}x^{2|\lambda|}X} \prod_{i=1}^{d(\lambda)} \frac{1 + v^{-1}s^{2\lambda_i - 2i+1}x^{2|\lambda|}X}{1 + v^{-1}s^{-2\lambda_i^\vee + 2i-1}x^{2|\lambda|}X}$$

for any Young diagram λ.

By theorem 4.3.6 see that $E_\lambda(X)$ derives from $E_\emptyset(X)$ by replacing every factor $(1+v^{-1}s^{2\lambda_i-2i+1}x^{2|\lambda|}X)$ of the denominator of $E_\emptyset(X)$ by $(1+v^{-1}s^{-2\lambda_i^\vee+2i-1}x^{2|\lambda|}X)$ for $i = 1, \ldots, d(\lambda)$.

From theorem 4.3.6 we immediately deduce that $E_{\lambda^\vee}(-X)E_\lambda(X)_{s \mapsto -s^{-1}}$. This gives a second, independent proof of lemma 3.6.2.

4.4 Hopf link with specialization $v = s^{-N}$

Given Young diagrams λ and μ and an integer $N \geq \max(l(\lambda), l(\mu))$, we prove a simple formula for the value of $\langle \lambda, \mu \rangle$ after the substitution $v = s^{-N}$.

Lemma 4.4.1 *We have*

$$\langle \lambda, \mu \rangle_N = s^{(1-N)(|\lambda|+|\mu|)}x^{2|\lambda||\mu|}s_\lambda(q^{N-i})s_\mu(q^{\lambda_k+N-k})$$

where $i = 1, \ldots, N$ and $k = 1, \ldots, N$, and λ and μ are any Young diagrams, and N is an integer, $N \geq \max(l(\lambda), l(\mu))$ and $q = s^2$.

Proof By lemma 4.1.5 we have that

$$\langle \lambda, \mu \rangle = \langle \lambda \rangle \, s_\mu(E_\lambda(X)) = s_\lambda(E_\emptyset(X))s_\mu(E_\lambda(X)).$$

By lemma 4.3.3 we have that

$$s_\mu(E_\lambda^N(X)) = s_\mu(\alpha q^{\lambda_i+N-i}) = \alpha^{|\mu|}s_\mu(q^{\lambda_i+N-i})$$

where $\alpha = s^{1-N}x^{2|\lambda|}$, $q = s^2$, and $i = 1, \ldots, N$. Hence

$$\begin{aligned} \langle \lambda, \mu \rangle_N &= (s^{1-N})^{|\lambda|}s_\lambda(q^{N-i})(s^{1-N}x^{2|\lambda|})^{|\mu|}s_\mu(q^{\lambda_k+N-k}) \\ &= s^{(1-N)(|\lambda|+|\mu|)}x^{2|\lambda||\mu|}s_\lambda(q^{N-i})s_\mu(q^{\lambda_k+N-k}) \end{aligned}$$

where $i = 1, \ldots, N$ and $k = 1, \ldots, N$. ♣

By extracting the factor q^N from each of the variables in lemma 4.4.1 we deduce that

$$\begin{aligned}\langle \lambda, \mu \rangle_N &= s^{(1-N)(|\lambda|+|\mu|)} x^{2|\lambda||\mu|} s_\lambda(q^{N-i}) s_\mu(q^{\lambda_k+N-k}) \\ &= s^{(N+1)(|\lambda|+|\mu|)} x^{2|\lambda||\mu|} s_\lambda(q^{-i}) s_\mu(q^{\lambda_k-k}).\end{aligned}$$

It is tempting to conjecture that

$$\langle \lambda, \mu \rangle = (v^{-1}s)^{(|\lambda|+|\mu|)} x^{2|\lambda||\mu|} s_\lambda(q^{-i}) s_\mu(q^{\lambda_k-k}).$$

But this is not true in general because in the case λ equal to a single cell and μ equal to the empty Young diagram the left hand side is simply the Homfly polynomial of the unknot which is equal to $(v^{-1}-v)/(s-s^{-1})$ whereas the right hand side is the product of a power of v^{-1} and a Laurent polynomial in s.

We proceed to give an appealing formula for $\langle \lambda, \mu \rangle_N$.

Theorem 4.4.2 *We have*

$$\langle \lambda, \mu \rangle_N = s^{(1-N)(|\lambda|+|\mu|)} x^{2|\lambda||\mu|} \frac{\det(q^{(\lambda_{N-i+1}+i-1)(\mu_{N-j+1}+j-1)})}{\det(q^{(i-1)(j-1)})}$$

where $i = 1, \ldots, N$ *and* $j = 1, \ldots, N$, *and* λ *and* μ *are any Young diagrams, and* N *is an integer,* $N \geq \max(l(\lambda), l(\mu))$ *and* $q = s^2$.

Proof In the proof of lemma 4.3.1 we found that

$$s_\mu(q^{\lambda_i+N-i}) s_\lambda(q^{N-i}) = \frac{\det(q^{(\lambda_i+N-i)(\mu_j+N-j)})}{\det(q^{(N-i)(N-j)})}.$$

Hence we deduce from lemma 4.4.1 that

$$\langle \lambda, \mu \rangle_N = s^{(1-N)(|\lambda|+|\mu|)} x^{2|\lambda||\mu|} \frac{\det(q^{(\lambda_i+N-i)(\mu_j+N-j)})}{\det(q^{(N-i)(N-j)})}$$

where $i = 1, \ldots, N$ and $j = 1, \ldots, N$. Since the determinant of a matrix is unchanged under the simultaneous reversal of the order of all rows and of all columns, we finally get

$$\langle \lambda, \mu \rangle_N = s^{(1-N)(|\lambda|+|\mu|)} x^{2|\lambda||\mu|} \frac{\det(q^{(\lambda_{N-i+1}+i-1)(\mu_{N-j+1}+j-1)})}{\det(q^{(i-1)(j-1)})}.$$

where $i = 1, \ldots, N$ and $j = 1, \ldots, N$. ♣

The determinants appearing in theorem 4.4.2 are derived from the following infinite Vandermonde matrix

$$\begin{aligned} V &= (q^{(i-1)(j-1)})_{1\le i,j} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & q & q^2 & q^3 & q^4 & \cdots \\ 1 & q^2 & q^4 & q^6 & q^8 & \cdots \\ 1 & q^3 & q^6 & q^9 & q^{12} & \cdots \\ 1 & q^4 & q^8 & q^{12} & q^{16} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned}$$

The matrix $(q^{(i-1)(j-1)})_{1\le i,j\le N}$ is the upper left $(N\times N)$ submatrix of V. It derives from V by choosing rows i and columns j for $i=1,\ldots,N$ and $j=1,\ldots,N$. The matrix $(q^{(\lambda_{N-i+1}+i-1)(\mu_{N-j+1}+j-1)})_{1\le i,j\le N}$ derives from V by choosing the rows $i+\lambda_{N-i+1}$ and the columns $j+\mu_{N-j+1}$ for $i=1,\ldots,N$ and $j=1,\ldots,N$.

For example, with $\lambda=(2,1,1)$, $\mu=(2,2)$ and $N=3$ we get

$$\begin{aligned} \langle\lambda,\mu\rangle_3 &= s^{-16}x^{32}\det\begin{vmatrix} 1 & q^3 & q^4 \\ 1 & q^6 & q^8 \\ 1 & q^{12} & q^{16} \end{vmatrix} / \det\begin{vmatrix} 1 & 1 & 1 \\ 1 & q & q^2 \\ 1 & q^2 & q^4 \end{vmatrix} \\ &= x^{32}q(q^2+1)(q^2+q+1)(q^4+q^3+1). \end{aligned}$$

In the case $N=2$, i.e. $v=s^{-2}$, we have a simple formula for the Homfly polynomial of the Hopf link decorated with Q_{d_a} and Q_{d_b} for row diagrams d_a and d_b of length a respectively b. We set $[k]=(s^k-s^{-k})/(s-s^{-1})$ for any integer k.

Lemma 4.4.3 *For integers $a\ge 0$ and $b\ge 0$ we have*

$$\langle d_a,d_b\rangle_2=(x^2s)^{ab}[(a+1)(b+1)].$$

Proof By the above calculations we have

$$\begin{aligned} \langle d_a,d_b\rangle_2 &= s^{-(a+b)}x^{2ab}\frac{\begin{vmatrix} 1 & 1 \\ 1 & q^{(a+1)(b+1)} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & q \end{vmatrix}} \\ &= s^{-(a+b)}x^{2ab}\frac{q^{(a+1)(b+1)}-1}{q-1} \\ &= s^{-(a+b)}x^{2ab}\frac{s^{(a+1)(b+1)}}{s}\frac{s^{(a+1)(b+1)}-s^{-(a+1)(b+1)}}{s-s^{-1}} \\ &= (x^2s)^{ab}[(a+1)(b+1)]. \end{aligned}$$

♣

Remark If we make the substitutions $v = s^{-2}$ and $x = s^{-\frac{1}{2}}$ then $\langle d_a, d_b \rangle$ becomes simply $[(a+1)(b+1)]$. This corresponds to the calculations of the $U_h(sl(2))$-quantum invariant in [19] and [15]. We remark that the row diagram d_a of length a indexes the $(a+1)$-dimensional irreducible representation of $U_h(sl(2))$.

Lemma 4.4.4 *Let λ and μ by Young diagrams, and let $n \geq 0$ be an integer. If $n < \max((l(\lambda), l(\mu))$ then $\langle \lambda, \mu \rangle_n = 0$. If $n \geq \max((l(\lambda), l(\mu))$ then $\langle \lambda, \mu \rangle_n$ can be written as the product of a power of s, a power of x, and a non-zero polynomial in $q = s^2$ with integer coefficients.*

Proof By lemma 4.1.5 we have

$$\langle \lambda, \mu \rangle = \langle \lambda \rangle \, s_\mu(E_\lambda(X)) = \langle \mu \rangle \, s_\lambda(E_\mu(X))$$

because $\langle \lambda, \mu \rangle = \langle \mu, \lambda \rangle$. Using the expression in theorem 4.3.6 for $E_\lambda(X)$ we can make the substitution $v = s^{-n}$ for any integer $n \geq 0$. If $n < \max((l(\lambda), l(\mu))$ then either $\langle \lambda \rangle$ or $\langle \mu \rangle$ becomes zero after substituting $v = s^{-n}$ by lemma 3.6.1, hence $\langle \lambda, \mu \rangle$ becomes zero after substituting $v = s^{-n}$.

If $n \geq \max((l(\lambda), l(\mu))$ then we have by lemma 4.4.1 that

$$\langle \lambda, \mu \rangle_n = s^{(1-N)(|\lambda|+|\mu|)} x^{2|\lambda||\mu|} s_\lambda(q^{n-i}) s_\mu(q^{\lambda_k+n-k})$$

where $i = 1, \ldots, n$ and $k = 1, \ldots, n$. Since a Schur function in finitely many variables is a (symmetric) integer polynomial in its variables, we have that the product $s_\lambda(q^{n-i}) s_\mu(q^{\lambda_k+n-k})$ is an integer polynomial in q. It remains to show that the two appearing Schur functions are non-zero. In fact, they are non-zero even after substituting $s = 1$. Our claim is that $s_\lambda(1, \ldots, 1)$ and $s_\mu(1, \ldots, 1)$ are non-zero where the number of variables is n. We recall that $s_\lambda(q^{n-i}) = \langle \lambda \rangle_n$ and we get by lemma 3.6.1 that

$$\begin{aligned} \langle \lambda \rangle_n &= \prod_{y \in \lambda} \frac{s^n s^{\mathrm{cn}(y)} - s^{-n} s^{-\mathrm{cn}(y)}}{s^{\mathrm{hl}(y)} - s^{-\mathrm{hl}(y)}} \\ &= \prod_{y \in \lambda} \frac{[n + \mathrm{cn}(y)]}{[\mathrm{hl}(y)]}. \end{aligned}$$

Since $[k] = (s^k - s^{-k})/(s - s^{-1}) = s^{k-1} + s^{k-3} + \cdots + s^{-k+1}$, we have

$$s_\lambda(1, 1, \ldots, 1) = \prod_{y \in \lambda} \frac{n + \mathrm{cn}(y)}{\mathrm{hl}(y)}.$$

Since we consider the case $n \geq \max(l(\lambda), l(\mu))$ we have that the content of any cell of λ and of μ is greater than $(-n)$. Hence $s_\lambda(q^{n-i})$ becomes a positive number after substituting $s = 1$ and is thus non-zero. The value of $s_\mu(q^{\lambda_k+n-k})$ after substituting $s = 1$ is equal to $s_\mu(1, 1, \ldots, 1)$ where the Schur function has n variables. This is non-zero by the same argument as for $s_\lambda(1, \ldots, 1)$. ♣

On first sight, Lemma 4.4.4 is surprising because the denominator of Q_λ is non-trivial as described in lemma 3.6.3. But in fact, the Homfly polynomial of any link with decorations of type Q_λ can be written as a Laurent polynomial in $s^{\frac{1}{N}}$ after the substitutions $x = s^{-\frac{1}{N}}$ and $v = s^{-N}$. This can be seen by an argument using the $U_h(sl(N))$-invariants.

Chapter 5

Roots of unity

5.1 Homfly polynomial at roots of unity

We fix integers $N \geq 2$ and $l \geq 1$. We fix a complex number ξ such that ξ^N is a primitve root of unity of order $2(l+N)$. We denote ξ^{-N} by ζ.

We shall work occasionally with the substitutions $x = \xi$, $s = \xi^{-N}$ and $v = \xi^{N^2}$. This can also be written as $x = \xi$, $s = \zeta$ and $v = s^{-N}$.

Lemma 5.1.1 *Let $L = L_1 \cup \ldots \cup L_k$ be a link diagram with k components. Let L' be the element of the skein of the plane derived from L by decorating one component with Q_{c_j} for some $j \geq N+1$, and all the other components decorated by elements of the skein of the annulus involving denominators only of the type $(s^i - s^{-i})$ for some $i \geq 1$.*

Then the Homfly polynomial of L' becomes zero after the substitution $v = s^{-N}$.

Proof Let L_1 be the component decorated by Q_{c_j} , $j \geq N+1$. We recall that Q_{c_j} is the closure of the idempotent $(1/\beta_j)b_j \in H_j$ in the skein of the annulus.

We arrange L as the closure of an $(1,1)$-tangle T in the plane so that the closing arc belongs to the component L_1. We now decorate the components of L. This turns T in a (j,j)-tangle T' involving denominators only of the type $(s^i - s^{-i})$. In the Hecke algebra H_j we have that the product of T' and b_j is a scalar multiple α of b_j, and the scalar involves denominators only of the type $(s^i - s^{-i})$. Hence, the Homfly polynomial of L' is the product of α and the Homfly polynomial of Q_{c_j}.

The Homfly polynomial of Q_{c_j} becomes zero after the substitution $v = s^{-N}$ because the factor for $t = N+1$ is equal to zero in

$$\left\langle Q_{c_j} \right\rangle = \prod_{t=1}^{j} \frac{v^{-1}s^{1-t} - vs^{t-1}}{s^t - s^{-t}} = \prod_{t=1}^{j} \frac{s^{N+1-t} - s^{-N-1+t}}{s^t - s^{-t}}$$

which follows from lemma 3.6.1 for $\lambda = c_j$. The scalar α is well defined after the substitution $v = s^{-N}$ and therefore the Homfly polynomial of L' becomes zero after the substitution $v = s^{-N}$. ♣

Corollary 5.1.2 *We are allowed to make the substitutions $x = \xi$, $v = s^{-N}$ and $s = \zeta$ in the Homfly polynomial of any link L whose components are decorated by any Q_λ.*

Proof Any Q_λ is a polynomial in the Q_{c_i}. The monomials including Q_{c_i} with $i \geq N+1$ can be neglected because any decoration with them evaluates to zero by lemma 5.1.1. The denominators of the remaining Q_{c_i} with $1 \leq i \leq N$ only involve $(s^i - s^{-i})$ for $1 \leq i \leq N$ which does not become zero for the substitution $s = \zeta$. The substitutions for x and v do not pose any problem. ♣

Lemma 5.1.3 *Let $L = L_1 \cup \ldots \cup L_k$ be a link diagram with k components. Let L' be the element of the skein of the plane derived from L by decorating one component with Q_{d_i} for some i, $l+1 \leq i \leq l+N-1$, and all the other components decorated by some $Q_\lambda, Q_\mu, \ldots$.*

Then the Homfly polynomial of L' becomes zero after first making the substitution $v = s^{-N}$ and then substituting s by ζ.

Proof First, we write all the decorations Q_λ as polynomials in $Q_{c_1}, Q_{c_2}, \ldots$. By lemma 5.1.1, the Homfly polynomials of all the summands involving some Q_{c_j} with $j \geq N+1$ become zero after the substitution $v = s^{-N}$. Hence, it is sufficient to prove that the Homfly polynomial of any link L with one component decorated by Q_{d_i} and all the other components decorated with Q_{c_k} for some $1 \leq k \leq N$ becomes zero after the substitutions $v = s^{-N}$ and $s = \zeta$.

By the same argument as in the proof of lemma 5.1.1 we write the decorated link L' as the closure of some (i,i)-tangle, and deduce that the Homfly polynomial is the product of a scalar α and the Homfly polynomial of Q_{d_i}. The denominators of α involve only $(s^j - s^{-j})$ with $1 \leq j \leq N$ because only the Young diagrams $c_1, \ldots, c_N$ are involved. Hence the substitution $v = s^{-N}$ and $s = \zeta$ is allowed for the scalar α, since the order of ζ is greater than $2N$.

The Homfly polynomial of Q_{d_i} after the substitution $v = s^{-N}$ and $s = \zeta$ is equal to

$$\langle Q_{d_i} \rangle = \prod_{t=1}^{i} \frac{v^{-1}s^{t-1} - vs^{t-1}}{s^t - s^{-t}} = \prod_{t=1}^{i} \frac{\zeta^{N+t-1} - \zeta^{-N-t+1}}{\zeta^t - \zeta^{-t}}$$

by lemma 3.6.1. None of the denominators is equal to zero because $1 \leq i < l+N$. The numerator for $t = l+1$ becomes zero. Hence this product is equal to zero. Hence the Homfly polynomial of L' which is the product of α and the Homfly polynomial of Q_{d_i} is equal to zero. ♣

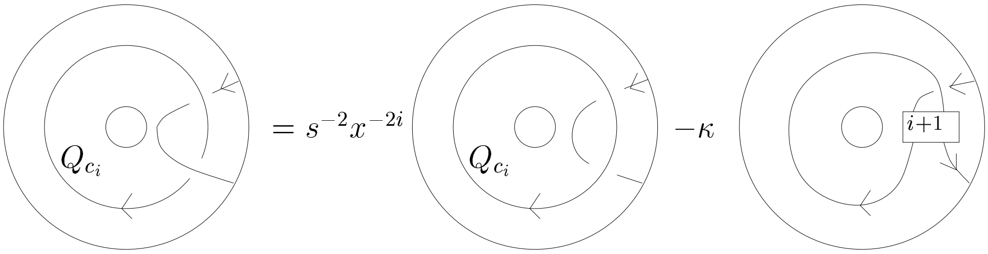


Figure 5.1: Pulling a string through Q_{c_i}.

Lemma 5.1.4 *For a link with decorations of type Q_λ on its components we can remove any component decorated by Q_{c_N} without changing the value of the Homfly polynomial provided we make the substitutions $x = s^{-\frac{1}{N}}$ and $v = s^{-N}$.*

Proof We recall that A_i resp. A'_i is the closure of the quasi-idempotent $a_i \in H_i$ in C_i resp. C'_i. Similarly, B_i resp. B'_i is the closure of the quasi-idempotent $b_i \in H_i$ in C_i resp. C'_i. By lemma 5.1.1 we can assume that only decorations Q_λ are chosen where λ is a column diagram of length up to N. We get

$$\frac{s^{-1}x^i}{\alpha_i}(A_i e) = \frac{sx^{-i}}{\alpha_i}(eA_i) + (s^{-1} - s)\frac{[i+1]}{\alpha_{i+1}}A'_{i+1}$$

if we eliminate $A'_i a$ from the first and the second equation in the proof of lemma 3.4.4. We get

$$\frac{-sx^i}{\beta_i}(B_i e) = \frac{-s^{-1}x^{-i}}{\beta_i}(eB_i) + (s^{-1} - s)(-1)^i\frac{[i+1]}{\beta_{i+1}}B'_{i+1}$$

by applying the map γ from subsection 2.4.1. This is equivalent to

$$\frac{1}{\beta_i}(B_i e) = \frac{s^{-2}x^{-2i}}{\beta_i}(eB_i) - s^{-1}x^{-i}(s^{-1} - s)(-1)^i\frac{[i+1]}{\beta_{i+1}}B'_{i+1}.$$

We apply to this equation the map to the variant skein of the annulus where the two boundary points are on the same component. We get the skein relation in figure 5.1 where $\kappa = s^{-1}x^{-i}(s^{-1} - s)(-1)^i[i+1]$. The box labelled $i+1$ stands for $(1/\beta_{i+1})b_{i+1} \in H_{i+1}$.

We recall that $(1/\beta_j)B_j = Q_{c_j}$ by definition. When we join the boundary points of $(1/\beta_{N+1})b_{N+1}$ by any tangle in $\mathbb{R}^2$ then the Homfly polynomial of the resulting skein element is a scalar multiple of $\langle Q_{c_{N+1}}\rangle$. This scalar involves only denominators of type $(s^i - s^{-i})$ for $1 \le i \le N$ coming from the other decorations of the kind Q_μ. Since $\langle Q_{c_{N+1}}\rangle$ becomes zero after the substitution $v = s^{-N}$, we

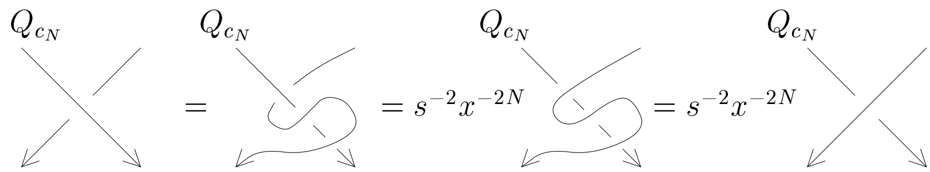


Figure 5.2: Pulling an oriented arc through a component decorated with Q_{c_n}.

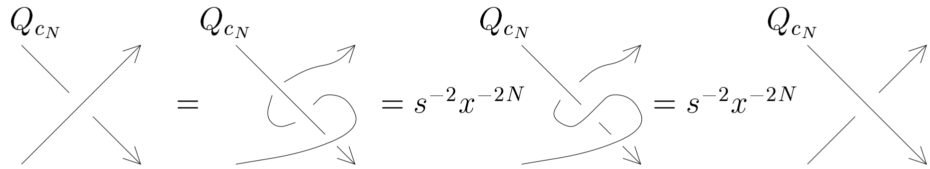


Figure 5.3: Pulling a differently oriented arc through a component decorated with Q_{c_N}.

see that the diagram at the very right in figure 5.1 can be neglected. The new relation is depicted in figure 5.2 where we used regular isotopy. Similarly, the relation in figure 5.3 follows. We note that these are not relations in the skein of the plane. The equalities is only valid after evaluating the Homfly polynomial and then making substitutions.

We can thus pull the component clear from the remaining link. We can furthermore switch all the self crossings of the link without affecting the Homfly polynomial after substitutions. We thus arrive at the unknot decorated with Q_{c_N}. Since the switch of a crossing changes the writhe by 2, we arrive at the unknot with writhe equal to either 0 or 1.

A positive curl decorated by Q_{c_N} may be removed by expense of the scalar $x^{N^2}v^{-N}s^{-N(N-1)}$ as described in theorem 17 in [2]. This becomes $x^{N^2}s^N$ after the substitution $v = s^{-N}$. When we substitute $x^N = s^{-1}$ then this scalar becomes equal to 1. We remark that the scalar $x^{N^2}s^N$ does not become 1 in general when we make the substitution $x^{2N} = s^{-2}$.

Finally, it follows from lemma 3.6.1 that the value of $\langle Q_{c_N}\rangle$ becomes equal to 1 after the substitution $v = s^{-N}$. We have thus removed the component decorated by Q_{c_N} without affecting the Homfly polynomial of the link modulo the substitutions. ♣

5.2 Skein of the annulus at roots of unity

We fix a complex number ξ such that ξ^N is a root of unity of order $2(l+N)$. Given this choice of ξ, we define a partial map $\alpha_{N\,l}$ from the rational functions in x, v and s to the complex numbers by making the substitutions $x = \xi$, $s = \xi^{-N}$, and $v = \xi^{N^2}$ whenever this is well defined. The order of the substitutions might affect whether it is well defined or not. The fraction $(vs^{2l+3N} - 1)/(s^{2(l+N)} - 1)$ becomes 1 after the substitution $v = s^{-N}$. Instead, the immediate substitution of s by a $2(l+N)$-th primitive root of unity leads to the denominator 0.

Definition Let v and w be rational functions in x, v and s. We write $v \doteq w$ if $\alpha_{N\,l}(v) = \alpha_{N\,l}(w)$.

Definition Let a and b be any elements of the skein C of the annulus over the scalars $\mathbb{C}[x, v, s, (s^i - s^{-i})^{-1}, i \geq 1]$. Let L be a framed link and L_1 one of its components. We decorate L_1 by a (or b) and all the other components by some $Q_\lambda, Q_\mu, \ldots$. We say that $a \doteq b$ if

$$\chi(L; a, Q_\lambda, Q_\mu, \ldots) \doteq \chi(L; b, Q_\lambda, Q_\mu, \ldots)$$

for all framed links L, for all components L_1 and for all Young diagrams $\lambda, \mu, \ldots$.

Lemmas 5.1.1, 5.1.3, 5.1.4 can be reformulated as

Corollary 5.2.1 *We have*

$$\begin{array}{ll} Q_{c_j} \doteq 0 & \text{for all } j \geq N+1, \\ Q_{d_i} \doteq 0 & \text{for all } l+1 \leq i \leq l+N-1, \\ Q_{c_N} \doteq \emptyset. & \end{array}$$

The relation $\doteq$ satisfies the following property.

Lemma 5.2.2 *Let a and b be elements of the skein C of the annulus such that $a \doteq b$. Then $aQ_\lambda \doteq bQ_\lambda$ for any Young diagram λ.*

Proof Let L be a link diagram and denote one of its components by L_1. Denote by L' the link diagram that is derived from L by taking the 2-parallel of the component L_1, i.e. L_1 becomes $L_1' \cup L_1''$. Then the decoration of L with aQ_λ on the component L_1 and Q_μ's for various Young diagrams μ on the other components is equal to the decoration of L' with a on L_1' and Q_λ on L_1'', and the Q_μ's on the other components. The definition of $a \doteq b$ implies that $aQ_\lambda \doteq bQ_\lambda$. ♣

The map from the algebra $\mathcal{Y}$ of Young diagrams to the skein of the annulus that maps a Young diagram λ to Q_λ is an algebra homomorphism as shown in theorem 3.5.6. From lemmas 5.1.1, 5.1.3, and 5.1.4 we deduce that this map factors through the ideal of $\mathcal{Y}$ which is generated by

$$\{d_{l+1}, \ldots, d_{l+N-1}, c_N - c_0, c_{N+1}, c_{N+2}, \ldots, \}$$

when we consider equivalence classes modulo '$\doteq$'. We start in chapter 6 a careful analysis of the algebra $\mathcal{Y}$ quotiented by this ideal.

Chapter 6

An ideal in the ring of Young diagrams

6.1 The ideal $I_{N,l}$

Throughout this chapter we fix integers $N \geq 2$ and $l \geq 1$, and we denote $l+N$ by m. The letter l stands for 'level'. We are considering rings, but all the results remain true when we consider in later chapters the rings to be algebras over an extension of $\mathbb{Z}$ in order to handle Homfly skeins involving the variables x, v and s and the scalars $\mathbb{C}$.

We define an ideal $I_{N,l}$ in the ring $\mathcal{Y}$ of Young diagrams. The ideal is generated by the row diagrams of lengths from $(l+1)$ to $(l+N-1)$, the column diagrams of length greater than N, and the difference between the empty diagram and the column diagram of length N,

$$I_{N,l} = \langle\langle d_{l+1}, \ldots, d_{m-1}, c_0 - c_N, c_{N+1}, c_{N+2}, \ldots \rangle\rangle.$$

We denote by $\mathcal{Y}_{N,l}$ the quotient ring $\mathcal{Y}/I_{N,l}$ and by ρ the quotient ring homomorphism from $\mathcal{Y}$ to $\mathcal{Y}_{N,l}$,

$$\rho : \mathcal{Y} \to \mathcal{Y}/I_{N,l} = \mathcal{Y}_{N,l}.$$

We defined an ideal I_N in section 1.3. We clearly have $I_N \subset I_{N,l}$, and thus the quotient map $\mathcal{Y} \to \mathcal{Y}/I_{N,l} = \mathcal{Y}_{N,l}$ factors through $\mathcal{Y}/I_N = \mathcal{Y}_N$.

We shall say that a Young diagram λ *lies (or is) in the* $(N-1) \times l$*-rectangle* if λ has at most $(N-1)$ rows and at most l columns. We shall prove in lemma 6.4.1 that for any Young diagram λ we have either $\rho(\lambda) = 0$ or there exists a Young diagram μ in the $(N-1) \times l$-rectangle so that $\rho(\lambda) = \pm\rho(\mu)$. In theorem 6.5.2 we shall prove that the set $\{\rho(\lambda) \mid \lambda \text{ lies in the } (N-1) \times l\text{-rectangle}\}$ is a linear basis for $\mathcal{Y}_{N,l}$.

For elements a and b of $\mathcal{Y}$ we shall say that a and b *are equal in* $\mathcal{Y}_{N,l}$ if $\rho(a) = \rho(b)$. Since the quotient map ρ is a ring homomorphism, we have that for a square matrix with entries in $\mathcal{Y}$, the determinant does not change in $\mathcal{Y}_{N,l}$ if we replace any entry a of the matrix by an element that is equal in $\mathcal{Y}_{N,l}$ to a.

Remark If we add the row diagram d_l to the generators of $I_{N,l}$ we get a larger ideal $I'_{N,l}$. In fact, $I'_{N,l} = \mathcal{Y}$ as we show now. If we added d_{l+N} instead of d_l to the generators of $I_{N,l}$ we still get $\mathcal{Y}$ because $d_{l+N} = (-1)^{N+1} d_l$ modulo $I_{N,l}$ by lemma 6.3.1.

Lemma 6.1.1 $I'_{N,l} = \mathcal{Y}$.

Proof We deduce from equation (1.2.2) for $r = N + i$ that

$$d_i = c_{N-1} d_{i+1} - c_{N-2} d_{i+2} + \cdots + (-1)^{N-1} d_{i+N}$$

modulo $I_{N,l}$ for any integer $i \geq 1$ because $c_N = c_0$ and $c_j = 0$ for $j \geq N+1$. Step by step we deduce from this equation that $d_i \in I'_{N,l}$ for all $i = l-1, l-2, \ldots, 1$. Hence, $d_1, \ldots, d_{l+N-1} \in I'_{N,l}$. In particular, $d_1, \ldots, d_N \in I'_{N,l}$ because $l \geq 1$.

From equation (1.2.2) (or from the Giambelli formula) we deduce that any c_j is a polynomial in $d_1, \ldots, d_j$. Hence, $c_j \in I'_{N,l}$ for $j = 1, \ldots, N$. Hence, $c_j \in I'_{N,l}$ for any $j \geq 1$. Hence, $I'_{N,l} = \mathcal{Y}$. ♣

6.2 Adding a row of length l

We recall the notation λ' for a Young λ with N rows from section 1.3. It denotes the Young diagram that derives from λ by removing all (initial) columns of length N.

We define a map σ on the set of Young diagrams in the $(N-1) \times l$-rectangle by adding an initial row of length l at the top of λ and then removing all columns of length N,

$$\sigma(\lambda_1, \ldots, \lambda_{N-1}) = (l - \lambda_{N-1}, \lambda_1 - \lambda_{N-1}, \ldots, \lambda_{N-2} - \lambda_{N-1}).$$

This map is extended linearly to the subspace spanned by the Young diagrams in the $(N-1) \times l$-rectangle. It is easy to check that $\sigma^N(\lambda) = \lambda$ for any Young diagram λ in the $(N-1) \times l$-rectangle. Figure 6.1 shows that $\sigma(4,3,2,2) = (5,2,1)$ for $N = 5$ and $l = 7$.

Lemma 6.2.1 *The elements $\sigma(\lambda)$ and $d_l \lambda$ are equal in $\mathcal{Y}_{N,l}$ for any Young diagram λ in the $(N-1) \times l$-rectangle.*

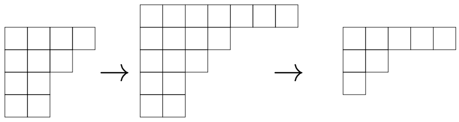


Figure 6.1: Adding an initial row of length l and removing all columns of length N.

Proof Let $\lambda = (\lambda_1, \ldots, \lambda_{N-1})$ be a Young diagram in the $(N-1) \times l$-rectangle. Denote $\eta = (l, \lambda_1, \ldots, \lambda_{N-1})$. Then $\eta' = \sigma(\lambda)$ by definition.

We have by lemma 1.3.1 that η and η' are equal $\mathcal{Y}_N$, hence they are equal in $\mathcal{Y}_{N,l}$. It remains to show that η and $d_l\lambda$ are equal in $\mathcal{Y}_{N,l}$.

The Giambelli formula applied to the Young diagram η gives

$$\eta = \begin{vmatrix} d_l & d_{l+1} & \cdots & d_{l+N-1} \\ d_{\lambda_1-1} & d_{\lambda_1} & \cdots & d_{\lambda_1+N-2} \\ \vdots & \vdots & \ddots & \vdots \\ d_{\lambda_{N-1}-N+1} & d_{\lambda_{N-1}-N+2} & \cdots & d_{\lambda_{N-1}} \end{vmatrix}.$$

When we consider this equality in $\mathcal{Y}_{N,l}$, we can replace $d_{l+1}, \ldots, d_{l+N-1}$ by zero. By developing the determinant by the first row we get

$$\begin{aligned} \eta &= d_l \begin{vmatrix} d_{\lambda_1} & \cdots & d_{\lambda_1+N-2} \\ \vdots & \ddots & \vdots \\ d_{\lambda_{N-1}-N+2} & \cdots & d_{\lambda_{N-1}} \end{vmatrix} \\ &= d_l\lambda \end{aligned}$$

in $\mathcal{Y}_{N,l}$. ♣

6.3 Row diagrams modulo $I_{N,l}$

We start by proving a useful relation for row diagrams in $\mathcal{Y}_{N,l}$.

Lemma 6.3.1 *We have*

$$d_{km+r} = (-1)^{(N+1)k} d_l^k d_r$$

in $\mathcal{Y}_{N,l}$ for any integer $k \geq 0$ and integer r, $0 \leq r \leq m-1$.

Proof By equation (1.2.3) we have

$$1 = \left(\sum_{i=0}^{\infty} (-1)^i c_i z^i\right) \left(\sum_{j=0}^{\infty} d_j z^j\right) \tag{6.3.1}$$

in $\mathcal{Y}$. Using the relations for $\mathcal{Y}_{N,l}$ we deduce that

$$1 = \left(\sum_{i=0}^{N}(-1)^i c_i z^i\right)\left(\sum_{j=0}^{l} d_j z^j + \sum_{j=m}^{\infty} d_j z^j\right)$$

in $\mathcal{Y}_{N,l}$. Looking at the exponents less than or equal to m we deduce that

$$1 = \left(\sum_{i=0}^{N}(-1)^i c_i z^i\right)\left(\sum_{j=0}^{l} d_j z^j\right) + d_m z^m, \tag{6.3.2}$$

hence

$$1 - d_m z^m = \left(\sum_{i=0}^{N}(-1)^i c_i z^i\right)\left(\sum_{j=0}^{l} d_j z^j\right).$$

Multiplication of both sides by $\sum_{k=0}^{\infty} d_m^k z^{mk}$ and the use of the relations for $\mathcal{Y}_{N,l}$ leads to

$$1 = \left(\sum_{i=0}^{\infty}(-1)^i c_i z^i\right)\left(\sum_{j=0}^{m-1} d_j z^j\right)\left(\sum_{k=0}^{\infty} d_m^k z^{mk}\right).$$

We remark that for any commutative algebra the inverse of a formal power series $a_0 + a_1 z + a_2 z^2 + \cdots$ with an invertible constant term a_0 is uniquely determined. Hence, by comparing the above equation with equation (6.3.1) we deduce that

$$\sum_{j=0}^{\infty} d_j z^j = \left(\sum_{r=0}^{m-1} d_r z^r\right)\left(\sum_{k=0}^{\infty} d_m^k z^{mk}\right).$$

This implies that for $k \geq 0$ and $0 \leq r \leq m-1$

$$d_{km+r} = d_r d_m^k.$$

Looking at the coefficient of z^m on both sides of equation (6.3.2), we see that $0 = (-1)^N c_N d_l + d_m$. Since $c_N = 1$ in $\mathcal{Y}_{N,l}$, we get $d_m = (-1)^{N+1} d_l$. Substituting this in the above equation yields $d_{km+r} = (-1)^{(N+1)k} d_l^k d_r$. If $l+1 \leq r \leq m-1$ then $d_r = 0$ in $\mathcal{Y}_{N,l}$, hence $d_{km+r} = 0$ in $\mathcal{Y}_{N,l}$. ♣

The Young diagrams $d_{l+1}, \ldots, d_{m-1}$ are equal to zero in $\mathcal{Y}_{N,l}$, and we thus get

Corollary 6.3.2 *We have*

$$d_{km+r} = \begin{cases} (-1)^{(N+1)k} d_l^k d_r & \text{if } 0 \leq r \leq l, \\ 0 & \text{if } l+1 \leq r \leq m-1 \end{cases}$$

in $\mathcal{Y}_{N,l}$ for any integer $k \geq 0$ and integer r, $0 \leq r \leq m-1$.

The combination of lemma 6.2.1 and Corollary 6.3.2 shows that in $\mathcal{Y}_{N,l}$ any row diagram is either equal to zero or it is equal up to a sign to a Young diagram in the $(N-1) \times l$-rectangle.

Corollary 6.3.3 *We have*

$$d_{km+r} = \begin{cases} (-1)^{(N+1)k}\sigma^k(d_r) & \text{if } 0 \le r \le l, \\ 0 & \text{if } l+1 \le r \le m-1 \end{cases}$$

in $\mathcal{Y}_{N,l}$ for any integer $k \ge 0$ and integer r, $0 \le r \le m-1$.

6.4 Reduction of a Young diagram

We shall extend Corollary 6.3.3 by proving that any Young diagram is up to a sign equal in $\mathcal{Y}_{N,l}$ to a Young diagram in the $(N-1) \times l$-rectangle.

Definition For integers $q_1, \dots, q_a$ we define an element $\begin{pmatrix} q_1 \\ \vdots \\ q_a \end{pmatrix}_G$ of $\mathcal{Y}$ by

$$\begin{pmatrix} q_1 \\ \vdots \\ q_a \end{pmatrix}_G = \begin{vmatrix} d_{q_1-(a-1)} & \cdots & d_{q_1-(a-j)} & \cdots & d_{q_1} \\ \vdots & & \vdots & & \vdots \\ d_{q_a-(a-1)} & \cdots & d_{q_a-(a-j)} & \cdots & d_{q_a} \end{vmatrix}$$

(where $d_r = 0$ for $r < 0$).

The letter G stands for 'Giambelli'. If $q_1 > \cdots > q_a \ge 0$ then this $(a \times a)$-determinant is equal to a Young diagram by the Giambelli formula. If $q_1, \dots, q_a$ are pairwise different non-negative integers then a permutation of rows shows that this determinant is equal to a Young diagram up to a sign. If $q_i = q_j$ for different indices i and j then this determinant is equal to zero. If some $q_i < 0$ then this determinant is equal to zero.

The Giambelli formula for a Young diagram $\lambda = (\lambda_1, \dots, \lambda_{N-1})$ takes the form

$$(\lambda_1, \lambda_2, \dots, \lambda_a) = \begin{pmatrix} \lambda_1 + a - 1 \\ \vdots \\ \lambda_i + a - i \\ \vdots \\ \lambda_a \end{pmatrix}_G .$$

By a permutation of rows we have for example

$$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}_G = -\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}_G = -c_2.$$

Definition For a Young diagram λ with at most $(N-1)$ rows we write

$$\lambda_i + N - 1 - i = k_i m + r_i$$

for (uniquely determined) integers $k_i \geq 0$ and $0 \leq r_i \leq m-1$, $i = 1, \ldots, N-1$. We set $K = k_1 + \cdots + k_{N-1}$. The *reduction* $\overline{\lambda}$ of a Young diagram λ is defined as

$$\overline{\lambda} = \begin{cases} (-1)^{(N+1)K} \sigma^K \begin{pmatrix} r_1 \\ \vdots \\ r_{N-1} \end{pmatrix}_G & \text{if } l(\lambda) \leq N-1, \text{ and } 0 \leq r_i \leq m-1 \\ & \text{for all } i = 1, \ldots, N-1, \\ 0 & \text{if } l(\lambda) \leq N-1 \text{ and } r_i = m-1 \\ & \text{for some } 1 \leq i \leq N-1, \\ \overline{\lambda'} & \text{if } l(\lambda) = N, \\ 0 & \text{if } l(\lambda) \geq N+1. \end{cases}$$

We see that the reduction of a Young diagram is either equal to zero or it is equal to a Young diagram inside the $(N-1) \times l$-rectangle up to a sign.

Example We consider the Young diagram $\lambda = (8, 6, 3, 2)$ for $N = 5$ and $l = 3$. We have $m = l + N = 8$.

We have

$$\begin{array}{rcl} \lambda_i + N - 1 - i & = & k_i m + r_i \\ 8 + 5 - 1 - 1 & = & 1 \cdot 8 + 3 \\ 6 + 5 - 1 - 2 & = & 1 \cdot 8 + 0 \\ 3 + 5 - 1 - 3 & = & 0 \cdot 8 + 4 \\ 2 + 5 - 1 - 4 & = & 0 \cdot 8 + 2. \end{array}$$

Hence,

$$k_1 = 1,\ k_2 = 1,\ k_3 = 0,\ k_4 = 0 \text{ and } r_1 = 3,\ r_2 = 0,\ r_3 = 4,\ r_4 = 2.$$

Hence, $K = k_1 + k_2 + k_3 + k_4 = 2$. None of the r_i is equal to $m-1$ which is equal to 7. We thus have

$$\overline{\lambda} = (-1)^{(5+1)2} \sigma^2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ 2 \end{pmatrix}_G = \sigma^2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ 2 \end{pmatrix}_G = -\sigma^2 \begin{pmatrix} 4 \\ 3 \\ 2 \\ 0 \end{pmatrix}_G$$

where the minus sign appears because we have permuted the rows of the determinant in order that they are decreasing downwards. In fact, they are strictly decreasing, and by the Giambelli formula this determinant is equal to a Young diagram ζ. The diagonal entries of the determinant are d_1, d_1, d_1, d_0. Hence, $\zeta = (1,1,1,0)$. We have $\sigma(\zeta) = (3,1,1,1)$. Adding a further row of length $l = 3$ at the top, we get the Young diagram $(3,3,1,1,1)$ with the first column of length $N = 5$. We thus have $\sigma^2(\zeta) = \sigma(3,1,1,1) = (2,2,0,0)$. We therefore finally have $\overline{\lambda} = -(2,2)$.

We remark that $\overline{\mu} = 0$ for $\mu = (8,6,3,2)$ with $N = 6$ and $l = 3$ (because $r_1 = r_4$ in this case).

For a Young diagram λ in the $(N-1) \times l$-rectangle we have $\overline{\lambda} = \lambda$ because $k_i = 0$ for every $i = 1, \ldots, N-1$. Hence the linear map $\mathcal{Y} \to \mathcal{Y}$ given by $\lambda \mapsto \overline{\lambda}$ is the projection of $\mathcal{Y}$ to the submodule spanned by the Young diagrams in the $(N-1) \times l$-rectangle.

Lemma 6.4.1 *We have $\rho(\lambda) = \rho(\overline{\lambda})$ for any Young diagram λ.*

Proof We first consider the case of a Young diagram λ with at most $N-1$ rows. We have

$$\begin{aligned}
\lambda &= \begin{pmatrix} k_1 m + r_1 \\ \vdots \\ k_{N-1} m + r_{N-1} \end{pmatrix}_G \\
&= \begin{vmatrix} d_{\lambda_1} & \cdots & d_{\lambda_1+N-2} \\ \vdots & & \vdots \\ d_{\lambda_i+1-i} & \cdots & d_{\lambda_i+N-1-i} \\ \vdots & & \vdots \\ d_{\lambda_{N-1}-N+2} & \cdots & d_{\lambda_{N-1}} \end{vmatrix} \\
&= \begin{vmatrix} d_{\lambda_1} & \cdots & & & & \cdots & d_{\lambda_1+N-2} \\ \vdots & & & & & & \vdots \\ d_{k_i m + r_i - (N-2)} & \cdots & d_{k_i m - 1} & d_{k_i m} & \cdots & d_{k_i m + r_i - 1} & d_{k_i m + r_i} \\ \vdots & & & & & & \vdots \\ d_{\lambda_{N-1}-N+2} & \cdots & & & & \cdots & d_{\lambda_{N-1}} \end{vmatrix}
\end{aligned} \tag{6.4.3}$$

where the above $(N-1) \times (N-1)$-determinant shows the i-th row in detail for some $1 \leq i \leq N-1$. The entry $d_{k_i m}$ may or may not occur, depending whether $0 \leq r_i \leq N-2$ or $N-1 \leq r_i \leq m-1$.

If $r_i = m-1$ then all entries of the i-th row become zero in $\mathcal{Y}_{N,l}$ by corollary 6.3.2. Hence the determinant becomes zero in $\mathcal{Y}_{N,l}$, i.e. $\rho(\lambda) = 0$. On the other hand, $\overline{\lambda} = 0$ in this case by definition. Hence $\rho(\lambda) = \rho(\overline{\lambda})$ in this case.

We assume from now on that $0 \leq r_i < m-1$ for all $i = 1, \dots, N-1$. By lemma 6.3.1 we can replace $d_{k_i m+j}$ by $(-1)^{(N+1)k_i} d_l^{k_i} d_j$ for all $j = 0, \dots, r_i$. Recall that the determinant is of size $(N-1) \times (N-1)$. Hence, there are at most $(N-1)$ elements to the left of $d_{k_i m}$ and so their indices lie between $l+1$ and $m-1$ modulo m. Hence all the entries to the left of $d_{k_i m}$ become zero in $\mathcal{Y}_{N,l}$ by lemma 6.3.1. Hence we have in $\mathcal{Y}_{N,l}$

$$\lambda = (-1)^{(N+1)k_i} d_l^{k_i} \begin{vmatrix} d_{\lambda_1} & \cdots & & & & \cdots & d_{\lambda_1+N-2} \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & d_0 & \cdots & d_{r_i-1} & d_{r_i} \\ \vdots & & & & & & \vdots \\ d_{\lambda_{N-1}-N+2} & \cdots & & & & \cdots & d_{\lambda_{N-1}} \end{vmatrix}.$$

By applying this argument to every row in equation (6.4.3), we see that in $\mathcal{Y}_{N,l}$

$$\lambda = (-1)^{(N+1)K} d_l^K \begin{pmatrix} r_1 \\ \vdots \\ r_{N-1} \end{pmatrix}_G$$

where $K = k_1 + k_2 + \cdots + k_{N-1}$. Since $r_1, \dots, r_{N-1}$ are all different from $(m-1)$, the above determinant is (up to a sign depending on a permutation of its rows) equal to a Young diagram in the $(N-1) \times l$-rectangle, or it is zero. We can therefore apply Lemma 6.2.1 and get

$$\lambda = (-1)^{(N+1)K} \sigma^K \begin{pmatrix} r_1 \\ \vdots \\ r_{N-1} \end{pmatrix}_G$$

in $\mathcal{Y}_{N,l}$. Hence, $\lambda = \overline{\lambda}$ in $\mathcal{Y}_{N,l}$ in the case that $0 \leq r_i \leq m-1$ for $i = 1, \dots, N-1$. Hence, $\lambda = \overline{\lambda}$ in $\mathcal{Y}_{N,l}$ for every Young diagram λ with at most $(N-1)$ rows.

If λ has N rows then $\overline{\lambda} = \overline{\lambda'}$ by definition, and $\rho(\overline{\lambda'}) = \rho(\lambda')$ by the above case for Young diagrams with at most $(N-1)$ rows. Hence,

$$\rho(\overline{\lambda}) = \rho(\overline{\lambda'}) = \rho(\lambda') = \rho(\lambda).$$

since $\rho(\lambda') = \rho(\lambda)$ by lemma 1.3.1.

If λ has at least $N+1$ rows then $\overline{\lambda} = 0$ by definition, and $\rho(\lambda) = 0$ by lemma 1.3.1. Hence, $\rho(\lambda) = \rho(\overline{\lambda})$. ♣

6.5 A basis for $\mathcal{Y}_{N,l}$

We define the $\mathbb{Z}$-submodule $L_{N,l}$ of $\mathcal{Y}$ to be the span of $(\lambda - \overline{\lambda})$ for all Young diagrams λ,

$$L_{N,l} = \langle \lambda - \overline{\lambda} \mid \lambda \text{ a Young diagram} \rangle.$$

We have

$$\mathcal{Y} = L_{N,l} \oplus \langle \lambda \mid \text{ Young diagram } \lambda \text{ lies in the } (N-1) \times l\text{-rectangle} \rangle \quad (6.5.4)$$

because, first, $\overline{\lambda}$ is either zero or up to a sign equal to a Young diagram in the $(N-1) \times l$-rectangle, and second, $\lambda = \overline{\lambda}$ if λ lies in the $(N-1) \times l$-rectangle.

The proof that $L_{N,l}$ is an ideal in $\mathcal{Y}$ depends on lemma 6.6.7 which will be proved later.

Lemma 6.5.1 *$L_{N,l}$ is an ideal in $\mathcal{Y}$.*

Proof Since the ring of Young diagrams is generated by all the column diagrams $c_1, c_2, \ldots$ it is sufficient to show that

$$c_i(\lambda - \overline{\lambda}) \in L_{N,l} \text{ for any } i \geq 1 \text{ and any Young diagram } \lambda.$$

We remark that $L_{N,l}$ contains all Young diagrams with more than N rows and all terms $(a - \overline{a})$ for $a \in \mathcal{Y}$.

Let $i > N$. Since c_i is a subdiagram of every summand of $c_i(\lambda - \lambda')$, they all have more than N rows. Hence, $c_i(\lambda - \overline{\lambda}) \in L_{N,l}$.

Let λ be a Young diagram with more than N rows, and let $i \geq 1$. We have $\overline{\lambda} = 0$ by definition. Since λ is a subdiagram of every summand of $c_i\lambda$, we have $c_i(\lambda - \overline{\lambda}) \in L_{N,l}$.

Let $1 \leq i \leq N$ and let λ be a Young diagram with less than N rows. We have trivially

$$c_i(\lambda - \overline{\lambda}) = (c_i\lambda - \overline{c_i\lambda}) + (\overline{c_i\lambda} - \overline{c_i\overline{\lambda}}) - (c_i\overline{\lambda} - \overline{c_i\overline{\lambda}}).$$

The first and the third summand lie in $L_{N,l}$ by definition. The second summand is equal to zero by lemma 6.6.7. Hence $c_i(\lambda - \overline{\lambda}) \in L_{N,l}$.

Let $1 \leq i \leq N$ and λ be a Young diagram with N rows. We have $\overline{\lambda} = \overline{\lambda'}$ by definition. We have trivially

$$c_i(\lambda - \overline{\lambda}) = c_i(\lambda - \lambda') + c_i(\lambda' - \overline{\lambda'}).$$

The first summand lies in $L_{N,l}$ because it lies in L_N by lemma 1.3.2. The second summand lies in $L_{N,l}$ by the previous case for Young diagrams with less than N rows. Hence $c_i(\lambda - \overline{\lambda}) \in L_{N,l}$. ♣

Theorem 6.5.2 *The set* $\{\rho(\lambda) \mid \lambda$ *lies in the* $(N-1) \times l -$ *rectangle*$\}$ *is a linear basis for* $\mathcal{Y}_{N,l}$.

Proof We have $\rho(\lambda) = \rho(\overline{\lambda})$ by lemma 6.4.1, hence $(\lambda - \overline{\lambda}) \in I_{N,l}$. Hence $L_{N,l}$ is a submodule of $I_{N,l}$. Since $L_{N,l}$ is an ideal in $\mathcal{Y}$, we have $L_{N,l} = I_{N,l}$. By equation (6.5.4) we see that the images of the Young diagrams in the $(N-1) \times l$-rectangle are a basis for $\mathcal{Y}_{N,l}$. ♣

6.6 Proof that $\overline{c_i \lambda} = \overline{c_i \overline{\lambda}}$

The combinatorial Littlewood-Richardson rule via counting the number of strict extensions is not suitable for algebraic computations. In order to prove lemma 6.6.7 we need a compact formula for the multiplication of a Young diagram by a column diagram. Such a formula is provided in the next lemma using the vector notation for Young diagrams. The essential simplification provided by this lemma is that we do not have to restrict the addition of cells of c_i to λ so that the resulting diagram is a Young diagram. If the resulting diagram is not a Young diagram then the corresponding summand is equal to zero.

Lemma 6.6.1 *Let* $q_1, \ldots, q_{N-1}$ *be non-negative integers and let* i *be an integer,* $1 \leq i \leq N$. *Then*

$$c_i \begin{pmatrix} q_1 \\ \vdots \\ q_{N-1} \end{pmatrix}_G = \sum_{\varepsilon_1 + \cdots + \varepsilon_N = i} \begin{pmatrix} q_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ q_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G \quad \text{modulo } L_{N,l}.$$

The variables $\varepsilon_1, \ldots, \varepsilon_N$ *are to have values in* $\{0, 1\}$.

Proof We start by proving the lemma for the case that $q_1 > q_2 > \cdots > q_{N-1}$ are non-negative integers. We have

$$\begin{pmatrix} q_1 \\ \vdots \\ q_{N-1} \end{pmatrix}_G = (q_1 - (N-2), q_2 - (N-1), \ldots, q_{N-1})$$

which is a Young diagram, say η. We know by the Littlewood-Richardson rule that the summands with up to N rows occurring in $c_i \eta$ in $\mathcal{Y}$ are all the Young diagrams

$$(\eta_1 + \varepsilon_1, \ldots, \eta_{N-1} + \varepsilon_{N-1}, \varepsilon_N) \tag{6.6.5}$$

where the variables $\varepsilon_1, \ldots, \varepsilon_N$ are to have values 0 or 1 and their sum is equal to i. This is because every summand in $c_i \eta$ derives from η by adding at most one cell to each row of η. By removing a possible first row of length N we transform the above Young diagram into

$$(\eta_1 + \varepsilon_1 - \varepsilon_N, \dots, \eta_{N-1} + \varepsilon_{N-1} - \varepsilon_N).$$

Writing the summands of $c_i\eta$ in determinantal form we get

$$c_i \begin{pmatrix} q_1 \\ \vdots \\ q_{N-1} \end{pmatrix}_G = \sum_{\varepsilon_1 + \cdots + \varepsilon_N = i} \begin{pmatrix} q_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ q_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G \tag{6.6.6}$$

up to summands with more than N rows and terms $(\eta - \eta')$ where η has N rows. The variables $\varepsilon_1, \dots, \varepsilon_N$ are to have values 0 or 1, *and* they have to satisfy the condition that the sequence in expression (6.6.5) is a Young diagram.

Now assume that for some $\varepsilon_1, \dots, \varepsilon_N$ the sequence in expression (6.6.5) is *not* a Young diagram, this means it is increasing at some point. Then we have for some j, $1 \leq j \leq N-1$, that $\eta_j + \varepsilon_j < \eta_{j+1} + \varepsilon_{j+1}$ where $\eta_N = 0$. Since ε_j and ε_{j+1} can only have values 0 or 1, and $\eta_j \geq \eta_{j+1}$ because η is a Young diagram, we deduce $\eta_j = \eta_{j+1}$ and $\varepsilon_j = 0$, $\varepsilon_{j+1} = 1$. Hence $q_j + \varepsilon_j - \varepsilon_N = q_{j+1} + \varepsilon_{j+1} - \varepsilon_N$. Then the corresponding determinant is equal to zero,

$$\begin{pmatrix} q_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ q_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G = 0,$$

because the rows j and $(j+1)$ are identical. Hence the right hand side of equation (6.6.6) is not altered by extending the sum of determinants to all $\varepsilon_1, \dots, \varepsilon_N$ so that $\varepsilon_1 + \cdots + \varepsilon_N = i$ and each variable ε_j has values in $\{0, 1\}$.

We have thus proved the lemma for the case $q_1 > q_2 > \cdots > q_{N-1} \geq 0$. The case that $q_1, \dots, q_{N-1}$ are pairwise different non-negative integers follows immediately by a permutation of the rows of the determinants.

To finally prove the lemma we consider from now on the case that $q_{j_1} = q_{j_2}$ for some $1 \leq j_1 < j_2 \leq N-1$. In this case the left hand side of equation (6.6.6) is equal to zero. We have to prove that the right hand side is equal to zero as well.

First we note that for a summand corresponding to $(\varepsilon_1, \dots, \varepsilon_N)$ with $\varepsilon_{j_1} = \varepsilon_{j_2}$ the determinant at the right hand side of equation (6.6.6) contains two identical rows, hence it is equal to zero. We can thus restrict to those summands with $\varepsilon_{j_1} = 0$ and $\varepsilon_{j_2} = 1$ and those summands with $\varepsilon_{j_1} = 1$ and $\varepsilon_{j_2} = 0$. We get a fixed point free permutation of these summands by interchanging the values of ε_{j_1} and ε_{j_2}. The determinants of two corresponding summands add up to zero because they differ by a transposition of the rows j_1 and j_2. Hence the whole sum adds up to zero. ♣

Applying the reduction map $\lambda \mapsto \overline{\lambda}$ to lemma 6.6.1 leads to

Corollary 6.6.2 *Let $q_1, \ldots, q_{N-1}$ be non-negative integers and let i be an integer, $1 \leq i \leq N$. Then*

$$\overline{c_i \begin{pmatrix} q_1 \\ \vdots \\ q_{N-1} \end{pmatrix}_G} = \sum_{\varepsilon_1 + \cdots + \varepsilon_N = i} \overline{\begin{pmatrix} q_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ q_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G}.$$

The variables $\varepsilon_1, \ldots, \varepsilon_N$ are to have values 0 or 1.

If for an N-tuple $(\varepsilon_1, \ldots, \varepsilon_N)$ we have $q_j + \varepsilon_j - \varepsilon_N \equiv m - 1 \bmod m$ for some $1 \leq j \leq N-1$ then this summand is equal to zero.

Proof The reduction of elements of $L_{N,l}$ is equal to zero because $\overline{\lambda - \overline{\lambda}} = \overline{\lambda} - \overline{\overline{\lambda}} = 0$ for any Young diagram λ. Hence

$$\overline{c_i \begin{pmatrix} q_1 \\ \vdots \\ q_{N-1} \end{pmatrix}_G} = \sum_{\varepsilon_1 + \cdots + \varepsilon_N = i} \overline{\begin{pmatrix} q_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ q_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G}.$$

If $q_j + \varepsilon_j - \varepsilon_N \equiv m - 1 \bmod m$ for some $1 \leq j \leq N-1$ then we have to consider two cases. Either $q_j + \varepsilon_j - \varepsilon_N = -1$ in which case the determinant is zero because the j-the row consists entirely of zeros. Or $q_j + \varepsilon_j - \varepsilon_N = k_j(N+l) + m - 1$ for some integer $k_j \geq 0$ in which case the reduction is defined to be equal to zero. ♣

There is a compact formulation for the operation of σ in terms of the vector notation for Young diagrams.

Lemma 6.6.3 *For integers $m - 2 \geq q_1 \geq q_2 \geq \cdots \geq q_{N-1} \geq 0$ we have*

$$\sigma \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N-1} \end{pmatrix}_G = \begin{pmatrix} m - 2 - q_{N-1} \\ q_1 - 1 - q_{N-1} \\ \vdots \\ q_{N-2} - 1 - q_{N-1} \end{pmatrix}_G.$$

Proof We denote elements in $\mathcal{Y}$,

$$\alpha = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N-1} \end{pmatrix}_G \quad \text{and} \quad \beta = \begin{pmatrix} m - 2 - q_{N-1} \\ q_1 - 1 - q_{N-1} \\ \vdots \\ q_{N-2} - 1 - q_{N-1} \end{pmatrix}_G.$$

If $m-2 \geq q_1 > q_2 > \cdots > q_{N-1} \geq 0$ then α is equal to a Young diagram in the $(N-1)\times l$-rectangle by the Giambelli formula. The claimed equality of $\sigma(\alpha)$ and β is the translation of the definition of $\sigma(\lambda)$ into determinantal form.

From now on we consider the remaining case $q_i = q_{i+1}$ for some $1 \leq i \leq N-2$. Then the determinant α is equal to zero because it has identical rows i and $i+1$. Hence $\sigma(\lambda) = 0$ as well. We shall show that $\beta = 0$, too.

If $q_{N-2} = q_{N-1}$, then $q_{N-2} - 1 - q_{N-1}$ is negative, hence the determinant β is equal to zero.

If $q_i = q_{i+1}$ for some $1 \leq i \leq N-3$ then the columns $(i+1)$ and $(i+2)$ of the determinant β are equal, hence $\beta = 0$. Therefore the statement of the lemma is also true in the case $q_i = q_{i+1}$ for some $1 \leq i \leq N-2$. ♣

The next lemma describes that for a Young diagram λ in the $(N-1)\times l$-rectangle the operation σ commutes with the operation of multiplication with c_i followed by reduction. Note that $c_i\sigma^f(\lambda)$ means $c_i(\sigma^f(\lambda))$.

Lemma 6.6.4 *Let λ be a Young diagram in the $(N-1)\times l$-rectangle, let f be a non-negative integer, and let $1 \leq i \leq N$. Then*

$$\overline{c_i\sigma^f(\lambda)} = \sigma^f(\overline{c_i\lambda}).$$

Proof By induction on f. Let λ be a Young diagram in the $(N-1)\times l$-rectangle. The case $f=0$ is trivial. The essential part of the proof is to show the statement of the lemma for $f=1$ since induction immediately shows for $f \geq 2$ that

$$\begin{aligned} \sigma^f(\overline{c_i\lambda}) &= \sigma\sigma^{f-1}(\overline{c_i\lambda}) \\ &= \sigma(\overline{c_i\sigma^{f-1}(\lambda)}) \\ &= \overline{c_i\sigma\sigma^{f-1}(\lambda)} \\ &= \overline{c_i\sigma^f(\lambda)}. \end{aligned}$$

We set $q_j = \lambda_j + N - 1 - j$ for $j = 1, \ldots, N-1$, and we have

$$\lambda = \begin{pmatrix} q_1 \\ \vdots \\ q_{N-1} \end{pmatrix}_G$$

and $m-2 \geq q_1 > q_2 > \cdots > q_{N-1} \geq 0$. By lemma 6.6.3 we have

$$\sigma(\lambda) = \begin{pmatrix} m-2-q_{N-1} \\ q_1 - 1 - q_{N-1} \\ \vdots \\ q_{N-2} - 1 - q_{N-1} \end{pmatrix}_G .$$

Therefore, using corollary 6.6.2,

$$\overline{c_i\sigma(\lambda)} = \sum_{\substack{\varepsilon_1+\cdots+\varepsilon_N=i \\ m-2-q_{N-1}+\varepsilon_1-\varepsilon_N\le m-2}} \begin{pmatrix} m-2-q_{N-1}+\varepsilon_1-\varepsilon_N \\ q_1-1-q_{N-1}+\varepsilon_2-\varepsilon_N \\ \vdots \\ q_{N-2}-1-q_{N-1}+\varepsilon_{N-1}-\varepsilon_N \end{pmatrix}_G$$

where the sum is restricted to those summands with $m-2-q_{N-1}+\varepsilon_1-\varepsilon_N \le m-2$ because only the first entry of the vector could be greater than $m-2$. (It could be at most $m-1$ in which case it reduces to zero.) The condition is equivalent to $q_{N-1}+\varepsilon_N-\varepsilon_1 \ge 0$. The only situation in which the second entry of the vector is not less than the first entry occurs if the first and the second entry are equal. Hence those summands are equal to zero. We can thus reduce the summands in the above sum to those with $q_1-1-q_{N-1}+\varepsilon_2-\varepsilon_N < m-2-q_{N-1}+\varepsilon_1-\varepsilon_N$ which is equivalent to $q_1+\varepsilon_2-\varepsilon_1 \le m-2$. Hence

$$\overline{c_i\sigma(\lambda)} = \sum_{\substack{\varepsilon_1+\cdots+\varepsilon_N=i \\ q_{N-1}+\varepsilon_N-\varepsilon_1\ge 0 \\ q_1+\varepsilon_2-\varepsilon_1\le m-2}} \begin{pmatrix} m-2-q_{N-1}+\varepsilon_1-\varepsilon_N \\ q_1-1-q_{N-1}+\varepsilon_2-\varepsilon_N \\ \vdots \\ q_{N-2}-1-q_{N-1}+\varepsilon_{N-1}-\varepsilon_N \end{pmatrix}_G . \tag{6.6.7}$$

On the other hand, we have by corollary 6.6.2

$$\overline{c_i\lambda} = \sum_{\substack{\beta_1+\cdots+\beta_N=i \\ q_1+\beta_1-\beta_N\le m-2}} \begin{pmatrix} q_1+\beta_1-\beta_N \\ \vdots \\ q_{N-1}+\beta_{N-1}-\beta_N \end{pmatrix}_G .$$

For the summands in the above sum we have

$$m-2 \ge q_1+\beta_1-\beta_N \ge q_2+\beta_2-\beta_N \ge \cdots \ge q_{N-1}+\beta_{N-1}-\beta_N.$$

By eclipsing the summands with $q_{N-1}+\beta_{N-1}-\beta_N < 0$, which are equal to zero anyway, we get by lemma 6.6.3

$$\sigma(\overline{c_i\lambda}) = \sum_{\substack{\beta_1+\cdots+\beta_N=i \\ q_{N-1}+\beta_{N-1}-\beta_N\ge 0 \\ q_1+\beta_1-\beta_N\le m-2}} \begin{pmatrix} m-2-q_{N-1}+\beta_N-\beta_{N-1} \\ q_1-1-q_{N-1}+\beta_1-\beta_{N-1} \\ \vdots \\ q_{N-2}-1-q_{N-1}+\beta_{N-2}-\beta_{N-1} \end{pmatrix}_G . \tag{6.6.8}$$

There is a bijection of the summands in equations (6.6.7) and (6.6.8) that respects the additional conditions imposed on the summands. The summand $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{N-1}, \varepsilon_N)$ of the sum in equation (6.6.7) agrees with the summand $(\beta_1, \ldots, \beta_N) = (\varepsilon_2, \ldots, \varepsilon_{N-1}, \varepsilon_N, \varepsilon_1)$ of the sum in equation (6.6.8). We therefore have $\overline{\sigma(c_i\lambda)} = \sigma(\overline{c_i\lambda})$. ♣

Now we are able to prove under minor conditions that for a Young diagram λ with less then N rows the reductions of $c_i\lambda$ and of $c_i\overline{\lambda}$ agree for $1 \leq i \leq N$.

Lemma 6.6.5 *Let $\lambda = (\lambda_1, \ldots, \lambda_{N-1})$ be a Young diagram with*

$$\lambda_j + N - 1 - j \not\equiv m - 1 \bmod m \quad \text{for all } j = 1, \ldots, N-1.$$

Then $\overline{c_i\lambda} = \overline{c_i\overline{\lambda}}$ for any integer i, $1 \leq i \leq N$.

Proof For $j = 1, \ldots, N-1$ we write $\lambda_j + N - 1 - j = k_j m + r_j$ with integers $k_j \geq 0$ and $0 \leq r_j \leq m-1$. Our assumption is that $0 \leq r_j < m-1$ for $j = 1, \ldots, N-1$. We denote

$$\zeta = \begin{pmatrix} r_1 \\ \vdots \\ r_{N-1} \end{pmatrix}_G .$$

Then the Young diagram λ reduces to

$$\overline{\lambda} = (-1)^{(N+1)K} \sigma^K(\zeta) \tag{6.6.9}$$

where $K = k_1 + \cdots + k_{N-1}$. We have

$$\overline{\begin{pmatrix} k_1 m + r_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ k_{N-1} m + r_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G} = (-1)^{(N+1)K} \sigma^K \begin{pmatrix} r_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ r_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G$$

provided that $0 \leq r_j + \varepsilon_j - \varepsilon_N \leq m-2$ for $j = 1, \ldots, N-1$.

Hence, by corollary 6.6.2,

$$\begin{aligned} \overline{c_i\lambda} &= \sum_{\substack{\varepsilon_1 + \cdots + \varepsilon_N = i \\ 0 \leq r_j + \varepsilon_j - \varepsilon_N \leq m-2}} (-1)^{(N+1)K} \sigma^K \begin{pmatrix} r_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ r_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G \\ &= (-1)^{(N+1)K} \sigma^K \left(\sum_{\substack{\varepsilon_1 + \cdots + \varepsilon_N = i \\ 0 \leq r_j + \varepsilon_j - \varepsilon_N \leq m-2}} \begin{pmatrix} r_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ r_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G \right). \end{aligned}$$

The last sum in the above equation is equal to $\overline{c_i\zeta}$ by corollary 6.6.2, and we thus get

$$\overline{c_i\lambda} = (-1)^{(N+1)K}\sigma^K(\overline{c_i\zeta}).$$

We apply lemma 6.6.4 and get

$$\begin{aligned}\overline{c_i\lambda} &= (-1)^{(N+1)K}\overline{c_i\sigma^K(\zeta)} \\ &= \overline{c_i(-1)^{(N+1)K}\sigma^K(\zeta)} \\ &= \overline{c_i\overline{\lambda}}.\end{aligned}$$

♣

The remaining special case will be proved now.

Lemma 6.6.6 *Let $\lambda = (\lambda_1, \ldots, \lambda_{N-1})$ be a Young diagram with*

$$\lambda_j + N - 1 - j \equiv m - 1 \bmod m \quad \textit{for some } 1 \leq j \leq N-1.$$

Then $\overline{c_i\lambda} = 0$ for any integer i, $1 \leq i \leq N$.

Proof Let λ be a Young diagram with less than N rows and let $1 \leq i \leq N$. We write $\lambda_j + N - 1 - j = k_j m + r_j$ with integers $k_j \geq 0$ and $0 \leq r_j \leq m-1$ for $j = 1, \ldots, N-1$. We have by corollary 6.6.2

$$\overline{c_i\lambda} = \sum_{\substack{\varepsilon_1+\cdots+\varepsilon_N=i \\ 0\leq r_j+\varepsilon_j-\varepsilon_N\leq m-2}} \overline{\begin{pmatrix} k_1 m + r_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ k_{N-1}m + r_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G}.$$

We consider first the case that $r_{j_1} = r_{j_2} = m-1$ for different indices j_1 and j_2. Because $|\varepsilon_{j_1} - \varepsilon_{j_2}|$ and ε_N are either equal to 0 or 1, we see that the terms $(k_{j_1}m + r_{j_1} + \varepsilon_{j_1} - \varepsilon_N)$ and $(k_{j_2}m + r_{j_2} + \varepsilon_{j_2} - \varepsilon_N)$ are either equal or at least one of them is congruent to $(m-1)$ modulo m. Hence any summand on the right hand side in the above equation reduces to zero.

We assume from now on that exactly one of $r_1, \ldots, r_{N-1}$ is equal to $m-1$, say r_{j_1}. For a summand $(\varepsilon_1, \ldots, \varepsilon_N)$ in the above sum with $\varepsilon_{j_1} = \varepsilon_N$ we have $r_{j_1} + \varepsilon_{j_1} - \varepsilon_N = m-1$. Hence this summand is equal to zero. Hence we can restrict the sum to the summands with $\varepsilon_{j_1} \neq \varepsilon_N$.

If $\varepsilon_{j_1} = 0$ and $\varepsilon_N = 1$ then

$$\overline{\begin{pmatrix} k_1 m + r_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ k_{N-1}m + r_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G} = (-1)^{(N+1)K}\sigma^K \begin{pmatrix} r_1 + \varepsilon_1 - 1 \\ \vdots \\ m-2 \\ \vdots \\ r_{N-1} + \varepsilon_{N-1} - 1 \end{pmatrix}_G$$

with $(m-2)$ as the j_1-th entry. If $r_j = m-2$ for some $1 \leq j \leq N-1$ and $\varepsilon_j = 1$ then the above term is equal to zero because the rows j and j_1 of the determinant on the right hand side would be identical.

If $\varepsilon_{j_1} = 1$ and $\varepsilon_N = 0$ then

$$\overline{\begin{pmatrix} k_1 m + r_1 + \varepsilon_1 - \varepsilon_N \\ \vdots \\ k_{N-1} m + r_{N-1} + \varepsilon_{N-1} - \varepsilon_N \end{pmatrix}_G} = (-1)^{(N+1)(K+1)} \sigma^{K+1} \begin{pmatrix} r_1 + \varepsilon_1 \\ \vdots \\ 0 \\ \vdots \\ r_{N-1} + \varepsilon_{N-1} \end{pmatrix}_G$$

with 0 as the j_1-th entry. If $r_j = 0$ for some $1 \leq j \leq N-1$ and $\varepsilon_j = 0$ then the above term is equal to zero because the rows j and j_1 of the determinant on the right hand side would be identical. Hence

$$\overline{c_i \lambda} = \sum_{\substack{\varepsilon_1 + \cdots + \varepsilon_N = i \\ \varepsilon_{j_1} = 0,\, \varepsilon_N = 1 \\ 0 \leq r_j + \varepsilon_j - \varepsilon_N \leq m-2}} (-1)^{(N+1)K} \sigma^K \begin{pmatrix} r_1 + \varepsilon_1 - 1 \\ \vdots \\ m-2 \\ \vdots \\ r_{N-1} + \varepsilon_{N-1} - 1 \end{pmatrix}_G$$

$$+ \sum_{\substack{\varepsilon_1 + \cdots + \varepsilon_N = i \\ \varepsilon_{j_1} = 1,\, \varepsilon_N = 0 \\ 0 \leq r_j + \varepsilon_j - \varepsilon_N \leq m-2}} (-1)^{(N+1)(K+1)} \sigma^{K+1} \begin{pmatrix} r_1 + \varepsilon_1 \\ \vdots \\ 0 \\ \vdots \\ r_{N-1} + \varepsilon_{N-1} \end{pmatrix}_G .$$

We shall prove that the summand $(\varepsilon_1, \ldots, \varepsilon_{j_1-1}, 0, \varepsilon_{j_1+1}, \ldots, \varepsilon_{N-1}, 1)$ from the first sum and the summand $(\varepsilon_1, \ldots, \varepsilon_{j_1-1}, 1, \varepsilon_{j_1+1}, \ldots, \varepsilon_{N-1}, 0)$ from the second sum of the above equation add up to zero, hence the whole sum adds up to zero. To prove this claim, it is sufficient to show that

$$\begin{pmatrix} r_1 + \varepsilon_1 - 1 \\ \vdots \\ m-2 \\ \vdots \\ r_{N-1} + \varepsilon_{N-1} - 1 \end{pmatrix}_G = (-1)^N \sigma \begin{pmatrix} r_1 + \varepsilon_1 \\ \vdots \\ 0 \\ \vdots \\ r_{N-1} + \varepsilon_{N-1} \end{pmatrix}_G$$

since the summands in question are iterated images of the same power of σ of these terms. By shifting the j_1-th row of the first determinant by $(j_1 - 1)$ rows upwards and the j_1-th row of the second determinant by $(N-1-j_1)$ rows downwards, the above equation is equivalent to

$$\begin{pmatrix} m-2 \\ r_1+\varepsilon_1-1 \\ \vdots \\ r_{N-1}+\varepsilon_{N-1}-1 \end{pmatrix}_G = \sigma \begin{pmatrix} r_1+\varepsilon_1 \\ \vdots \\ r_{N-1}+\varepsilon_{N-1} \\ 0 \end{pmatrix}_G .$$

This is true by lemma 6.6.3 which can be applied after a suitable permutation of rows. Hence $\overline{c_i\lambda} = 0$. ♣

Lemma 6.6.7 *We have $\overline{c_i\lambda} = \overline{c_i\overline{\lambda}}$ for any Young diagram λ with less than N rows and any $1 \le i \le N$.*

Proof If $\lambda_j + N - 1 - j \equiv m - 1 \bmod m$ for some $1 \le j \le N-1$ then $\overline{\lambda} = 0$ by definition, hence $c_i\overline{\lambda} = 0$. Hence, by lemma 6.6.6, $\overline{c_i\lambda} = 0 = \overline{c_i\overline{\lambda}}$.

If $\lambda_j + N - 1 - j \not\equiv m - 1 \bmod m$ for all $j = 1, \dots, N-1$ then $\overline{c_i\lambda} = \overline{c_i\overline{\lambda}}$ by lemma 6.6.5. ♣

6.7 Useful results

Recall that m was defined as $l + N$.

Lemma 6.7.1 *A Young diagram $\lambda = (\lambda_1, \dots, \lambda_{N-1}, \lambda_N)$ with $\lambda_N = 0$ reduces to zero if and only if*

$$\lambda_i - \lambda_j \equiv i - j \bmod m \quad \text{for some } 1 \le i < j \le N.$$

Proof We set $\lambda_j + N - 1 - j = k_j m + r_j$ for $j = 1, \dots, N-1$ with $k_j \ge 0$ and $0 \le r_j \le m-1$. The reduction is equal to zero if either $r_j = m-1$ for some index j, or if $r_i = r_j$ for different indices i and j.

The case $r_i = r_j$ occurs if and only if $\lambda_i + N - 1 - i \equiv \lambda_j + N - 1 - j \bmod m$. This is equivalent to $\lambda_i - \lambda_j \equiv i - j \bmod m$.

The case $r_j = m-1$ occurs if and only if $\lambda_j + N - 1 - j \equiv m - 1 \bmod m$, i.e. $\lambda_j \equiv j - N \bmod m$. This can be written as $\lambda_j - \lambda_N \equiv j - N \bmod m$. ♣

Lemma 6.7.2 *Let $\lambda = (\lambda_1, \dots, \lambda_N)$ be a Young diagram with $\lambda_N = 0$ that satisfies*

$$\lambda_i - \lambda_j \equiv i - j \bmod m \text{ for some } 1 \le i < j \le N.$$

If $(\lambda_1, \dots, \lambda_i + b, \dots, \lambda_N)$ and $(\lambda_1, \dots, \lambda_j + b, \dots, \lambda_N)$ are Young diagrams for an integer $b \ge 0$ then the reductions of these two Young diagrams add up to zero.

Proof Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be a Young diagram with $\lambda_N = 0$ that satisfies

$$\lambda_i - \lambda_j \equiv i - j \bmod (N+l) \text{ for some } 1 \leq i < j \leq N$$

and furthermore

$$\mu = (\lambda_1, \ldots, \lambda_i + b, \ldots, \lambda_N) \text{ and } \zeta = (\lambda_1, \ldots, \lambda_j + b, \ldots, \lambda_N)$$

are Young diagrams for some integer $b \geq 0$.

First we consider the case $1 \leq i < j \leq N-1$. The case $j = N$ will be considered later. We set $q_f = \lambda_f + N - 1 - f$ and write $q_f = k_f m + r_f$ with $k_f \geq 0$ and $0 \leq r_f \leq m-1$ for $f = 1, \ldots, N-1$. Our assumption is that $r_i = r_j$.

We have $q_i + b = (k_i + a)m + s$ and $q_j + b = (k_j + a)m + s$ for integers $a \geq 0$ and $0 \leq s \leq m-1$. With $K = k_1 + \cdots + k_{N-1}$ we have by definition

$$\overline{\mu} = (-1)^{(N+1)(K+a)} \sigma^{(K+a)} \begin{pmatrix} r_1 \\ \vdots \\ s \\ \vdots \\ r_{N-1} \end{pmatrix}_G$$

with s as the i-th entry, and

$$\overline{\zeta} = (-1)^{(N+1)(K+a)} \sigma^{(K+a)} \begin{pmatrix} r_1 \\ \vdots \\ s \\ \vdots \\ r_{N-1} \end{pmatrix}_G$$

with s as the j-th entry. Since the corresponding determinants differ by a transposition of rows, $\overline{\mu}$ and $\overline{\zeta}$ differ by the scalar (-1) as claimed.

Now we prove the case $j = N$ by induction on b. Let $1 \leq i \leq N-1$. The induction hypothesis for b is that for any Young diagram $\tau = (\tau_1, \ldots, \tau_{N-1})$ with $\tau_i \equiv i - N \bmod m$ we have that the reduction of $\gamma = (\tau_1, \ldots, \tau_i + e, \ldots, \tau_{N-1})$ and the reduction of $\delta = (\tau_1 - e, \ldots, \tau_{N-1} - e)$ add up to zero for any $e = 0, 1, \ldots, b-1$ provided that γ and δ are Young diagrams.

The induction hypothesis for $b = 0$ is true by lemma 6.7.1.

We assume that the induction hypothesis is true for an integer $b \geq 0$. We shall deduce from this the induction hypothesis for $(b+1)$.

Let $\tau = (\tau_1, \ldots, \tau_{N-1})$ be a Young diagram that satisfies $\tau_i \equiv i - N \bmod m$, and that $(\tau_1, \ldots, \tau_i + b + 1, \ldots, \tau_{N-1})$ and $(\tau_1 - (b+1), \ldots, \tau_{N-1} - (b+1))$ are Young diagrams. We denote the Young diagrams

$$\beta = (\tau_1, \ldots, \tau_i + b, \ldots, \tau_{N-1}) \text{ and } \eta = (\tau_1 - b, \ldots, \tau_{N-1} - b).$$

We have $\overline{\beta} + \overline{\eta} = 0$ by induction hypothesis for b.

For $1 \leq r \leq N$ the addition of a cell to the r-th row of β gives a Young diagram if and only if the addition of a cell to the r-th row of η gives a Young diagram, except in one case. If $\tau_i = \tau_{i+1}$ and $b \geq 1$ then the addition of a cell to the $(i+1)$-st row of η does not give a Young diagram, but the addition of a cell to the $(i+1)$-st row of β gives a Young diagram. (In this case $i \neq N-1$ because $\tau_{N-1} \geq b+1 \geq 1$ and $\tau_N = 0$.) But this Young diagram, say ν, reduces to zero by lemma 6.7.1 because $\nu_{i+1} \equiv i+1-N \bmod m$.

Let $r \neq i$ and $r \neq N$. If both of

$$(\tau_1, \ldots, \tau_r + 1, \ldots, \tau_i + b, \ldots, \tau_{N-1})$$

and

$$(\tau_1 - b, \ldots, \tau_r - b + 1, \ldots, \tau_i - b, \ldots, \tau_{N-1} - b)$$

are Young diagrams then their reductions add up to zero. This follows from the induction hypothesis for $(b-1)$ applied to

$$(\tau_1, \ldots, \tau_r + 1, \ldots, \tau_i, \ldots, \tau_{N-1}).$$

Hence, only the terms for $r = i$ and $r = N$ appear in the following equation. Remark that for $r = N$ we have to remove a column of length N.

$$\begin{aligned}
\overline{c_1\beta} + \overline{c_1\eta} &= \overline{(\tau_1, \ldots, \tau_i + b + 1, \ldots, \tau_{N-1})} \\
&\quad + \overline{(\tau_1 - 1, \ldots, \tau_i + b - 1, \ldots, \tau_{N-1} - 1)} \\
&\quad + \overline{(\tau_1 - b, \ldots, \tau_i - b + 1, \ldots, \tau_{N-1} - b)} \\
&\quad + \overline{(\tau_1 - b - 1, \ldots, \tau_i - b - 1, \ldots, \tau_{N-1} - b - 1)}.
\end{aligned}$$

Recall that $\overline{\beta} + \overline{\eta} = 0$. By lemma 6.6.7 we get

$$\overline{c_1\beta} + \overline{c_1\eta} = \overline{c_1\overline{\beta}} + \overline{c_1\overline{\eta}} = \overline{c_1\overline{\beta} + c_1\overline{\eta}} = \overline{c_1(\overline{\beta} + \overline{\eta})} = 0.$$

We thus get

$$\begin{aligned}
0 &= \overline{(\tau_1, \ldots, \tau_i + b + 1, \ldots, \tau_{N-1})} \\
&\quad + \overline{(\tau_1 - 1, \ldots, \tau_i + b - 1, \ldots, \tau_{N-1} - 1)} \\
&\quad + \overline{(\tau_1 - b, \ldots, \tau_i - b + 1, \ldots, \tau_{N-1} - b)} \\
&\quad + \overline{(\tau_1 - b - 1, \ldots, \tau_i - b - 1, \ldots, \tau_{N-1} - b - 1)}.
\end{aligned} \tag{6.7.10}$$

For $b = 0$, this equation becomes

$$0 = 2\overline{(\tau_1, \ldots, \tau_i + 1, \ldots, \tau_{N-1})} + 2\overline{(\tau_1 - 1, \ldots, \tau_{N-1} - 1)}$$

which is (up to the negligible scalar 2) the induction hypothesis for τ with $b = 1$.

For $b \geq 1$, the induction hypothesis for $(b-1)$ applied to the Young diagram $(\tau_1 - 1, \dots, \tau_i, \dots, \tau_{N-1} - 1)$ shows that the second and the third summand in equation (6.7.10) add up to zero. The remaining equation is the induction hypothesis for $(b+1)$ applied to the Young diagram τ. ♣

Sometimes, the ideal $I_{N,l}$ appears with a different set of generators.

Lemma 6.7.3 *Let $N \geq 2$ and $l \geq 1$. Denote by P the ideal of $\mathcal{Y}$ generated by all Young diagrams with $l+1$ columns and less than N rows. Denote by Q the ideal of $\mathcal{Y}$ generated by all row diagrams $d_{l+1}, d_{l+2}, \dots, d_{m-1}$. Then $P = Q$.*

Proof By the Giambelli formula for a Young diagram λ we have

$$\lambda = \det(d_{\lambda_i + j - i})_{1 \leq i,j \leq l(\lambda)}.$$

If $\lambda_1 = l+1$ then the first row reads $d_{l+1}, d_{l+2}, \dots, d_{l+l(\lambda)}$. If λ has less than N rows then all these elements lie in Q, hence by developing the determinant by the first row we see that λ lies in Q. Hence $P \subset Q$.

Denote by $\mu_{i,j}$ the hook diagram with j cells in the first row and i cells in the first column. The number of cells of $\mu_{i,j}$ is $i+j-1$. For $i \geq 1$ and $j \geq 1$ we have by the Littlewood-Richardson rule

$$\mu_{i,j} = c_i d_{j-1} - \mu_{i+1,j-1}. \tag{6.7.11}$$

In particular, for any $r \geq 1$,

$$d_{l+r} = c_1 d_{l+r-1} - \mu_{2,l+r-1}.$$

Applying successively equation (6.7.11) to the above equation we get

$$d_{l+r} = c_1 d_{l+r-1} - c_2 d_{l+r-2} + \cdots + (-1)^r c_{r-1} d_{l+1} + (-1)^{r+1} \mu_{r,l+1}.$$

From this we deduce inductively that $d_{l+1}, \dots, d_{l+r}$ lie in the ideal generated by $\mu_{1,l+1}, \mu_{2,l+1}, \dots, \mu_{r,l+1}$. If $r \leq N-1$ then $\mu_{r,l+1}$ lies in P. Hence all of $d_{l+1}, \dots, d_{m-1}$ lie in P, hence $Q \subset P$. Hence $P = Q$. ♣

Chapter 7

A lattice model for Young diagrams

7.1 The lattice

For an integer $N \geq 2$ we consider a vector space $V(N)$ over $\mathbb{R}$ with a basis $\varepsilon_1, \ldots, \varepsilon_N$ and an inner product on $V(N)$ given by $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq N$. We define elements α_i in $V(N)$,

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$$

for $i = 1, \ldots, N-1$. They are linearly independent. We denote by $V'(N)$ the vector subspace spanned by $\alpha_1, \ldots, \alpha_{N-1}$. There are unique elements $\Lambda_1, \ldots, \Lambda_{N-1}$ of $V'(N)$ so that

$$\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$$

for any $1 \leq i, j \leq N-1$. Explicitly, these elements are given by

$$\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{N}(\varepsilon_1 + \cdots + \varepsilon_N)$$

for $i = 1, \ldots, N-1$. For notational purposes we set $\Lambda_0 = 0$ and $\Lambda_N = 0$. We denote by $P(N)$ the integral lattice spanned by $\Lambda_1, \ldots, \Lambda_{N-1}$,

$$P(N) = \{a_1\Lambda_1 + \cdots + a_{N-1}\Lambda_{N-1} \mid a_i \in \mathbb{Z} \text{ for } i = 1, \ldots, N-1\}.$$

We denote by $P_+(N)$ the cone in $P(N)$,

$$P_+(N) = \{a_1\Lambda_1 + \cdots + a_{N-1}\Lambda_{N-1} \mid a_i \in \mathbb{Z},\ a_i \geq 0 \text{ for } i = 1, \ldots, N-1\}.$$

Since $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$, we have that an element v of $V'(N)$ lies in $P(N)$ (resp. in $P_+(N)$) if and only if $\langle v, \alpha_j \rangle$ is integral (resp. integral and non-negative) for $j = 1, \ldots, N-1$.

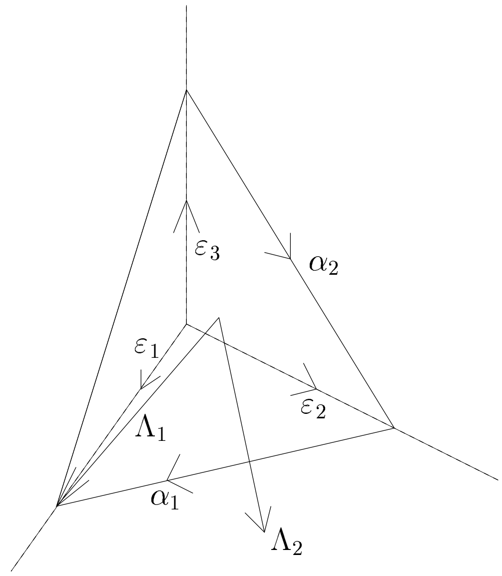


Figure 7.1: The vector space $V(3)$.

The integral lattice spanned by $\alpha_1, \ldots, \alpha_{N-1}$ is a sublattice of $P(N)$ because $\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}$, and thus any α_i lies in $P(N)$. The restriction of the inner product to $P(N)$ is not necessarily integral, in fact

$$\langle \Lambda_i, \, \Lambda_j \rangle = \min(i, j) - \frac{ij}{N} \quad \text{for } 1 \leq i, j \leq N.$$

We can write

$$\Lambda_i = i \left(\frac{1}{i}(\varepsilon_1 + \cdots + \varepsilon_i) - \frac{1}{N}(\varepsilon_1 + \cdots + \varepsilon_N) \right) \quad \text{for } 1 \leq i \leq N-1.$$

This means that Λ_i lies in the direction of the line that joins the centres of the simplices with vertices $\varepsilon_1, \ldots, \varepsilon_i$ respectively $\varepsilon_1, \ldots, \varepsilon_N$. Figure 7.1 shows in $V(3)$ the affine plane parallel to $V'(3)$ containing ε_1, ε_2 and ε_3.

7.2 Relation between $V(N)$ and $sl(N)$

The set of diagonal matrices in $sl(N)$ is a Cartan subalgebra h of $sl(N)$. The Cartan-Killing form induces an inner product B on the dual $h^\star$ of h. One can choose primitive positive roots $\beta_1, \ldots, \beta_{N-1}$ and corresponding fundamental weights $\omega_1, \ldots, \omega_{N-1}$ in $h^\star$ so that there is an isomorphism between the $\mathbb{R}$-vector space spanned by the primitive positive roots and $V'(N)$ mapping β_i to α_i and

ω_i to Λ_i. Furthermore, this isomorphism respects (up to the scalar $2N$) the inner products on $h^\star$ and $V'(N)$. The relation between non-negative integral linear combinations of fundamental weights and the irreducible representations of $sl(N)$ shows us how to relate Young diagrams and elements of $P_+(N)$. We explain this now.

7.3 The lattice and Young diagrams

We describe a bijection between Young diagrams with less than N rows and the cone $P_+(N) \subset P(N)$.

To $a_1\Lambda_1+\cdots+a_{N-1}\Lambda_{N-1}$ in $P_+(N)$ we associate the Young diagram that has a_1 columns of length 1, a_2 columns of length 2, ..., and a_{N-1} columns of length $N-1$. For example, $a_1\Lambda_1$ corresponds to a single row of length a_1, and $2\Lambda_1 + 3\Lambda_3 + \Lambda_4$ corresponds to the Young diagram $(6, 4, 4, 1)$. In general, $a_1\Lambda_1 + \cdots + a_{N-1}\Lambda_{N-1}$ corresponds to the Young diagram $\lambda = (\lambda_1, \ldots, \lambda_{N-1})$ with

$$\lambda_i = a_i + a_{i+1} + \cdots + a_{N-1} \tag{7.3.1}$$

for $i = 1, \ldots, N-1$.

Lemma 7.3.1 *Let $\lambda = (\lambda_1, \ldots, \lambda_{N-1}, \lambda_N)$ be a Young diagram with $\lambda_N = 0$. Denote its corresponding element in $P_+(N)$ by p. Then*

$$\lambda_i - \lambda_j = \langle \varepsilon_i - \varepsilon_j,\, p\rangle$$

for any $1 \le i < j \le N$.

Proof We have $p = a_1\Lambda_1 + \cdots + a_{N-1}\Lambda_{N-1}$ for some non-negative integers $a_1, \ldots, a_{N-1}$. From the above equation (7.3.1) we have

$$\lambda_i - \lambda_j = a_i + \cdots + a_{j-1}$$

for any $1 \le i < j \le N$. On the other hand,

$$\begin{aligned} \langle \varepsilon_i - \varepsilon_j,\, p\rangle &= \langle \alpha_i + \cdots + \alpha_{j-1},\, a_1\Lambda_1 + \cdots + a_{N-1}\Lambda_{N-1}\rangle \\ &= a_i + \cdots + a_{j-1}. \end{aligned}$$

because $\langle \alpha_k,\, \Lambda_m\rangle = \delta_{km}$ for any $1 \le k, m \le N-1$. ♣

From lemmas 6.7.1 and 7.3.1 we immediately deduce

Lemma 7.3.2 *A Young diagram $(\lambda_1, \ldots, \lambda_N)$ with $\lambda_N = 0$ reduces to zero if and only if it corresponds to an element of $P_+(N)$ that lies in a hyperplane*

$$H_{i,j,c} = \{x \in V(N) \mid \langle x,\, \varepsilon_i - \varepsilon_j\rangle = i - j + c(N + l)\}$$

for some $1 \le i < j \le N$ and integer c.

We shall denote the family of all the hyperplanes $H_{i,j,c}$ with $1 \le i < j \le N$ and integer c by $\mathcal{H}$.

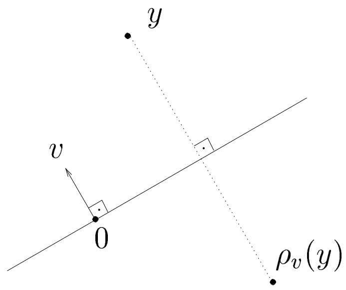


Figure 7.2: A point y and its mirror image $\rho_v(y)$.

7.4 Hyperplanes and reflections

A non-zero element v of $V(N)$ determines an $(N-1)$-dimensional hyperplane

$$\{x \in V(N) \mid \langle x,\, v\rangle = 0\}.$$

The reflection ρ_v in this hyperplane maps $y \in V(N)$ to $\rho_v(y)$ so that $\rho_v(y) - y$ is a scalar multiple of v, and $(\rho_v(y) + y)/2$ lies in this hyperplane (see figure 7.2). We deduce

$$\rho_v(y) = y - 2\frac{\langle y,\, v\rangle}{\langle v,\, v\rangle}v.$$

More general, for any $r \in \mathbb{R}$, the reflection $\rho_{v,r}$ in the hyperplane

$$\{x \in V(N) \mid \langle x,\, v\rangle = r\}$$

is given by

$$\rho_{v,r}(y) = y + 2\frac{r - \langle y,\, v\rangle}{\langle v,\, v\rangle}v. \tag{7.4.2}$$

This says that the reflections ρ_v and $\rho_{v,r}$ differ by a translation in the direction of v.

Lemma 7.4.1 *Let v and w be non-zero elements of $V(N)$ and let r and t be real numbers. The reflection in the hyperplane $\{x \in V(N) \mid \langle x,\, v\rangle = r\}$ maps the hyperplane $\{y \in V(N) \mid \langle y,\, w\rangle = t\}$ to the hyperplane*

$$\left\{z \in V(N) \mid \left\langle z,\, w - 2\frac{\langle v,\, w\rangle}{\langle v,\, v\rangle}v\right\rangle = t - 2r\frac{\langle v,\, w\rangle}{\langle v,\, v\rangle}\right\}.$$

Proof The mirror image of the hyperplane is given by

$$\{z \in V(N) \mid \langle \rho_{v,r}(z),\, w\rangle = t\}.$$

A simple application of the above formula for $\rho_{v,r}$ gives the explicit form of this hyperplane. ♣

Lemma 7.4.2 *The set $V'(N)$ and the set $P(N)$ are invariant under reflection in any hyperplane of $\mathcal{H}$.*

Proof Let us consider a hyperplane $H_{i,j,c}$ of $\mathcal{H}$. From equation (7.4.2) we deduce that the reflection $\rho_{\varepsilon_i-\varepsilon_j,c}$ in the hyperplane $H_{i,j,c}$ is given by

$$\rho_{v,c}(w) = w + (r - \langle w,\, v\rangle)v$$

where $r = i - j + c(N + l)$ and $v = \varepsilon_i - \varepsilon_j$ and thus $\langle v,\, v\rangle = 2$.

We have $\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}$, and thus $v \in P(N) \subset V'(N)$. Hence, we have $\rho_{v,c}(w) \in V'(N)$ for any w of $V'(N)$. This describes the invariance of $V'(N)$ under reflection in $H_{i,j,c}$.

Let w be an element of $P(N)$, i.e. w lies in $V'(N)$ and $\langle w,\, \alpha_k\rangle$ is integral for any $k = 1, \ldots, N-1$. Then $\langle w,\, v\rangle$ is integral because

$$\begin{aligned}\langle w,\, v\rangle &= \langle w,\, \varepsilon_i - \varepsilon_j\rangle \\ &= \langle w,\, \alpha_i + \cdots + \alpha_{j-1}\rangle \\ &= \langle w,\, \alpha_i\rangle + \cdots + \langle w,\, \alpha_{j-1}\rangle\,.\end{aligned}$$

Hence, $\rho_{v,c}(w)$ lies in $P(N)$. ♣

Lemma 7.4.3 *The family $\mathcal{H}$ of hyperplanes is invariant under reflection in any hyperplane of $\mathcal{H}$.*

Proof The essential tool is lemma 7.4.1 by which we know that the reflection in the hyperplane $H_{i,j,e}$ maps the hyperplane $H_{k,m,f}$ to the hyperplane

$$\{z \in V(N) \mid \langle z,\, w - \langle v,\, w\rangle\, v\rangle = d - c\,\langle v,\, w\rangle\} \tag{7.4.3}$$

with $v = \varepsilon_i - \varepsilon_j$, $w = \varepsilon_k - \varepsilon_m$, $c = i - j + e(N + l)$ and $d = k - m + f(N + l)$. We have that $\langle v,\, v\rangle = \langle \varepsilon_i - \varepsilon_j,\, \varepsilon_i - \varepsilon_j\rangle = 2$. Hence, all we have to know are the values of $\langle v,\, w\rangle$ which are equal to $\langle \varepsilon_i - \varepsilon_j,\, \varepsilon_k - \varepsilon_m\rangle$ for $1 \leq i < j \leq N$ and $1 \leq k < m \leq N$.

If i, j, k and m are pairwise different then $\langle v,\, w\rangle = 0$ and thus formula 7.4.3 tells us that $H_{k,m,f}$ is invariant under reflection in $H_{i,j,e}$.

In the remaining five cases we have

$$\langle \varepsilon_i - \varepsilon_j,\, \varepsilon_k - \varepsilon_m\rangle = \begin{cases} 2 & \text{if } i = k \text{ and } j = m, \\ 1 & \text{if } i = k \text{ and } j \neq m, \\ 1 & \text{if } i \neq k \text{ and } j = m, \\ -1 & \text{if } j = k, \\ -1 & \text{if } m = i. \end{cases}$$

We thus have

$$w - \langle v, w, v\rangle = \begin{cases} \varepsilon_i - \varepsilon_j & \text{if } i = k \text{ and } j = m, \\ \varepsilon_j - \varepsilon_m & \text{if } i = k \text{ and } j \neq m, \\ \varepsilon_k - \varepsilon_i & \text{if } i \neq k \text{ and } j = m, \\ \varepsilon_i - \varepsilon_m & \text{if } j = k, \\ \varepsilon_k - \varepsilon_j & \text{if } m = i. \end{cases}$$

In these five cases we get by equation (7.4.3) the hyperplanes consisting of all $z \in V(N)$ that satisfy

$$\begin{array}{rcll} \langle z, \varepsilon_j - \varepsilon_i\rangle & = & j - i + (f - 2e)(N + l) & \text{if } i = k,\ j = m \\ \langle z, \varepsilon_j - \varepsilon_m\rangle & = & j - m + (f - e)(N + l) & \text{if } i = k,\ j \neq m \\ \langle z, \varepsilon_k - \varepsilon_i\rangle & = & k - i + (f - e)(N + l) & \text{if } i \neq k,\ j = m \\ \langle z, \varepsilon_i - \varepsilon_m\rangle & = & i - m + (f + e)(N + l) & \text{if } j = k \\ \langle z, \varepsilon_k - \varepsilon_j\rangle & = & k - j + (f + e)(N + l) & \text{if } m = i. \end{array}$$

These hyperplanes are again of the form $H_{a,b,c}$ with integers a, b, and c such that $1 \leq a \leq N$, $1 \leq b \leq N$ and $a \neq b$. To ensure $a < b$, we have to multiply both sides of the above equations by (-1) if necessary. ♣

7.5 The decomposition of $V(N)$ by $\mathcal{H}$

We can write the family of hyperplanes $\mathcal{H}$ as the union of $\binom{N}{2}$ locally finite sets of hyperplanes,

$$\mathcal{H} = \bigcup_{1\leq i<j\leq N} \bigcup_{c\in\mathbb{Z}} H_{i,j,c}.$$

Hence, $\mathcal{H}$ is a locally finite set of hyperplanes. Thus, $\mathcal{H}$ induces a polyhedral decomposition of the N-dimensional Euclidean space $V(N)$ which is invariant under reflection in any hyperplane of $\mathcal{H}$. We denote the decomposition by D. The polyhedra of D are not necessarily compact.

Every hyperplane $H_{i,j,c}$ determines two half-spaces of $V(N)$. We denote

$$H^+_{i,j,c} = \{x \in V(N) \mid \langle x, \varepsilon_i - \varepsilon_j\rangle \geq i - j + c(l + N)\}$$

and $H^-_{i,j,c}$ is the other half-space. For a subset $B \subset V(N)$ we denote by $\mathring{B}$ the set of interior points of B with respect to the topology induced by the Euclidean metric.

We denote

$$H_\cap = \bigcap_{i=1}^{N-1} H^+_{i,i+1,0}$$

which is a closed unbounded convex subset of $V(N)$. It is the union of (infinitely many) polyhedra of the decomposition D.

Lemma 7.5.1

$$P(N) \cap \mathring{H}_\cap = P_+(N).$$

Proof Let p be an element of $P(N)$. Then $p \in \mathring{H}_\cap$ if and only if

$$\langle p, \varepsilon_i - \varepsilon_{i+1} \rangle > -1$$

for $i = 1, \ldots, N-1$. Since $\varepsilon_i - \varepsilon_{i+1} = \alpha_i$ and $p \in P(N)$ we have that $\langle p, \varepsilon_i - \varepsilon_{i+1} \rangle$ is an integer. The above condition on p is therefore equivalent to $\langle p, \alpha_i \rangle \geq 0$ for $i = 1, \ldots, N-1$. The element p satisfies this if and only if $p \in P_+(N)$. ♣

Lemma 7.5.2 *The set*

$$S = H^-_{1,N,1} \cap H_\cap$$

is an N-dimensional polyhedron of the decomposition D.

Proof We have to show that the interior of S is disjoint to any hyperplane of $\mathcal{H}$ and that it is not empty. The interior $\mathring{S}$ of S is given by

$$\begin{aligned} \mathring{S} \;&=\; \{x \in V(N) \mid \langle x, \varepsilon_1 - \varepsilon_N \rangle < l + 1\} \cap \\ &\quad\; \{x \in V(N) \mid \langle x, \varepsilon_i - \varepsilon_{i+1} \rangle > -1 \;\text{ for } i = 1, \ldots, N-1\}. \end{aligned} \tag{7.5.4}$$

Assume that an element x of $\mathring{S}$ lies in a hyperplane $H_{j,k,c}$ for some $1 \leq j < k \leq N$ and integer c. We have by equation (7.5.4)

$$\begin{aligned} \langle x, \varepsilon_j - \varepsilon_k \rangle &= \langle x, \varepsilon_j - \varepsilon_{j+1} \rangle + \cdots + \langle x, \varepsilon_{k-1} - \varepsilon_k \rangle \\ &> (-1) + \cdots + (-1) \\ &= j - k. \end{aligned}$$

Hence c has to be greater than zero, i.e. $c \geq 1$. Thus

$$\langle \varepsilon_j - \varepsilon_k, p \rangle \geq j - k + N + l.$$

Hence

$$\begin{aligned} \langle x, \varepsilon_1 - \varepsilon_N \rangle &= \langle x, \varepsilon_1 - \varepsilon_2 \rangle + \cdots + \langle x, \varepsilon_{j-1} - \varepsilon_j \rangle + \langle x, \varepsilon_j - \varepsilon_k \rangle \\ &\quad + \langle x, \varepsilon_k - \varepsilon_{k+1} \rangle + \cdots + \langle x, \varepsilon_{N-1} - \varepsilon_N \rangle \\ &> (-1) + \cdots + (-1) + j - k + N + l + (-1) + \cdots + (-1) \\ &= (-1)(j-1) + j - k + N + l + (-1)(N-k) \\ &= l + 1. \end{aligned}$$

The inequality $\langle x, \varepsilon_1 - \varepsilon_N \rangle > l + 1$ is in contradiction to equation (7.5.4). Hence the interior of S is disjoint to any hyperplane of $\mathcal{H}$.

The interior of S is not empty because it contains e.g. $(\varepsilon_1 + \cdots + \varepsilon_N)$ because $\langle \varepsilon_1 + \cdots + \varepsilon_N, \varepsilon_a - \varepsilon_b \rangle = 0$ for any $1 \leq a, b \leq N$. ♣

Lemma 7.5.3 *For any two N-dimensional polyhedra R and T of the decomposition D of $V(N)$ there exists a sequence of N-dimensional polyhedra of D, say $S_1, S_2, \ldots, S_k$ so that $S_1 = R$, $S_k = T$, and the polyhedra S_j and S_{j+1} differ by a reflection in a hyperplane of $\mathcal{H}$ for $j = 1, \ldots, k-1$.*

If R and T lie in $H_\cap$ then we can choose $S_2, \ldots, S_{k-1}$ to lie in $H_\cap$, too.

Proof We choose a point r in the interior of R, and a point t in the interior of T. Since $V(N)$ is a connected N-dimensional manifold, we can find a path in $V(N)$ connecting r and t which intersects the $(N-1)$-dimensional polyhedra of D transversally and which is disjoint to the $(N-2)$-skeleton of D. The sequence of N-dimensional polyhedra through which the path from r to t is going satisfies the condition of the statement of the lemma. This is because the invariance of the decomposition of D under reflection in hyperplanes of $\mathcal{H}$ implies that any two polyhedra of the decomposition D with a common $(N-1)$-dimensional side differ by a reflection in the hyperplane spanned by this side.

If R and T lie in $H_\cap$ then we can choose the above path to lie in the interior of $H_\cap$ because the interior of $H_\cap$ is connected, even after removing the $(N-2)$-skeleton of D. ♣

Lemma 7.5.4 *For any element p of $P_+(N)$ which does not lie on any hyperplane of $\mathcal{H}$ there exists a sequence of elements of $P_+(N)$, $p = p_1, p_2, \ldots, p_r$ so that p_r lies in $\mathring{S}$, and p_j and p_{j+1} differ by a reflection in a hyperplane of $\mathcal{H}$ for $j = 1, \ldots, r-1$.*

Proof Let p be an element of $P_+(N)$ that does not lie in a hyperplane of $\mathcal{H}$. Then p lies in the interior of an N-dimensional polyhedron R of the decomposition D of $V(N)$. By lemma 7.5.3 there exists a sequence $R = S_1, S_2, \ldots, S_k = S$ of N-dimensional polyhedra which all lie in $H_\cap$ so that S_i and S_{i+1} differ by a reflection in a hyperplane of $\mathcal{H}$. The successive mirror images of p are disjoint from $\mathcal{H}$, hence they lie in $\mathring{H}_\cap$. They lie in $P(N)$ by lemma 7.4.2. Therefore, they lie in $P_+(N)$ by lemma 7.5.1. The final element of this sequence of points lies in $\mathring{S}$ as required. ♣

Lemma 7.5.5 *Let p and q be two elements of $P_+(N)$ so that there exists a hyperplane of $\mathcal{H}$ with respect to which q is the mirror image of p. Then the Young diagrams λ and μ corresponding to p resp. q satisfy $\overline{\lambda} + \overline{\mu} = 0$.*

Proof Let us consider two elements p and q of $P(N)$ that are mirror images of each other with respect to a hyperplane $H_{i,j,c}$ of $\mathcal{H}$. By equation (7.4.2) we have

$$q - p = b(\varepsilon_i - \varepsilon_j) \quad \text{where } b = i - j + c(N + l) - \langle \varepsilon_i - \varepsilon_j,\, p \rangle$$

because $\langle \varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j \rangle = 2$. By interchanging p and q we may assume that $b \geq 0$. We have $\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}$. Since $p \in P_+(N)$ we deduce that $\langle p, \varepsilon_i - \varepsilon_j \rangle$ is an integer. Hence b is a non-negative integer.

We denote the element $p + b(\Lambda_{j-1} - \Lambda_j)$ by y. This element y lies in the hyperplane $H_{i,j,c}$ because

$$\begin{aligned}
\langle y, \varepsilon_i - \varepsilon_j \rangle &= \langle p + b(\Lambda_{j-1} - \Lambda_j), \varepsilon_i - \varepsilon_j \rangle \\
&= \langle p, \varepsilon_i - \varepsilon_j \rangle + b \langle \Lambda_{j-1} - \Lambda_j, \varepsilon_i - \varepsilon_j \rangle \\
&= \langle p, \varepsilon_i - \varepsilon_j \rangle + b \langle \Lambda_{j-1} - \Lambda_j, \alpha_i + \cdots + \alpha_{j-1} \rangle \\
&= \langle p, \varepsilon_i - \varepsilon_j \rangle + b \\
&= \langle p, \varepsilon_i - \varepsilon_j \rangle + i - j + c(N + l) - \langle \varepsilon_i - \varepsilon_j, p \rangle \\
&= i - j + c(N + l).
\end{aligned}$$

From $\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}$ and $\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}$ we deduce that $\varepsilon_i - \varepsilon_j = -\Lambda_{i-1} + \Lambda_i + \Lambda_{j-1} - \Lambda_j$. Hence $y = q + b(\Lambda_{i-1} - \Lambda_i)$. We claim that y lies in $P_+(N)$. Since y lies in $P(N)$, we have to show that $\langle y, \alpha_k \rangle \geq 0$ for $k = 1, \ldots, N-1$. Since p and q lie in $P_+(N)$ we have that $\langle p, \alpha_k \rangle \geq 0$ and $\langle q, \alpha_k \rangle \geq 0$ for $k = 1, \ldots, N-1$. From

$$\langle y, \alpha_k \rangle = \langle p, \alpha_k \rangle + \langle b(\Lambda_{j-1} - \Lambda_j), \alpha_k \rangle$$

we deduce that $\langle y, \alpha_k \rangle \geq 0$ for $k = 1, \ldots, N-1$ except $k = j$. From

$$\langle y, \alpha_k \rangle = \langle q, \alpha_k \rangle + \langle b(\Lambda_{i-1} - \Lambda_i), \alpha_k \rangle$$

we deduce the missing case $\langle y, \alpha_j \rangle \geq 0$. Hence y lies in $P_+(N)$ and we denote the corresponding Young diagram by λ.

We have $p = y + b(\Lambda_j - \Lambda_{j-1})$ and $q = y + b(\Lambda_i - \Lambda_{i-1})$. The Young diagram corresponding to q is $(\lambda_1, \ldots, \lambda_i + b, \ldots, \lambda_{N-1})$. The Young diagram corresponding to p is $(\lambda_1, \ldots, \lambda_j + b, \ldots, \lambda_{N-1})$ if $1 \leq j \leq N-1$, and it is $(\lambda_1 - b, \ldots, \lambda_{N-1} - b)$ if $j = N$.

The reductions of $(\lambda_1 - b, \ldots, \lambda_{N-1} - b)$ and $(\lambda_1, \ldots, \lambda_{N-1}, b)$ agree by the definition of the reduction. Hence, for any $1 \leq j \leq N$ the reduction of p is equal to the reduction of $(\lambda_1, \ldots, \lambda_j + b, \ldots, \lambda_{N-1}, \lambda_N)$ with $\lambda_N = 0$.

By lemma 6.7.2 we deduce that the reductions of the Young diagrams corresponding to p and q add up to zero. ♣

Lemma 7.5.6 *The intersection of $P_+(N)$ with the interior of the polyhedron S from Lemma 7.5.2 corresponds to the set of Young diagrams inside the $(N-1) \times l$-rectangle.*

Proof From equation (7.5.4) we deduce that an element p of $P_+(N)$ lies in $\mathring{S}$ if and only if $\langle p, \varepsilon_1 - \varepsilon_N \rangle < l+1$ and $\langle p, \varepsilon_i - \varepsilon_{i+1} \rangle > -1$ for $i = 1, \dots, N-1$. By lemma 7.3.1 this is equivalent to $\lambda_1 - \lambda_N < l+1$ and $\lambda_i - \lambda_{i+1} > -1$ for $i = 1, \dots, N-1$. Since λ is a Young diagram, the only non-trivial condition is $\lambda_1 < l+1$. This means that λ lies in the $(N-1) \times l$-rectangle. ♣

Remark The normal vector $(\varepsilon_i - \varepsilon_j)$ of any hyperplane $H_{i,j,c}$ of $\mathcal{H}$ lies in $V'(N)$. This implies that the decomposition D' of $V'(N)$ induced by $\mathcal{H}$ is the orthogonal projection along $(\varepsilon_1 + \cdots + \varepsilon_N)$ of the decomposition D of $V(N)$. Therefore, the polyhedra of the decomposition D are non-compact prisms. The intersection of S with $V'(N)$ is a compact $(N-1)$-simplex. Therefore any polyhedron of the decomposition D' is a compact simplex.

7.6 Resumé

We have identified the Young diagrams with less than N rows that reduce to zero to be the intersection of $P_+(N)$ with a family $\mathcal{H}$ of hyperplanes of the N-dimensional Euclidean vector space $V(N)$. This family of hyperplanes splits the $(N-1)$-dimensional Euclidean space $V'(N) \subset V(N)$ up into $(N-1)$-simplices that can be transformed into each other by successive reflection in these hyperplanes. If two elements p_1 and p_2 of $P_+(N)$ differ by a reflection in a hyperplane $H_{i,j,c}$ then the reductions of their corresponding Young diagrams λ_1 and λ_2 respectively differ by the scalar (-1). Furthermore, if $j \neq N$ then λ_1 and λ_2 differ by a shift of cells between the rows i and j hence λ_1 and λ_2 have the same number of cells.

As a fundamental simplex we choose the simplex next to the origin whose elements correspond to the Young diagrams that lie in the $(N-1) \times l$-rectangle. Hence we have found another way to show that any Young diagram with at most $N-1$ rows is up to a sign congruent modulo $I_{N,l}$ to a Young diagram in the $(N-1) \times l$-rectangle. We can now interpret the sign as the parity of the number of reflections that we need in order to bring an element of $P_+(N)$ into the simplex next to the origin.

Figure 7.3 shows the situation for $N = 3$ and $l = 3$. There are three classes of parallel hyperplanes in $\mathcal{H}$. Their intersection with $P(N)$ are the lattice points $p = a_1\Lambda_1 + a_2\Lambda_2$ that satisfy $\langle p, \varepsilon_i - \varepsilon_j \rangle = c(l+N)$ for some $1 \le i < j \le 3$ and integer c. Since $\langle p, \varepsilon_i - \varepsilon_j \rangle = \langle a_1\Lambda_1 + a_2\Lambda_2, \alpha_i + \cdots + \alpha_{j-1} \rangle = a_i + \cdots + a_{j-1}$, the three classes are

$$\begin{aligned} a_1 &= -1 + 6c \quad \text{for } i = 1 \text{ and } j = 2, \\ a_2 &= -1 + 6c \quad \text{for } i = 2 \text{ and } j = 3, \\ a_1 + a_2 &= -2 + 6c \quad \text{for } i = 1 \text{ and } j = 3. \end{aligned}$$

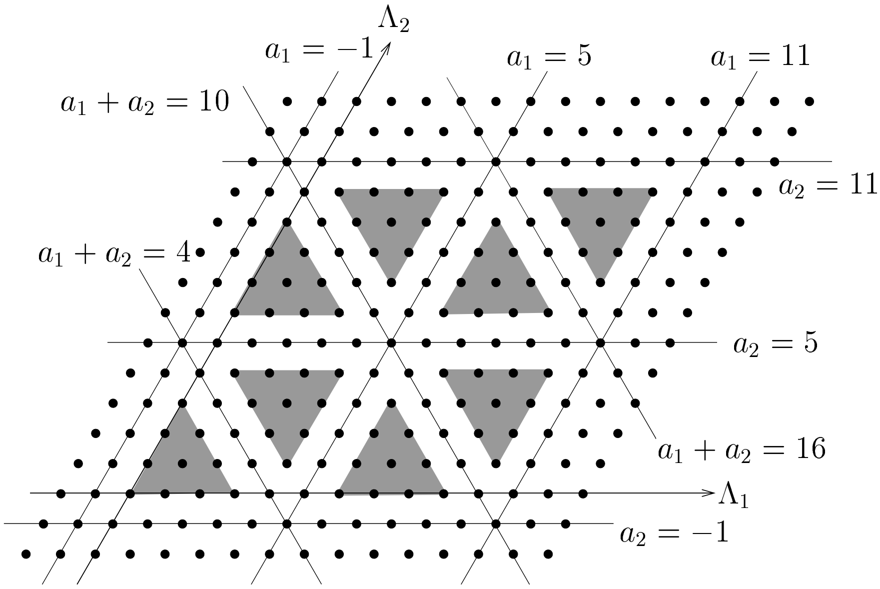


Figure 7.3: The lattice $P(3) \subset V'(3)$ of elements $a_1\Lambda_1 + a_2\Lambda_2$.

Each shaded triangles is the convex hull of the intersection of $P_+(N)$ with the interior of a 3-dimensional polyhedron of the decomposition D of $V(3)$. The angle between Λ_1 and Λ_2 is $\arccos\left(\langle\Lambda_1,\,\Lambda_2\rangle/\sqrt{\langle\Lambda_1,\,\Lambda_1\rangle\langle\Lambda_2,\,\Lambda_2\rangle}\right)$ which is equal to $\pi/3$.

Chapter 8

Invertibility of the Hopf matrix at roots of unity

We start with algebraic results in $\mathcal{Y}_{N,l}$. We recall that the dual λ^* of a Young diagram λ has been introduced in subsection 1.3.2.

8.1 Multiplication in $\mathcal{Y}_{N,l}$

Since the Young diagrams in the $(N-1) \times l$-rectangle are a basis for $\mathcal{Y}_{N,l}$, we can write the product of any two Young diagrams as a linear combination of these basis elements. Since taking the dual is a bijection of these basis elements, we can write

$$\lambda\mu = \sum_{\nu} b_{\lambda\mu\nu} \nu^*$$

in $\mathcal{Y}_{N,l}$ for integers $b_{\lambda\mu\nu}$, and the summation is over all Young diagrams in the $(N-1) \times l$-rectangle. It is easy to compute these integers. The product $\lambda\mu$ is a linear combination of Young diagrams in $\mathcal{Y}$ by the Littlewood-Richardson rule. Then one replaces each of these summands by its reduction as described in section 6.4.

Obviously, $b_{\lambda\mu\nu} = b_{\mu\lambda\nu}$ for any Young diagrams λ, μ and ν because the multiplication of Young diagrams is commutative. Interestingly, we will prove in lemma 8.1.5 that $b_{\lambda\mu\nu} = b_{\lambda\nu\mu}$ which implies that any permutation of the indices leaves $b_{\lambda\mu\nu}$ invariant. This result explains our motivation to define $b_{\lambda\mu\nu}$ as the coefficient of ν^* in the product $\lambda\mu$ instead of referring to the coefficient of ν in the product $\lambda\mu$.

For non-negative integers a and b we denote the rectangular Young diagram with a rows and b columns by (b^a).

			1	1	1
	1	1	2	2	2
	2	2	3	3	3
1	3	3	4	4	4

Figure 8.1: A strict extension ζ of $\lambda = (6,3,1,1)$ by $\mu = (6,5,5,3)$ to $(6^5) = (6,6,6,6,6)$.

Lemma 8.1.1 *Let λ be a Young diagram with at most $N-1$ rows. The only Young diagram μ for which the summand (λ_1^N) appears as a summand in the product $\lambda\mu$ in $\mathcal{Y}$ is the dual of λ. The Young diagram (λ_1^N) appears as a summand with multiplicity 1 in $\lambda\lambda^*$ in $\mathcal{Y}$.*

Proof We assume that μ is a Young diagram such that (λ_1^N) appears as a summand in the product $\lambda\mu$. Then there exists a strict extension ζ of λ by μ to (λ_1^N).

In a first step, we prove by induction on the length of the first row of λ that for every column of ζ the labelled cells read $1,2,3,\ldots$ downwards as shown in figure 8.1. This is clear in the case $\lambda_1 = 0$ for the empty Young diagram.

Let $\lambda_1 \geq 1$. The top label of the last column is 1 because the word $w(\zeta)$ starts with 1. Assume that the last column of ζ does not read $1,2,3,\ldots$ downwards, i.e. it reads $1,2,\ldots,i-1,i,j,\ldots$ with $j \neq i+1$. Then j has to be greater than $i+1$ because the last row has to be strictly increasing downwards. This implies that the label $i+1$ appears later than the label j in the word $w(\zeta)$ (because the rows are weakly increasing from left to right). But this is a contradiction to the conditions on strict extensions. We have thus proved that the last column reads $1,2,\ldots,l(\mu)$ downwards.

We denote by $\hat{\lambda}$ the Young diagram that derives from λ by removing its last column. We remove the last column of ζ, and we get an extension $\hat{\zeta}$ of $\hat{\lambda}$ by a Young diagram μ' (which derives from μ by removing the first column). This extension is easily seen to be strict because the word $w(\hat{\zeta})$ derives from $w(\zeta)$ by deleting the first appearance of each label $1,\ldots,l(\mu)$. By the induction hypothesis we know that every column of $\hat{\zeta}$ reads $1,2,3,\ldots$ downwards. Hence, every column of ζ reads $1,2,3,\ldots$ downwards.

In a second step, we count the number of occurrences of each label in ζ. Let i be a label. The label i occurs in the j-th column of ζ if and only if $\lambda_j^\vee + i \leq N$. The number of columns of ζ in which the label i appears is therefore quickly identified as $\lambda_1 - \lambda_{N-i+1}$.

This implies that $\mu_i = \lambda_1 - \lambda_{N-i+1}$ because the number of labels i in ζ is equal to the length of the i-th row of μ. Hence, $\mu = \lambda^*$.

We have thus proved that if there exists a strict extension ζ of λ by μ to (λ_1^N) then ζ reads $1, 2, 3, \ldots$ in every column downwards, and $\mu = \lambda^*$. In fact, this extension of λ by λ^* is easily seen to be strict. Its uniqueness implies that the rectangular Young diagram (λ_1^N) appears exactly once as a summand of $\lambda\lambda^*$. ♣

Lemma 8.1.2 *The empty Young diagram $\emptyset$ is the only Young diagram that lies in the $(N-1) \times 2l$-rectangle which reduces to either $\emptyset$ or to $-\emptyset$ in $\mathcal{Y}_{N,l}$.*

Proof Let λ be a Young diagram that fits in the $(N-1) \times 2l$-rectangle and that reduces to either $\emptyset$ or $-\emptyset$. We write $\lambda_i + N - 1 - i = k_i(l + N) + r_i$ with integers $k_i \geq 0$ and $0 \leq r_i \leq l + N - 1$ for $i = 1, \ldots, N-1$. Since $\overline{\lambda}$ is non-zero, we have that none of the r_i is equal to $l + N - 1$. Since $\lambda_i \leq 2l$ we have that k_i is equal to either 0 or 1. Hence $0 \leq K = k_1 + \cdots + k_{N-1} \leq N - 1$.

The case $K = 0$ appears if and only if either $\lambda_1 = l + 1$ (which is not possible since $r_i \neq l + N - 1$), or $\lambda_1 \leq l$ in which case $\overline{\lambda} = \lambda$, and therefore $\overline{\lambda} = \pm\emptyset$ implies that $\lambda = \emptyset$.

From now on we consider the case $1 \leq K \leq N - 1$, i.e. $k_j = 1$ for at least one index j. In order that $\overline{\lambda} = \pm\emptyset$, we need that

$$\sigma^K \begin{pmatrix} r_1 \\ \vdots \\ r_{N-1} \end{pmatrix}_G = \pm\emptyset.$$

This is equivalent to

$$\begin{pmatrix} r_1 \\ \vdots \\ r_{N-1} \end{pmatrix}_G = (l^{(N-K)})$$

where $(l^{(N-K)})$ denotes the Young diagram that consists of $(N-K)$ rows of length l. This rectangular Young diagram can be written as

$$\begin{pmatrix} l + N - 2 \\ l + N - 3 \\ \vdots \\ l + k - 1 \\ K - 2 \\ K - 3 \\ \vdots \\ 0 \end{pmatrix}_G$$

with the notation from section 6.4. Hence, if λ reduces to $\pm\emptyset$ then

$$\{r_1, \ldots, r_{N-1}\} = \{l+N-2, l+N-3, \ldots, l+K-1, K-2, K-3, \ldots, 0\}.$$

$k_i = 1$ for some index i implies that $r_i \leq l-2$ because $\lambda_i + N - 1 - i \leq 2l + N - 2$. Hence, the above equality of the two sets implies that $k_i = 1$ for at most $K-1$ indices i. Since $K = k_1 + \ldots + k_{N-1}$ we get $K \leq K-1$ which is a contradiction. Hence, there exists no Young diagram in the $(N-1) \times 2l$-rectangle that reduces to $\pm\emptyset$ in the case $1 \leq K \leq N-1$. ♣

The quotient map from the ring of Young diagrams $\mathcal{Y}$ to $\mathcal{Y}_{N,l}$ factors through $\mathcal{Y}_N$. The quotient map from $\mathcal{Y}$ to $\mathcal{Y}_N$ maps every Young diagram either to zero or to a Young diagram with less than N rows. The quotient map from $\mathcal{Y}$ to $\mathcal{Y}_{N,l}$ maps every Young diagram either to zero or (up to a sign) to a Young diagram in the $(N-1) \times l$-rectangle.

Lemma 8.1.3 *Let λ and μ be Young diagrams with at most $(N-1)$ rows. If the empty Young diagram appears as a summand of $\lambda\mu$ in $\mathcal{Y}_N$ then $\mu = \lambda^*$.*

Proof If the empty Young diagram appears as a summand of $\lambda\mu$ in $\mathcal{Y}_N$ then this summand comes from a summand η of $\lambda\mu$ in $\mathcal{Y}$ which becomes the empty Young diagram in $\mathcal{Y}_N$, i.e. η is an $(N \times k)$-rectangle for some k. Since λ is a subdiagram of any summand of $\lambda\mu$ in $\mathcal{Y}$, we have that $k \geq \lambda_1$. In fact, k cannot be greater than λ_1 since any column of any (strict) extension of λ by μ has at most $l(\mu)$ labelled cells, and $l(\mu) \leq N-1$ by assumption. Hence, η is the $(N \times \lambda_1)$-rectangle, and by lemma 8.1.1 we deduce that $\mu = \lambda^*$. ♣

Lemma 8.1.4 *Let λ and μ be Young diagrams in the $(N-1) \times l$-rectangle. If the empty Young diagram appears as a summand of $\lambda\mu$ in $\mathcal{Y}_{N,l}$ then $\mu = \lambda^*$ in which case the multiplicity of the empty Young diagram is equal to 1.*

Proof We know that in $\mathcal{Y}_N$ we can write the product $\lambda\mu$ uniquely as a linear combination of Young diagrams with at most $(N-1)$ rows. Since λ and μ have at most l columns, the summands appearing in $\lambda\mu$ in $\mathcal{Y}$ have at most $2l$ columns and the same is true in $\mathcal{Y}_N$.

By lemma 8.1.2 we know that the empty Young diagram is the only Young diagram in the $(N-1) \times 2l$ rectangle that reduces to $\pm\emptyset$ in $\mathcal{Y}_{N,l}$. Hence, the empty Young diagram appears in $\mathcal{Y}_{N,l}$ as a summand of $\lambda\mu$ if and only if the empty diagram appears as a summand of $\lambda\mu$ in $\mathcal{Y}_N$. This happens by lemma 8.1.3 if and only if $\mu = \lambda^*$ in which case the multiplicity of the empty Young diagram is equal to 1. ♣

Lemma 8.1.5 *The coefficients $b_{\lambda\mu\nu}$ do not change under any permutation of their indices.*

Proof Let λ, μ and η be any Young diagrams in the $(N-1)\times l$-rectangle. We have (by definition of the integers $b_{\lambda\mu\nu}$)

$$\lambda\mu = \sum_{\nu} b_{\lambda\mu\nu}\nu^*$$

in $\mathcal{Y}_{N,l}$, and therefore

$$\lambda\mu\eta = \sum_{\nu} b_{\lambda\mu\nu}\nu^*\eta.$$

When we write the right hand side of the above equation as a linear combination of Young diagrams in the $(N-1)\times l$-rectangle then the coefficient of the empty Young diagram is equal to $b_{\lambda\mu\eta}$. This is because $\nu^*\eta$ does not involve the empty Young diagram unless $\nu=\eta$, and then the coefficient of the empty diagram is equal to 1 as seen in lemma 8.1.4.

The left hand side of the above equation is symmetric under permutation of λ, μ and η because $\mathcal{Y}_{N,l}$ is Abelian. Hence $b_{\lambda\mu\eta}$ is symmetric under any permutation of its indices. ♣

Let $\varrho:\mathcal{Y}_{N,l}\to\mathcal{Y}_{N,l}$ be a ring endomorphism. We define an element

$$\Omega_\varrho = \sum_{\lambda} \varrho(\lambda^*)\lambda \in \mathcal{Y}_{N,l}$$

where the sum is over all Young diagrams λ that lie in the $(N-1)\times l$-rectangle. Obviously, Ω_ϱ depends on N and l, but this shall not lead to confusion because we fix N and l throughout.

In section 8.2 we shall consider $\mathcal{Y}_{N,l}$ as an algebra over $\mathbb{C}$ and construct Ω_ϱ for an algebra homomorphism $\varrho:\mathcal{Y}_{N,l}\to\mathbb{C}$.

Theorem 8.1.6 *Let $\varrho:\mathcal{Y}_{N,l}\to\mathcal{Y}_{N,l}$ be a ring endomorphism and let μ be any Young diagram. Then $\mu\Omega_\varrho=\varrho(\mu)\Omega_\varrho$ in $\mathcal{Y}_{N,l}$.*

Proof It is sufficient to prove the statement for elements μ of a basis of $\mathcal{Y}_{N,l}$. Hence, let μ be any Young diagram in the $(N-1)\times l$-rectangle. We have

$$\begin{aligned}
\mu\Omega_\varrho &= \sum_{\lambda} \varrho(\lambda^*)\mu\lambda \\
&= \sum_{\lambda} \varrho(\lambda^*)\sum_{\nu} b_{\mu\lambda\nu}\upsilon^* \\
&= \sum_{\nu} \nu^*\sum_{\lambda} b_{\mu\lambda\nu}\varrho(\lambda^*)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_\nu \nu^* \varrho\left(\sum_\lambda b_{\mu\lambda\nu}\lambda^*\right) \\
&= \sum_\nu \nu^* \varrho\left(\sum_\lambda b_{\mu\nu\lambda}\lambda^*\right) \\
&= \sum_\nu \nu^* \varrho(\mu\nu) \\
&= \sum_\nu \nu^* \varrho(\mu)\varrho(\nu) \\
&= \varrho(\mu)\sum_\nu \varrho(\nu)\nu^* \\
&= \varrho(\mu)\Omega_\varrho
\end{aligned}
$$

where we used that $b_{\mu\lambda\nu} = b_{\mu\nu\lambda}$ and that taking the dual induces a permutation of the Young diagrams in the $(N-1)\times l$-rectangle. ♣

8.2 The Hopf matrix

We recall the results and the notation from chapter 5. We consider the skein of the annulus C with coefficients $\mathbb{C}[x^{\pm 1}, v^{\pm 1}, s^{\pm 1}, (s^i - s^{-i})^{-1}, i \geq 1]$ and its submodule C_+. We fix a complex number ξ such that ξ^N is a root of unity of order $2(l+N)$. We denote the substitution $x = \xi$, $v = \xi^{N^2}$ and $s = \xi^{-N}$ in a rational function from $\mathbb{C}(x, v, s)$ by $\alpha_{N\,l}$. We recall that we denote $\langle Q_\lambda \rangle$ by $\langle \lambda \rangle$ occasionally.

We proved in corollary 5.2.1 that $Q_{c_i} \doteq 0$ for $i \geq N+1$, $Q_{c_N} \doteq \emptyset$, and $Q_{d_j} \doteq 0$ for any j with $l+1 \leq j \leq l+N-1$. Since $a \doteq b$ implies that $aQ_\lambda \doteq bQ_\lambda$ for any Young diagram λ, we looked in chapter 6 at the ideal $I_{N,l}$ of the ring of Young diagrams $\mathcal{Y}$ generated by $c_N - c_0$, c_i for $i \geq N+1$, and d_j for $l+1 \leq j \leq l+N-1$.

In particular, the map $\lambda \mapsto \alpha_{N\,l}(\langle \lambda \rangle)$ from $\mathcal{Y}$ to $\mathbb{C}$ factors through $\mathcal{Y}_{N,l}$, where $\langle \lambda \rangle$ denotes the Homfly polynomial of Q_λ as a subset of $\mathbb{R}^2$. We consider $\mathcal{Y}$ as an algebra over $\mathbb{C}$, and the map $\lambda \mapsto \alpha_{N\,l}(\langle \lambda \rangle)$ as the algebra endomorphism of $\mathcal{Y}$ given by $\lambda \mapsto \alpha_{N\,l}(\langle \lambda \rangle)c_0$. We define

$$\Omega = \sum_\lambda \alpha_{N\,l}(\langle \lambda^* \rangle)\lambda \in C_+$$

where the sum is over all Young diagrams λ in the $(N-1)\times l$-rectangle. We can apply theorem 8.1.6 and get

Lemma 8.2.1 *$Q_\lambda\Omega \doteq \langle Q_\lambda \rangle\,\Omega$ for any Young diagram λ in the $(N-1)\times l$-rectangle.*

We defined $\overline{\lambda}$ for any Young diagram λ in section 6.4. This is either equal to zero or up to a sign equal to a Young diagram μ, $\overline{\lambda} = \varepsilon\mu$ where $\varepsilon^2 = 1$. We define $Q_{\overline{\lambda}} = \varepsilon Q_\mu$ in this case. Lemma 6.4.1 shows that $\overline{\lambda} = \lambda$ modulo the ideal of the algebra of Young diagrams generated by $c_N - c_0$, c_i for $i \geq N+1$, and d_j for $l+1 \leq j \leq l+N-1$. We therefore get

Lemma 8.2.2 *$Q_\lambda \doteq Q_{\overline{\lambda}}$ for any Young diagram λ.*

We can extend lemma 8.2.1 now to all Young diagrams λ.

Lemma 8.2.3 *$Q_\lambda \Omega \doteq \langle Q_\lambda \rangle \Omega$ for any Young diagram λ.*

Proof From lemma 8.2.2 we deduce that $\langle Q_\lambda \rangle \doteq \langle Q_{\overline{\lambda}} \rangle$ by looking at the evaluation on the unknot. We use lemma 5.2.2 and lemma 8.2.1 to get

$$Q_\lambda \Omega \doteq Q_{\overline{\lambda}} \Omega \doteq \langle Q_{\overline{\lambda}} \rangle \Omega \doteq \langle Q_\lambda \rangle \Omega.$$

♣

The set of Q_λ for all Young diagrams λ is a linear basis for C_+ over the scalars $\mathbb{C}[x^{\pm 1}, v^{\pm 1}, s^{\pm 1}, (s^i - s^{-i})^{-1}, i \geq 1]$. We therefore have $y\Omega \doteq \langle y \rangle \Omega$ for any element y of C_+ over the scalars $\mathbb{C}[x^{\pm 1}, v^{\pm 1}, s^{\pm 1}, (s^i - s^{-i})^{-1}, i \geq 1]$ whenever the substitution $\alpha_{N\,l}$ is defined for $\langle y \rangle$.

We consider an oriented link diagram $L_1 \cup L_2$ in the annulus as depicted in figure 8.2. In fact, this lies in the subalgebra C_+ of the skein of the annulus. When we decorate L_1 by Q_λ and L_2 by Ω then the resulting element of the skein of the annulus lies again in C_+. This element is a scalar multiple $p_{\lambda\,\Omega}$ of Q_λ by equation (2.4.2). This is similar to lemma 2.4.7. We remark that the orientation of the decoration is now different. The following lemma appeared in [3] with a different proof.

Lemma 8.2.4 *We have $p_{\lambda\,\Omega} \doteq 0$ for any Young diagram λ in the $(N-1) \times l$-rectangle different from the empty diagram provided we choose ξ to be a primitive root of unity of order $2N(l+N)$.*

Proof Let λ be a Young diagram in the $(N-1) \times l$-rectangle.

We decorate the Hopf link depicted in figure 8.2. We decorate the component L_1 with Q_λ and the component L_2 with the product of Q_{c_i} and Ω as depicted in figure 8.3. We denote the resulting element of C_+ by T. Each of the two loops can be removed at the expense of a scalar, hence T is equal to $p_{\lambda\,c_i} p_{\lambda\,\Omega} Q_\lambda$ in C_+.

On the other hand, we have $Q_{c_i}\Omega \doteq \langle Q_{c_i} \rangle \Omega$ by lemma 8.2.1. Hence the loop decorated by Q_{c_i} can be swallowed at the expense of the scalar $\langle Q_{c_i} \rangle$, while the loop decorated with Ω is swallowed at the expense of the scalar $p_{\lambda\Omega}$ as before. We thus get $T \doteq \langle Q_{c_i} \rangle p_{\lambda\,\Omega} Q_\lambda$. When we decorate the unknot by these two elements of C_+, we get by definition of $\doteq$ that

$$\langle \langle Q_{c_i} \rangle p_{\lambda\,\Omega} Q_\lambda \rangle \doteq \langle p_{\lambda\,c_i} p_{\lambda\,\Omega} Q_\lambda \rangle,$$

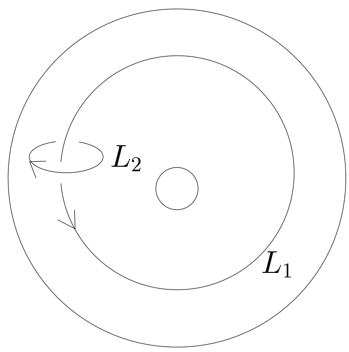


Figure 8.2: The Hopf link $L_1 \cup L_2$ in the annulus.

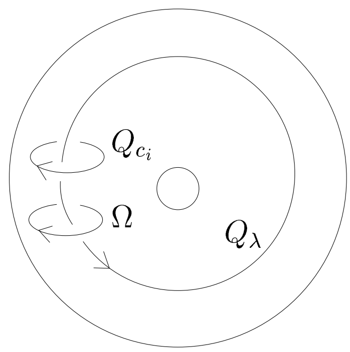


Figure 8.3: Decorated Hopf link in the annulus.

which is equivalent to

$$\langle Q_{c_i} \rangle \, p_{\lambda\,\Omega} \, \langle Q_\lambda \rangle \doteq c_{\lambda\,c_i} p_{\lambda\,\Omega} \, \langle Q_\lambda \rangle \, .$$

If λ lies in the $(N-1) \times l$-rectangle then $\alpha_{N\,l}(\langle Q_\lambda \rangle)$ is different from zero by lemma 3.6.1. Hence,

$$\langle Q_{c_i} \rangle \, p_{\lambda\,\Omega} \doteq p_{\lambda\,c_i} p_{\lambda\,\Omega}.$$

From now on let λ be such a Young diagram in the $(N-1) \times l$-rectangle for which $\alpha_{N\,l}(p_{\lambda\,\Omega})$ is different from zero. The above equation then implies that $p_{\lambda\,Q_{c_i}} \doteq \langle Q_{c_i} \rangle$ for any $i \geq 0$. This implies that $\langle Q_\lambda, Q_{c_i} \rangle \doteq \langle Q_\lambda \rangle \langle Q_{c_i} \rangle$ where $\langle Q_\lambda, Q_{c_i} \rangle$ is the Homfly polynomial of the Hopf link with decorations Q_λ and Q_{c_i}.

When we look at the definition of $E_\lambda(X)$ from section 4.1 we see that the equality $\langle Q_\lambda, Q_{c_i} \rangle \doteq \langle Q_\lambda \rangle \langle Q_{c_i} \rangle$ implies that $E_\lambda(X)$ agrees with $E_\emptyset(X)$ after the substitution $\alpha_{N\,l}$. Lemma 4.3.3 gives explicit formulas for $E_\lambda(X)$ and for $E_\emptyset(X)$ after the substitution $v = s^{-N}$. We thus deduce from $E_\lambda(X) = E_\emptyset(X)$ that

$$\prod_{k=1}^{N} (1 + s^{N+2\lambda_k - 2k+1} x^{2|\lambda|} X) \doteq \prod_{j=1}^{N} (1 + s^{N-2j+1} X)$$

which is equivalent to

$$\prod_{k=1}^{N} (1 + s^{2\lambda_k - 2k} x^{2|\lambda|} X) \doteq \prod_{j=1}^{N} (1 + s^{-2j} X)$$

since $\alpha_{N\,l}(s^{N+1})$ is non-zero. By the definition of $\doteq$ this is equivalent to

$$\{\xi^{-N(2\lambda_k - 2k)} \xi^{2|\lambda|} \mid k = 1, \ldots, N\} = \{\xi^{2Nj} \mid j = 1, \ldots, N\}.$$

In particular, the value for $k = N$ on the left hand side has to appear in the set on the right hand side. This means that $\xi^{2N^2} \xi^{2|\lambda|} = \xi^{2Nj}$ for some $1 \leq j \leq N$.

Equivalently, $\xi^{2(|\lambda|+N^2-Nj)} = 1$. Since λ lies in the $(N-1) \times l$-rectangle we have $0 \leq |\lambda| \leq (N-1)l$. We have $0 \leq N^2 - Nj < N^2$ for $1 \leq j \leq N$. Hence,

$$0 \leq 2(|\lambda| + N^2 - Nj) < 2(N-1)l + 2N^2 = 2N(l+N) - 2l < 2N(l+N).$$

We chose ξ to be a root of unity of order $2N(l+N)$, hence $\xi^{2(|\lambda|+N^2-Nj)} = 1$ implies that $2(|\lambda| + N^2 - Nj) = 0$ which implies that $|\lambda| = 0$, hence λ is the empty diagram.

Our assumption that $\alpha_{N\,l}(p_{\lambda\,\Omega})$ is different from zero for some Young diagram λ in the $(N-1)\times l$-rectangle has led us to the result that λ is the empty diagram. This implies that $p_{\lambda\,\Omega} \doteq 0$ for any Young diagram λ in the $(N-1) \times l$-rectangle different from the empty diagram. ♣

We immediately deduce from lemma 8.2.4 that

Corollary 8.2.5 $\langle \Omega, Q_\lambda \rangle \doteq 0$ *for every Young diagram λ in the $(N-1) \times l$-rectangle different from the empty Young diagram.*

The following lemma settles the case $\lambda = \emptyset$ which is not covered by lemma 8.2.4. We obviously have $p_{\emptyset\,\Omega} = \langle \Omega \rangle$.

Lemma 8.2.6 $\langle \Omega \rangle$ *becomes a positive real number after substituting $v = s^{-N}$ and then substituting s by any complex number of norm equal to 1.*

Proof We denote by P the complex number derived from $\langle Q_\lambda \rangle$ by first substituting $v = s^{-N}$ and then substituting s by a complex number τ of norm equal to 1. By lemma 4.1.5 and equation (4.3.9) we have that $\langle Q_\lambda \rangle$ becomes after the substitution $v = s^{-N}$ the Schur function in the variables $s^{-N+1}, s^{-N+3}, \ldots, s^{N-1}$. Hence, P is the Schur function s_λ in the variables $\tau^{-N+1}, \tau^{-N+3}, \ldots, \tau^{N-1}$. The conjugate of τ is equal to τ^{-1} because the norm of τ is equal to 1. Hence, conjugation induces a permutation of the variables of the Schur function s_λ. Since the Schur function is symmetric in its variables, the conjugate of P is equal to P. Hence, P is a real number and P^2 is a non-negative real number.

We have $\Omega = \sum_\lambda \langle Q_\lambda \rangle Q_\lambda$, hence

$$\langle \Omega \rangle = \sum_\lambda \langle Q_\lambda \rangle^2$$

where the summation is over all Young diagram in the $(N-1) \times l$-rectangle. Hence, $\langle \Omega \rangle$ becomes a non-negative real number after first substituting $v = s^{-N}$ and then $s = \tau$. In fact, this sum is positive because the summand for the empty Young diagram is equal to 1. ♣

Remark One can prove that $\alpha_{N\,l}(\langle Q_\lambda \rangle)$ is a real number directly from lemma 3.6.1 because each fraction appearing as a factor in the formula is self-conjugate and therefore real. The denominators do not become zero because the hook length of any cell of any Young diagram in the $(N-1)\times l$-rectangle is smaller than $l+N$. Even though this alternate proof is more straightforward, the above proof gives a more detailed view on $\langle Q_\lambda \rangle$.

Lemma 8.2.7 *$\langle \lambda \rangle = \langle \lambda^* \rangle$ for any Young diagram λ with at most $(N-1)$ rows after the substitution $v = s^{-N}$.*

Proof This is checked quickly by lemma 3.6.1 by substituting $v = s^{-N}$. In fact, it is sufficient to show this for λ equal to all column diagrams, i.e. $\langle c_i \rangle = \langle c_{N-i} \rangle$. Lemma 1.3.4 then ensures that $\langle \lambda \rangle = \langle \lambda^* \rangle$ for any Young diagram λ with at most $N-1$ rows. ♣

Lemma 8.2.8 *Let λ and μ be Young diagrams with at most $N-1$ rows. Then $\langle \lambda^*, \mu \rangle$ is the complex conjugate of $\langle \lambda, \mu \rangle$ after the substitutions $s^{2(l+N)=1}$, $v = s^{-N}$ and $x^N = s^{-1}$.*

Proof We have $\lambda^*_i = \lambda_1 - \lambda_{N-i+1}$ for $i = 1, \ldots, N-1$, and $|\lambda^*| = N\lambda_1 - |\lambda|$. By lemma 4.3.3 we get

$$
\begin{aligned}
E^N_{\lambda^*}(X) &= \prod_{i=1}^{N} (1 + s^{N+2\lambda^*_i - 2i+1} x^{2|\lambda^*|} X) \\
&= \prod_{i=1}^{N} (1 + s^{N+2(\lambda_1 - \lambda_{N-i+1}) - 2i+1} x^{2(\lambda_1 N - |\lambda|)} X) \\
&= \prod_{i=1}^{N} (1 + s^{N-2\lambda_{N-i+1} - 2i+1} x^{-2|\lambda|} X) \\
&= \prod_{j=1}^{N} (1 + s^{-N-2\lambda_j + 2j-1} x^{-2|\lambda|} X)
\end{aligned}
$$

where we used that $x^N = s^{-1}$. Hence, $E^N_{\lambda^*}(X)$ is the complex conjugate of $E^N_\lambda(X)$. Lemma 4.1.5 implies that

$$\frac{1}{\lambda^*} \langle \lambda^*, \eta \rangle = \overline{\frac{1}{\langle \lambda \rangle} \langle \lambda, \eta \rangle}.$$

Lemma 8.2.7 finally implies that $\langle \lambda^*, \eta \rangle = \overline{\langle \lambda, \eta \rangle}$. ♣

We fix from now on an arbitrary total ordering of all the Young diagrams that lie in the $(N-1)\times l$-rectangle. The indices of any of the following square matrices are ordered in this way. We denote by H the matrix whose entry indexed by Young diagrams λ and μ is the Homfly polynomial $\langle\lambda,\mu\rangle$ of the Hopf link (with framing zero and linking number 1) decorated by Q_λ and Q_μ. Clearly, H is symmetric. We denote the identity matrix by E.

Theorem 8.2.9 *We have*

$$H\overline{H} = \alpha_{N\,l}(\langle\Omega\rangle)E$$

after the substitutions $s = x^{-N}$, $v = s^{-N}$, and x by a root of unity of order $2N(l+N)$.

Proof The entry $k_{\lambda\,\mu}$ of $H\overline{H}$ indexed by λ and μ is equal to

$$\sum_\eta \langle\lambda,\eta\rangle\langle\eta,\mu^*\rangle$$

where the summation is over all Young diagrams η that lie in the $(N-1)\times l$-rectangle. By lemma 4.1.3 we have that $\langle\lambda,\eta\rangle\langle\eta,\mu\rangle = \langle\eta\rangle\,\langle\eta,\lambda\mu\rangle$, hence

$$\begin{aligned} k_{\lambda\,\mu} &= \sum_\eta \langle\eta\rangle\,\langle\eta,\lambda\mu^*\rangle \\ &= \langle\Omega,\lambda\mu^*\rangle \\ &\doteq \langle\Omega,\overline{\lambda\mu^*}\rangle \end{aligned}$$

where we used lemma 8.2.3 in the last equality. We can write $\overline{\lambda\mu^*}$ as a linear combination of Young diagrams in the $(N-1)\times l$-rectangle. By corollary 8.2.5 we see that only the multiplicity of the empty Young diagram makes a non-zero contribution.

We know by lemma 8.1.4 that the empty Young diagram appears as a summand in $\overline{\lambda\mu^*}$ if and only if $\mu=\lambda$ in which case it appears with multiplicity equal to 1. Hence, $k_{\lambda\,\lambda} = \alpha_{N\,l}(\langle\Omega\rangle)$ and $k_{\lambda\,\mu} = 0$ if $\mu \neq \lambda$. ♣

Chapter 9

Homfly polynomials at roots of unity and $\mathcal{Y}_{N,l}$

We fix integers $N \geq 2$ and $l \geq 1$. We consider the substitutions of s by a primitive root of unity of order $2(l+N)$, x by an N-th root of s^{-1}, and v by s^{-N}. We shall abbreviate this by $s^{2(N+l)} = 1$, $x^N = s^{-1}$, and $v = s^{-N}$. We denote the Homfly polynomial after these substitutions by $\chi_{N,l}$.

9.1 Homfly polynomials at roots of unity

Lemma 9.1.1 $\langle Q_{d_l} \rangle = 1$ *after the substitutions* $v = s^{-N}$ *and* $s^{2(l+N)} = 1$.

Proof We have

$$\begin{aligned} s^{N+i} - s^{-N-i} &= s^{N+l}\big(s^{i-l} - s^{-2N-i-l}\big) \\ &= -s^{N+l}\big(s^{l-i}s^{-2(l+N)} - s^{i-l}\big) \end{aligned}$$

for any integer i. If s is a primitive root of unity of order $2(l+N)$ then s^{l+N} is equal to -1 and therefore $s^{N+i} - s^{-N-i} = s^{l-i} - s^{i-l}$. Lemma 3.6.1 gives a formula for $\langle Q_{d_l} \rangle$ with substitutions $v = s^{-N}$ and $s^{2(l+N)} = 1$,

$$\begin{aligned} \langle Q_{d_l} \rangle &= \frac{v^{-1} - v}{s - s^{-1}} \frac{v^{-1}s - vs^{-1}}{s^2 - s^{-2}} \cdots \frac{v^{-1}s^{l-1} - vs^{-l+1}}{s^l - s^{-l}} \\ &= \frac{s^N - s^{-N}}{s - s^{-1}} \frac{s^{N+1} - s^{-N-1}}{s^2 - s^{-2}} \cdots \frac{s^{N+l-1} - s^{-N-l+1}}{s^l - s^{-l}} \\ &= \frac{s^l - s^{-l}}{s - s^{-1}} \frac{s^{l-1} - s^{-l+1}}{s^2 - s^{-2}} \cdots \frac{s - s^{-1}}{s^l - s^{-l}} \\ &= 1 \end{aligned}$$

where we used that $s^{N+i} - s^{-N-i} = s^{l-i} - s^{i-l}$. ♣

Figure 9.1: Switching crossings at the expense of the scalar $(s^2x^{-2l})^{\pm 1}$.

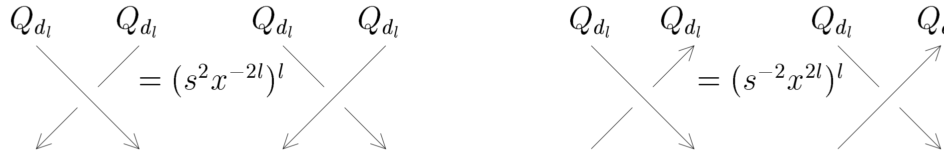


Figure 9.2: Switching crossings at the expense of the scalar $(s^2x^{-2l})^{\pm l}$.

Whenever we have locally a component decorated with Q_{d_l} overpassing a simple arc in a crossing of sign ε then we can switch the crossing at the expense of the scalar $(s^{-2}x^{2l})^{\varepsilon}$ as shown in figure 9.1 provided we make the substitutions $x^N = s^{-1}$, $v = s^{-N}$ and $s^{2(l+N)} = 1$. The argument is virtually the same as in lemma 5.1.4 and the formula can be derived as well from figures 5.2 and 5.3 by applying the map γ from subsection 2.4.1 that replaces s by $-s^{-1}$.

We get the skein relations in figure 9.2 by applying the corresponding skein relations in figure 9.1 l-times to each summand of Q_{d_l}. This is possible because Q_{d_l} can be written as a sum of diagrams each looking like l parallel arcs near the crossing.

We know by theorem 17 in [2] that in the Hecke algebra H_l we can remove a curl decorated by the idempotent corresponding to d_l at the expense of a scalar f that is given by

$$f = x^{l^2}v^{-l}s^{l(l-1)}.$$

We define p by

$$p = -s^{-1}x^l.$$

We have $p^l = f$ when we substitute $v = s^{-N}$ and $s^{2(l+N)} = 1$ because

$$p^l = (s^{l+N}s^{-1}x^l)^l = x^{l^2}s^{lN}s^{l(l-1)}$$

where we used that $s^{l+N} = -1$.

The scalars appearing in figure 9.1 are p^{-2} and p^2.

Remark By connecting in figure 9.2 the arcs with a straight line at the right, we see that changing from a positive curl to a negative curl means multiplication with the scalar $(s^{-2}x^{2l})^l$, i.e. $f^2 = (s^{-2}x^{2l})^l$. But this determines the value of f only up to a sign. To get the exact value of f we need the computation from [2] as mentioned above.

Lemma 9.1.2 *We have* $\chi_{N,l}(K; Q_{d_l}) = p^{\mathrm{wr}(K)l}$ *for any framed knot* K.

Proof We consider a diagram of K with blackboard framing. It is possible by switching some, say r, of the crossings of K to get a diagram K' of the unknot. Among these r switches there are a switches that transform a positive crossing into a negative crossing, and b switches that transform a negative crossing into a positive crossing, $r = a + b$. We have $\mathrm{wr}(K') = \mathrm{wr}(K) + 2b - 2a$. We have

$$\chi_{N,l}(K; Q_{d_l}) = (p^{2l})^a (p^{-2l})^b \chi_{N,l}(K'; Q_{d_l})$$

by the skein relation in figure 9.2.

Using regular isotopy we can transform K' into a circle $\mathcal{O}$ plus a number of positive and negative curls, say c resp. d. We have $\mathrm{wr}(K') = c - d$. A positive (resp. negative) curl may be removed by introducing the scalar f (resp. f^{-1}). Therefore,

$$\chi_{N,l}(K'; Q_{d_l}) = f^c f^{-d} \chi_{N,l}(\mathcal{O}; Q_{d_l}) = f^{\mathrm{wr}(K')} = f^{\mathrm{wr}(K)+2b-2a}$$

where we used the result $\chi_{N,l}(\mathcal{O}; Q_{d_l}) = 1$ from lemma 9.1.1.

We merge the above two lines of equations and get

$$\begin{aligned} \chi_{N,l}(K; Q_{d_l}) &= p^{2l(a-b)} \chi_{N,l}(K'; Q_{d_l}) \\ &= p^{2l(a-b)} f^{\mathrm{wr}(K)+2b-2a} \\ &= p^{\mathrm{wr}(K)l} \end{aligned}$$

because $f = p^l$ after the substitutions $v = s^{-N}$ and $x^N = s^{-1}$. ♣

9.2 Linking matrix and σ-operations

The linking number v_{ij} between different components L_i and L_j of a link diagram L is defined as the sum of the signs of all overpasses of L_i over L_j. It is easily seen to be invariant under all Reidemeister moves and is therefore an invariant of links under ambient isotopy. One verifies the symmetry $v_{ij} = v_{ji}$ by looking at the diagram L first from above and then from below.

We define the self linking number of a knot diagram K to be the linking number between the two components of the blackboard 2-parallel of K. In the context of framed knots, this is the linking number between the knot and a parallel that represents the framing. It is clear that this agrees with the writhe of K. For a link diagram L we denote by v_{ii} the self linking number of the component L_i.

Lemma 9.2.1 *Given a framed link $L = L_1 \cup L_2 \cup \ldots \cup L_t$ and Young diagrams $\lambda^2, \ldots, \lambda^t$. Then*

$$\chi_{N,l}(L_1 \cup L_2 \cup \ldots \cup L_t; Q_{d_l}, Q_{\lambda^2}, \ldots, Q_{\lambda^t}) = \\ \chi_{N,l}(L_2 \cup \ldots \cup L_t; Q_{\lambda^2}, \ldots, Q_{\lambda^t}) p^{lv_{11}+2\sum_{i=2}^{t}|\lambda^i|v_{1i}}.$$

Proof We consider a diagram of L with blackboard framing. We look at a crossing of L where the component L_1 crosses over another component L_i, $i \neq 1$. We denote the sign of this crossing by ε. We switch this crossing to an underpass for L_1. We denote the resulting link by L'. We have

$$\chi_{N,l}(L; Q_{d_l}, Q_{\lambda^2}, \ldots, Q_{\lambda^t}) = p^{2\varepsilon|\lambda^i|}\chi_{N,l}(L'; Q_{d_l}, Q_{\lambda^2}, \ldots, Q_{\lambda^t}).$$

because Q_{λ^i} can be written as a sum of diagrams each of which looks like $|\lambda^i|$ parallel arcs near the crossing. Applying the (left for $\varepsilon = -1$ resp. right for $\varepsilon = 1$) skein relation in figure 9.1 $|\lambda^i|$-times gives the result. Doing this for all overpasses of L_1 with all the other components we separate the decorated component L_1 and get

$$\chi_{N,l}(L; Q_{d_l}, Q_{\lambda^2}, \ldots, Q_{\lambda^t}) = \\ p^{2\sum_{i=2}^{t}|\lambda^i|v_{1i}}\chi_{N,l}(L_1; Q_{d_l})\chi_{N,l}(L_2 \cup \ldots \cup L_t; Q_{\lambda^2}, \ldots, Q_{\lambda^t}).$$

We use lemma 9.1.2 and get

$$\chi_{N,l}(L; Q_{d_l}, Q_{\lambda^2}, \ldots, Q_{\lambda^t}) = p^{lv_{11}+2\sum_{i=2}^{t}|\lambda^i|v_{1i}}\chi_{N,l}(L_2 \cup \ldots \cup L_t; Q_{\lambda^2}, \ldots, Q_{\lambda^t}).$$

♣

Lemma 9.2.2 *Given a framed link $L = L_1 \cup \ldots \cup L_t$, Young diagrams $\lambda^1, \ldots, \lambda^t$, and non-negative integers $n_1, \ldots, n_t$. Then*

$$\chi_{N,l}(L; Q_{\lambda^1}Q_{d_l}^{n_1}, \ldots, Q_{\lambda^t}Q_{d_l}^{n_t}) = \chi_{N,l}(L; Q_{\lambda^1}, \ldots, Q_{\lambda^t})p^{\phi(n_1,\ldots,n_t,|\lambda^1|,\ldots,|\lambda^t|,\{v_{ij}\})}$$

where

$$\phi(a_1, \ldots, a_t, b_1, \ldots, b_t, \{v_{ij}\}) = \sum_{1 \leq i,j \leq t} a_i v_{ij}(2b_j + la_j).$$

Proof By induction on $n = n_1 + \cdots + n_t$. We proved the case $n = 1$ in lemma 9.2.1.

We consider the case $n \geq 2$. We renumber the components so that $n_1 \geq 1$. Instead of decorating the component L_i with $Q_{\lambda^i}Q_{d_l}^{n_i}$, for all $i = 1, \ldots, t$, we can consider the (n_i+1)-parallel of L_i and decorate the components by $Q_{\lambda^i}, Q_{d_l}, \ldots, Q_{d_l}$.

We can use lemma 9.2.1 to remove one of the components of $L_1^{n_1+1}$ decorated by Q_{d_l}. We get

$$\begin{aligned}
&\chi_{N,l}(L; Q_{\lambda^1}Q_{d_l}^{n_1}, \ldots, Q_{\lambda^t}Q_{d_l}^{n_t}) \\
&\quad = \chi_{N,l}(L_1^{n_1+1} \cup \ldots \cup L_t^{n_t+1}; Q_{\lambda^1}, \underbrace{Q_{d_l}, \ldots, Q_{d_l}}_{n_1}, \ldots\ldots, Q_{\lambda^t}, \underbrace{Q_{d_l}, \ldots, Q_{d_l}}_{n_t}) \\
&\quad = \chi_{N,l}(L_1^{n_1} \cup \ldots \cup L_t^{n_t+1}; Q_{\lambda^1}, \underbrace{Q_{d_l}, \ldots, Q_{d_l}}_{n_1}, \ldots\ldots, Q_{\lambda^t}, \underbrace{Q_{d_l}, \ldots, Q_{d_l}}_{n_t})p^{\kappa} \\
&\quad = \chi_{N,l}(L; Q_{\lambda^1}Q_{d_l}^{n_1-1}, \ldots, Q_{\lambda^t}Q_{d_l}^{n_t})p^{\kappa}
\end{aligned}$$

where

$$\begin{aligned}
\kappa &= lv_{11} + 2\left(|\lambda^1|v_{11} + (n_1-1)lv_{11} + \sum_{j=2}^{t}(|\lambda^j|v_{1j} + n_j l v_{1j})\right) \\
&= v_{11}(2|\lambda^1| + l(2n_1-1)) + 2\sum_{j=2}^{t} v_{1j}(|\lambda^j| + n_j l).
\end{aligned}$$

The remaining part of the proof is algebraic. Our induction hypothesis is that

$$\chi_{N,l}(L; Q_{\lambda^1}Q_{d_l}^{n_1-1}, \ldots, Q_{\lambda^t}Q_{d_l}^{n_t}) = \chi_{N,l}(L; Q_{\lambda^1}, \ldots, Q_{\lambda^t})p^{\phi(n_1-1,\ldots,n_t,|\lambda^1|,\ldots,|\lambda^t|,\{v_{ij}\})}.$$

To accomplish the induction step we have to prove that

$$\phi(n_1, \ldots, n_t, |\lambda^1|, \ldots, |\lambda^t|, \{v_{ij}\}) = \phi(n_1-1, \ldots, n_t, |\lambda^1|, \ldots, |\lambda^t|, \{v_{ij}\}) + \kappa.$$

We have

$$\begin{aligned}
\phi(a_1, \ldots, a_t, b_1, \ldots, b_t, \{v_{ij}\}) &= \sum_{i=2}^{t} a_i v_{i1}(2b_1 + la_1) + \sum_{j=2}^{t} a_1 v_{1j}(2b_j + la_j) \\
&\quad + a_1 v_{11}(2b_1 + la_1) + \sum_{2\le i,j\le t} a_i v_{ij}(2b_j + la_j)
\end{aligned}$$

and the last summand is not affected by the value of a_1. Therefore

$$\begin{aligned}
&\phi(a_1, a_2, \ldots, a_t, b_1, \ldots, b_t, \{v_{ij}\}) - \phi(a_1-1, a_2, \ldots, a_t, b_1, \ldots, b_t, \{v_{ij}\}) \\
&\quad = \sum_{i=2}^{t} a_i v_{i1}(2b_1 + la_1) + \sum_{j=2}^{t} a_1 v_{1j}(2b_j + la_j) + a_1 v_{11}(2b_1 + la_1) \\
&\quad\quad - \Big(\sum_{i=2}^{t} a_i v_{i1}(2b_1 + l(a_1-1)) + \sum_{j=2}^{t}(a_1-1)v_{1j}(2b_j + la_j) \\
&\quad\quad + (a_1-1)v_{11}(2b_1 + l(a_1-1))\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=2}^{t} a_i v_{i1} l + \sum_{j=2}^{t} v_{1j}(2b_j + la_j) + v_{11}(2b_1 + l(2a_1 - 1)) \\
&= v_{11}(2b_1 + l(2a_1 - 1)) + 2\sum_{j=2}^{t} v_{1j}(b_j + la_j).
\end{aligned}
$$

Substituting $a_i = n_i$ and $b_j = |\lambda^j|$ for $i = 1, \ldots, t$ and $j = 1, \ldots, t$ we get from the above equation that

$$
\phi(n_1, \ldots, n_t, |\lambda^1|, \ldots, |\lambda^t|, \{v_{ij}\}) - \phi(n_1 - 1, \ldots, n_t, |\lambda^1|, \ldots, |\lambda^t|, \{v_{ij}\}) = \kappa
$$

as claimed. This completes the induction step. ♣

Lemma 9.2.3 *We have*

$$
p^{\phi(a_1, a_2, \ldots, a_t, b_1, \ldots, b_t, \{v_{ij}\})} = p^{\phi(a_1+N, a_2, \ldots, a_t, b_1, \ldots, b_t, \{v_{ij}\})}
$$

for any integers $a_1, \ldots, a_t, b_1, \ldots, b_t$ *and* $\{v_{ij}\}$.

Proof We have

$$
\begin{aligned}
&\phi(a_1 + N, a_2, \ldots, a_t, b_1, \ldots, b_t, \{v_{ij}\}) - \phi(a_1, a_2, \ldots, a_t, b_1, \ldots, b_t, \{v_{ij}\}) \\
&= lv_{11}((a_1 + N)^2 - a_1^2) + 2l\sum_{j=2}^{t} v_{1j} a_j (a_1 + N - a_1) + 2\sum_{j=1}^{t} v_{1j} b_j (a_1 + N - a_1) \\
&= lv_{11}(N^2 + 2a_1 N) + 2Nl\sum_{j=2}^{t} v_{1j} a_j + 2N\sum_{j=1}^{t} v_{1j} b_j
\end{aligned}
$$

which is an integer linear combination of $2N$ and N^2. In order to complete the proof we mention that

$$
p^{2N} = (-s^{-1}x^l)^{2N} = s^{-2N}x^{2lN} = s^{2l}x^{2lN} = (sx^N)^{2l} = 1
$$

and

$$
\begin{aligned}
p^{N^2} &= (-s^{-1}x^l)^{N^2} = (-1)^{N^2}(s^{Nl}x^{N^2 l})s^{-Nl-N^2} \\
&= (-1)^{N^2}s^{-N(N+l)} = (-1)^{N^2-N} = (-1)^{N(N-1)} = 1
\end{aligned}
$$

are both equal to 1. ♣

Lemma 9.2.3 follows in the case of non-negative integers $a_1, \ldots, a_n$ immediately from the fact that $d_l^N = \emptyset$ in the ring $\mathcal{Y}_{N,l}$.

Theorem 9.2.4 *Given a framed link $L = L_1 \cup \ldots \cup L_t$, Young diagrams $\lambda^1, \ldots, \lambda^t$, and integers $n_1, \ldots, n_t$. Then*

$$\chi_{N,l}(L; Q_{\sigma^{n_1}(\lambda^1)}, \ldots, Q_{\sigma^{n_t}(\lambda^t)}) = \chi_{N,l}(L; Q_{\lambda^1}, \ldots, Q_{\lambda^t}) p^{\phi(n_1,\ldots,n_t,|\lambda^1|,\ldots,|\lambda^t|,\{v_{ij}\})}$$

where

$$\phi(a_1, \ldots, a_t, b_1, \ldots, b_t, \{v_{ij}\}) = \sum_{1 \le i,j \le t} a_i v_{ij}(2b_j + l a_j).$$

Proof We know that $d_l \lambda = \sigma(\lambda)$ in $\mathcal{Y}_{N,l}$ for any Young diagram λ by lemma 6.2.1. We know by the remarks at the end of section 5.2 that the Homfly polynomial does not distinguish between decorations Q_η and Q_μ if $\eta = \mu$ in $\mathcal{Y}_{N,l}$ provided one substitutes $v = s^{-N}$, $x^N = s^{-1}$ and $s^{2(l+N)} = 1$. Hence, lemma 9.2.2 can be restated with $Q_{\sigma^{n_i}(\lambda^i)}$ in place of $Q_{\lambda^i} Q_{d_l}^{n_i}$. Since $\sigma^N(\lambda) = \lambda$ for any Young diagram λ and by the result of lemma 9.2.3 we can admit negative n_i, too. ♣

With the substitution

$$s = \exp\left(\frac{\pi i}{l+N}\right) \quad \text{and} \quad x = \exp\left(-\frac{\pi i}{(l+N)N}\right)$$

we get

$$p = -s^{-1}x^l = -x^{N+l} = -\exp\left(-\frac{\pi i}{N}\right).$$

We denote $\phi = \phi(a_1, \ldots, a_t, b_1, \ldots, b_t, \{v_{ij}\})$. We have $\phi \equiv l v_{ii} a_i^2 \bmod 2$ because $v_{ij} = v_{ji}$. We thus get

$$\begin{aligned} (-1)^\phi &= \exp(\pi i \phi) \\ &= \exp\left(\pi i l \sum_{i=1}^t v_{ii} a_i^2\right) \\ &= \exp\left(\frac{\pi i}{N} N l \sum_{i=1}^t v_{ii} a_i^2\right). \end{aligned}$$

We thus get

$$\begin{aligned} p^\phi &= \left(-\exp(-\frac{\pi i}{N})\right)^\phi \\ &= \exp\left(\frac{\pi i}{N}\left(Nl \sum_{i=1}^t v_{ii} a_i^2 - \phi\right)\right) \\ &= \exp\left(\frac{\pi i}{N}\left(Nl \sum_{i=1}^t v_{ii} a_i^2 - \sum_{1 \le i,j \le t} a_i v_{ij}(2b_j + l a_j)\right)\right). \end{aligned} \tag{9.2.1}$$

If b_i (which is the number of cells of λ^i) is congruent to zero modulo N for all $i = 1, \ldots, t$ then $2\sum_{1\leq i,j\leq t} v_{ij}a_ib_j \equiv 0 \bmod 2N$, and thus

$$\begin{aligned} p^{\phi} &= \exp\left(\frac{\pi i}{N}\left(Nl\sum_{i=1}^{t} v_{ii}a_i^2 - l\sum_{1\leq i,j\leq t} a_i v_{ij} a_j\right)\right) \\ &= \exp\left(\frac{\pi i}{N}(N-1)l\sum_{1\leq i,j\leq t} a_i v_{ij} a_j\right) \end{aligned} \tag{9.2.2}$$

because $p^{2N} = 1$ and

$$\sum_{i=1}^{t} v_{ii}a_i^2 \equiv \sum_{1\leq i,j\leq t} a_i v_{ij} a_j \bmod 2.$$

Equations (9.2.1) and (9.2.2) are given in proposition 3.2.1 in [16]. We remark that Kohno and Takata are using the letter k rather than l.

9.3 Transposing and conjugation, one way

We denote by $\overline{L}$ the mirror image of a link diagram with blackboard framing. We denote the complex conjugate of a complex number by an overline as well.

Lemma 9.3.1 *Let $L = L_1 \cup \cdots \cup L_t$ be a link diagram with blackboard framing, and let $\lambda, \ldots, \mu$ be Young diagrams. Then*

$$\chi_{N,l}(\overline{L}; Q_\lambda, \ldots, Q_\mu) = \overline{\chi_{N,l}(L; Q_\lambda, \ldots, Q_\mu)}.$$

Proof We apply the map ρ from subsection 2.4.1 to the link diagram L decorated by $Q_\lambda, \ldots, Q_\mu$. This leaves every Q_λ invariant, because Q_λ is a polynomial in Q_{d_i}'s which are invariant under ρ by lemma 2.4.4. The map ρ maps L to its mirror image $\overline{L}$.

This tells us in the skein of the plane that ρ maps L decorated with $Q_\lambda, \ldots, Q_\mu$ to $\overline{L}$ decorated by $Q_\lambda, \ldots, Q_\mu$. Therefore, the Homfly polynomial (which is a rational function in x, v and s) of L decorated with $Q_\lambda, \ldots, Q_\mu$ is mapped to the Homfly polynomial of $\overline{L}$ decorated by $Q_\lambda, \ldots, Q_\mu$. We have by definition that $\rho(s) = s^{-1}$, $\rho(x) = x^{-1}$ and $\rho(v) = v^{-1}$. Since s, x and v are roots of unity, and the conjugate of any complex number with absolute value 1 is equal to its inverse, we have

$$\chi_{N,l}(\overline{L}; Q_\lambda, \ldots, Q_\mu) = \overline{\chi_{N,l}(L; Q_\lambda, \ldots, Q_\mu)}.$$

♣

Lemma 9.3.1 relates the Homfly polynomial of a link L decorated with $Q_\lambda, \dots, Q_\mu$ to the Homfly polynomial of its mirror image with the same decorations.

We now relate the Homfly polynomial of a link L decorated with $Q_\lambda, \dots, Q_\mu$ to the Homfly polynomial of L decorated with $Q_{\tilde\lambda}, \dots, Q_{\tilde\mu}$ where $\tilde\lambda$ lies in the same σ-orbit as the transposed Young diagram $\lambda^\vee$ of λ. The Homfly polynomials will turn out to be the complex conjugate of each other.

Given a Young diagram λ in the $(N-1)\times l$-rectangle we see that the transposed Young diagram $\lambda^\vee$ lies in the $l \times (N-1)$-rectangle, and in $\mathcal{Y}_{l,N}$ it is therefore equal to a Young diagram in the $(l-1) \times N$-rectangle by removing all initial columns of length l.

Given a link L and decorations $Q_\lambda, \dots, Q_\mu$ on its components, the Homfly polynomial of this decorated link is a rational function $p(x, v, s)$ in x, v and s. The Homfly polynomial of L with decorations $Q_{\lambda^\vee}, \dots, Q_{\mu^\vee}$ is a rational function $q(x, v, s)$ in x, v and s. We have $q(x, v, s) = p(-x, -v, s^{-1})$ and $q(x, v, s) = p(x, v, -s^{-1})$ by lemma 3.6.2.

We want $q(x, v, s)$ to be the conjugate complex number of $p(x, v, s)$ after substitutions or something similar. We have to be careful about the substitution. We want the $2(l+N)$-th root of unity ω to be substituted for s to be the same in the context of $\mathcal{Y}_{N,l}$ and $\mathcal{Y}_{l,N}$.

The value to be substituted for v in the context of $\mathcal{Y}_{N,l}$ is s^{-N}. In the context of $\mathcal{Y}_{l,N}$ we substitute v by s^{-l}. We denote $v_1 = s^{-N}$ and $v_2 = s^{-l}$.

The value for x involves a choice. In the context of $\mathcal{Y}_{N,l}$ the condition is $x^N = s^{-1}$ and our choice x_1 is therefore determined up to an N-th root of unity.

In the context of $\mathcal{Y}_{l,N}$ the condition is $x^l = s^{-1}$, and our choice x_2 is therefore determined up to an l-th root of unity.

The problem with the approach $q(x, v, s) = p(x, v, -s^{-1})$ is that the complex conjugate of s is rather s^{-1} than $-s^{-1}$.

The approach $q(x, v, s) = p(-x, -v, s^{-1})$ seems to be appropriate, since s^{-1} is the conjugate of s, and $-v$ in the context $\mathcal{Y}_{l,N}$ is the conjugate of v in the context $\mathcal{Y}_{N,l}$ because $-v_2 = -s^{-l} = s^N = v_1^{-1} = \overline{v_1}$ since $s^{N+l} = -1$. A problem occurs for x, since $-x_2$ is hardly ever the conjugate of x_1. (Well, sometimes it is, as described in section 9.4). We take account of this problem with x by choosing a suitable element in the σ-orbit of the transposed Young diagram. First, we consider the approach via $q(x, v, s) = p(x, v, -s^{-1})$.

We denote the Homfly polynomial after the substitutions $v = v_1$, $x = x_1$ and $s = \omega$ by $\chi_{N,l}$. We denote the Homfly polynomial after the substitutions $v = v_2$, $x = x_2$ and $s = \omega$ by $\chi_{l,N}$.

9.3.1 Transposing from $\mathcal{Y}_{N,l}$ to $\mathcal{Y}_{l,N}$

The definition of the σ-operation in section 6.2 was given in the context of $\mathcal{Y}_{N,l}$, i.e. for Young diagrams in the $(N-1)\times l$-rectangle. Here, we denote this operation by σ_l. In the context of $\mathcal{Y}_{l,N}$, i.e. for Young diagrams in the $(l-1)\times N$-rectangle, we denote the addition of an initial row of length N to a Young diagram λ and then removing all columns of length l by $\sigma_N(\lambda)$. We have $\sigma_l^N(\mu)=\mu$ and $\sigma_l^N(\lambda)=\lambda$ for any Young diagrams μ and λ in the $(N-1)\times l$-rectangle resp. $(l-1)\times N$-rectangle.

First, we make the meaning of transposing precise. Consider maps,

$$\begin{aligned} F: \quad & \{\text{Young diagrams in } (N-1)\times l\text{-rectangle}\} \to \\ & \qquad \{\text{Young diagrams in } (l-1)\times N\text{-rectangle}\} \end{aligned}$$

given by transposing the Young diagram and then removing all initial columns of length l. Similarly:

$$\begin{aligned} G: \quad & \{\text{Young diagrams in } (l-1)\times N\text{-rectangle}\} \to \\ & \qquad \{\text{Young diagrams in } (N-1)\times l\text{-rectangle}\} \end{aligned}$$

given by transposing the Young diagram and then removing all initial columns of length N. It is clear that

$$G(\sigma_N^{\lambda_{N-1}}(F(\sigma_l(\lambda))))=\lambda$$

and

$$\sigma_l^j(GF(\lambda))=\lambda$$

where j is the number of initial rows of length l in λ. We have $\sigma_l^N(\lambda)=\lambda$ for any Young diagram λ in the $(N-1)\times l$-rectangle. The above equations imply that G and F induce a bijection of the σ_l-orbits and the σ_N-orbits. This bijection will be revisited in lemma 10.1.3.

The equality $|\sigma(\lambda)|=|\lambda|+l-N\lambda_{N-1}$ implies that $|\sigma(\lambda)|\equiv|\lambda|+l \mod N$. If N and l are coprime then there exists exactly one element in each σ-orbit whose number of cells is divisible by N. If N and l are not coprime then the existence of such Young diagrams is not guaranteed. If $|\lambda|$ is divisible by N then $\sigma_N^{-\frac{|\lambda|}{N}}(\lambda^\vee)$ is a Young diagram in the $(l-1)\times N$-rectangle whose number of cells is divisible by l. The following theorem was motivated by Proposition 3.3.2 in [16]

Theorem 9.3.2 *Let $N\geq 2$ and $l\geq 1$. Let $\lambda^1,\ldots,\lambda^t$ be Young diagrams in the $(N-1)\times l$-rectangle such that N divides the number of cells of each λ^i. Denote $\mu^i=\sigma_N^{-\frac{|\lambda_i|}{N}}\left((\lambda^i)^\vee\right)$. Then, for any framed link L,*

$$\overline{\chi_{N,l}(L;Q_{\lambda^1},\ldots,Q_{\lambda^t})}=\chi_{l,N}(L;Q_{\mu^1},\ldots,Q_{\mu^t}).$$

Proof We have

$$\chi(L; Q_{\lambda^1}, \ldots, Q_{\lambda^t}) = x^y \chi(L; Q_{\lambda^1}, \ldots, Q_{\lambda^t})_{x=1} \tag{9.3.3}$$

where y is the writhe of the diagram where every component L_i is replaced by its $|\lambda^i|$-parallel, $i = 1, \ldots, t$. This is a straightforward extension of corollary 4.1.2. Here,

$$y = \sum_{1 \le i < j \le t} 2v_{ij}|\lambda^i||\lambda^j| + \sum_{i=1}^{t} v_{ii}|\lambda^i|^2 \tag{9.3.4}$$

because for $i \neq j$, we have that v_{ij} is half the sum of the signed crossings between the components L_i and L_j of L. Furthermore, v_{ii} is the writhe of the component L_i. Considering the parallels, any crossing between components L_i and L_j becomes $|\lambda^i||\lambda^j|$ crossings of the same sign. This establishes equation (9.3.4).

We have by theorem 9.2.4 that

$$\chi_{l,N}(L; Q_{\sigma^{a_1}(\mu^1)}, \ldots, Q_{\sigma^{a_t}(\mu^t)}) = p_2^w \chi_{l,N}(L; Q_{\mu^1}, \ldots, Q_{\mu^t})$$

where

$$w = \sum_{1 \le i,j \le t} a_i v_{ij}(2|\mu_i| + N a_j).$$

Since the number of cells of each μ^j is divisible by l and $p_2^{2l} = 1$, we can use

$$w' = \sum_{1 \le i,j \le t} a_i a_j v_{ij} N$$

instead of w in the above equation. In particular, for $a_i = |\lambda^i|/N$ we have

$$\chi_{l,N}(L; Q_{(\lambda^1)^\vee}, \ldots, Q_{(\lambda^t)^\vee}) = p_2^{w'} \chi_{l,N}(L; Q_{\mu^1}, \ldots, Q_{\mu^t})$$

where $w' = y/N$ in this case.

We have

$$\chi(L; Q_{(\lambda^1)^\vee}, \ldots, Q_{(\lambda^t)^\vee}) = \chi(L; Q_{\lambda_1}, \ldots, Q_{\lambda^t})_{x \mapsto -x,\ v \mapsto -v,\ s \mapsto s^{-1}}$$

by lemma 3.6.2. Hence, by equation (9.3.3),

$$\chi(L; Q_{(\lambda^1)^\vee}, \ldots, Q_{(\lambda^t)^\vee}) = (-x)^y \chi(L; Q_{\lambda^1}, \ldots, Q_{\lambda^t})_{x=1, v \mapsto -v, s \mapsto s^{-1}}.$$

Making the substitutions in the context of $\mathcal{Y}_{l,N}$ we get

$$\chi_{l,N}(L; Q_{(\lambda^1)^\vee}, \ldots, Q_{(\lambda^t)^\vee}) = (-x_2)^y \chi(L; Q_{\lambda^1}, \ldots, Q_{\lambda^t})_{x=1, v \mapsto -\omega^{-l}, s \mapsto \omega^{-1}}.$$

From equation (9.3.3) and the above equations we get

$$\begin{aligned}
\overline{\chi_{N,l}(L; Q_{\lambda^1}, \ldots, Q_{\lambda^t})} &= \overline{x_1^y \chi_{N,l}(L; Q_{\lambda_1}, \ldots, Q_{\lambda^t})_{x=1}} \\
&= x_1^{-y} \chi(L; Q_{\lambda^1}, \ldots, Q_{\lambda^t})_{x=1,\ v \mapsto \overline{\omega^{-N}},\ s \mapsto \overline{\omega}} \\
&= x_1^{-y} \chi(L; Q_{\lambda^1}, \ldots, Q_{\lambda^t})_{x=1,\ v \mapsto -\omega^{-l},\ s \mapsto \omega^{-1}} \\
&= x_1^{-y} (-x_2)^{-y} \chi_{l,N}(L; Q_{(\lambda^1)^\vee}, \ldots, Q_{(\lambda^t)^\vee}) \\
&= (-x_1 x_2)^{-y} p_2^{w'} \chi_{l,N}(L; Q_{\mu^1}, \ldots, Q_{\mu^t}).
\end{aligned}$$

We thus have to prove that

$$(-x_1 x_2)^y = p_2^{w'}.$$

Since $p_2 = -\omega^{-1} x_2^N = -x_2^{N+l}$, the above equation is equivalent to

$$(-x_1 x_2)^y = (-x_2^{N+l})^{\frac{y}{N}}.$$

Since N divides any $|\lambda^i|$, $i = 1, \ldots, t$, we have that $y = cN^2$ for some integer c. We thus have to prove that

$$(-x_1 x_2)^{cN^2} = (-x_2^{N+l})^{cN}$$

for any integer c. We have that $(-1)^{cN^2} = (-1)^{cN}$, and it is therefore sufficient to prove that

$$(x_1 x_2)^N = x_2^{(N+l)}.$$

This is equivalent to

$$x_1^N = x_2^l$$

which is true since x_1 is an N-th root of ω, and x_2 is an l-th root of ω. ♣

9.4 Transposing and conjugation, the other way

We set

$$s = \exp\left(2\pi i \frac{k}{2(l+N)}\right)$$

where k is an integer coprime to $2(l+N)$, and $1 \leq k \leq 2(l+N)$. We set

$$x_1 = \exp\left(-\frac{2\pi i k}{2(l+N)N} + \frac{2\pi i r}{N}\right)$$

for some $0 \leq r \leq N-1$. We set

$$x_2 = \exp\left(-\frac{2\pi i k}{2(l+N)l} + \frac{2\pi i q}{l}\right)$$

for some $0 \leq q \leq l-1$.

The other way to achieve that $q(x, v, s)$ $(= p(-x, -v, s^{-1}))$ is the conjugate of $p(x, v, s)$ is by choosing the substitutions for x, v and s in such a way that the conjugate of s is equal s^{-1}, that the conjugate of v_1 is equal to $-v_2$, and that the conjugate of x_1 is equal to $-x_2$.

Since s, v and x are roots of unity after the substitutions we have that their conjugates are equal to their inverses. Hence, the conjugate of s is equal to s^{-1}. The conjugate of v_1 is equal to v_1^{-1}, and

$$v_1^{-1} = \left(s^{-N}\right)^{-1} = s^N = s^{N+l}s^{-l} = -s^{-l} = -v_2$$

satisfies the above condition. The only remaining condition is that $x_1^{-1} = -x_2$. This is equivalent to $x_1x_2 = -1$. The rest of this section solves the question when x_1x_2 is equal to -1. It turns out that there are unique solutions for x_1 and x_2 provided that l and N are coprime odd integers.

Remark We have that $q(x, v, s) = p(x, v, -s^{-1})$, too, but there are no choices for k, r and q such that the conjugate of x_1 is equal to x_2. This is because $x_1x_2 = 1$ leads to the equation $2(rl + qN) - k \equiv 2Nl \bmod \mathbb{Z}$ which implies that k is even. This contradicts the condition that k and $2(l + N)$ are coprime.

9.4.1 When is $x_1x_2 = -1$?

The equation $x_1x_2 = -1$ is equivalent to

$$\exp\left(-\frac{2\pi ik}{2(l+N)N} + \frac{2\pi ir}{N}\right)\exp\left(-\frac{2\pi ik}{2(l+N)l} + \frac{2\pi iq}{l}\right) = \exp(\pi i)$$

by our above notation. This equation is equivalent to

$$-\frac{k}{2(l+N)N} + \frac{r}{N} + \left(-\frac{k}{2(l+N)l} + \frac{q}{l}\right) \equiv \frac{1}{2}$$

where congruence means here and in the following congruence modulo $\mathbb{Z}$. This congruence is equivalent to

$$\frac{2(rl + qN) - k}{2Nl} \equiv \frac{1}{2}. \tag{9.4.5}$$

In particular, Nl is a divisor of $2(rl + qN) - k$, and there exists an integer a such that

$$2(rl + qN) - k = aNl.$$

This implies that the greatest common divisor g.c.d.(l, N) of l and N is a divisor of k. Since k is supposed to be coprime to $2(l + N)$, and g.c.d.(l, N) is a divisor of $(l+N)$, we deduce that g.c.d.(l,N) has to be equal to 1, i.e. l and N are coprime.

If N or l is even then the equation $2(rl+qN)-k=aNl$ implies that k is even. This is in contradiction to the condition that k and $2(l+N)$ are coprime. We have proved so far

Lemma 9.4.1 *Let l, N and k be positive integers so that k is coprime to $2(l+N)$, and $1 \leq k \leq 2(l+N)$. There exists a solution to r and q to the condition (9.4.5) only if l and N are coprime odd integers.*

We described in subsection 9.3.1 a relation between the Young diagrams in the $(N-1)\times l$-rectangle and the Young diagrams in the $(l-1)\times N$-rectangle. The relation is induced by transposing a Young diagram and then reducing it to its representative in the $(l-1)\times N$-rectangle. But in order to be a bijection, we have to consider the σ-orbits of the Young diagrams.

The effect of the σ-operation can be controlled by theorem 9.2.4. In order to be able to neglect the influence of the σ-operation we want p to be equal to 1 in the context of $\mathcal{Y}_{N,l}$ and $\mathcal{Y}_{l,N}$. We recall that $p_1=-s^{-1}x_1^l$ in the context of $\mathcal{Y}_{N,l}$, and $p_2=-s^{-1}x_2^N$ in the context of $\mathcal{Y}_{l,N}$.

The condition $-s^{-1}x_1^l=1$ is equivalent to $s^{-1}x_1^l=-1$ which is equivalent by the above equations to

$$-\frac{k}{2(l+N)}+l\left(-\frac{k}{2(l+N)N}+\frac{r}{N}\right)\equiv\frac{1}{2}.$$

This equation can be written as

$$\frac{2rl-k}{2N}\equiv\frac{1}{2}. \tag{9.4.6}$$

We want $p_2=-s^{-1}x_2^N=1$ as well. This is equivalent to

$$\frac{2qN-k}{2l}\equiv\frac{1}{2} \tag{9.4.7}$$

which differs from condition (9.4.6) by interchanging l and N and interchanging r and q.

Given coprime odd integers N and l, and an integer k coprime to $2(l+N)$, and $1\leq k\leq 2(l+N)$, we are looking for solutions for r and q that satisfy conditions (9.4.5), (9.4.6) and (9.4.7).

If r and r' are solutions to condition (9.4.6) then $2N$ divides $2(r-r')l$ and thus $r\equiv r' \mod N$. Similarly, any solution q to (9.4.6) is unique up to congruence modulo l.

Since l and N are coprime and odd we have that $2l$ and N are coprime. Hence, there exist integers c and d such that

$$2cl + dN = 1,$$

and in particular d is odd. We deduce that

$$\frac{2ckl - k}{N} = -kd$$

and therefore

$$r = ck$$

is a solution for condition (9.4.6) because $-kd$ is odd since k and d are odd.

Since $2cl + dN = 1$, we have $(d + l)N - 1 = l(N - 2c)$ where $N - 2c$ is odd, and $d + l$ is even. Therefore

$$\frac{k(d + l)N - k}{l} = k(N - 2)c,$$

and therefore

$$q = \frac{k(d + l)}{2}$$

is a solution to condition (9.4.7).

We have to check condition (9.4.5) for these solutions. We have

$$\begin{aligned}
\frac{2(rl + qN) - k}{Nl} &= \frac{2(ckl + \frac{k(d+l)}{2}N) - k}{Nl} \\
&= k\frac{2cl + (d + l)N - 1}{Nl} \\
&= k\frac{(2cl + dN) - 1 + lN}{Nl} \\
&= k
\end{aligned}$$

which is an odd integer and thus condition (9.4.5) is satisfied. We can summarize our considerations.

Lemma 9.4.2 *Given positive integers l and N, there exists an integer k coprime to $2(l + N)$ and integers r and q satisfying conditions (9.4.5) only if l and N are coprime odd integers.*

Given positive coprime odd integers l and N and an integer k coprime to $2(l + N)$. Let c and d be integers that satisfy $2cl + dN = 1$. The there exist an integer $r = ck$ (unique up to congruence modulo N) and an integer $q = k(d+l)/2$ (unique up to congruence modulo l) that satisfy conditions (9.4.5), (9.4.6) and (9.4.7).

Remark For the solution $r = ck$ we get by our construction

$$x_1 = \exp\left(\pi i \frac{k(2c-d)}{l+N}\right) \quad \text{and} \quad x_2 = x^{-1}.$$

We finally show that the solutions r and q are symmetric, i.e. if we interchange N and l then the corresponding solutions are $r' = q$ up to congruence modulo N, and $q' = r$ up to congruence modulo l.

Lemma 9.4.3 *The interchange of N and l interchanges the solutions r and q.*

Proof Given coprime odd integers l and N we have

$$\begin{aligned} 2cl + dN &= 1 \text{ and} \\ 2c'N + d'l &= 1 \end{aligned}$$

for some integers c, c', d and d'. The solutions we found are

$$\begin{aligned} r &= kc, \; q = k\frac{d+l}{2} \text{ and, symmetrically} \\ r' &= kc', \; q' = k\frac{d'+N}{2}. \end{aligned}$$

We have to show that

$$kc' \equiv k\frac{d+l}{2} \bmod l \quad \text{and} \quad kc \equiv k\frac{d'+N}{2} \bmod N$$

for any integer k coprime to $2(l+N)$. In fact, we show that

$$c' \equiv \frac{d+l}{2} \bmod l, \quad \text{and} \quad c \equiv \frac{d'+N}{2} \bmod N.$$

We have by the above equation that

$$2cl + dN = 2c'N + d'l,$$

hence

$$(2c - d')l = (2c' - d)N.$$

Since l and N are coprime, we deduce that N divides $2c - d'$, and l divides $(2c' - d)$, hence

$$2c \equiv d' \bmod N \quad \text{and} \quad 2c' \equiv d \bmod l.$$

This implies that $2c \equiv d' + N \bmod N$. The integer d' is odd because $2c'N + d'l = 1$. Hence, the sum of two odd integers $d' + N$ is even. Since N is odd, we have

$$c \equiv \frac{d'+N}{2} \bmod N.$$

Similarly, we have that $c' \equiv (d+l)/2 \bmod l$, and this completes our proof that $r' \equiv q \bmod N$ and $q' \equiv r \bmod l$. ♣

Chapter 10

Young-solutions

We fix integers $N \geq 2$ and $l \geq 1$. We fix ζ, a primitive root of unity of order $l+N$. We denote $E_{l+N} = \{1, \zeta, \dots, \zeta^{l+N-1}\}$, the set of all $(l+N)$-th roots of 1. We denote $\xi = \exp(2\pi i/N)$. But in section 10.4 we shall denote by ξ another primitive N-th root of unity.

10.1 Encoding Young diagrams in the unit circle

To every Young diagram λ in the $(N-1) \times l$-rectangle we assign a set T_λ of N points on the unit circle in the complex plane,

$$T_\lambda = \{1, \zeta^{\lambda_{N-1}+1}, \zeta^{\lambda_{N-2}+2}, \dots, \zeta^{\lambda_1+N-1}\}.$$

This describes a bijection between the Young diagrams in the $(N-1) \times l$-rectangle and the set

$$T = \{\{1, \zeta^{a_1}, \dots, \zeta^{a_{N-1}}\} \mid 1 \leq a_1 < \cdots < a_{N-1} \leq l+N-1\}.$$

In particular, we see that the number of Young diagrams in the $(N-1) \times l$-rectangle is equal to $\binom{l+N-1}{N-1}$. We denote $\zeta^{\lambda_{N-k}+k}$ as the *k-th element* of T_λ, $0 \leq k \leq N-1$.

The group of symmetries (i.e. Euclidean isometries) of the set E_{l+N} is the dihedral group $\mathbb{Z}_{l+N} \propto \mathbb{Z}_2$ which is generated by the reflection in the x-axis (i.e. conjugation) and the rotation by the angle $2\pi/(l+N)$ (i.e. multiplication by $\exp(2\pi i/(l+N))$).

The successive rotations by the angle $2\pi/(l+N)$ do not act on T because every T_λ has to contain the element 1.

But there is an operation of the cyclic group $\mathbb{Z}_N = (a \mid a^N = 1)$ on T. The element a^k of $\mathbb{Z}_N$, $1 \leq k \leq N-1$, acts on T_λ as the rotation of the unit circle that brings the k-th element of T_λ to 1. The element b of $\mathbb{Z}_2 = (b \mid b^2 = 1)$ acts as the reflection in the x-axis, i.e. complex conjugation.

We have that $bab = a^{-1}$ because $\overline{\gamma \overline{x}} = \gamma^{-1} x$ for any complex number x where $\gamma = \zeta^{-(\lambda_{N-1}+1)}$. This means that the conjugation by $b \in \mathbb{Z}_2$ acts as the inversion on $\mathbb{Z}_N$, and therefore the dihedral group $\mathbb{Z}_N \propto \mathbb{Z}_2$ acts on T. We remark that the action of $\mathbb{Z}_N \propto \mathbb{Z}_2$ on T is not free in general.

We describe now the action of $\mathbb{Z}_N \propto \mathbb{Z}_2$ more accurately. We refer for the σ-operation to section 6.2 and for the concept of the dual Young diagram λ^* to subsection 1.3.2.

Lemma 10.1.1 *The generators a and b of $\mathbb{Z}_N \propto \mathbb{Z}_2$ act as*

$$\begin{aligned} a(T_\lambda) &= T_{\sigma(\lambda)}, \\ b(T_\lambda) &= T_{\sigma^{-1}(\lambda^*)} \end{aligned}$$

for any Young diagram λ in the $(N-1) \times l$-rectangle.

Proof The action of a transforms $T_\lambda = \{1, \zeta^{\lambda_{N-1}+1}, \ldots, \zeta^{\lambda_1+N-1}\}$ via the rotation $\zeta^{-(\lambda_{N-1}+1)}$ into the set

$$\begin{aligned} a(T_\lambda) &= \zeta^{-(\lambda_{N-1}+1)} T_\lambda \\ &= \{\zeta^{-(\lambda_{N-1}+1)}, 1, \zeta^{\lambda_{N-2}+2-(\lambda_{N-1}+1)}, \ldots, \zeta^{\lambda_1+N-1-(\lambda_{N-1}+1)}\} \\ &= \{1, \zeta^{\lambda_{N-2}+2-(\lambda_{N-1}+1)}, \ldots, \zeta^{\lambda_1+N-1-(\lambda_{N-1}+1)}, \zeta^{-(\lambda_{N-1}+1)}\} \\ &= \{1, \zeta^{\lambda_{N-2}-\lambda_{N-1}+1}, \ldots, \zeta^{\lambda_1-\lambda_{N-1}+N-2}, \zeta^{l-\lambda_{N-1}+N-1}\} \\ &= T_{\sigma(\lambda)} \end{aligned}$$

because $\sigma(\lambda) = (l-\lambda_{N-1}, \lambda_1 - \lambda_{N-1}, \ldots, \lambda_{N-2} - \lambda_{N-1})$ and $\zeta^{l+N} = 1$. The action of b transforms $T_\lambda = \{1, \zeta^{\lambda_{N-1}+1}, \ldots, \zeta^{\lambda_1+N-1}\}$ via conjugation into the set

$$\begin{aligned} b(T_\lambda) &= \{1, \zeta^{-(\lambda_{N-1}+1)}, \zeta^{-(\lambda_{N-2}+2)}, \ldots, \zeta^{-(\lambda_1+N-1)}\} \\ &= \{1, \zeta^{-(\lambda_1+N-1)}, \ldots, \zeta^{-(\lambda_{N-2}+2)}, \zeta^{-(\lambda_{N-1}+1)}\} \\ &= \{1, \zeta^{l+N-(\lambda_1+N-1)}, \ldots, \zeta^{l+N-(\lambda_{N-2}+2)}, \zeta^{l+N-(\lambda_{N-1}+1)}\} \\ &= \{1, \zeta^{l-\lambda_1+1}, \ldots, \zeta^{l-\lambda_{N-2}+N-2}, \zeta^{l-\lambda_{N-1}+N-1}\} \\ &= T_\mu \end{aligned}$$

where $\mu = (\mu_1, \ldots, \mu_{N-1})$ is the Young diagram with $\mu_i = l - \lambda_{N-i}$. Hence,

$$\begin{aligned} \sigma(\mu) &= (l - \mu_{N-1}, \mu_1 - \mu_{N-1}, \ldots, \mu_{N-2} - \mu_{N-1}) \\ &= (\lambda_1, \lambda_1 - \lambda_{N-1}, \ldots, \lambda_1 - \lambda_2) \\ &= \lambda^*. \end{aligned}$$

We have thus proved that $b(T_\lambda) = T_\mu$ with $\sigma(\mu) = \lambda^*$, hence $b(T_\lambda) = T_{\sigma^{-1}(\lambda^*)}$. ♣

We remark that lemma 10.1.1 gives a second proof that $bab = a^{-1}$.

Furthermore, we see that two elements λ and μ from the $(N-1) \times l$-rectangle lie in the same σ-orbit if and only if T_λ and T_μ differ by a rotation. We recall that $\xi = \exp(2\pi i/N)$.

Lemma 10.1.2 *The cardinality of the σ-orbit of λ is equal to the cardinality of the set*

$$\{\xi^j T_\lambda \mid j = 0, \ldots, N-1\}$$

for any Young diagram λ in the $(N-1) \times l$-rectangle.

Proof The rotations that keep T_λ invariant form a finite subgroup of S^1. Any finite subgroup of S^1 is cyclic and therefore there is a unique rotation by a positive angle α, $0 < \alpha \leq 2\pi$, that generates all the rotations that keep T_λ invariant. The cardinality of the σ-orbit of λ is then equal to $\frac{N\alpha}{2\pi}$ by lemma 10.1.1.

The rotation by α induces a permutation of the N points of T_λ. This permutation is a power of an N-cycle. Hence, the rotation by $N\alpha$ induces the identity permutation and thus $N\alpha$ is an integer multiple of 2π. Therefore, there exists a unique integer j_0, $1 \leq j_0 \leq N$, such that $\alpha = \frac{2\pi j_0}{N}$. No other rotation $\xi^{j'}$ with $1 \leq j' < j_0$ keeps T_λ invariant. The cardinality of

$$\{\xi^j T_\lambda \mid j = 0, \ldots, N-1\}$$

is therefore equal to j_0.

On the other hand, the cardinality of the σ-orbit of λ is equal to $\frac{N\alpha}{2\pi}$ (as stated above) which is equal to j_0. ♣

10.1.1 The unit circle and the outline of Young diagrams

We describe now a relation between the outline of a Young diagram and the set T_λ on the unit circle. We position a Young diagram λ that lies in the $(N-1) \times l$-rectangle in an actual $N \times l$-rectangle and remove the lower and the right edge of this rectangle. An example is shown in figure 10.1. We refer to the solid line in this figure as the *outline of the Young diagram.*

We define a word $w(\lambda)$ with the letters 'full' and 'empty' by reading the sequence $1, \zeta, \zeta^2, \ldots, \zeta^{l+N-1}$ and we write 'full' if the element lies in T_λ, and we write 'empty' if it does not lie in T_λ. This word $w(\lambda)$ can be read directly off the Young diagram λ in the following way.

We start at the bottom left and follow the outline of λ to the top right. Whenever we go vertically we write 'full', and whenever we go horizontally we write 'empty'. We start with 'full' because $\lambda_N = 0$. On the other hand, 1 lies in T_λ. Whenever $\lambda_{i+1} = \lambda_i$, i.e. we go one step vertically, then the elements $\zeta^{\lambda_{i+1}+N-(i+1)}$ and ζ^{λ_i+N-i} are consecutive in the sequence $1, \zeta, \zeta^2, \ldots, \zeta^{l+N-1}$.

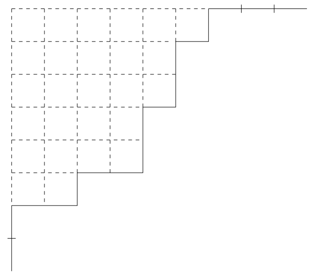


Figure 10.1: Young diagram $\lambda = (6, 5, 5, 4, 4, 2)$ with extended lines in the case $N = 8$ and $l = 9$.

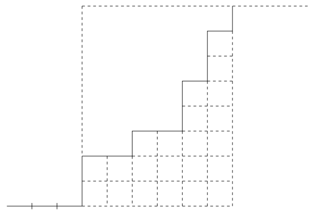


Figure 10.2: The dual Young diagram upside down.

Whenever $\lambda_i = \lambda_{i+1} + k$ for some $k > 0$, then on the one hand we go k steps vertically, and on the other hand the complement $E_{l+N} \backslash T_\lambda$ contains the k consecutive elements $\zeta^{\lambda_{i+1}+N-(i+1)+1}, \ldots, \zeta^{\lambda_i+N-i-1}$. Walking along λ we thus read $w(\lambda)$.

This visualization of the word $w(\lambda)$ leads to a nice interpretation of the result from lemma 10.1.1 that $b(T_\lambda) = T_{\sigma^{-1}(\lambda^*)}$.

Figure 10.2 derives from figure 10.1 by taking the complement of λ in the $N \times \lambda_1$-rectangle. The upper right spoke in figure 10.1 becomes the lower left spoke in figure 10.2 because λ^* derives from this diagram after the rotation by π and thus we also have to rotate the two bounding edges of the $N \times l$-rectangle.

Walking along the solid outline of λ^* and the solid spokes from the top right to the bottom left, we read the reverse word of $w(\lambda)$ up to the cyclic shift of length $l - \lambda_1$ due to the horizontal spoke. Hence, up to rotation (i.e. σ-operations), $b(T_\lambda)$ is equal to T_{λ^*}.

This technique of reading the word $w(\lambda)$ allows us to present a relation between T_λ and $T_{\lambda^\vee}$ as explained in the following.

Given any subset S of E_{l+N} with N elements, we can rotate this set by some angle so that 1 lies in this set. This is well defined up to some rotation by $\mathbb{Z}_N$, and thus S determines a Young diagram up to σ-operation. Furthermore, the complement of S consists of l points. These determine a Young diagram in the

$(l-1) \times N$-rectangle up to σ-operation. We remark that this σ-operation refers to the $(l-1) \times N$-rectangle which means adding a row of length N and removing all columns of length l. We avoid the notations σ_l and σ_N for the σ-operations in the $(N-1) \times l$-rectangle resp. $(l-1) \times N$-rectangle.

It is obvious that every σ-orbit of Young diagrams in the $(N-1) \times l$-rectangle contains a representative that lies in the $(N-1) \times (l-1)$-rectangle. Therefore, it is not a strong restriction to consider Young diagrams in the $(N-1) \times (l-1)$-rectangle.

Lemma 10.1.3 *The sets $E_{l+N} \backslash T_\lambda$ and $T_{(\lambda^\vee)^*}$ differ by a rotation for any Young diagram λ in the $(N-1) \times (l-1)$-rectangle.*

Proof We remark that $\lambda^\vee$ lies in the $(l-1) \times N$-rectangle because λ lies in the $(N-1) \times (l-1)$-rectangle.

When we transpose λ in figure 10.1 we see that the word $w(\lambda^\vee)$ derives from the word $w(\lambda)$ by first reversing the order of its letters and then switching the letters 'empty' and 'full'.

This means that ζ^{i-1} lies in T_λ if and only if ζ^{l+N-i} does not lie in $T_{\lambda^\vee}$ for $i = 1, \ldots, l+N$. The complex conjugate of ζ^{l+N-i} is equal to ζ^i, and therefore ζ^{i-1} lies in T_λ if and only if ζ^i does not lie in $\overline{T_\lambda}$ for $i = 1, \ldots, l+N$. This means that

$$\zeta T_\lambda \uplus \overline{T_{\lambda^\vee}} = E_{l+N}.$$

Complex conjugation transforms the set $T_{\lambda^\vee}$ into $T_{(\lambda^\vee)^*}$ up to rotation by lemma 10.1.1. Hence, $E_{l+N} \backslash T_\lambda$ and $T_{(\lambda^\vee)^*}$ are equal up to rotation. ♣

The example in figure 10.1 for $N = 8$ and $l = 9$ leads to the set

$$T_\lambda = \{1, \zeta, \zeta^4, \zeta^7, \zeta^8, \zeta^{10}, \zeta^{11}, \zeta^{13}\}.$$

The complement is

$$E_{l+N} \backslash T_\lambda = \{\zeta^2, \zeta^3, \zeta^5, \zeta^6, \zeta^9, \zeta^{12}, \zeta^{14}, \zeta^{15}, \zeta^{16}\}.$$

Replacing ζ by ζ^{-1}, i.e. conjugation, transforms this sequence into

$$\{\zeta, \zeta^2, \zeta^3, \zeta^5, \zeta^8, \zeta^{11}, \zeta^{12}, \zeta^{14}, \zeta^{15}\}$$

since $\zeta^{l+N} = \zeta^{17} = 1$. Rotation by ζ^{-1} leads to the set

$$\{1, \zeta, \zeta^2, \zeta^4, \zeta^7, \zeta^{10}, \zeta^{11}, \zeta^{13}, \zeta^{14}\}$$

which is equal to T_μ with $\mu = (6, 6, 5, 5, 3, 1) = \lambda^\vee$.

10.2 Young-solutions

We defined in section 6.1 the quotient ring $\mathcal{Y}_{N,l} = \mathcal{Y}/I_{N,l}$. We recall that the ring of Young diagrams $\mathcal{Y}$ is freely generated as an Abelian ring by the column diagrams $c_1, c_2, \ldots$. Hence, a ring homomorphism $\phi : \mathcal{Y} \to \mathbb{C}$ factors through $\mathcal{Y}_{N,l}$ if and only if

$$\begin{aligned} \phi(c_0) &= \phi(c_N), \\ \phi(c_i) &= 0 \;\text{ for } i \geq N+1, \text{ and} \\ \phi(d_j) &= 0 \;\text{ for } l+1 \leq j \leq l+N-1. \end{aligned}$$

Since the empty Young diagram c_0 is the unit for the multiplication, we have $\phi(c_0) = 1$. Hence, ϕ factors through $\mathcal{Y}_{N,l}$ if and only if

$$\begin{aligned} \phi(c_N) &= 1, \\ \phi(c_i) &= 0 \;\text{ for } i \geq N+1, \text{ and} \\ \phi(d_j) &= 0 \;\text{ for } l+1 \leq j \leq l+N-1. \end{aligned} \tag{10.2.1}$$

In particular, if ϕ factors through $\mathcal{Y}_{N,l}$ then ϕ is determined by $\phi(c_1), \ldots, \phi(c_{N-1})$.

An $(N-1)$-tuple $(\gamma_1, \ldots, \gamma_{N-1})$ of complex numbers is called a *Young-solution* if the map $\phi : \mathcal{Y} \to \mathbb{C}$ given by

$$\begin{aligned} \phi(c_i) &= \gamma_i \text{ for } 1 \leq i \leq N-1 \\ \phi(c_N) &= 1, \\ \phi(c_i) &= 0 \text{ for } i \geq N+1 \end{aligned}$$

factors through $\mathcal{Y}_{N,l}$.

Lemma 10.2.1 *There is bijection between Young-solutions and the family of sets of pairwise different complex numbers $\{y_1, \ldots, y_N\}$ that satisfy*

$$\begin{aligned} & y_i^{l+N} = y_j^{l+N} \textit{ for any } 1 \leq i, j \leq N, \\ & y_1^{(l+N)N} = 1, \\ & y_1 y_2 \cdots y_N = 1. \end{aligned}$$

The bijection is given by assigning to γ_i the i-th elementary symmetric function in $y_1, \ldots, y_N$.

Proof We define $\gamma_0 = \gamma_N = 1$. We define a polynomial $C(Z)$ in the variable Z,

$$C(Z) = \sum_{i=0}^{N} (-1)^i \gamma_i Z^i$$

for any $(N-1)$-tuple $(\gamma_1, \ldots, \gamma_{N-1})$ of complex numbers. We define $D(Z)$ to be the inverse power series of $C(Z)$,

$$D(Z) = C^{-1}(Z) = \sum_{j=0}^{\infty} \delta_j Z^j$$

where the complex numbers δ_j depend on $\gamma_1, \ldots, \gamma_{N-1}$. By equations (1.1.1) and (10.2.1) we see that $(\gamma_1, \ldots, \gamma_{N-1})$ is a Young-solution if and only if $\delta_j = 0$ for all $j = l+1, \ldots, l+N-1$.

Let $(\gamma_1, \ldots, \gamma_{N-1})$ be a Young-solution. Then

$$D(Z) = \sum_{j=0}^{l} \delta_j Z^j + \sum_{j=l+N}^{\infty} \delta_j Z^j,$$

and we denote the first summand (which is a polynomial) by $D'(Z)$, and the second summand (which is a power series) by $D''(Z)$. We have

$$C(Z)D'(Z) + C(Z)D''(Z) = 1.$$

The maximal degree in Z of $C(Z)D'(Z)$ is equal to $l+N$, and the minimal degree in Z of $C(Z)D''(Z)$ is equal to $l+N$, too. The term of degree $l+N$ in $C(Z)D''(Z)$ is equal to $\delta_{l+N}Z^{l+N}$. Hence,

$$C(Z)D'(Z) + \delta_{l+N}Z^{l+N} = 1.$$

Equivalently,

$$C(Z)D'(Z) = 1 - \beta Z^{l+N}$$

where $\beta = \delta_{l+N}$. The complex number β is non-zero because $C(Z)$ is a polynomial of degree N, and $D'(Z)$ has constant term 1.

Any root α of $C(Z)$ satisfies $\alpha^{l+N} = \beta^{-1}$ by the above equation. The N roots $\alpha_1, \ldots, \alpha_N$ of $C(Z)$ are pairwise different because the roots of $1 - \beta Z^{l+N}$ are pairwise different. We have $\alpha_1 \cdots \alpha_N = 1$ because the constant term of $C(Z)$ is equal to 1, and the coefficient of the highest term Z^N of $C(Z)$ is equal to $(-1)^N$.

We have

$$\begin{aligned} C(Z) &= (-1)^N \prod_{i=1}^{N} (Z - \alpha_i) \\ &= (-1)^N \prod_{i=1}^{N} \alpha_i(\alpha_i^{-1}Z - 1) \\ &= (-1)^N \prod_{i=1}^{N} (\alpha_i^{-1}Z - 1) \\ &= \prod_{i=1}^{N} (1 - \alpha_i^{-1}Z). \end{aligned}$$

This means that the coefficient γ_i of $(-1)^i Z^i$ in $C(Z)$ is the i-th elementary symmetric function in $\alpha_1^{-1}, \ldots, \alpha_N^{-1}$ which are the inverses of the roots of $C(Z)$. We have that $(\alpha_i^{-1})^{l+N} = \beta$ for all $i = 1, \ldots, N$ as mentioned above. The equation $\alpha_1 \cdots \alpha_N = 1$ implies that

$$(\alpha_1^{-1} \cdots \alpha_N^{-1})^{l+N} = 1,$$

hence $\beta^N = 1$. (Another way to see this is the following. We have by lemma 6.3.1 that $d_{l+N} = (-1)^{N+1} d_l$ in $\mathcal{Y}_{N,l}$. Hence, $d_{l+N}^N = (-1)^{(N+1)N} d_l^N = \sigma^N(c_0) = c_0 = 1$ by lemma 6.2.1.)

This means that for any Young-solution there exists a unique set of pairwise different complex numbers $y_1, \ldots, y_N$ $(= \alpha_1^{-1}, \ldots, \alpha_N^{-1})$ such that $y_1 \cdots y_N = 1$, $y_1^{N(l+N)} = 1$, and $y_i^{l+N} = y_j^{l+N}$ for any $1 \leq i, j \leq N$. The uniqueness derives from the fact that $y_1, \ldots, y_N$ are the inverses of the roots of $C(Z)$.

Conversely, let γ_i be the i-th elementary symmetric function of pairwise different complex numbers $y_1, \ldots, y_N$ with the properties as stated in the lemma. Denote $\tau = y_1^{-(l+N)}$. Aside from $y_1, \ldots, y_N$ there are l other $(l+N)$-th roots of τ^{-1}, say $x_1, \ldots, x_l$. We have

$$C(Z) = \prod_{i=1}^{N} (1 - y_i Z).$$

Then its inverse power series

$$\begin{aligned} D(Z) &= C^{-1}(Z) \\ &= \frac{1}{1 - \tau Z^{l+N}} \prod_{j=1}^{l} (1 - x_j Z) \\ &= (1 + \tau Z^{l+N} + \tau^2 Z^{2(l+N)} + \cdots) \prod_{j=1}^{l} (1 - x_j Z) \end{aligned}$$

has zero as coefficient of Z^k for $k = l+1, \ldots, l+N-1$. Hence, $(\gamma_1, \ldots, \gamma_{N-1})$ is a Young-solution. ♣

10.3 Young-solutions and the unit-circle

Our notation does not distinguish between a Young-solution and the set of N complex numbers assigned to it by lemma 10.2.1. We recall that ζ is a fixed primitive root of unity of order $l + N$.

Let $\{y_1, \ldots, y_N\}$ be a Young-solution. The y_i are pairwise different, and their $(N + l)$-th powers are all equal. Hence, there exist integers $a_1, \ldots, a_{N-1}$ with

$1 \leq a_1 < \cdots < a_{N-1} \leq N + l - 1$ so that

$$\{1, y_1^{-1}y_2, \ldots, y_1^{-1}y_N\} = \{1, \zeta^{a_1}, \ldots, \zeta^{a_{N-1}}\}.$$

Therefore,

$$y_1^{-1}\{y_1, \ldots, y_N\} = T_\lambda$$

for some Young diagram λ in the $(N-1) \times l$-rectangle. If we had chosen y_2 instead of y_1 then

$$y_2^{-1}\{y_1, \ldots, y_N\} = T_\mu$$

for some Young diagram μ in the $(N-1) \times l$-rectangle. Since the sets

$$y_1^{-1}\{y_1, \ldots, y_N\} \text{ and } y_2^{-1}\{y_1, \ldots, y_N\}$$

differ by a rotation a rotation of the unit circle, we know by lemma 10.1.1 that μ and λ lie in the same σ-orbit. The assignment of the σ-orbit of λ to the Young-solution $\{y_1, \ldots, y_N\}$ is therefore well defined.

Lemma 10.3.1 *The number of Young-solutions that are assigned the same σ-orbit is equal to the number of Young diagrams in this orbit.*

Proof Let λ be a Young diagram in the $(N-1) \times l$-rectangle. We denote $a_i = \lambda_{N-i} + i$ for $i = 1, \ldots, N-1$. By lemma 10.2.1 we see that the Young-solutions that are assigned the σ-orbit of λ are

$$\{y_0, y_0\zeta^{a_1}, \ldots, y_0\zeta^{a_{N-1}}\}$$

(which is equal to y_0T_λ) where y_0 has to satisfy the conditions

$$y_0^{l+N} = (y_0\zeta^{a_1})^{l+N} = \ldots = (y_0\zeta^{a_{N-1}})^{l+N}, \quad y_0^N\zeta^{(a_1+\cdots+a_{N-1})} = 1, \; y_0^{(l+N)N} = 1.$$

These conditions are equivalent to

$$y_0^N\zeta^{(a_1+\cdots+a_{N-1})} = 1 \quad \text{and} \quad y_0^{(l+N)N} = 1$$

which is equivalent to

$$y_0^N\zeta^{(a_1+\cdots+a_{N-1})} = 1. \tag{10.3.2}$$

There are N solutions for y_0 in the last equation. We choose one solution y_0', and then the other solutions for this equation are $y_0'\xi, y_0'\xi^2, \ldots, y_0'\xi^{N-1}$ where $\xi = \exp(2\pi i/N)$.

Our claim is that the cardinality of the following set of Young-solutions

$$\{y_0'\xi^jT_\lambda \mid j = 0, \ldots, N-1\}$$

is equal to the cardinality of the σ-orbit of λ. Since the rotation by y_0' does not influence the cardinality of this set, we have to show that the cardinality of

$$\{\xi^jT_\lambda \mid j = 0, \ldots, N-1\}$$

is equal to the cardinality of the σ-orbit of λ. This is true by lemma 10.1.2. ♣

10.4 Hopf link and Young-solutions

In chapter 4 we were considering the Homfly polynomial $\langle \lambda, \mu \rangle$ of the Hopf link with decorations Q_λ and Q_μ on its components. This is a rational function in x, v and s. We considered in previous parts the substitution of s by a primitive root of unity of order $2(l+N)$, the substitution of x by an N-th root of s^{-1}, and the substitution of v by s^{-N}.

Here, it will be necessary to restrict the substitutions. We will choose x to be a primitive root of unity of order $2N(l+N)$ and we shall fix this choice unless stated otherwise. We will substitute s by x^{-N}, and we will substitute v by x^{N^2}. This is necessary because we shall want x^{-N} to be a primitive root of unity of order $2(l+N)$ as usual, but additionally, we shall want $\xi = x^{2(l+N)}$ to be a primitive root of unity of order N.

To any Young diagram λ in the $(N-1) \times l$-rectangle we assign the set of complex numbers

$$\underline{c}(\lambda) = \{\alpha, \alpha\zeta^{\lambda_{N-1}+1}, \ldots, \alpha\zeta^{\lambda_1+N-1}\} = \alpha T_\lambda$$

where $\alpha = x^{N(N-1)+2|\lambda|}$ and $\zeta = x^{-2N}$. This is a Young-solution because

$$\begin{aligned}
\alpha^N \zeta^{(\lambda_{N-1}+1)+\cdots+(\lambda_1+N-1)} &= x^{N^2(N-1)+2N|\lambda|}\zeta^{|\lambda|+\frac{N(N-1)}{2}} \\
&= x^{N^2(N-1)+2N|\lambda|}x^{-2N\left(|\lambda|+\frac{N(N-1)}{2}\right)} \\
&= 1
\end{aligned}$$

and thus the condition from equation (10.3.2) is satisfied.

Lemma 10.4.1 *We have $\underline{c}(\sigma(\lambda)) = x^{2(N+l)}\underline{c}(\lambda)$ for any Young diagram λ in the $(N-1) \times l$-rectangle.*

Proof The Young-solution assigned to $\sigma(\lambda)$ is by the above definition

$$\underline{c}(\sigma(\lambda)) = \beta T_{\sigma(\lambda)}$$

where $\beta = x^{N(N-1)+2|\sigma(\lambda)|}$.

We have $|\sigma(\lambda)| = |\lambda| + l - N\lambda_{N-1}$ because $\sigma(\lambda)$ derives from λ by adding a row of length l and then removing all (i.e. λ_{N-1}) columns of length N. Hence,

$$\begin{aligned}
\beta &= x^{N(N-1)+2(|\lambda|+l-N\lambda_{N-1})} \\
&= \alpha x^{2l-2N\lambda_{N-1}} \\
&= \alpha x^{2l}\zeta^{\lambda_{N-1}}.
\end{aligned}$$

By lemma 10.1.1 we know that

$$T_{\sigma(\lambda)} = a(T_\lambda) = \zeta^{-(\lambda_{N-1}+1)}T_\lambda.$$

Hence,

$$\begin{aligned}\underline{c}(\sigma(\lambda)) &= \beta T_{\sigma(\lambda)}\\ &= \alpha x^{2l}\zeta^{\lambda_{N-1}}\zeta^{-(\lambda_{N-1}+1)}T_\lambda\\ &= x^{2l}\zeta^{-1}\alpha T_\lambda\\ &= x^{2(l+N)}\underline{c}(\lambda).\end{aligned}$$

♣

The complex conjugate of the set $\underline{c}(\lambda)$ is again a Young-solution because the condition from equation (10.3.2) is satisfied. We know by lemma 10.1.1 that complex conjugation of T_λ leads to the σ-orbit of the dual λ^* of λ. Hence, $\overline{\underline{c}(\lambda)}$ corresponds to the σ-orbit of λ^*, i.e. $\underline{c}(\lambda^*) = \xi^k\overline{\underline{c}(\lambda)}$ for some k, $0 \leq k \leq N-1$, and $\xi = x^{2(l+N)}$. It turns out that $k = 0$.

Lemma 10.4.2 *We have $\underline{c}(\lambda^*) = \overline{\underline{c}(\lambda)}$ for any Young diagram λ in the $(N-1)\times l$-rectangle.*

Proof Let λ be a Young diagram in the $(N-1)\times l$-rectangle. We recall that the dual λ^* is up to rotation the complement of λ in the $N \times \lambda_1$-rectangle. Hence, $\lambda_i^* = \lambda_1 - \lambda_{N-i+1}$ for $i = 1,\ldots,N-1$, and $|\lambda^*| = N\lambda_1 - |\lambda|$. We thus get

$$\begin{aligned}\underline{c}(\lambda^*) &= \beta\{1, \zeta^{\lambda^*_{N-1}+1}, \ldots, \zeta^{\lambda^*_1+N-1}\}\\ &= \beta\{1, \zeta^{\lambda_1-\lambda_2+1}, \ldots, \zeta^{\lambda_1-\lambda_N+N-1}\}\\ &= \beta\zeta^{\lambda_1+N-1}\{\zeta^{-(\lambda_1+N-1)}, \zeta^{-(\lambda_2+N-2)}, \ldots, \zeta^{-(\lambda_{N-1}+1)}, 1\}\\ &= \beta\zeta^{\lambda_1+N-1}\{1, \zeta^{-(\lambda_{N-1}+1)}, \ldots, \zeta^{-(\lambda_2+N-2)}, \zeta^{-(\lambda_1+N-1)}\}\end{aligned}$$

where $\beta = x^{N(N-1)+2|\lambda^*|}$. We have that

$$\begin{aligned}\beta\zeta^{\lambda_1+N-1} &= s^{1-N}x^{2(N\lambda_1-|\lambda|)}\zeta^{\lambda_1+N-1}\\ &= s^{1-N}\zeta^{-\lambda_1}x^{-2|\lambda|}\zeta^{\lambda_1+N-1}\\ &= s^{1-N}x^{-2|\lambda|}\zeta^{N-1}\\ &= s^{N-1}x^{-2|\lambda|}\end{aligned}$$

where we used that $\zeta = x^{-2N}$ and $s = x^{-N}$. We thus get

$$\underline{c}(\lambda^*) = s^{N-1}x^{-2|\lambda|}\{1, \zeta^{-(\lambda_{N-1}+1)}, \ldots, \zeta^{-(\lambda_2+N-2)}, \zeta^{-(\lambda_1+N-1)}\}.$$

On the other hand, we have by definition

$$\underline{c}(\lambda) = s^{1-N}x^{2|\lambda|}\{1, \zeta^{\lambda_{N-1}+1}, \ldots, \zeta^{\lambda_1+N-1}\}.$$

The conjugate of x is equal to x^{-1} because the norm of x is equal to 1. Therefore, the conjugate of $s = x^{-N}$ is equal to s^{-1}. We thus derive from the above equations that

$$\begin{aligned} \underline{c}(\lambda^*) &= s^{N-1} x^{-2|\lambda|} \{1, \zeta^{-(\lambda_{N-1}+1)}, \ldots, \zeta^{-(\lambda_2+N-2)}, \zeta^{-(\lambda_1+N-1)}\} \\ &= \overline{\underline{c}(\lambda)}. \end{aligned}$$

♣

The terms $\lambda_i + N - i$ for $i = 1, \ldots, N-1$ appear in the sets T_λ and $\underline{c}(\lambda)$ because of the relation between the ring of Young diagrams and Schur functions. We can exploit this by relating Schur functions, Young-solutions and the Hopf link by lemmas 4.1.5 and 4.3.3. Their combination implies that $\langle \lambda, \mu \rangle / \langle \lambda \rangle$ is the Schur function s_μ in infinitely many variables which are to be substituted by $s^{N+2\lambda_i-2i+1} x^{2|\lambda|}$ for $i = 1, \ldots, N$, and all the other variables are to be substituted by zero. This result is true under the condition that λ has at most N rows, and that v is to be substituted by s^{-N}.

If μ has more than N rows then the Schur function s_μ becomes zero after the above substitution of N variables by $s^{N+2\lambda_i-2i+1} x^{2|\lambda|}$ for $i = 1, \ldots, N$ and all other variables are substituted by zero. Therefore, we restrict to the interesting case that μ has at most N rows, and we thus can regard s_μ as the Schur function in N variables.

We can write

$$s^{N+2\lambda_i-2i+1} x^{2|\lambda|} = s^{1-N} x^{2|\lambda|} s^{2(\lambda_i+N-i)}$$

for $i = 1, \ldots, N$. We restrict λ to Young diagrams in the $(N-1) \times l$-rectangle and we thus have $\lambda_N = 0$. Hence,

$$\frac{\langle \lambda, \mu \rangle}{\langle \lambda \rangle} = s_\mu \left(\gamma, \gamma s^{2(\lambda_{N-1}+1)}, \ldots, \gamma s^{2(\lambda_1+N-1)} \right)$$

where $\gamma = s^{1-N} x^{2|\lambda|}$. We see that γ is equal to the value of α that we have chosen in order to define $\underline{c}(\lambda)$. In fact, this was our motivation for the definition of $\underline{c}(\lambda)$. We thus have proved that

Lemma 10.4.3 *We have*

$$s_\mu(\underline{c}(\lambda)) = \frac{\langle \lambda, \mu \rangle}{\langle \lambda \rangle}$$

for any Young diagram λ in the $(N-1) \times l$-rectangle and any Young diagram μ with at most N rows, after the substitutions of x by a primitive root of unity of order $2N(l+N)$, $s = x^{-N}$, and $v = s^{-N} = x^{N^2}$.

We finish our study of Young-solutions by describing the effect of the σ-operation to the statement of lemma 10.4.3.

We know by lemma 10.4.1 that $\underline{c}(\sigma^k(\lambda)) = \xi^k \underline{c}(\lambda)$ for any $k = 1, \ldots, N-1$, where $\xi = x^{2(l+N)}$. Since s_μ is a homogeneous polynomial of degree $|\mu|$ in N variables, we have that

$$s_\mu(\underline{c}(\sigma^k(\lambda))) = \xi^{k|\mu|} s_\mu(\underline{c}(\lambda)).$$

Lemma 10.4.3 implies that

$$\frac{\langle \sigma^k(\lambda), \mu \rangle}{\langle \sigma^k(\lambda) \rangle} = \xi^{k|\mu|} \frac{\langle \lambda, \mu \rangle}{\langle \lambda \rangle} \tag{10.4.3}$$

for any Young diagrams λ and μ in the $(N-1) \times l$-rectangle.

We can deduce this result as well from lemma 9.2.2. The lemma implies that after the substitutions for s, x and v we have

$$\langle \sigma^k(\lambda), \mu \rangle = p^{2k|\mu|} \langle \lambda, \mu \rangle$$

and

$$\left\langle \sigma^k(\lambda) \right\rangle = \langle \lambda \rangle$$

for any non-negative integer k because the linking number of the two components of the Hopf link is equal to 1, and the self linking number of any component is equal to zero. Here, $p = -s^{-1} x^l$. Since we make the substitution $s = x^{-N}$, we get $p^2 = x^{2(l+N)}$. This is our preferred primitive N-th root of unity ξ, and we thus deduce equation 10.4.3.

Chapter 11

Quantum invariants and Homfly polynomial

In this chapter we consider algebras over a commutative ring k. The cases for k we are interested in are either $\mathbb{C}$, the field $\mathbb{C}(q)$ of rational functions in a variable q, or the algebra $\mathbb{C}[[h]]$ of power series in a variable h. The case $k = \mathbb{C}[[h]]$ is rather tricky since so-called 'completions' of modules over $\mathbb{C}[[h]]$ are necessary to deal with the scalars. Furthermore, $\mathbb{C}[[h]]$ is given the h-adic topology. This is described in chapter XVI of [12].

For a complex semi-simple Lie algebra g we define in section 11.3 the Quantum enveloping algebra $U_h(g)$ over $\mathbb{C}[[h]]$. There exists a simpler version $U_q(g)$ over $\mathbb{C}[q]$ whose theory is somehow parallel to $U_h(g)$ as mentioned at the end of section XVII.2 of [12]. The translation between these two algebras is $q = e^h$. The disadvantage of $U_q(g)$ is the lack of a universal R-matrix. The exposition given here deals with $U_h(g)$ but without mentioning explicitly the technical difficulties arising for tensor products of $U_h(g)$-modules.

We remark that the variable h in $U_h(sl(N))$ is not the same in [12] and [4], one differs from the other by the factor 2. Furthermore, they are considering different Hopf algebra structures on this algebra, but lemma 11.3.3 will show that they are equivalent.

11.1 Ribbon Hopf algebras

Whenever we are considering the tensor product of two algebras A and B over a commutative ring k, we understand the tensor product to be over k and we abbreviate $A \otimes_k B$ by $A \otimes B$.

Definition A *ribbon Hopf algebra* $\mathcal{A}$ is both an algebra and a coalgebra over a commutative ring k, i.e. there are maps $\zeta : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ (called multiplication),

$\iota : k \to \mathcal{A}$ (called the unit), $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (called comultiplication), and $\varepsilon : \mathcal{A} \to k$ (called the counit) which satisfy

$$\begin{aligned} \zeta(\mathrm{id}_{\mathcal{A}} \otimes \iota) &= \zeta(\iota \otimes \mathrm{id}_{\mathcal{A}}) = \mathrm{id}_{\mathcal{A}}, \\ \zeta(\zeta \otimes \mathrm{id}_{\mathcal{A}}) &= \zeta(\mathrm{id}_{\mathcal{A}} \otimes \zeta) \end{aligned}$$

(i.e. $\mathcal{A}$ is an algebra), and

$$\begin{aligned} (\mathrm{id}_{\mathcal{A}} \otimes \varepsilon)\Delta &= (\varepsilon \otimes \mathrm{id}_{\mathcal{A}})\Delta = \mathrm{id}_{\mathcal{A}}, \\ (\Delta \otimes \mathrm{id}_{\mathcal{A}})\Delta &= (\mathrm{id}_{\mathcal{A}} \otimes \Delta)\Delta, \end{aligned}$$

(i.e. $\mathcal{A}$ is a coalgebra). Furthermore, multiplication and unit are homomorphisms of coalgebras, and, equivalently, comultiplication and counit are homomorphisms of algebras. Furthermore, we require the existence of an anti-homomorphism $S : \mathcal{A} \to \mathcal{A}$ (called the antipode) that satisfies

$$\zeta(S \otimes \mathrm{id}_{\mathcal{A}})\Delta = \iota\varepsilon = \zeta(\mathrm{id}_{\mathcal{A}} \otimes S)\Delta.$$

Furthermore, we require the existence of an invertible element $R \in \mathcal{A} \otimes \mathcal{A}$ (called a universal R-matrix) and an invertible and central element $v \in \mathcal{A}$ such that

$$\begin{aligned} \Delta^{\mathrm{op}}(x) &= R\Delta(x)R^{-1} \text{ for all } x \in \mathcal{A}, \\ (\Delta \otimes \mathrm{id}_{\mathcal{A}})(R) &= R_{13}(1 \otimes R), \\ (\mathrm{id}_{\mathcal{A}} \otimes \Delta)(R) &= R_{13}(R \otimes 1), \\ v^2 &= uS(u), \\ \Delta(v) &= (R_{21}R)^{-1}(v \otimes v) \\ \varepsilon(v) &= 1 \\ S(v) &= v \end{aligned}$$

where $R = \sum_i s_i \otimes t_i$, $u = \sum_i S(t_i)s_i$, $R_{13} = \sum_i s_i \otimes 1 \otimes t_i$, $R_{21} = \sum_i t_i \otimes s_i$, and $\Delta^{\mathrm{op}} = \tau_{\mathcal{A},\mathcal{A}}\Delta$ where $\tau_{\mathcal{A},\mathcal{A}}$ is the flip of the components of $\mathcal{A} \otimes \mathcal{A}$. We shall denote uv^{-1} by μ which is sometimes called the *ribbon element.*

We remark that a ribbon Hopf algebra may contain several universal R-matrices. We also remark that μ satisfies the equation $\Delta(\mu) = \mu \otimes \mu$.

The tensor product of any $\mathcal{A}$-modules V and W is an $\mathcal{A} \otimes \mathcal{A}$-module by defining $(a_1 \otimes a_2) \cdot (v \otimes w) = (a_1 v \otimes a_2 w)$. The Hopf structure allows us to turn $V \otimes W$ into an $\mathcal{A}$-module by defining $a \cdot (v \otimes w) = \Delta(a) \cdot (v \otimes w)$.

The dual $V^* = \mathrm{Hom}_k(V, k)$ of an $\mathcal{A}$-module V becomes an $\mathcal{A}$-module by defining $\langle a \cdot \xi, v\rangle = \langle \xi, S(a) \cdot v\rangle$ where $a \in \mathcal{A}$, $\xi \in V^*$, $v \in V$, and $\langle\ ,\ \rangle$ is the natural pairing between V^* and V.

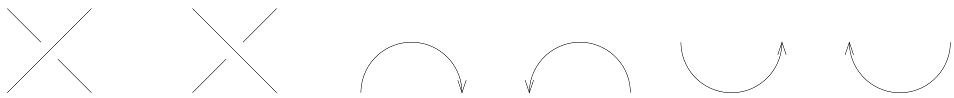


Figure 11.1: The homomorphisms $\rho, \overline{\rho}, F_1, F_2, F_3, F_4$ (from left to right).

11.2 An invariant of ribbon tangles

We consider a special case of ribbon tangles. An (m, n)-ribbon tangle is a diagram of oriented arcs and oriented simple closed curves in the square $[0, 1] \times [0, 1]$ of the Euclidean plane such that all the boundary points of the arcs belong either to the m points at the top $[0, 1] \times 1$ of the square, or to the n points at the bottom $[0, 1] \times 0$. (This implies that $m + n$ has to be even). The Euclidean plane is assigned the standard orientation. We consider only diagrams for which the blackboard framing agrees with the actual framing of the tangle as explained in section 2.1. (The blackboard framing is the diagram together with its regular neighbourhood (respecting crossings) in the plane).

We consider a diagram of an (m, n)-ribbon tangle T with $\mathcal{A}$-modules $V_1, \dots, V_k$ assigned to its components. We shall also refer to this assignment as a *colouring*. The boundary points at the top of T belong to arcs that are coloured by, say, $V_{i_1}, \dots, V_{i_m}$ from left to right. At the bottom we read, say, $V_{j_1}, \dots, V_{j_n}$ from left to right. At each of these endpoints, the corresponding arc is locally oriented either top-down or bottom-up. If the module we read off at an endpoint is, say, V then we denote a module V' by saying that V' is equal to V if the corresponding local orientation is top-down, and V' is equal to the dual module V^* if the corresponding local orientation is bottom-up.

A coloured ribbon tangle then determines a module homomorphism $J(T)$,

$$J(T) : V'_{j_1} \otimes \cdots \otimes V'_{j_n} \to V'_{i_1} \otimes \cdots \otimes V'_{i_m}.$$

$J(T)$ is defined by dissecting T into stripes in which we have either a single crossing, a single cap, or a single cup as shown in figure 11.1. For these basic pieces we define the corresponding module homomorphisms now. The map $J(T)$ is then the composition of these maps read from the bottom to the top of the diagram.

Consider the crossing at the very left of figure 11.1. For this diagram, $J(T)$ is a map $V \otimes W \to W \otimes V$ for modules V and W depending on the colouring and the local orientations of the two arcs. We denote $J(T)$ by $\rho_{V,W}$ (or ρ) in this case. We define this map $\rho_{V,W}$ as first multiplying with the universal R-matrix R and then switching the factors of $V \otimes W$. This map is $\mathcal{A}$-linear.

For the second crossing in figure 11.1, the map $J(T) : V \otimes W \to W \otimes V$ is denoted by $\overline{\rho}_{V,W}$ (or $\overline{\rho}$). It is defined by $\overline{\rho}_{V,W} = \rho^{-1}_{W,V}$.

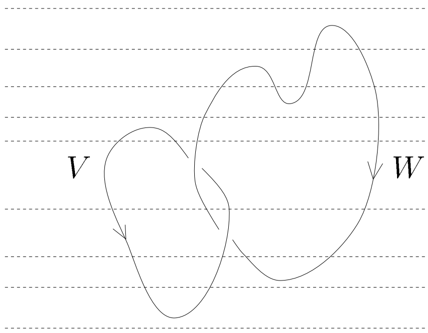


Figure 11.2: Decomposition of an oriented link diagram into simple pieces.

When the cap resp. cup arcs in figure 11.1 are coloured by a module V then the corresponding module homomorphisms are (from left to right)

$$\begin{aligned}
&F_1 : V^* \otimes V \to k, \quad F_1(g \otimes v) = g(v),\\
&F_2 : V \otimes V^* \to k, \quad F_2(v \otimes g) = g(\mu v),\\
&F_3 : k \to V \otimes V^*, \quad F_3(1) = \sum_m v_m \otimes v^m,\\
&F_4 : k \to V^* \otimes V, \quad F_4(1) = \sum_m v^m \otimes (\mu v_m)
\end{aligned}$$

where $\{v_m\}$ is a basis for V, and $\{v^m\}$ is the corresponding dual basis for V^*.

Finally, a straight vertical line determines the identity map, and the juxtaposition of diagrams is handled by the tensor product of the involved modules.

Reshetikhin and Turaev show in [22] that this map $J(T)$ is an isotopy-invariant of ribbon tangles. Any coloured $(0,0)$-ribbon tangle T (i.e. framed link) determines an $\mathcal{A}$-linear map $J(T) : k \to k$ which is the multiplication by an element of k. This scalar is invariant under isotopy of ribbon tangles, and it is called the *A-invariant* of the coloured framed link.

An example is shown in figure 11.2. The components of the Hopf link are coloured by $\mathcal{A}$-modules V resp. W. The linear map from k to k is given by

$$\begin{aligned}
k \quad &\xrightarrow{F_3} \quad V \otimes V^*\\
&\xrightarrow{F_4} \quad V \otimes V^* \otimes W^* \otimes W\\
&\xrightarrow{\rho_{V^*,W^*}} \quad V \otimes W^* \otimes V^* \otimes W\\
&\xrightarrow{\rho_{W^*,V^*}} \quad V \otimes V^* \otimes W^* \otimes W\\
&\xrightarrow{F_2} \quad k \otimes W^* \otimes W = W^* \otimes W\\
&\xrightarrow{F_3} \quad W^* \otimes W \otimes W^* \otimes W\\
&\xrightarrow{F_1} \quad k \otimes W^* \otimes W = W^* \otimes W\\
&\xrightarrow{F_1} \quad k.
\end{aligned}$$

Remark The action of $\mathcal{A}$ on the trivial module k is given by $a \cdot t = \varepsilon(a)t$. Furthermore, the tensor product of any number of copies of k is again k. For any tangle T, the homomorphism $J(T)$ for the trivial module k is the identity of k because $\varepsilon uv^{-1} = 1$ and $(\varepsilon\otimes\varepsilon)(R) = (\varepsilon\otimes\varepsilon)(R^{-1}) = 1$. (In fact $(\varepsilon\otimes\mathrm{id})(R) = 1\otimes 1$). In particular, the $\mathcal{A}$-invariant of any framed link coloured on all of its components by the trivial module k is equal to 1.

11.3 q-deformed universal enveloping algebras

Let $A = (a_{ij})_{i,j=1,\ldots,n}$ be a generalized Cartan matrix, i.e. $a_{ii} = 2$ and $a_{ij} \leq 0$ for all $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. Furthermore, A has to be symmetrizable, i.e. there exists a diagonal $(n\times n)$-matrix D with coprime integer diagonal entries $d_1, \ldots, d_n$ such that DA is symmetric and positive definite. (It turns out that D is unique.)

We define for an indeterminate q and an integer $j \geq 0$

$$\begin{aligned}
[j]_q &= \frac{q^j - q^{-j}}{q - q^{-1}} \quad (j \geq 1), \\
[0]_q &= 1, \\
[j]_q! &= [j]_q[j-1]_q \cdots [1]_q \quad (j \geq 1), \\
[0]_q! &= 1, \\
\begin{bmatrix} m \\ j \end{bmatrix}_q &= \frac{[m]_q!}{[j]_q![m-j]_q!} \quad (m \geq j \geq 0).
\end{aligned}$$

We denote by $\mathbb{C}[[h]]$ the ring of formal power series in the variable h. The invertible elements of $\mathbb{C}[[h]]$ are those power series that have a non-zero constant term.

We remark that $[k]_{e^{th}}$ is well defined and invertible in $\mathbb{C}[[h]]$ for any complex number t and integer k. This is because $e^{hr} - e^{-hr} = 2hr + \frac{r^3}{3}h^3 + \cdots$ and $e^h - e^{-h} = 2h + \frac{1}{3}h^3 + \cdots$, and after cancellation of the factor h, the denominator $e^h - e^{-h}$ becomes invertible.

A generalized Cartan matrix determines a Lie algebra that we denote by g. Our single application will be with the Lie algebra $sl(N)$ of traceless $(N\times N)$-matrices with complex entries. (The Lie bracket is given by the commutator $[A,B] = AB - BA$ which is traceless since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.)

The set of diagonal matrices of $sl(N)$ forms a Cartan subalgebra. The Cartan matrix A for $sl(N)$ is of size $(N-1)\times(N-1)$, with entries $a_{ii} = 2$, $a_{ij} = -1$ for $|i-j| = 1$, and $a_{ij} = 0$ for $|i-j| > 1$, where $i,j = 1,\ldots,N-1$. It is symmetric and positive definite, and therefore D is the identity matrix.

Definition Given a generalized $(n \times n)$-Cartan matrix A, we define $U_h(g)$ as the algebra over $\mathbb{C}[[h]]$ topologically generated by elements H_i, X_i^+ and X_i^- for $i = 1, \ldots, n$ with the following relations:

$$[H_i, H_j] = 0,\ [H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm},\ [X_i^+, X_j^-] = \delta_{ij} \frac{e^{d_i h H_i} - e^{-d_i h H_i}}{e^{d_i h} - e^{-d_i h}} \text{ and}$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{e^{d_i h}} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} = 0 \quad \text{for } i \neq j$$

where $[x, y] = xy - yx$, and δ_{ij} is the Kronecker-delta, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. The last equation is called the *Quantum-Serre-relation.* We remark that $(e^{d_i h H_i} - e^{-d_i h H_i})/(e^{d_i h} - e^{-d_i h})$ is defined over $\mathbb{C}[[h]]$ because the factor h in the denominator cancels with a factor h in the numerator.

Lemma 11.3.1 *We have*

$$\begin{aligned} e^{thH_i} X_j^+ &= e^{tha_{ij}} X_j^+ e^{thH_i} \text{ and} \\ e^{thH_i} X_j^- &= e^{-tha_{ij}} X_j^- e^{thH_i} \end{aligned}$$

in $U_h(g)$ for any complex number t and any integers $1 \leq i, j \leq n$.

Proof We have $[H_i, X_j^+] = a_{ij} X_j^+$, hence $H_i X_j^+ = X_j^+(H_i + a_{ij})$. Inductively we deduce that

$$H_i^k X_j^+ = X_j^+ (H_i + a_{ij})^k$$

for any integer $k \geq 0$. Hence,

$$\begin{aligned} e^{thH_i} X_j^+ &= \sum_{k \geq 0} \frac{1}{k!} (thH_i)^k X_j^+ \\ &= X_j^+ \sum_{k \geq 0} \frac{1}{k!} (th)^k (H_i + a_{ij})^k \\ &= X_j^+ e^{th(H_i + a_{ij})} \\ &= e^{tha_{ij}} X_j^+ e^{thH_i}. \end{aligned}$$

The result for X_j^- is proved similarly. ♣

There are two ways to turn $U_h(g)$ into a topological Hopf algebra over $\mathbb{C}[[h]]$. One way is to define the comultiplication Δ_h as

$$\begin{aligned} \Delta_h(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \\ \Delta_h(X_i^+) &= X_i^+ \otimes e^{d_i h H_i} + 1 \otimes X_i^+, \\ \Delta_h(X_i^-) &= X_i^- \otimes 1 + e^{-d_i h H_i} \otimes X_i^-, \end{aligned}$$

and the antipode S_h defined by

$$S_h(H_i) = -H_i, \quad S_h(X_i^+) = -X_i^+ e^{-d_i h H_i}, \quad S_h(X_i^-) = -e^{d_i h H_i} X_i^-, \quad (11.3.1)$$

and the counit ε_h defined by

$$\varepsilon_h(H_i) = \varepsilon_h(X_i^\pm) = 0.$$

The other way is to define the comultiplication Δ'_h as

$$\begin{aligned} \Delta'_h(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \\ \Delta'_h(X_i^\pm) &= X_i^\pm \otimes e^{\frac{d_i h H_i}{2}} + e^{-\frac{d_i h H_i}{2}} \otimes X_i^\pm \end{aligned}$$

and the antipode S'_h by

$$S'_h(H_i) = -H_i, \quad S'_h(X_i^+) = -e^{d_i h} X_i^+, \quad S'_h(X_i^-) = -e^{d_i h} X_i^-,$$

and the counit ε'_h by

$$\varepsilon'_h(H_i) = \varepsilon'_h(X_i^\pm) = 0.$$

The first definition corresponds to definition 6.5.1 of [4], the second corresponds to definition XVII.2.3 of [12]. In fact, this is not exactly the definition of Kassel, because he uses a variable h which corresponds to $2h$ in our setting. This means, one has to replace the h in our definition by $h/2$ in order to get Kassel's definition.

The Hopf algebras are in fact isomorphic. To prove this, we first look at the level of the algebra.

Lemma 11.3.2 *The map f given by $X_i^+ \mapsto X_i^+ e^{\frac{d_i h H_i}{2}}$, $X_i^- \mapsto e^{-\frac{d_i h H_i}{2}} X_i^-$ and $H_i \mapsto H_i$ extends to an algebra isomorphism of $U_h(g)$.*

Proof We have to check that the relations are preserved. We have

$$\begin{aligned} [f(H_i), f(X_j^+)] &= [H_i, X_j^+ e^{\frac{d_j h H_j}{2}}] \\ &= H_i X_j^+ e^{\frac{d_j h H_j}{2}} - X_j^+ e^{\frac{d_j h H_j}{2}} H_i \\ &= [H_i, X_j^+] e^{\frac{d_j h H_j}{2}} \\ &= a_{ij} X_j^+ e^{\frac{d_j h H_j}{2}} \\ &= f(a_{ij} X_j^+) \end{aligned}$$

where we used that $[H_i, H_j] = 0$. The case for X_j^- is checked similarly.

We have

$$\begin{aligned}
[f(X_i^+), f(X_j^-)] &= X_i^+ e^{\frac{d_i h H_i}{2}} e^{-\frac{d_j h H_j}{2}} X_j^- - e^{-\frac{d_j h H_j}{2}} X_j^- X_i^+ e^{\frac{d_i h H_i}{2}} \\
&= X_i^+ X_j^- e^{\frac{d_i h H_i}{2}} e^{-\frac{d_j h H_j}{2}} e^{-\frac{d_i h a_{ij}}{2}} e^{d_j h} \\
&\quad - X_j^- X_i^+ e^{\frac{d_i h H_i}{2}} e^{-\frac{d_j h H_j}{2}} e^{-\frac{d_j h a_{ji}}{2}} e^{d_j h} \\
&= [X_i^+, X_j^-] e^{\frac{d_i h H_i}{2}} e^{-\frac{d_j h H_j}{2}} e^{-\frac{d_i h a_{ij}}{2}} e^{d_j h}
\end{aligned}$$

where we used lemma 11.3.1 and the fact that $d_i a_{ij} = d_j a_{ji}$ since DA is a symmetric matrix and that $a_{jj} = 2$.

If $i \neq j$ then $[X_i^+, X_j^-] = 0$ in $U_h(g)$, hence the above equation implies that $[f(X_i^+), f(X_j^-)] = f([X_i^+, X_j^-])$.

If $i = j$ then $a_{ij} = 2$ and trivially $d_i = d_j$, hence

$$[f(X_i^+), f(X_i^-)] = [X_i^+, X_i^-].$$

Since $[X_i^+, X_i^-] = (e^{d_i h H_i} - e^{-d_i h H_i})/(e^{d_i h} - e^{-d_i h})$ is a relation for $U_h(g)$, and $f(H_i) = H_i$, it follows from the above equation that

$$[f(X_i^+), f(X_i^-)] = f([X_i^+, X_i^-]).$$

We have therefore

$$[f(X_i^+), f(X_j^-)] = f([X_i^+, X_j^-])$$

for any $i, j = 1, \ldots, n$.

Finally, the map f respects the Quantum-Serre-relation because

$$f\left((X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k}\right)$$

turns out to be a multiple of $(X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k}$, and the factor depends only on i and j (and not on k). In fact, let t and r by any non-negative integers. Then

$$\begin{aligned}
& f\left((X_i^+)^t X_j^+ (X_i^+)^r\right) \\
&\quad = \left(X_i^+ e^{\frac{d_i h H_i}{2}}\right)^t \left(X_j^+ e^{\frac{d_j h H_j}{2}}\right) \left(X_i^+ e^{\frac{d_i h H_i}{2}}\right)^r \\
&\quad = (X_i^+)^t X_j^+ (X_i^+)^r e^{\frac{d_i h H_i (t+r)}{2}} e^{\frac{d_j h H_j}{2}} e^{d_i h (1+2+\cdots+(t+r-1))} e^{\frac{d_i h a_{ij} t}{2}} e^{\frac{d_j h a_{ji} r}{2}}
\end{aligned}$$

where we shifted (using lemma 11.3.1) t-times a factor $e^{\frac{d_i h H_i}{2}}$ past X_j^+, r-times a factor $e^{\frac{d_j h H_j}{2}}$ past X_i^+, and $(1 + 2 + \cdots + (t + r - 1))$-times a factor $e^{\frac{d_i h H_i}{2}}$ past X_i^+. Since $d_i a_{ij} = d_j a_{ji}$, we get

$$f\left((X_i^+)^t X_j^+ (X_i^+)^r\right) = (X_i^+)^t X_j^+ (X_i^+)^r e^{\frac{d_i h H_i (t+r)}{2}} e^{\frac{d_j h H_j}{2}} e^{d_i h \frac{(t+r)(t+r-1)}{2}} e^{\frac{d_i h a_{ij} (t+r)}{2}}.$$

Hence,

$$f\left((X_i^+)^k X_j^+ (X_i^+)^{1-a_{ij}-k}\right) = (X_i^+)^k X_j^+ (X_i^+)^{1-a_{ij}-k}\beta_{ij}$$

where

$$\beta_{ij} = e^{\frac{d_i h H_i (1-a_{ij})}{2}} e^{\frac{d_j h H_j}{2}} e^{d_i h \frac{(1-a_{ij})(1-a_{ij}-1)}{2}} e^{\frac{d_i h a_{ij}(1-a_{ij})}{2}}$$

which is independent of k. We denote

$$T_{ij} = \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{e^{d_i h}} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k}.$$

We have $f(T_{ij}) = T_{ij}\beta_{ij}$ for any i and j. Hence, $f(T_{ij}) = 0$ for any $i \neq j$ since $T_{ij} = 0$ for any $i \neq j$. Hence, f respects the Quantum-Serre-relation. The case for X_i^- and X_j^- is proved similarly.

The map f is bijective since $e^{\frac{d_i h H_i}{2}}$ is invertible with inverse $e^{-\frac{d_i h H_i}{2}}$. ♣

Lemma 11.3.3 *The algebra isomorphism $f : U_h(g) \to U_h(g)$ is an isomorphism of Hopf algebras $(U_h(g), \Delta_h, \varepsilon_h, S_h)$ and $(U_h(g), \Delta'_h, \varepsilon'_h, S'_h)$.*

Proof First, we show that f respects the antipode. We have

$$\begin{aligned} fS_h(X_i^+) &= f(-X_i^+ e^{-d_i h H_i}) \\ &= -X_i^+ e^{\frac{d_i h H_i}{2}} e^{-d_i h H_i} \\ &= -X_i^+ e^{-\frac{d_i h H_i}{2}} \end{aligned}$$

and

$$\begin{aligned} S'_h(f(X_i^+)) &= S'_h\left(X_i^+ e^{\frac{d_i h H_i}{2}}\right) \\ &= S'_h\left(e^{\frac{d_i h H_i}{2}}\right) S'_h(X_i^+) \\ &= e^{-\frac{d_i h H_i}{2}}(-e^{d_i h} X_i^+) \\ &= -e^{d_i h} e^{-d_i h} X_i^+ e^{-\frac{d_i h H_i}{2}} \\ &= -X_i^+ e^{-\frac{d_i h H_i}{2}} \end{aligned}$$

where we used that the antipode is an anti-homomorphism and we used lemma 11.3.1. Hence, $fS_h(X_i^+) = S'_h(f(X_i^+))$. Similarly, $fS_h(X_i^-) = S'_h(f(X_i^-))$ Finally,

$$fS_h(H_i) = f(-H_i) = -H_i = S'_h(H_i) = S'_h(f(H_i))$$

which completes the proof that f respects the antipode, i.e. $fS_h = S'_h f$.

In order to show that f respects the comultiplication, we make the observation that $\Delta'_h(e^{thH_i}) = e^{thH_i} \otimes e^{thH_i}$ for any complex number t and any $1 \leq i \leq n$. This follows immediately from $\Delta'_h(H_i) = H_i \otimes 1 + 1 \otimes H_i$ by mimicking the proof that $e^{x+y} = e^x e^y$ for any complex numbers x and y. We therefore get

$$\begin{aligned}
(f \otimes f)\Delta_h(X_i^+) &= (f \otimes f)(X_i^+ \otimes e^{d_i hH_i} + 1 \otimes X_i^+) \\
&= f(X_i^+) \otimes f(e^{d_i hH_i}) + f(1) \otimes f(X_i^+) \\
&= X_i^+ e^{\frac{d_i hH_i}{2}} \otimes e^{d_i hH_i} + 1 \otimes X_i^+ e^{\frac{d_i hH_i}{2}}
\end{aligned}$$

and hence

$$\begin{aligned}
\Delta'_h(f(X_i^+)) &= \Delta'_h\left(X_i^+ e^{\frac{d_i hH_i}{2}}\right) \\
&= \Delta'_h(X_i^+)\Delta'_h\left(e^{\frac{d_i hH_i}{2}}\right) \\
&= \left(X_i^+ \otimes e^{\frac{d_i hH_i}{2}} + e^{-\frac{d_i hH_i}{2}} \otimes X_i^+\right)\left(e^{\frac{d_i hH_i}{2}} \otimes e^{\frac{d_i hH_i}{2}}\right) \\
&= X_i^+ e^{\frac{d_i hH_i}{2}} \otimes e^{d_i hH_i} + 1 \otimes X_i^+ e^{\frac{d_i hH_i}{2}}.
\end{aligned}$$

Hence, $(f \otimes f)\Delta_h(X_i^+) = \Delta'_h(f(X_i^+))$, and the case for X_i^- is proved similarly. Finally, we have

$$(f \otimes f)\Delta_h(H_i) = (f \otimes f)(H_i \otimes 1 + 1 \otimes H_i) = H_i \otimes 1 + 1 \otimes H_i = \Delta'_h(f(H_i))$$

hence $(f \otimes f)\Delta_h = \Delta'_h f$.

Finally, it is trivial to see that f respects the counit. ♣

11.3.1 The ribbon element

It is interesting to note that the algebra homomorphisms $(S_h)^2$ and $(S'_h)^2$ are equal (it is sufficient to verify this for the generators $H_i, X_i^{\pm}$). One can show that the square of the antipode is always equal to the conjugation by the element $u = \sum_i S(t_i)s_i$ (where the universal R-matrix $R = \sum_i s_i \otimes t_i$) which appears in the definition of a ribbon Hopf algebra (see e.g. Proposition VIII.4.1 in [12]). But we can find another element μ of $U_h(g)$ such that $S_h^2(a) = \mu a \mu^{-1}$ for any $a \in U_h(g)$ by following the approach indicated in section XVII.2 of [12]. We try to find a μ of the form

$$\mu = e^{h(\mu_1 H_1 + \cdots + \mu_n H_n)}$$

for integers $\mu_1, \ldots, \mu_n$. We then have $(S'_h)^2(H_i) = H_i = \mu H_i \mu^{-1}$.

We have by lemma 11.3.1

$$e^{h(\mu_1 H_1 + \cdots + \mu_n H_n)} X_j^+ e^{-h(\mu_1 H_1 + \cdots + \mu_n H_n)} = e^{h(\mu_1 a_{1j} + \cdots + \mu_n a_{nj})} X_j^+.$$

We have $(S_h')^2(X_j^+) = e^{2d_j h}X_j^+$ by definition. Hence, the only condition on μ is that

$$\mu_1 a_{1j} + \cdots + \mu_n a_{nj} = 2d_j \tag{11.3.2}$$

for $j = 1, \ldots, n$. If equation (11.3.2) is satisfied then

$$\begin{aligned}
\mu X_j^- \mu^{-1} &= e^{h(\mu_1 H_1 + \cdots + \mu_n H_n)} X_j^- e^{-h(\mu_1 H_1 + \cdots + \mu_n H_n)} \\
&= e^{-h(\mu_1 a_{1j} + \cdots + \mu_n a_{nj})} X_j^- \\
&= e^{-2d_j h} X_j^- \\
&= (S_h')^2(X_j^-),
\end{aligned}$$

and hence $S_h^2(a) = \mu a \mu^{-1}$ for any a in $U_h(g)$.

We solve equation (11.3.2) now. This equation is equivalent to

$$A^t \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = \begin{pmatrix} 2d_1 \\ \vdots \\ 2d_n \end{pmatrix}.$$

Hence,

$$\begin{aligned}
\mu_i &= 2\sum_{j=1}^{n} ((A^t)^{-1})_{ij} d_j \\
&= 2\sum_{j=1}^{n} (A^{-1})_{ji} d_j.
\end{aligned}$$

We are thus led to compute the inverse of the Cartan matrix for $sl(N)$. We denote $n = N - 1$. The $(n \times n)$-Cartan matrix A for $sl(N)$ is given by

$$A = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & & & \vdots \\
0 & -1 & 2 & \ddots & \ddots & & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\
\vdots & & \ddots & \ddots & 2 & -1 & 0 \\
\vdots & & & 0 & -1 & 2 & -1 \\
0 & \cdots & \cdots & 0 & 0 & -1 & 2
\end{pmatrix}.$$

This matrix is symmetric and positive definite which implies that the diagonal entries $d_1, \ldots, d_n$ of D are all equal to 1. The determinant of A is equal to $n + 1$

which is proved by induction on the size of the matrix, n. (Develop A by the first column, and develop one of the appearing summands by the first row).

We define the $(n \times n)$-matrix $B = (B_{ij})_{i,j=1,\ldots,n}$,

$$B_{ij} = \min(i,j)(n+1-\max(i,j)).$$

Lemma 11.3.4 *$\frac{1}{n+1}B$ is the inverse matrix of A.*

Proof We have

$$\begin{aligned}
(AB)_{ij} &= \sum_{k=1}^{n} A_{ik}B_{kj} \\
&= \sum_{k=i-1,i,i+1} A_{ik}B_{kj} \\
&= -B_{i-1\ j} + 2B_{ij} - B_{i+1\ j}.
\end{aligned}$$

For $i+1 \leq j$ we have that $B_{i-1\ j} = (i-1)(n+1-j)$, $B_{ij} = i(n+1-j)$, and $B_{i+1\ j} = (i+1)(n+1-j)$. Hence, $(AB)_{ij} = 0$. This means that all entries of AB above the main diagonal are equal to zero. Since AB is symmetric, all off-diagonal entries are equal to zero.

For $i = j$ we have that $B_{i-1\ j} = (i-1)(n+1-i)$, $B_{ij} = i(n+1-i)$, and $B_{i+1\ j} = i(n+1-(i+1))$. Hence, the diagonal entries of AB are

$$\begin{aligned}
(AB)_{ii} &= -(i-1)(n+1-i) + 2i(n+1-i) - i(n+1-(i+1)) \\
&= n+1.
\end{aligned}$$

Hence, AB is equal to $(n+1)$-times the identity matrix. ♣

We have

$$\begin{aligned}
\sum_{j=1}^{n} B_{ji} &= \sum_{j=1}^{n} \min(i,j)(n+1-\max(i,j)) \\
&= \sum_{j=1}^{i} j(n+1-i) + \sum_{j=i+1}^{n} i(n+1-j) \\
&= (n+1-i)\sum_{j=1}^{i} j + i\sum_{k=1}^{n-i} k \\
&= (n+1-i)\frac{i(i+1)}{2} + i\frac{(n-i)(n-i+1)}{2} \\
&= \frac{1}{2}i(n+1-i)(n+1).
\end{aligned}$$

Hence, we can compute the value of μ_i from equation 11.3.3 in the case of $U_h(sl(N))$.

$$
\begin{aligned}
\mu_i &= 2\sum_{j=1}^{n}(A^{-1})_{ji} \\
&= \frac{2}{n+1}\sum_{j=1}^{n} B_{ji} \\
&= \frac{2}{2(n+1)} i(n+1-i)(n+1) \\
&= i(n+1-i).
\end{aligned}
$$

We have thus proved

Lemma 11.3.5 *The square of the antipode S_h in $U_h(sl(N))$ from equation 11.3.1 is the conjugation by $\mu = e^{\rho h}$ where*

$$\rho = \sum_{i=1}^{n} \mu_i H_i = \sum_{i=1}^{n} i(N-i)H_i.$$

Kassel proves in chapter XVII.3 that $\mu = e^{\rho h}$ is a ribbon element.

There is another way to get the ribbon element μ following Chari and Pressley (chapter 8.3.F of [4]). We express the sum of the positive roots as a linear combination of simple positive roots, $\sum_{k=1}^{N-1} t_k \alpha_k$. Then we get a ribbon element $e^{h\sum_{k=1}^{N-1} t_k H_k}$.

For $sl(N)$ we have the positive roots $\varepsilon_i - \varepsilon_j$ for all $1 \leq i < j \leq N$. The simple positive roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, N-1$. We have

$$\sum_{1 \leq i < j \leq N} \varepsilon_i - \varepsilon_j = \sum_{1 \leq i < j \leq N} \alpha_i + \cdots + \alpha_{j-1}.$$

The term α_k appears as a summand in the sum on the right side of the above equation for some i, j if and only if $i \leq k \leq j-1$. There are k possibilities for i, namely $1 \leq i \leq k$, and $N-k$ possibilities for j, namely $k+1 \leq j \leq N$. Hence, α_k appears $k(N-k)$ times. We thus get

$$\sum_{1 \leq i < j \leq N} \varepsilon_i - \varepsilon_j = \sum_{k=1}^{N-1} k(N-k)\alpha_k$$

which gives the same ribbon element as by Kassel's approach.

11.3.2 The fundamental module of $U_h(sl(N))$

The fundamental module $V_\square$ of $U_h(sl(N))$ has a basis $v_1, \ldots, v_N$ on which the elements H_i and $X_j^\pm$ act naturally as matrices. The matrix E_{ij} denotes the $(N \times N)$-matrix whose entries are zero except the entry 1 at the place (i, j). The matrix corresponding to X_i^+ is $E_{i\,i+1}$, to X_i^- corresponds $E_{i+1\,i}$ and to H_i corresponds $E_{ii} - E_{i+1\,i+1}$ for $i = 1, \ldots, N-1$.

Lemma 11.3.6 *The action of $\mu = e^{\rho h}$ on the fundamental module $V_\square$ is given by*

$$e^{\rho h}(v_j) = e^{h(N+1-2j)} v_j$$

for all $j = 1, \ldots, n$.

Proof The action of H_i on the fundamental module is given by $H_i(v_i) = v_i$, $H_i(v_{i+1}) = -v_{i+1}$, and $H_i(v_j) = 0$ for $j \neq i$ and $j \neq i+1$.

We have

$$\begin{aligned} e^{h\rho} &= \prod_{i=1}^{N-1} e^{h\mu_i H_i} \\ &= \prod_{i=1}^{N-1} \left(\sum_{k \geq 0} \frac{\mu_i^k}{k!} h^k H_i^k \right). \end{aligned}$$

For the action of $e^{h\rho}$ on a basis element v_j we only have to look at powers of H_j and H_{j-1}. We get

$$\begin{aligned} e^{h\rho}(v_j) &= \left(\sum_{k \geq 0} \frac{\mu_j^k}{k!} h^k H_j^k \right) \left(\sum_{r \geq 0} \frac{\mu_{j-1}^r}{r!} h^r H_{j-1}^r \right) (v_j) \\ &= \left(\sum_{k \geq 0} \frac{\mu_j^k}{k!} h^k \right) \left(\sum_{r \geq 0} \frac{\mu_{j-1}^r}{r!} h^r (-1)^r \right) (v_j) \\ &= e^{\mu_j h} e^{-\mu_{j-1} h} v_j \\ &= e^{h(\mu_j - \mu_{j-1})} v_j \end{aligned}$$

where we have to interpret μ_0 and μ_N as being equal to zero which just extends our result that $\mu_i = i(N-i)$ for $i = 1, \ldots, N-1$. The above equation implies that any v_j is an eigenvector of $e^{h\rho}$ with eigenvalue

$$\begin{aligned} \gamma_j &= e^{h(\mu_j - \mu_{j-1})} \\ &= e^{h(j(N-j)-(j-1)(N-(j-1)))} \\ &= e^{h(N+1-2j)}. \end{aligned}$$

♣

The construction of a universal R-matrix is described in chapter 8.3.G of [4]. Chari and Pressley describe the action of this universal R-matrix on $V_\square \otimes V_\square$ as

$$R = x\left[s\sum_{1\le a\le N} E_{aa}\otimes E_{aa} + \sum_{1\le a\ne b\le N} E_{aa}\otimes E_{bb} + (s-s^{-1})\sum_{1\le a<b\le N} E_{ab}\otimes E_{ba}\right].$$

In this formula, $s = e^h$, $x = e^{-\frac{h}{N}}$, and $E_{ab}(v_i) = \delta_{bi}v_a$, i.e. E_{ab} corresponds to the $(N\times N)$-matrix which is everywhere zero except the single entry 1 in the a-th row and b-th column. We remark that we changed the notation $q = e^h$ given there to $s = e^h$. The above formula was given by Drinfeld in [5].

Remark Let V be an $\mathcal{A}$-module for a ribbon Hopf algebra $\mathcal{A}$. The multiplication with a universal R-matrix followed by switching the factors is an automorphism of $V\otimes V$ which satisfies the Yang-Baxter equation. Any scalar multiple of a solution of the Yang-Baxter equation is again a solution, but a non-trivial scalar multiple of a universal R-matrix is no longer a universal R-matrix because R has to satisfy $(\Delta\otimes \mathrm{id}_{\mathcal{A}})(R) = R_{13}(1\otimes R)$. This explains why Turaev could neglect the factor $e^{-\frac{h}{N}}$ in section 4.2 of [24] because he only needed a solution of the Yang-Baxter equation.

We define the k-linear endomorphism $\check{R}$ of $V_\square\otimes V_\square$ as the composition of R and the flip P of the components. This coincides with the map $\rho_{V,W}$ from section 11.2 for $V = W = V_\square$. We remark that $\check{R}$ is in fact $U_h(sl(N))$-linear. We have

$$\begin{aligned} P\circ(E_{ab}\otimes E_{kl})(v_i\otimes v_j) &= P(\delta_{bi}v_a\otimes\delta_{lj}v_k)\\ &= \delta_{lj}v_k\otimes\delta_{bi}v_a\\ &= \delta_{bi}v_k\otimes\delta_{lj}v_a\\ &= (E_{kb}\otimes E_{al})(v_i\otimes v_j). \end{aligned}$$

We thus have

$$P\circ(E_{ab}\otimes E_{kl}) = E_{kb}\otimes E_{al}$$

for any $1\le a,b,k,l\le N$. We thus get from the above equation for R that

$$\check{R} \;=\; x\left[s\sum_{a=1}^{N} E_{aa}\otimes E_{aa} + \sum_{1\le a\ne b\le N} E_{ba}\otimes E_{ab} + (s-s^{-1})\sum_{1\le a<b\le N} E_{bb}\otimes E_{aa}\right]. \tag{11.3.3}$$

The action of $x^{-1}\check{R}$ on the basis elements is therefore given by

$$x^{-1}\check{R}(v_i\otimes v_j) = \begin{cases} s(v_i\otimes v_i) & i=j\\ v_j\otimes v_i & i<j\\ v_j\otimes v_i + (s-s^{-1})v_i\otimes v_j & i>j. \end{cases} \tag{11.3.4}$$

Applying $x^{-1}\breve{R}$ twice gets

$$
\begin{aligned}
x^{-2}\breve{R}^2(v_i\otimes v_j) &= \begin{cases} s^2(v_i\otimes v_i) & i=j \\ v_i\otimes v_j+(s-s^{-1})v_j\otimes v_i & i<j \\ v_i\otimes v_j+(s-s^{-1})(v_j\otimes v_i+(s-s^{-1})v_i\otimes v_j) & i>j \end{cases} \\
&= \begin{cases} s^2(v_i\otimes v_i) & i=j \\ v_i\otimes v_j+(s-s^{-1})v_j\otimes v_i & i<j \\ (1+(s-s^{-1})^2)v_i\otimes v_j+(s-s^{-1})v_j\otimes v_i & i>j. \end{cases}
\end{aligned}
$$

We immediately verify by the above equations that

$$x^{-2}\breve{R}^2(v_i\otimes v_j)=(s-s^{-1})x^{-1}\breve{R}(v_i\otimes v_j)+v_i\otimes v_j$$

in every case $i=j$, $i<j$, or $i>j$. Hence,

$$x^{-2}\breve{R}^2=(s-s^{-1})x^{-1}\breve{R}+\mathrm{id}.$$

Equivalently,

$$x^{-1}\breve{R}-x\breve{R}^{-1}=(s-s^{-1})\mathrm{id}. \tag{11.3.5}$$

The identity map of $V_\square\otimes V_\square$ can be written as $\mathrm{id}=\sum_{1\le a,b\le N}E_{aa}\otimes E_{bb}$. This leads to an explicit formula for $\breve{R}^{-1}$,

$$\breve{R}^{-1}=x^{-1}\left[s^{-1}\sum_{a=1}^{N}E_{aa}\otimes E_{aa}+\sum_{1\le a\ne b\le N}E_{ba}\otimes E_{ab}+(s^{-1}-s)\sum_{1\le b<a\le N}E_{bb}\otimes E_{aa}\right]$$

which is well known.

We now compute the curl-factor for the fundamental module $V_\square$.

Lemma 11.3.7 *The $U_h(sl(N))$-linear endomorphism of the fundamental module $V_\square$ given by the curl in figure 11.3 is the multiplication with the scalar $e^{\left(N-\frac{1}{N}\right)h}$.*

Proof The endomorphism ϕ of $V_\square$ determined by the $(1,1)$-tangle in figure 11.3 is the composition of three maps,

$$\phi=(\mathrm{id}_{V_\square}\otimes F_2)\circ(\breve{R}\otimes\mathrm{id}_{V_\square^*})\circ(\mathrm{id}_{V_\square}\otimes F_3).$$

The maps F_2 and F_3 are given in section 11.1, and the map $\breve{R}$ is given in equation 11.3.4. We consider an element v_i of the canonical basis of $V_\square$ for some $1\le i\le N$. The effect of the cup-map $\mathrm{id}_{V_\square}\otimes F_3$ on $v_i\otimes 1$ is

$$v_i\otimes 1\mapsto v_i\otimes\sum_{k=1}^{N}v_k\otimes v^k=\sum_{k=1}^{N}v_i\otimes v_k\otimes v^k.$$

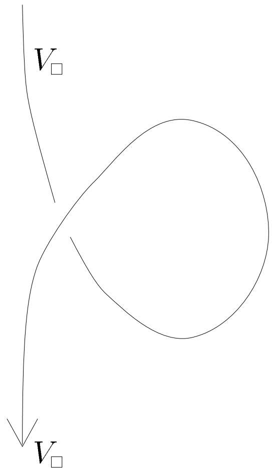


Figure 11.3: A positive curl.

$\check{R} \otimes \mathrm{id}_{V_\square^*}$ maps this element to

$$s(v_i \otimes v_i \otimes v^i) + \sum_{k=i+1}^{N} v_k \otimes v_i \otimes v^k + \sum_{k=1}^{i-1} \Big(v_k \otimes v_i \otimes v^k + (s - s^{-1}) v_i \otimes v_k \otimes v^k \Big)$$

apart from the scalar x. The cap-map $\mathrm{id}_{V_\square} \otimes F_2$ applied to this element then gives

$$s(v_i \otimes v^i(\mu v_i)) + \sum_{k=i+1}^{N} v_k \otimes v^k(\mu v_i) + \sum_{k=1}^{i-1} \Big(v_k \otimes v^k(\mu v_i) + (s - s^{-1}) v_i \otimes v^k(\mu v_k) \Big)$$

apart from the scalar x. We have by lemma 11.3.6 that v_i is an eigenvector of the multiplication by μ with eigenvalue $\gamma_i = s^{N+1-2i}$, hence $v^k(\mu v_i) = 0$ for $k \neq i$. The above expression for $\phi(v_i)$ is therefore equal to

$$\begin{aligned}
\phi(v_i) &= x \left[s(v_i \otimes v^i(\mu v_i)) + \sum_{k=1}^{i-1} (s - s^{-1}) v_i \otimes v_k(\mu v_k) \right] \\
&= x \left[s\gamma_i + \sum_{k=1}^{i-1} (s - s^{-1}) \gamma_k \right] v_i \\
&= x \left[s^{N+2-2i} + (s - s^{-1}) \sum_{k=1}^{i-1} s^{N+1-2k} \right] \\
&= x s^N v_i
\end{aligned}$$

for any $1 \leq i \leq N$. Hence, ϕ is the multiplication by the scalar $xs^N = e^{\left(N - \frac{1}{N}\right)h}$.
♣

Lemma 11.3.8 *The $U_h(sl(N))$-invariant of the zero-framed unknot coloured by the fundamental module is equal to $[N]_{e^h}$.*

Proof One can position the unknot with framing zero so that its diagram is a simple circle with anti-clockwise orientation. This diagram determines the composition of the cup- and cap-maps F_3 and F_2 which map

$$1 \mapsto \sum_{i=1}^{N} v_i \otimes v^i \mapsto \sum_{i=1}^{N} v^i(\mu v_i) = \sum_{i=1}^{N} \gamma_i.$$

The invariant of the unknot with framing zero coloured by the fundamental module is therefore equal to

$$\begin{aligned}\sum_{i=1}^{N} \gamma_i &= \sum_{i=1}^{N} s^{N+1-2i} \\ &= \frac{s^N - s^{-N}}{s - s^{-1}} \\ &= [N]_s\end{aligned}$$

where $s = e^h$. ♣

11.4 $U_h(sl(N))$ and the Homfly polynomial

We recall that H_k is the Hecke algebra of (k,k)-ribbon tangles with top-down orientations at its boundary points. The set of scalars is the ring $\mathbb{Z}[s, v, x, \delta]$ modulo the relation $\delta(s - s^{-1}) = v^{-1} - v$.

Definition The *variant Hecke-algebra* $\tilde{H}_k^N$ is defined in the same way as H_k with the only difference that the ring of scalars is $\mathbb{C}[[h]]$ and that in the defining relations we replace s by e^h, x by $e^{-\frac{h}{N}}$, and v by e^{-hN}.

This definition immediately provides a ring homomorphism $\nu : H_k \to \tilde{H}_k^N$ which is the substitution of s by e^h, x by $e^{-\frac{h}{N}}$, and v by e^{-hN}.

Lemma 11.4.1 *Let T be any (k,k)-ribbon tangle with top-down orientations at its boundary points. We colour all its components by the fundamental module $V_\square$. Then the map ϕ_k given by $T \mapsto J(T)$ induces an algebra homomorphism*

$$\phi_k : \tilde{H}_k^N \to \mathrm{End}_{U_h(sl(N))}(V_\square^{\otimes k}).$$

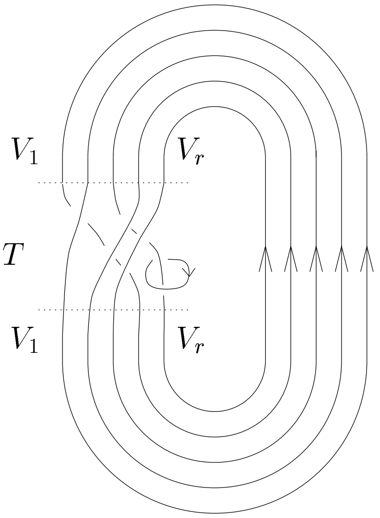


Figure 11.4: The $\mathcal{A}$-invariant of $\hat{T}$ can be computed as the trace of $\mu J(T)$.

Proof Let T be a diagram of a (k,k)-ribbon tangle. Since the assigned module endomorphism $J(T)$ is an invariant of ribbon tangles, it is in particular invariant under regular isotopy of T.

The skein relation in figure 2.1 is satisfied because $\check{R}$ satisfies the quadratic relation in equation (11.3.5). Furthermore, the skein relation for the curl in figure 2.2 is mapped to zero by ϕ_k because of the result for the positive curl in lemma 11.3.7. Finally, we have to check that T together with a split unknot with framing zero induces the endomorphism $[N]_{e^h}\phi_k(T)$. This is true by lemma 11.3.8. ♣

By looking at the case $k = 0$ we immediately deduce from lemma 11.4.1

Corollary 11.4.2 *Let L be a framed link. We colour all of its components by the fundamental module $V_\square$. The $U_h(sl(N))$-invariant of L is equal to the Homfly polynomial of L after the substitutions of s by e^h, x by $e^{-\frac{h}{N}}$, and v by e^{-hN}.*

Lemma 11.4.3 *Let $\mathcal{A}$ be a ribbon Hopf algebra over a commutative ring k. Let T be an (r,r)-ribbon tangle with top-down orientations at its boundary points. We consider a colouring of the closure of T and denote the modules assigned to the components of $\hat{T}$ by $V_1, \ldots, V_r$ as we read them at the boundary points of T from left to right (see figure 11.4). T induces a module endomorphism $J(T)$ of $V_1 \otimes \cdots \otimes V_r$. Then the $\mathcal{A}$-invariant of the closure of T with this colouring is equal to the trace of the linear endomorphism $\mu J(T)$ of $V_1 \otimes \cdots \otimes V_r$.*

Proof We choose a basis $\{v_{i_m}\}$ for every module V_m, $1 \leq m \leq r$, where i_m is running through some finite index set depending on m. The r cup-maps at the bottom of figure 11.4 map the trivial module k to the module $V_1 \otimes \cdots \otimes V_r \otimes (V_r)^* \otimes \cdots \otimes (V_1)^*$, and they map

$$1 \mapsto \sum_{i_1,\ldots,i_r} v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v^{i_r} \otimes \cdots \otimes v^{i_1}.$$

The map $J(T)$ on the first r factors is a k-linear map in particular. Hence,

$$J(T)(v_{i_1} \otimes \cdots \otimes v_{i_r}) = \sum_{j_1,\ldots,j_r} g_{i_1\cdots i_r}^{j_1\cdots j_r} v_{j_1} \otimes \cdots \otimes v_{j_r}$$

for scalars $g_{i_1\cdots i_r}^{j_1\cdots j_r} \in k$. Hence, the composition of the cup-maps and $J(T)$ maps

$$1 \mapsto \sum_{i_1,\ldots,i_r} \sum_{j_1,\ldots,j_r} g_{i_1\cdots i_r}^{j_1\cdots j_r} v_{j_1} \otimes \cdots \otimes v_{j_r} \otimes v^{i_r} \otimes \cdots \otimes v^{i_1}.$$

Finally, the r cap-maps map this to the scalar

$$\sum_{i_1,\ldots,i_r} \sum_{j_1,\ldots,j_r} g_{i_1\cdots i_r}^{j_1\cdots j_r} v^{i_1}(\mu v_{j_1}) \cdots v^{i_r}(\mu v_{j_r})$$

which is by definition the $\mathcal{A}$-invariant of the framed link $\hat{T}$ for the specific colouring with $V_1, \ldots, V_r$.

On the other hand, since $\Delta_h(\mu) = \mu \otimes \mu$, the map $\mu J(T)$ can be written as the composition $\mu^{\otimes k} J(T)$ and thus

$$\mu J(T)(v_{i_1} \otimes \cdots \otimes v_{i_r}) = \sum_{j_1,\ldots,j_r} g_{i_1\cdots i_r}^{j_1\cdots j_r} (\mu v_{j_1}) \otimes \cdots \otimes (\mu v_{j_r}).$$

Hence the normal trace of this linear map is equal to

$$\mathrm{tr}(\mu J(T)) = \sum_{i_1,\ldots,i_r} \sum_{j_1,\ldots,j_r} g_{i_1\cdots i_r}^{j_1\cdots j_r} v^{i_1}(\mu v_{j_1}) \cdots v^{i_r}(\mu v_{j_r})$$

which agrees with the above $\mathcal{A}$-invariant of $\hat{T}$. ♣

Lemma 11.4.3 motivates a definition. Given an $\mathcal{A}$-module V and an $\mathcal{A}$-module endomorphism $V \to V$, we define the *quantum trace* $\mathrm{tr}_q(f)$ as the trace of the k-linear endomorphism $\mu f : V \to V$,

$$\mathrm{tr}_q(f) = \mathrm{tr}(\mu f),$$

where μ is the ribbon element.

Lemma 11.4.4 *Let V be a finite-dimensional $U_h(sl(N))$-module. Let f and g be $\mathbb{C}[[h]]$-linear endomorphisms of V and α be a scalar in $\mathbb{C}[[h]]$. Then*

$$\mathrm{tr}_q(f+g) = \mathrm{tr}_q(f) + \mathrm{tr}_q(g), \quad \mathrm{tr}_q(\alpha f) = \alpha \mathrm{tr}_q(f)$$

Proof The proof is the same as for the normal trace. ♣

We recall that ν is the specialization $H_k \to \tilde{H}_k^N$, and phi_k is the natural map $\tilde{H}_k^N \to \mathrm{End}_{U_h(sl(N))}(V_\square^{\otimes k})$ as described in lemma 11.4.1.

Let λ be a Young diagram and denote the number of its cells by k. The element $\phi_k(\nu(e_\lambda))$ is a quasi-idempotent of $\mathrm{End}_{U_h(sl(N))}(V_\square^{\otimes k})$. This is because $e_\lambda e_\lambda = \alpha_\lambda e_\lambda$ in H_k for some scalar α_λ. Furthermore, the specialization $\nu : H_k \to \tilde{H}_k^N$ is a ring homomorphism and ϕ_k is an algebra homomorphism. Hence, we have $\phi_k(\nu(e_\lambda))\phi_k(\nu(e_\lambda)) = \nu(\alpha_\lambda)\phi_k(\nu(e_\lambda))$.

The interesting question is whether $\nu(\alpha_\lambda)$ is invertible in $\mathbb{C}[[h]]$, i.e. whether the constant term of $\nu(\alpha_\lambda)$ is non-zero.

Lemma 11.4.5 *$\nu(\alpha_\lambda)$ is invertible in $\mathbb{C}[[h]]$.*

Proof The constant term of $\nu(\alpha_\lambda)$ is equal to the limit $h \to 1$ (i.e. $x \to 1$) of the rational function which derives from α_λ by substituting $\delta = (v^{-1} - v)/(s - s^{-1})$ and then $s = x^{-N}$ and $v = x^{N^2}$.

The limit for $x \to 1$ of the Homfly polynomial of $\hat{e}_\lambda$ after the substitutions $\delta = (v^{-1} - v)/(s - s^{-1})$ and then $s = x^{-N}$ and $v = x^{N^2}$ is well defined. This is because the only possible problem is the denominator of δ. But a careful look reveals that this problem does not occur because $\lim_{x\to 1} \delta$ is well defined since

$$\lim_{x\to 1} \delta = \lim_{x\to 1} \frac{v^{-1} - v}{s - s^{-1}} = \lim_{x\to 1} \frac{x^{-N^2} - x^{N^2}}{x^{-N} - x^{N}} = \lim_{x\to 1} \frac{-N^2 x^{-N^2-1} - N^2 x^{N^2-1}}{-N x^{-N-1} - N x^{N-1}} = N$$

by l'Hôpital's rule. On the other hand, the limit for $x \to 1$ of the Homfly polynomial of $\hat{y}_\lambda$ after the substitutions $\delta = (v^{-1} - v)/(s - s^{-1})$ and then $s = x^{-N}$ and $v = x^{N^2}$ is well defined by lemma 3.6.1 (we have $\hat{y}_\lambda = Q_\lambda$ by definition). Since $y_\lambda = (1/\alpha_\lambda)e_\lambda$, we have that the limit for $x \to 1$ of α_λ after the substitutions $\delta = (v^{-1} - v)/(s - s^{-1})$ and then $s = x^{-N}$ and $v = x^{N^2}$ cannot be zero. ♣

It will not lead to confusion if we denote $\nu(y_\lambda) \in \tilde{H}_k^N$ by y_λ, too. We have that $\phi_{|\lambda|}(y_\lambda)$ is an idempotent of the $U_h(sl(N))$-endomorphism ring of $V_\square^{\otimes|\lambda|}$.

Lemma 11.4.6 *The endomorphism $\phi_{|\lambda|}(y_\lambda)$ of $V_\square^{\otimes|\lambda|}$ is a projection to a submodule for any Young diagram λ.*

Proof The essential observation is that $\phi_{|\lambda|}(y_\lambda)$ is an idempotent. Let g be an endomorphism of a module W over any commutative ring such that g satisfies $g^2 = g$. We can write any element w of W as $w = (w - g(w)) + g(w)$. Since $g^2(w) = g(w)$ we have that $w - g(w)$ lies in the kernel $ker(g)$ of g. Clearly, $g(w)$ lies in the image $im(g)$ of g. Hence, any element $w \in W$ lies in $ker(g) \oplus im(g)$. Since the only element of W that lies in the kernel and in the image of g is the element 0, we have that $W = ker(g) \oplus im(g)$. Hence, g is a projection of W to the submodule $im(g)$. ♣

We define W_λ to be the image of $\phi_{|\lambda|}(y_\lambda)$ in $V_\square^{\otimes|\lambda|}$.

Lemma 11.4.7 *Let λ be any Young diagram, and let C be any framed knot. The $U_h(sl(N))$-invariant of C coloured by the module W_λ is equal to the Homfly polynomial of C decorated by Q_λ after the substitutions of $s = e^h$, $x = e^{-\frac{h}{N}}$, and $v = e^{-Nh}$.*

Proof Let the framed knot C be represented as an oriented knot with blackboard framing. C can be positioned by regular isotopy as the closure of a braid β' such that all of its strings are oriented downwards. We denote the number of strings by d'. We now have to ensure that the blackboard framing of β' agrees with the framing of C. To do this, we multiply β' by $\sigma_{d'}\sigma_{d'+1}\cdots\sigma_{d'+j}$ or by $\sigma_{d'}^{-1}\sigma_{d'+1}^{-1}\cdots\sigma_{d'+j}^{-1}$. For a unique j, the blackboard framing of the closure of this $(d'+j+1)$-braid is a diagram of the framed knot C. We denote this braid by β, and denote the number of strings by d.

We denote by k the number of cells of λ. We denote by $\beta^{(k)}$ the k-fold blackboard parallel of β. The decoration of C by Q_λ is then the closure of the element $y_\lambda^{\otimes d}\beta^{(k)}$ of H_{kd}, where k is the number of cells of λ. This is because $y_\lambda = (y_\lambda)^d$ in H_k, and each factor y_λ can be slid along the closure of β to the top of the braid β. This is depicted in figure 11.5. To be precise in the following arguments, the y_λ's have to be at slightly different levels.

By lemmas 11.4.1 and 11.4.3 and the linearity of the quantum-trace we have that the Homfly-polynomial of C decorated by Q_λ after the substitutions for s, x and v is equal to the quantum trace of the endomorphism $\phi_{kd}(y_\lambda^{\otimes d}\beta^{(k)})$ of $(V_\square^{\otimes k})^{\otimes d}$.

On the other hand, the $U_h(sl(N))$-invariant of C coloured by W_λ is the quantum-trace of the endomorphism $J(\beta)$ of $W_\lambda^{\otimes d}$ by lemma 11.4.3. We thus have to prove that

$$\mathrm{tr}_q(J(\beta)) = \mathrm{tr}_q(\phi_{kd}(y_\lambda^{\otimes d}\beta^{(k)})),$$

or, equivalently,

$$\mathrm{tr}(\mu \cdot J(\beta)) = \mathrm{tr}(\mu \cdot \phi_{kd}(y_\lambda^{\otimes d})J(\beta^{(k)})) \tag{11.4.6}$$

where the trace on the left hand side refers to $W_\lambda^{\otimes d}$, and the trace on the right hand side refers to $(V_\square^{\otimes k})^{\otimes d}$.

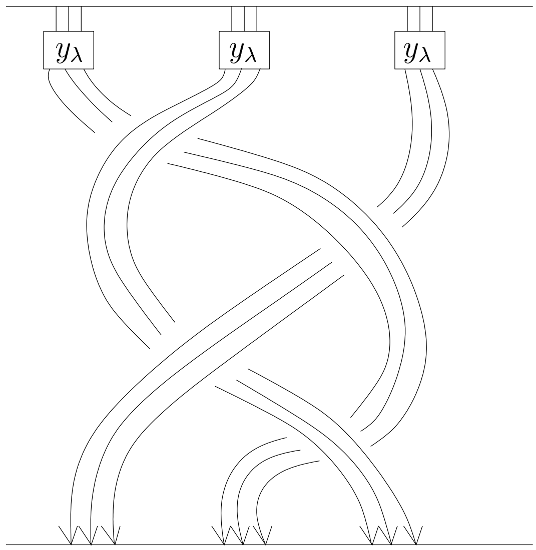


Figure 11.5: The element $y_\lambda^{\otimes d}\beta^{(k)}$ of H_{kd} in the case $\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ corresponding to the figure-eight knot with zero-framing, with $|\lambda| = k = 3$ and $d = 3$.

$$\begin{tikzcd}
W_\lambda^{\otimes d} \arrow[r, hook, "\iota^{\otimes d}"] & (V_\square^{\otimes k})^{\otimes d} \\
W_\lambda^{\otimes d} \arrow[r, hook, "\iota^{\otimes d}"] \arrow[u, "\mu\cdot"] & (V_\square^{\otimes k})^{\otimes d} \arrow[u, "\mu\cdot"'] \\
W_\lambda^{\otimes d} \arrow[r, hook, "\iota^{\otimes d}"] \arrow[u, "\mathrm{id}"] & (V_\square^{\otimes k})^{\otimes d} \arrow[u, "(\phi_k(y_\lambda))^{\otimes d}"'] \\
W_\lambda^{\otimes d} \arrow[r, hook, "\iota^{\otimes d}"] \arrow[u, "J(\beta)"] & (V_\square^{\otimes k})^{\otimes d} \arrow[u, "J(\beta^{(k)})"']
\end{tikzcd}$$

Figure 11.6: A commutative diagram.

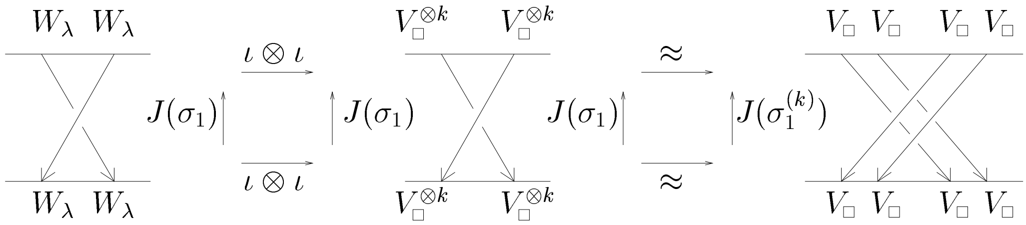


Figure 11.7: Homomorphisms arising at a crossing σ_1 in the braid β and the corresponding multiple crossings $\sigma_1^{(k)}$ in the braid $\beta^{(k)}$ shown in the case $k = 2$.

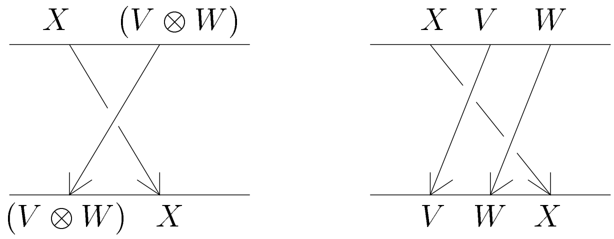


Figure 11.8: The braids σ_1 and $\sigma_1\sigma_2$ give the same map $V \otimes W \otimes X \to X \otimes V \otimes W$.

We claim that we have commutative diagram as shown in figure 11.6 where ι is the inclusion of W_λ to $V_\square^{\otimes k}$. It is then clear that equation (11.4.6) is true.

We recall that the maps $J(\beta)$, $J(\beta^{(k)})$, $\phi_k(y_\lambda)$ and ι are module homomorphisms, whereas the multiplication by μ is only a $\mathbb{C}[[h]]$-linear map. The top square in figure 11.6 commutes because ι is the inclusion. The middle square commutes because $\phi_k(y_\lambda)$ the restriction of $\phi_k(y_\lambda)$ to W_λ is the identity of W_λ.

It remains to prove the commutativity of the bottom square.

We consider a crossing of the braid β. Figure 11.7 depicts two commuting diagrams that relate three braids and the module homomorphism which they induce. The map $J(\sigma_1)$ (or $J(\sigma_j)$ for some $1 \leq j \leq d-1$, depending on the position of the crossing in β) is the multiplication by R followed by the flip of the factors of $W_\lambda \otimes W_\lambda$ resp. $V_\square^{\otimes k} \otimes V_\square^{\otimes k}$.

The maps in the left diagram commute because ι is the inclusion.

The maps in the right diagram commute because of the general behaviour depicted in figure 11.8. There, both braids induce the same map from $V \otimes W \otimes X$ to $V \otimes W \otimes X$ up to the obvious isomorphism between $(V \otimes W) \otimes X$ and $V \otimes W \otimes X$. A short proof of this observation is given e.g. in the proof of Lemma 3.10 in [15]. Repeated application of this result shows that the maps in the right diagram of figure 11.7 commute. A corresponding results holds for a negative crossing of β.

Hence, we have commuting diagrams as we move from the bottom to the top of β, and they form the commuting diagram at the bottom of figure 11.6. ♣

Lemma 11.4.8 *Let $L = L_1 \cup \cdots \cup L_r$ be a framed link whose components are coloured with modules $W_{\lambda^1}, \ldots, W_{\lambda^r}$. Then the $U_h(sl(N))$-invariant of this link is equal to the Homfly polynomial of the framed link L with decorations $Q_{\lambda^1}, \ldots, Q_{\lambda^r}$ on its components $L_1, \ldots, L_r$ after the substitutions $x = e^{-\frac{h}{N}}$, $v = e^{-Nh}$, and $s = e^h$.*

Proof We are able to represent L as the closure of a braid β with top-down orientations. To get the framing right, we introduce an additional straight string between points i and $i+1$ that lies above any strings of β. We add at the bottom a (positive or negative) crossing between this string and the string $i+1$. By doing this successively at suitable places, we adjust the blackboard framing to become the framing of L. We then proceed in exactly the same way as in the proof of lemma 11.4.7. The only difference is that the notation gets awkward because the modules that we read at the top and bottom of the braid β are some permutation of $W_{\lambda^1}, \ldots, W_{\lambda^r}$ with multiplicities that depend on the choice of β. Furthermore, the number of cells of the Young diagrams $\lambda^1, \ldots, \lambda^r$ may vary, and this makes the notation worse. But apart from the notation, the proof of lemma 11.4.7 extends in a straightforward way to the case of links. ♣

11.4.1 $W_\lambda \approx V_\lambda$

Let A be an algebra over a commutative ring k such that the dimension (over k) of any A-module is well defined. An A-module V is called *simple* if it has no other submodules than $\{0\}$ and V. It is called *semi-simple* if it is isomorphic to a direct sum of simple A-modules. We note that all finite-dimensional $U_h(sl(N))$-modules are semi-simple.

We fix the rank $N \geq 2$ of the quantum group $U_h(sl(N))$. For a Young diagram λ with at most N rows we shall denote by V_λ the simple module indexed by λ. Modules V_λ and V_μ are isomorphic if and only if λ and μ differ by initial columns of length N. For a Young diagram with more then N rows we set V_λ equal to the zero-module. The map $\lambda \mapsto V_\lambda$ induces a ring isomorphism from $\mathcal{Y}_N$ to the representation ring of $U_h(sl(N))$ (see e.g. chapter XVII of [12] or chapter 7 of [13]). This is due to the similarity of the representation theory of $U_h(sl(N))$ and $sl(N)$. The latter is described in [7].

Recall that the quantum trace $\mathrm{tr}_q(f)$ of a module endomorphism $f : V \to V$ is the trace of the $\mathbb{C}[[h]]$-linear map $\mu \cdot f : V \to V$. The *quantum dimension* $\dim_q(V)$ of the module V is defined as $\mathrm{tr}_q(\mathrm{id}_V)$,

$$\dim_q(V) = \mathrm{tr}_q(\mathrm{id}_V).$$

The fact that isomorphic modules have the same quantum dimension will be of importance. The zero-module has quantum trace equal to zero. We are not yet in the position to state that it is the only module of quantum dimension zero.

Lemma 11.4.9 *Let V and W be finite-dimensional modules over a ring R, and let f and g be module endomorphisms of V resp. W. Furthermore, we require that $f^2 = f$ and $g^2 = g$. Then*

$$im(f) \otimes im(g) \approx im(f \otimes g).$$

Proof The module homomorphism $\varphi : im(f) \otimes im(g) \to im(f \otimes g) \subset V \otimes W$ given by $f(x) \otimes g(y) \mapsto (f \otimes g)(x \otimes y) = f(x) \otimes g(y)$ is well defined and surjective.

We have $V = im(f) \oplus T$ and $W = im(g) \oplus U$ where $T = ker(f)$ and $U = ker(g)$ because $f^2 = f$ and $g^2 = g$. We thus have

$$\begin{aligned} V \otimes W &= (im(f) \oplus T) \otimes (im(g) \oplus U) \\ &\approx (im(f) \otimes im(g)) \oplus (im(f) \otimes U) \oplus (T \otimes im(g)) \oplus (T \otimes U). \end{aligned}$$

Since φ is the restriction of this isomorphism to $im(f) \otimes im(g)$ we have that φ is injective, too. Hence, φ is a bijective module homomorphism. ♣

Lemma 11.4.10 *Let V and W be a finite-dimensional $U_h(sl(N))$-modules. Let f and g be $\mathbb{C}[[h]]$-linear endomorphism of V resp. W. Then*

$$\mathrm{tr}_q(f \otimes g) = \mathrm{tr}_q(f)\mathrm{tr}_q(g).$$

Proof The same proof as for the normal trace applies. The only point to be careful about is that μ operates on $V \otimes W$ as $(\mu \otimes \mu)$ because $\Delta_h(\mu) = (\mu \otimes \mu)$. ♣

Lemma 11.4.11 *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a Young diagram with r rows and denote its transposed diagram by $\lambda^\vee = (\lambda_1^\vee, \ldots, \lambda_m^\vee)$, $m = \lambda_1$. Any submodule of $V_{d_{\lambda_1}} \otimes \cdots \otimes V_{d_{\lambda_r}}$ which is isomorphic to a submodule of $V_{c_{\lambda_1^\vee}} \otimes \cdots \otimes V_{c_{\lambda_m^\vee}}$ is either the zero-module or it is isomorphic to V_λ.*

Proof We first look at the level of Young diagrams. We consider the lexicographic order on the set of Young diagrams, i.e. for Young diagrams μ and η we define $\mu > \eta$ if $\mu_i = \eta_i$ for $i = 1, 2, \ldots, k$, and $\mu_{k+1} > \eta_{k+1}$ for some k. We define $\mu \geq \eta$ if either $\mu = \eta$ or $\mu > \eta$.

By the multiplication rule for Young diagrams it is easy to confirm that any summand η of $d_{\lambda_1} d_{\lambda_2} \cdots d_{\lambda_r}$ satisfies $\eta \geq \lambda = (\lambda_1, \ldots, \lambda_r)$. Similarly, any summand μ of $c_{\lambda_1^\vee} c_{\lambda_2^\vee} \cdots c_{\lambda_m^\vee}$ satisfies $\mu^\vee \geq \lambda^\vee$. It is easy to check that the only Young diagram μ with $|\lambda|$ cells that satisfies $\mu \geq \lambda$ and $\mu^\vee \geq \lambda^\vee$ is λ. Hence, the only Young diagram that could appear in both of these products is λ. It appears indeed with multiplicity one.

Going from Young diagrams to $U_h(sl(N))$-modules we have to be sure that there are no summands η of $d_{\lambda_1} d_{\lambda_2} \cdots d_{\lambda_r}$ and μ of $c_{\lambda_1^\vee} c_{\lambda_2^\vee} \cdots c_{\lambda_m^\vee}$ that differ by initial columns of length N. This is clear because every summand has the same number of cells $|\lambda|$. Hence, if $l(\lambda) \leq N$ then V_λ is the only irreducible module that is isomorphic to a submodule of both $V_{d_{\lambda_1}} \otimes \cdots \otimes V_{d_{\lambda_r}}$ and $V_{c_{\lambda_1^\vee}} \otimes \cdots \otimes V_{c_{\lambda_m^\vee}}$. If $l(\lambda) \geq N+1$ then V_λ is the zero-module and there is no irreducible submodule that occurs as a summand in both of the tensor products. ♣

Lemma 11.4.12 *Let g be a module endomorphism of a $U_h(sl(N))$-module V such that $g^2 = g$. Let W be a submodule of V. Then $g(W)$ is isomorphic to a submodule of W.*

Proof $g^2 = g$ implies that $W = im(g|_W) \oplus ker(g|_W)$. ♣

We recall that $W_\lambda = im(\phi_{|\lambda|}(y_\lambda)) \subset V_\Box^{\otimes|\lambda|}$.

Lemma 11.4.13 *For any Young diagram λ we have*

$$\dim_q(im(\phi_{|\lambda|}(y_\lambda))) = \prod_{c \in \lambda} \frac{s^{N+\mathrm{cn}(c)} - s^{-N-\mathrm{cn}(c)}}{s^{\mathrm{hl}(c)} - s^{-\mathrm{hl}(c)}},$$

where $s = e^h$. This quantum dimension is equal to zero if and only if $l(\lambda) \geq N+1$.

Proof We denote the unknot with framing zero by $\mathcal{O}$. It is the closure of the trivial 1-braid. By lemma 11.4.3 we thus know that the $U_h(sl(N))$-invariant of $\mathcal{O}$ coloured by W_λ is equal to $\mathrm{tr}_q(\mathrm{id}_{W_\lambda}) = \dim_q(W_\lambda)$. By lemma 11.4.7 we know that the $U_h(sl(N))$-invariant of $\mathcal{O}$ coloured by W_λ is equal to the Homfly polynomial of $\mathcal{O}$ decorated with Q_λ after the substitutions $s = e^h$, $x = e^{-\frac{h}{N}}$ and $v = e^{-Nh}$. Hence,

$$\dim_q(W_\lambda) = \langle Q_\lambda \rangle$$

with substitutions $s = e^h$, $x = e^{-\frac{h}{N}}$ and $v = e^{-Nh}$. The formula for $\langle Q_\lambda \rangle$ from lemma 3.6.1 with these substitutions thus gives the claimed formula for $\dim_q(W_\lambda)$.

This term becomes zero if and only if there exists a cell in λ with content 0. This happens if and only if $l(\lambda) \geq N+1$. ♣

We denote the row diagram with two cells by $\Box\Box$, and we denote the column diagram with two cells by $\boxminus$. We recall that $y_{\Box\Box} \in \tilde{H}_2^N$ is the idempotent derived from a_2, and $y_{\boxminus} \in \tilde{H}_2^N$ is the idempotent derived from b_2.

Lemma 11.4.14 *Either*

$$im(\phi_2(y_{\Box\Box})) \approx V_{\Box\Box} \ \ and \ \ im(\phi_2(y_{\boxminus})) \approx V_{\boxminus},$$

or

$$im(\phi_2(y_{\Box\Box})) \approx V_{\boxminus} \ \ and \ \ im(\phi_2(y_{\boxminus})) \approx V_{\Box\Box}.$$

Proof By lemma 11.4.13 we deduce

$$\dim_q(im(\phi_2(y_{\substack{\square\\\square}}))) \;=\; \mathrm{tr}_q(\phi_2(y_{\substack{\square\\\square}})) = \frac{s^N - s^{-N}}{s^2 - s^{-2}} \frac{s^{N-1} - s^{-N+1}}{s - s^{-1}} \tag{11.4.7}$$

and

$$\dim_q(im(\phi_2(y_{\square\square}))) \;=\; \mathrm{tr}_q(\phi_2(y_{\square\square})) = \frac{s^N - s^{-N}}{s^2 - s^{-2}} \frac{s^{N+1} - s^{-N-1}}{s - s^{-1}}. \tag{11.4.8}$$

Since $N \geq 2$, both of these values are different from zero. Hence neither $\phi_2(y_{\substack{\square\\\square}})$ nor $\phi_2(y_{\square\square})$ is the zero map. We have in the Hecke algebra H_2 the equation $y_{\substack{\square\\\square}} y_{\square\square} = 0$, hence $\phi_2(y_{\substack{\square\\\square}})\phi_2(y_{\square\square})$ is the zero map. Hence neither $\phi_2(y_{\substack{\square\\\square}})$ nor $\phi_2(y_{\square\square})$ is the identity map of $V_{\square}^{\otimes 2}$.

From equations (11.4.7) and (11.4.8) we also deduce that $\phi_2(y_{\substack{\square\\\square}})$ and $\phi_2(y_{\square\square})$ have different quantum traces, hence $im(\phi_2(y_{\substack{\square\\\square}}))$ and $im(\phi_2(y_{\square\square}))$ are not isomorphic.

We have proved so far that the submodules $im(\phi_2(y_{\substack{\square\\\square}}))$ and $im(\phi_2(y_{\square\square}))$ are non-trivial submodules of $V_{\square}^{\otimes 2}$, and they are non-isomorphic. Since $V_{\square}^{\otimes 2}$ decomposes by the Littlewood-Richardson rule as $V_{\square}^{\otimes 2} \approx V_{\substack{\square\\\square}} \oplus V_{\square\square}$ we have that either $im(\phi_2(y_{\square\square})) \approx V_{\square\square}$ and $im(\phi_2(y_{\substack{\square\\\square}})) \approx V_{\substack{\square\\\square}}$, or we have that $im(\phi_2(y_{\square\square})) \approx V_{\substack{\square\\\square}}$ and $im(\phi_2(y_{\substack{\square\\\square}})) \approx V_{\square\square}$. ♣

It would be natural to compute the quantum dimensions of $V_{\substack{\square\\\square}}$ and $V_{\square\square}$ to settle the ambiguity in lemma 11.4.15. This would involve the computation of the action of the ribbon element μ on $V_{\substack{\square\\\square}}$ or $V_{\square\square}$. But these computations can be avoided because lemma 11.4.16 shows that $im(\phi_2(y_{\substack{\square\\\square}})) \approx V_{\substack{\square\\\square}}$ by using the same approach as in the proof of lemma 11.4.15.

Lemma 11.4.15 *If $im(\phi_2(y_{\square\square})) \approx V_{\square\square}$ then $im(\phi_{|\lambda|}(y_\lambda)) \approx V_\lambda$ for any Young diagram λ.*

Proof By induction on $|\lambda|$, the number of cells of λ.

If λ is the empty Young diagram then $y_\emptyset$ is the empty diagram in H_0, hence $\phi_0(y_\emptyset) = \mathrm{id} : \mathbb{C}[[h]] \to \mathbb{C}[[h]]$, hence $im(\phi_0(y_\emptyset)) = \mathbb{C}[[h]] = V_\emptyset$.

There is only one Young diagram with a single cell, and $y_{\square}$ is the single string in H_1. Hence $\phi_1(y_{\square})$ is the identity map of $V_{\square}$ and thus $im(\phi_1(y_{\square})) = V_{\square}$.

The hypothesis of the lemma is that $im(\phi_2(y_{\square\square})) \approx V_{\square\square}$. Then $im(\phi_2(y_{\substack{\square\\\square}})) \approx V_{\substack{\square\\\square}}$ by lemma 11.4.14. Hence the statement of lemma 11.4.15 is true for all Young diagrams λ with at most 2 cells.

Let $k \geq 3$. The induction hypothesis is that $im(\phi_{|\mu|}) \approx V_\mu$ for any Young diagram μ with less than k cells provided that $im(\phi_2(y_{\square\square})) \approx V_{\square\square}$. From this we shall deduce that $im(\phi_{|\lambda|}) \approx V_\lambda$ for any Young diagram λ with k cells.

We remark that $im(\phi_{|\lambda|}(y_\lambda)) = im(\phi_{|\lambda|}(e_\lambda))$ for any Young diagram λ because y_λ and e_λ differ in $\tilde{H}_k^N$ by an invertible non-zero scalar.

We consider first a Young diagram $\lambda = (\lambda_1, \dots, \lambda_r)$ with k cells and r rows and λ different from d_k and c_k. We denote the transposed Young diagram by $\lambda^\vee = (\lambda_1^\vee, \dots, \lambda_m^\vee)$, $m = \lambda_1$.

By definition, we have $e_\lambda = \alpha w_\pi \beta w_\pi^{-1}$ with

$$\alpha = a_{\lambda_1} \otimes \cdots \otimes a_{\lambda_r} \text{ and } \beta = b_{\lambda_1^\vee} \otimes \cdots \otimes b_{\lambda_m^\vee}$$

where the tensor product denotes the juxtaposition $H_i \otimes H_j \subset H_{i+j}$. By the definition of ϕ and using $e_{d_i} = a_i$ and $e_{c_j} = b_j$, we get

$$\phi_k(\alpha) = \phi_{\lambda_1}(e_{d_{\lambda_1}}) \otimes \cdots \otimes \phi_{\lambda_r}(e_{d_{\lambda_r}}) \text{ and } \phi_k(\beta) = \phi_{\lambda_1^\vee}(e_{c_{\lambda_1^\vee}}) \otimes \cdots \otimes \phi_{\lambda_m^\vee}(e_{c_{\lambda_m^\vee}}).$$

Since λ is neither a single row nor a single column diagram, the rows and columns of λ and $\lambda^\vee$ have lengths less than k, hence we know by induction hypothesis and lemma 11.4.9 that

$$\begin{aligned} im(\phi_k(\alpha)) &\approx im(\phi_{\lambda_1}(y_{d_{\lambda_1}})) \otimes \cdots \otimes im(\phi_{\lambda_r}(y_{d_{\lambda_r}})) \approx V_{d_{\lambda_1}} \otimes \cdots \otimes V_{d_{\lambda_r}} \text{ and} \\ im(\phi_k(\beta)) &\approx im(\phi_{\lambda_1^\vee}(y_{c_{\lambda_1^\vee}})) \otimes \cdots \otimes im(\phi_{\lambda_m^\vee}(y_{c_{\lambda_m^\vee}})) \approx V_{c_{\lambda_1^\vee}} \otimes \cdots \otimes V_{c_{\lambda_m^\vee}}. \end{aligned}$$

Because $e_\lambda = \alpha w_\pi \beta w_\pi^{-1}$ and thus $\phi_k(e_\lambda) = \phi_k(\alpha)\phi_k(w_\pi \beta w_\pi^{-1})$, we have that $im(\phi_k(e_\lambda))$ is a submodule of $im(\phi_k(\alpha))$.

On the other hand, by lemma 11.4.12, $im(\phi_k(e_\lambda))$ is isomorphic to a submodule of $im(\phi_k(w_\pi \beta w_\pi^{-1}))$. The positive permutation braid w_π has an inverse in H_k (and in $\tilde{H}_k^N$) and therefore $\phi_k(w_\pi)$ is a module automorphism of $V_\square^{\otimes k}$. Hence, $im(\phi_k(e_\lambda)) \approx im(\phi_k(\beta))$, and hence $im(\phi_k(e_\lambda))$ is isomorphic to a submodule of $im(\phi_k(\beta))$.

Hence, by lemma 11.4.11, $im(\phi_k(e_\lambda))$ is either isomorphic to V_λ or it is the zero-module. Hence $im(\phi_k(y_\lambda))$ is either isomorphic to V_λ or it is the zero-module.

We have $\dim_q(im(\phi_k(y_\lambda))) = \mathrm{tr}_q(\phi_k(y_\lambda))$, and by Lemma 11.4.13 this value is zero if and only if $l(\lambda) \geq N+1$. Hence $im(\phi_k(y_\lambda))$ is not the zero module if $r = l(\lambda) \leq N$. Hence $im(\phi_k(y_\lambda)) \approx V_\lambda$ if $r \leq N$. On the other hand, if $r \geq N+1$, then V_λ is equal to the zero module anyway, hence $im(\phi_k(y_\lambda))$ is the zero module. We have thus proved the induction step for any Young diagram λ with k cells which is different from a single row and a single column diagram.

We now consider the row diagram $\lambda = d_k$. We have $e_{d_k} = a_k$, and $(a_{k-1} \otimes a_1)a_k$ is in H_k a non-zero scalar multiple of a_k by lemma 2.4.2. For the normalized idempotents in $\tilde{H}_k^N$ we have $(y_{d_{k-1}} \otimes y_\square)y_{d_k} = y_{d_k}$. Hence,

$$\phi_k(y_{d_{k-1}} \otimes y_\square)\phi_k(y_{d_k}) = \phi_k(y_{d_k}).$$

We thus see that $im(\phi_k(y_{d_k}))$ is a submodule of $im(\phi_k(y_{d_{k-1}} \otimes y_\Box))$. We have by lemmas 11.4.10 and 11.4.13 that

$$\begin{aligned}
\dim_q(im(\phi_k(y_{d_{k-1}} \otimes y_\Box))) &= \operatorname{tr}_q(\phi_k(y_{d_{k-1}} \otimes y_\Box)) \\
&= \operatorname{tr}_q(\phi_{k-1}(y_{d_{k-1}}) \otimes \phi_1(y_\Box)) \\
&= \operatorname{tr}_q(\phi_{k-1}(y_{d_{k-1}}))\operatorname{tr}_q(\phi_1(y_\Box)) \\
&= \frac{s^N - s^{-N}}{s^{k-1} - s^{-k+1}} \cdots \frac{s^{N+k-2} - s^{-N-k+2}}{s - s^{-1}} \frac{s^N - s^{-N}}{s - s^{-1}} \\
&\neq \frac{s^N - s^{-N}}{s^k - s^{-k}} \frac{s^{N+1} - s^{-N-1}}{s^{k-1} - s^{-k+1}} \cdots \frac{s^{N+k-1} - s^{-N-k+1}}{s - s^{-1}} \\
&= \operatorname{tr}_q(\phi_k(y_{d_k})) \\
&= \dim_q(im(\phi_k(y_{d_k}))).
\end{aligned}$$

(The above inequality is equivalent to $(s^{N-1} - s^{-N+1})(s^{k-1} - s^{-k+1}) \neq 0$ which is true due to $N \geq 2$ and $k \geq 2$). Hence $im(\phi_k(y_{d_k}))$ is not the whole of $im(\phi_k(y_{d_{k-1}} \otimes y_\Box))$. Furthermore, we see that $im(\phi_k(y_{d_k}))$ is not the zero-module since $\dim_q(im(\phi_k(y_{d_k})))$ is different from zero. Hence $im(\phi_k(y_{d_k}))$ is a non-trivial submodule of $im(\phi_k(y_{d_{k-1}} \otimes y_\Box))$. By lemma 11.4.9 and the induction hypothesis for Young diagrams with less than k cells, we deduce that

$$im(\phi_k(y_{d_{k-1}} \otimes y_\Box)) \approx im(\phi_{k-1}(y_{d_{k-1}})) \otimes im(\phi_1(y_\Box)) \approx V_{d_{k-1}} \otimes V_\Box.$$

Hence $im(\phi_k(y_{d_{k-1}} \otimes y_\Box)) \approx V_{d_k} \oplus V_{(k-1,1)}$. Hence, $im(\phi_k(y_{d_k}))$ is either isomorphic to V_{d_k} or isomorphic to $V_{(k-1,1)}$.

We have already proved the induction step in the case $\lambda = (k-1, 1)$, hence the quantum dimension of $V_{(k-1,1)}$ is equal to the quantum trace of $\phi_k(y_{(k-1,1)})$. We have

$$\begin{aligned}
\dim_q(im(\phi_k(y_{(k-1,1)}))) &= \operatorname{tr}_q(\phi_k(y_{(k-1,1)})) \\
&= \frac{s^N - s^{-N}}{s^k - s^{-k}} \frac{s^{N+1} - s^{-N-1}}{s^{k-2} - s^{-k+2}} \cdots \\
&\quad \cdots \frac{s^{N+k-2} - s^{-N-k+2}}{s - s^{-1}} \frac{s^{N-1} - s^{-N+1}}{s - s^{-1}} \\
&\neq \frac{s^N - s^{-N}}{s^k - s^{-k}} \frac{s^{N+1} - s^{-N-1}}{s^{k-1} - s^{-k+1}} \cdots \frac{s^{N+k-1} - s^{-N-k+1}}{s - s^{-1}} \\
&= \operatorname{tr}_q(\phi_k(y_{d_k})) \\
&= \dim_q(im(\phi_k(y_{d_k}))).
\end{aligned}$$

Hence $im(\phi_k(y_{d_k}))$ is not isomorphic to $V_{(k-1,1)}$ and therefore isomorphic to V_{d_k}.

The last remaining case in the proof of the induction step is for $\lambda = c_k$. This is very similar to the case $\lambda = d_k$. But some hazards occur if $k \geq N+1$ because by lemma 11.4.13 it can happen that $\mathrm{tr}_q(\phi_{|\eta|}(y_\eta))$ is non-zero whereas $\mathrm{tr}_q(\phi_{|\eta|}(y_{\eta^\vee}))$ is equal to zero.

By the same argument as for $\lambda = d_k$, we have that $im(\phi_k(y_{c_k}))$ is a submodule of $im(\phi_k(y_{c_{k-1}} \otimes y_\square))$. Hence, $im(\phi_k(y_{c_k}))$ is by induction hypothesis isomorphic to a submodule of

$$im(\phi_{k-1}(y_{c_{k-1}})) \otimes im(\phi_1(y_\square)) \approx V_{c_{k-1}} \otimes V_\square \approx V_{c_k} \oplus V_{(2,1^{k-2})} \tag{11.4.9}$$

Here and in the following, $(2, 1^{k-2})$ denotes the Young diagram that has a first row of length 2 and $(k-2)$ rows of length 1, i.e. it is the transposed Young diagram of $(k-1, 1)$.

If $k \leq N$ then we prove that $im(\phi_k(y_{c_k})) \approx V_{c_k}$ by verifying via lemma 11.4.13 that

$$\begin{aligned}
\mathrm{tr}_q(\phi_k(y_{c_k})) &\neq 0, \\
\mathrm{tr}_q(\phi_k(y_{c_k})) &\neq \mathrm{tr}_q(\phi_k(y_{(2,1^{k-2})})), \text{ and} \\
\mathrm{tr}_q(\phi_k(y_{c_k})) &\neq \mathrm{tr}_q(\phi_{k-1}(y_{c_{k-1}}))\mathrm{tr}_q(\phi_1(y_\square)).
\end{aligned}$$

If $k = N + 1$ then $V_{c_k} \approx 0$ because a module V_η indexed by a Young diagram η with more than N rows is the zero-module. Equation 11.4.9 implies that $im(\phi_k(y_{c_k}))$ is either the zero module or it is isomorphic to $V_{(2,1^{k-2})}$. We already know from the induction step in the case of the Young diagram $\lambda = (2, 1^{k-2})$ with k cells that $V_{(2,1^{k-2})} \approx im(\phi_k(y_{(2,1^{k-2})}))$, hence $\dim_q(V_{(2,1^{k-2})}) = \mathrm{tr}_q(\phi_k(y_{(2,1^{k-2})}))$ and this term is non-zero by lemma 11.4.13. On the other hand, $\mathrm{tr}_q(\phi_k(y_{c_k})) = 0$ for $k = N+1$ by lemma 11.4.13. Hence, $im(\phi_k(y_{c_k}))$ is not isomorphic to $V_{(2,1^{k-2})}$, hence $im(\phi_k(y_{c_k})) \approx V_{c_k} \approx 0$ for $k = N + 1$.

If $k \geq N + 2$ then both of V_{c_k} and $V_{(2,1^{k-2})}$ are the zero-module, hence $im(\phi_k(c_k))$ is the zero-module as well, hence $im(\phi_k(y_{c_k})) \approx V_{c_k}$. ♣

Lemma 11.4.16 *The image of* $\phi_2(y_{\square\square}) : V_\square^{\otimes 2} \to V_\square^{\otimes 2}$ *is isomorphic to* $V_{\square\square}$.

Proof We assume from now on that $im(\phi_2(y_{\square\square}))$ is not isomorphic to $V_{\square\square}$ and we shall derive a contradiction from this assumption.

Under the assumption that $im(\phi_2(y_{\square\square})) \not\approx V_{\square\square}$ we shall prove by induction (similar to the proof of lemma 11.4.15) that

$$im(\phi_k(y_{d_k})) \approx V_{c_k} \text{ and } im(\phi_k(y_{(k-1,1)})) \approx V_{(2,1^{k-2})}$$

for any $k \geq 2$.

In the case $k = 2$, the isomorphisms $im(\phi_2(y_{\square\square})) \approx V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ and $im(\phi_2(y_{\begin{smallmatrix}\square\\\square\end{smallmatrix}})) \approx V_{\square\square}$ follow from lemma 11.4.14.

The isomorphisms $im(\phi_i(y_{d_i})) \approx V_{c_i}$ and $im(\phi_i(y_{(i-1,1)})) \approx V_{(2,1^{i-2})}$ for any i with $2 \leq i < k$ are our induction hypothesis. We shall prove them for $i = k$.

First, we prove the induction step for the Young diagram $(k-1,1)$. With the notation of the proof of lemma 11.4.15 we have $e_{(k-1,1)} = \alpha w_\pi \beta w_\pi^{-1}$ with

$$\alpha = a_{k-1} \otimes a_1 \text{ and } \beta = b_2 \otimes b_1^{\otimes(k-2)}.$$

We get

$$im(\phi_k(\alpha)) \approx im(\phi_{k-1}(y_{d_{k-1}})) \otimes im(\phi_1(y_\square)) \approx V_{c_{k-1}} \otimes V_\square$$

and

$$im(\phi_k(\beta)) \approx im(\phi_2(y_{\square\square})) \otimes im(\phi_1(y_\square))^{\otimes(k-2)} \approx V_{\square\square} \otimes V_\square^{\otimes(k-2)}.$$

We have that $im(\phi_k(y_{(k-1,1)}))$ is isomorphic to a submodule of $im(\phi_k(\alpha))$ and to a submodule of $im(\phi_k(\beta))$. By lemma 11.4.11 (or by a direct calculation via the Littlewood-Richardson rule) we see that $V_{(2,1^{k-2})}$ is the only non-zero module which is isomorphic to a submodule of $V_{c_{k-1}} \otimes V_\square$ and to a submodule of $V_{\square\square} \otimes V_\square^{\otimes(k-2)}$. Hence, $im(\phi_k(y_{(k-1,1)})) \approx V_{(2,1^{k-2})}$, or $im(\phi_k(y_{(k-1,1)}))$ is the zero module. Since the quantum trace of $\phi_k(y_{(k-1,1)})$ is non-zero for any k by lemma 11.4.13, we have $im(\phi_k(y_{(k-1,1)})) \approx V_{(2,1^{k-2})}$.

Now consider d_k. By the same argument as given in the proof of lemma 11.4.15 we see that $im(\phi_k(y_{d_k}))$ is a non-trivial submodule of $im(\phi_k(y_{d_{k-1}} \otimes y_\square))$. Now

$$im(\phi_k(y_{d_{k-1}} \otimes y_\square)) \approx V_{c_{k-1}} \otimes V_\square \approx V_{c_k} \oplus V_{(2,1^{k-2})}$$

by the induction hypothesis. We proved above that $V_{(2,1^{k-2})} \approx im(\phi_k(y_{(k-1,1)}))$. Since $\mathrm{tr}_q(\phi_k(y_{d_k})) \neq \mathrm{tr}_q(\phi_k(y_{(k-1,1)}))$, we deduce $im(\phi_k(y_{d_k})) \not\approx V_{(2,1^{k-2})}$. Hence $im(\phi_k(y_{d_k}))$ has to be isomorphic to V_{c_k}. This completes the induction step.

A consequence of this result is that $im(\phi_{N+1}(y_{d_{N+1}}))$ is isomorphic to $V_{c_{N+1}}$, hence $\mathrm{tr}_q(\phi_{N+1}(y_{d_{N+1}}))$ is equal to the quantum dimension of $V_{c_{N+1}}$. But the quantum trace of $\phi_{N+1}(y_{d_{N+1}})$ is seen by lemma 11.4.13 to be different from zero, whereas $V_{c_{N+1}}$ is the zero module and therefore has a quantum dimension equal to zero. This contradiction implies that our assumption $im(\phi_2(y_{\square\square})) \not\approx V_{\square\square}$ was wrong. ♣

By the combination of lemmas 11.4.15, 11.4.16 and 11.4.6 we have thus proved that $W_\lambda \approx V_\lambda$.

Theorem 11.4.17 *The map $\phi_{|\lambda|}(y_\lambda)$ is a projection of $V_\square^{\otimes|\lambda|}$ to a submodule isomorphic to V_λ for any Young diagram λ,*

Lemma 11.4.8 can now be restated.

Theorem 11.4.18 *Given a framed link $L = L_1 \cup \cdots \cup L_r$ whose components are coloured with irreducible $U_h(sl(N))$-modules $V_{\lambda^1}, \ldots, V_{\lambda^r}$. Then the $U_h(sl(N))$-invariant of this link is equal to the Homfly polynomial of the link L with decorations $Q_{\lambda^1}, \ldots, Q_{\lambda^r}$ on its components $L_1, \ldots, L_r$ after the substitutions $x = e^{-\frac{h}{N}}$, $v = e^{-Nh}$, and $s = e^h$.*

Bibliography

[1] A. K. Aiston. *Skein theoretic idempotents of Hecke algebras and quantum group invariants.* PhD thesis, University of Liverpool, 1996.

[2] A. K. Aiston and H. R. Morton. Idempotents of Hecke algebras of type A. *J. Knot Theory Ramif.*, 7(4):463–487, 1998.

[3] C. Blanchet. Hecke algebras, modular categories and 3-manifolds quantum invariants. *Topology*, 39(1):193–223, 2000.

[4] V. Chari and A. Pressley. *A Guide To Quantum Groups.* Cambridge University Press, 1998.

[5] V. G. Drinfeld. Quantum groups. In *Proc. Int. Cong. Math. (Berkeley, 1986)*, pages 798–820. Amer. Math. Soc., 1987.

[6] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millet, and A. Ocneanu. A new polynomial invariant of knots and links. *Bulletin of the AMS (N.S.)*, 12(2):239–246, 1985.

[7] W. Fulton and J. Harris. *Representation Theory. A first Course.* Springer-Verlag, 1991.

[8] P. M. Gilmer and J. Zhong. On the Homflypt skein module of $S^1 \times S^2$. Available at *www.arXiv.org*, GT/0007125, 2000.

[9] A Gyoja. A q-analogue of Young symmetrizer. *Osaka J. Math.*, 23(4):841–852, 1986.

[10] I. N. Herstein. *Noncommutative Rings.* The Mathematical Association of America, 1968. Published as The Carus Mathematical Monographs, 15.

[11] V. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math (2)*, 126(2):335–388, 1987.

[12] C. Kassel. *Quantum Groups.* Springer-Verlag, 1995.

[13] C. Kassel, M. Rosso, and V. Turaev. *Quantum Groups and Knot Invariants.* Société Mathématique de France, 1997.

[14] K. Kawagoe. On the skeins in the annulus and applications to invariants of 3-manifolds. *J. Knot Theory Ramif.*, 7(2):187–203, 1998.

[15] R. Kirby and P. Melvin. The 3-manifold invariants of Witten and Reshetikhin-Turaev for sl(2,C). *Invent. math.*, 105(3):473–545, 1991.

[16] T. Kohno and T. Takata. Level-rank duality of Witten's 3-manifold invariants. In *Progress in algebraic combinatorics (Fukuoka, 1993)*, pages 243–264, Tokyo, 1996. Math. Soc. Japan. Published as *Adv. Stud. Pure Math., 24.*

[17] I. G. Macdonald. *Symmetric Functions and Hall Polynomials.* Oxford University Press, 1979.

[18] H. R. Morton. Skein theory and the Murphy operators. Available at *www.arXiv.org*, GT/0102098, 2001.

[19] H. R. Morton and P. Strickland. Jones polynomial invariants for knots and satellites. *Math. Proc. Camb. Phil. Soc.*, 109(1):83–103, 1991.

[20] H. R. Morton and P. Traczyk. Knots and algebras. In *Contribuciones Matematicas en homenaje al profesor D. Antonio Plans Sanz de Bremond*, pages 201–220. University of Zaragoza, 1990.

[21] J. Przytycki and P. Traczyk. Invariants of links of Conway type. *Kobe J. Math.*, 4(2):115–139, 1987.

[22] N. Reshetikhin and V.G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. math.*, 103(3):547–597, 1991.

[23] V. G. Turaev. The Conway and Kauffman modules of a solid torus. *Issled. Topol.*, 6:79–89, 1988.

[24] V. G. Turaev. The Yang-Baxter equation and invariants of links. *Invent. Math.*, 92(3):527–553, 1988.

[25] H. Wenzl. Hecke algebras of type A_n and subfactors. *Invent. Math.*, 92(2), 1988.

[26] Y. Yokota. Skeins and quantum SU(n) invariants of 3-manifolds. *Math. Ann.*, 307(1), 1997.

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