Skein theoretic idempotents of Hecke algebras and quantum group invariants.

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Abstract.

This thesis examines the connections between the quantum group invariants for the quantum groups $U_q(sl(N))$ and the Homfly skein of the annulus.

A Gyoja [G] constructed idempotent elements of the Hecke algebras of type A which specialise to the Young symmetrisers of the group algebra of the symmetric group $\mathbb{C}S_n$. We construct skein theoretic versions of these idempotents. For each Young diagram λ we construct an element of the Homfly skein of the annulus Q_λ for which

$$J(L; V_{\lambda_1}, \ldots, V_{\lambda_k}) = \mathcal{X}_N(L_1 * Q_{\lambda_1} \sqcup \cdots \sqcup L_k * Q_{\lambda_k})$$

where $J(L; V_{\lambda_1}, \ldots, V_{\lambda_k})$ is the $U_q(sl(N))$-invariant of a link L coloured by the representations V_{λ_1}, ..., V_{λ_k}, $\mathcal{X}_N$ is obtained from the framed Homfly polynomial by making (N dependent) substitutions for the variables of the Homfly polynomial in terms of q and $L_1 * Q_{\lambda_1} \sqcup \cdots \sqcup L_k * Q_{\lambda_k}$ is a satellite of the link L. We show that if we evaluate $\mathcal{X}_N$ when q a primitive root of unity we can restrict our attention to a limited set of colours among which we identify an element Ω_r. If every component of a link is coloured by Ω_r then we can normalise $\mathcal{X}_N$ to produce a 3-manifold invariant.

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Introduction.

The motivation for this thesis is to describe algebraic results about quantum group invariants combinatorially using the language of linear skein theory.

Quantum groups were discovered in 1985 by Drinfel'd [D1] and independently by Jimbo [Ji1]. They can be thought of as 1-parameter deformations of Lie algebras. In this thesis we are interested only with the quantum groups $U_q(sl(N))$, derived from the Lie algebras $sl(N)$. Kulish and Reshetikhin [KR] produced link invariants which generalised the Jones polynomial, using representations of the quantum group $U_q(sl(2))$. Reshetikhin and Turaev [RT1] generalised the method, producing knot invariants from any quantum group. These invariants depend on the isotopy class of the link and a choice of colouring. A colour is a representation of the quantum group and a colouring is a choice of colour for each component of the link. We identify a fundamental colour, an irreducible representation for which every other irreducible representation is a summand of some tensor power.

Meanwhile, several groups [FYHLMO, PT] had introduced a 2-variable polynomial, now called the Homfly polynomial. The Jones polynomial can be obtained from the Homfly polynomial by making substitutions for the two variables. The Homfly polynomial can be described combinatorially but turns out to be closely related to the quantum group invariants for the quantum group $U_q(sl(N))$. Turaev [T1] showed that upon appropriate substitutions for the variables, the $U_q(sl(N))$-invariants of a link can be calculated directly from the Homfly polynomial if the fundamental colour is applied to every component of the link. Thus, we have a connection between the algebraic world of quantum groups and the combinatorial world of the Homfly polynomial.

In this thesis we calculate patterns (link diagrams in an annulus) that allow us to calculate the $U_q(sl(N))$ invariants of a link L with any colouring by calculating the Homfly polynomial of an appropriate satellite of L.

The contents of this thesis can be summarised as follows.

Chapter 1 covers some of the definitions of classical knot theory relevant to this thesis.

Chapter 2 is an introduction to the combinatorial theory underpinning the results of subsequent chapters. We define the Homfly polynomial and derive the framed Homfly polynomial. We discuss the skein theory associated with the framed Homfly polynomial. The second half of this chapter is devoted to the combinatorial properties of Young diagrams. These will be used in Chapter 3 to describe the representation ring of the quantum group $U_q(sl(N))$ for generic values of q.

In Chapter 3, Hopf algebras are discussed and the quantum group $U_q(sl(N))$ is defined. The irreducible representations of $U_q(sl(N))$ are indexed by the Young diagrams with fewer than N rows. We demonstrate how to construct knot invariants from quantum groups. We finish by stating Turaev's theorem which relates the quantum invariants of $U_q(sl(N))$ with fundamental colouring and the framed Homfly polynomial.

Chapter 4 is the main body of the thesis. Here we discuss Hecke algebras of type A and their connections with the Homfly skein theory of Chapter 2. Gyoja [G] gave an algebraic construction for idempotent elements of the Hecke algebra and we use the connections with skein theory to give versions of these elements as linear combinations of braids. We prove that these idempotents, interpreted as endomorphisms of tensor powers of the fundamental representation of $U_q(sl(N))$, are the projection maps onto the irreducible summands. We denote the closure of the projector onto the irreducible module indexed by the Young diagram λ by Q_λ. The Q_λ provide the patterns required to calculate the the $U_q(sl(N))$-invariant of a knot coloured by the irreducible module V_λ in terms of the framed Homfly polynomial of the satellite knot $K * Q_\lambda$. We define $\mathcal{X}_N$ to be the framed Homfly polynomial with the substitution of variables required to equate it with the quantum invariants for $U_q(sl(N))$. We show that the Q_λ work for all N at once. We demonstrate that the Q_λ satisfy the product rules of the ring of Young diagrams, in particular, that they are given by the Giambelli formula for λ.

In Chapter 5 we extend link invariants to invariants of 3-manifolds. It was shown by Lickorish [Li1] that every 3-manifold can be obtained from a framed link by surgery. We evaluate $\mathcal{X}_N$ when q is a primitive rth root of unity. We demonstrate that we can work with a restricted set of linear combinations of colours which form a finite dimensional vector space. The finite dimensionality allows us to define a colour Ω_r for which $\mathcal{X}_N$ behaves nicely under the Kirby moves. Hence, we can derive a 3-manifold invariant from $\mathcal{X}_N$ by appropriate normalisation. When $N = 2$, Morton and Strickland [MS] showed that evaluation at a root of unity gives rise to a 3-manifold invariant which is equivalent to the 3-manifold invariant of Reshetikhin and Turaev [RT2]. This chapter extends the ideas of [MS] to all values of N.

Chapter 1

The preliminaries.

1.1 Introduction.

In this chapter we look at some of the classical theory of knots. Much of the detail can be found in the books by Burde and Zieschang [BZ], Rolfsen [Ro] and Adams [Ad]. The first two books have the flavour of algebraic topology, whereas the third book is more elementary.

We introduce the concepts required for the subsequent chapters.

1.2 Basic Definitions.

1.2.1 Definitions.

A topological *knot* is an embedding of the circle, S^1 into $\mathbb{R}^3$ or S^3.

A *link* L, with $|L|$ components, is an embedding of $|L|$ copies of S^1 into $\mathbb{R}^3$ or S^3. We will denote a link with k components by $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_k$. We call the link *oriented* if each component of L has an orientation.

Given two embeddings f_0,$f_1 : S^1 \to S^3$ we say that they are *isotopic* if there exists an embedding $F : S^1 \times I \to S^3 \times I$ such that $F(x,t) = (f(x,t),t)$, for $x \in S^1$, $t \in I = [0,1]$ with $f(x,0) = f_0(x)$ and $f(x,1) = f_1(x)$.

The two embeddings are *ambient isotopic* if there is a level preserving isotopy

$H : S^3 \times I \to S^3 \times I$, with $H(y,t) = (h_t(y), t)$, where $f_1 = h_1 \circ f_0$ and $h_0 = id$. If the knots are oriented we further require that the isotopy preserves orientation.

These definitions are extended to links in the obvious way.

We will call two links *equivalent* if they are ambient isotopic.

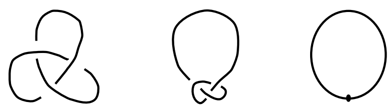
To avoid pathological behaviour, we shall restrict ourselves to the category of piecewise linear (p-l) links, i.e. to links which are ambient isotopic to a collection of simple closed polygons in $\mathbb{R}^3$ or S^3. Such links are called *tame*.

We adjust our definitions of isotopy and ambient isotopy to insist that the mappings $f(x,t)$ and h_t must be p-l embeddings for all t.

We shall call two p-l knots *p-l equivalent* if they are p-l ambient isotopic.

1.2.2 Notes.

The definition of ambient isotopy prevents us from "unknotting" the knot by pulling it tight, as in the diagram below.



Obviously if we were allowed to do this then the subject of knot theory would be much simpler.

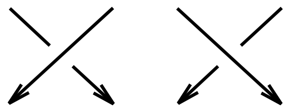
Two tame knots are ambient isotopic if and only if they are p-l ambient isotopic. Hence, for tame knots the two relations are the same.

From now on, we shall assume that we are working in the piecewise linear category. The prefix piecewise linear will be omitted.

1.2.3 Definitions.

A *link diagram* is a projection of a link, along a given direction, on to a plane. The image must have only a finite number of singular points, each a transverse double point with the over and under crossings distinguished.

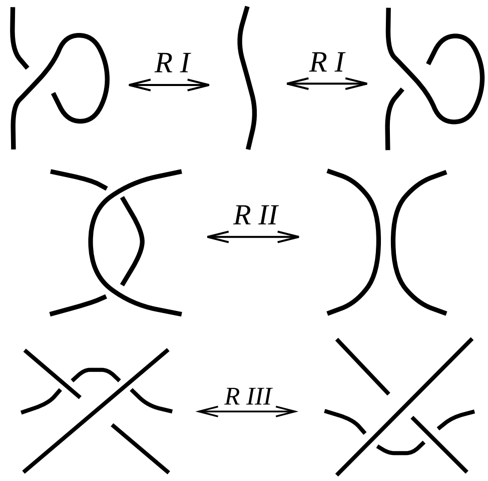
We will define the *sign of a crossing*, $\varepsilon(c)$, by the prescription



$\varepsilon = +1 \qquad \varepsilon = -1$

1.2.4 Theorem.[Re]

Let D and D' be diagrams of links L and L'. The links L and L' are equivalent if and only if D' can be obtained from D by applying a finite number of the moves described below.



We shall call these the Reidemeister moves I, II and III.

Proof. A proof of this Theorem can be found in [BZ]. ■

1.2.5 Definition.

Fix a particular diagram of an oriented link L and let L_1 and L_2 be two components of L. The *linking number*, $\mathrm{lk}(L_1, L_2)$, of the two components is defined to be

$$\mathrm{lk}(L_1, L_2) = \sum_c \varepsilon(c)\,.$$

where the sum is taken over the crossings, c, of L_1 *over* L_2 and $\varepsilon(c) = \pm 1$ is the sign of the crossing, c.

1.2.6 Proposition.

Linking number is an ambient isotopy invariant of oriented links and

$$\mathrm{lk}(L_1, L_2) = \mathrm{lk}(L_2, L_1)\,.$$

The linking number is unchanged if we reverse the orientation of both components and switches sign if we reverse the orientation of one of the components.

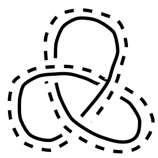
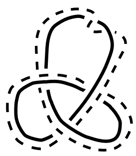
Proof. Consider how the linking number changes under the three Reidemeister moves. The RI move only involves one component, so the linking number is unchanged. The number of crossings and their signs are unchanged under $RIII$ and therefore so is linking number. With the RII move, two crossings are created (or removed). However, they appear with opposite sign so there is no net change in the linking number.

To see the second statement, view the diagram from below. Each crossing of L_1 over L_2 becomes a crossing of L_2 over L_1 with the same sign.

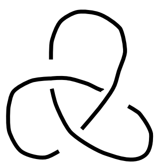
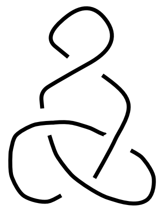
The last two statements can be shown by considering what happens to the sign of a crossing if the orientation of one or both strings is reversed. ∎

1.2.7 Definition.

A *framed link* is a link L with a chosen parallel curve for each component. If the link is oriented the parallel curves inherit their orientations from the components. Below we give two examples of framed trefoil knots.

In this thesis all framed link diagrams will be assumed to have *blackboard framing* (i.e. the parallel curve will be assumed to lie in the plane of the paper). With this assumption a framed knot will be uniquely determined by its diagram. Our previous examples will therefore be drawn as follows:

  .

1.2.8 Definition.

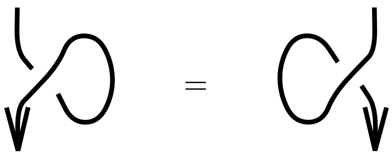
Two framed knots are said to be *regularly isotopic* (or framed equivalent) if they are ambient isotopic and the linking numbers of the parallel curves with the link components agree.

Obviously, although our two examples are ambient isotopic they are not framed equivalent, since the linking number of the knot and its parallel is $+3$ for the knot on the left and $+2$ for the one on the right.

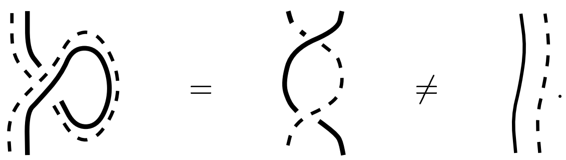
Most of the link invariants we are concerned with in this thesis are regular isotopy invariants. The following Proposition expresses regular isotopy in terms of Reidemeister moves. This interpretation of regular isotopy is the one that we will use in subsequent chapters.

1.2.9 Proposition.

Two framed links are equivalent if their diagrams are related by a series of RII and $RIII$ moves and the following move



Proof. Firstly we rule out the first Reidemeister move, on the grounds that it changes the linking number of the component with its parallel. From the pictures below it is easy to see that adding the full curl adjusts the linking number by ± 1.



The linking number remains unchanged under RII and $RIII$ using similar reasoning to that in the proof of Proposition 1.2.6.

In fact we can switch a curl from one side of the string to the other using RII and $RIII$.

We use the Whitney trick, shown in Figure 1.1, to cancel curls of opposite sign which occur on the opposite sides of a string.

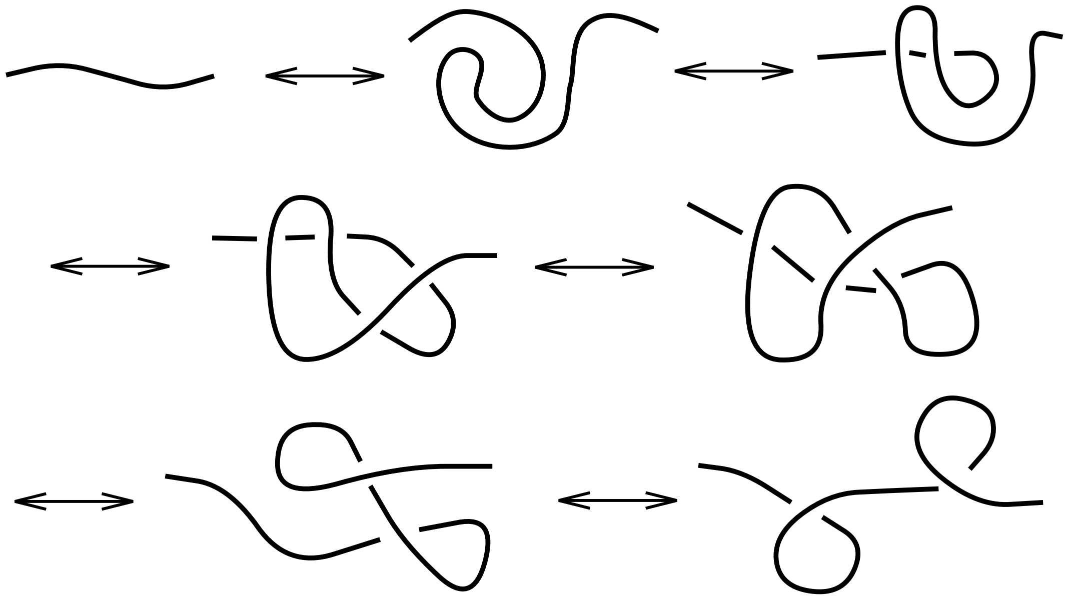
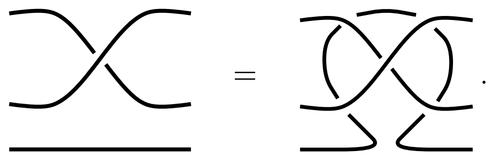


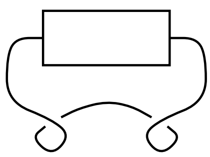
Figure 1.1: The Whitney trick for introducing or cancelling curls.

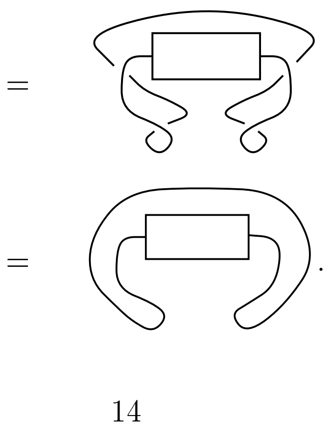
If we have two curls of opposite sign on the same side of a string, then we can cancel them by considering the link as a whole. We can pass the string under every other crossing in the link using RII and $RIII$, as shown below.



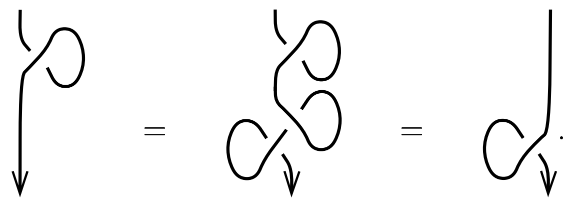
Therefore, if L is a link, with two adjacent curls of opposite sign on the same side of the string, we can proceed as follows

L =





Note that we have used only RII and $RIII$ moves to achieve this. We can, therefore add or remove pairs of curls with opposite sign at will and preserve the regular isotopy class of a link diagram. Hence,



Thus, moving a curl from one side of the string to the other can be achieved in a finite number of RII and $RIII$ moves. Therefore if two link diagrams are regular isotopic, we can obtain one from the other in a finite number of RII and $RIII$ moves.

■

1.2.10 Definition.

Let D be a diagram of a link L. The *writhe*, $\omega(D)$, is the signed crossing number of the diagram, i.e. $\omega(D) = \sum_c \varepsilon(c)$ over the crossings in D.

Writhe is an invariant of framed links. The proof of this fact is very similar to that for Proposition 1.2.6 concerning linking numbers. For a knot diagram the writhe is equal to the linking number of the framed knot with its parallel curve. This number is sometimes called the *framing* of the knot.

1.3 Braids.

1.3.1 Definition.

Let I denote the unit interval $[0, 1]$. Consider the cube $I \times I \times I$. Fix n pairs of points, namely $(1/2, i/(n+1), 0)$ and $(1/2, i/(n+1), 1)$ for $i = 1 \ldots n$. We define a *braid on n strings* to be the following. Attach a string to each of the n points $(1/2, i/n+1, 1)$. The other end of the string is attached to $(1/2, j/n+1, 0)$ for some $j = 1 \ldots n$. We insist no two strings are attached to the same point. We further insist that the strings can't turn back on themselves at any point i.e. for any $t \in [0, 1]$ each string intersects the plane $z = t$ exactly once.

Two braids are equivalent if one can be obtained from the other by a finite number of RII and $RIII$ moves, relative to the fixed boundary points.

1.3.2 **Proposition.**[A]

The set of all n-string braids forms a group, B_n, under product. The product of two braids, B and C, is given by concatenation, as in the figure below.

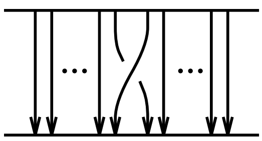
$$BC \quad = \quad \begin{array}{|c|}\hline B \\ \hline C \\ \hline \end{array}$$

Notice that we are taking multiplication on the right rather than the left.

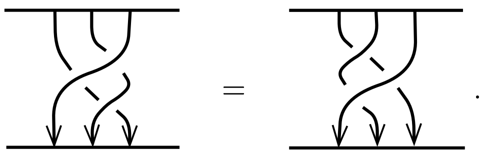
The braid group has a presentation

$$\left\langle \quad \sigma_i \quad i = 1, \ldots n-1 \qquad \middle| \qquad \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \qquad |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \quad \right\rangle$$

where σ_i corresponds to the braid where the $(i+1)$st string crosses over the ith string and no other strings cross.



Pictorially, the second set of relations is a version of the third Reidemeister move,



1.3.3 **Definition.**

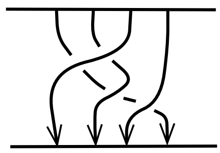
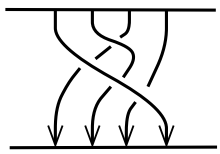
For each permutation $\pi \in S_n$, we will define an n-braid ω_π called a *positive permutation braid.* It is uniquely determined by the following properties :

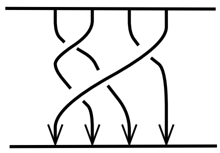
1. All strings are oriented from top to bottom.
2. String i joins the point numbered i at the top of the braid to the point numbered $\pi(i)$ at the bottom of the braid.
3. All the crossings occur with positive sign.
4. The ith and the jth strings cross at most once.

We can think of the braid as sitting in layers, with the first string at the back and the nth string at the front.

We will define the *negative permutation braid* in exactly the same manner as the positive permutation braid except that we will demand that all the crossing be negative. We shall denote this braid by ω_π^{-1}.

Note that, perversely, $\omega_\pi \omega_\pi^{-1} \neq 1$ in general but, $\omega_\pi \omega_{\pi^{-1}}^{-1}$ will always be the identity braid. For example if $\pi = (431) \in S_4$ then

ω_π =  ω_π^{-1} = 

$\omega_{\pi^{-1}}$ = 

1.3.4 Comment.

The positive permutation braids were first identified by Elrifai and Morton [EM]. Morton and Traczyk [MT] showed that they form a basis for the Hecke algebra. They will be the elements from which we will build our idempotent elements in Chapter 4.

1.4 New knots from old.

1.4.1 Definition.

Let P be a knot contained in a standardly embedded solid torus T. A disc in T is called a *meridinal disc* if it bounds a meridian. We require P to be essential in T (P must intersect every meridinal disc of T). Let C be a knot in S^3 and let T' be a tubular neighbourhood of C. Let $h : T \to T'$ be a homeomorphism. Then $h(P)$ is a *satellite knot* with *companion* C and *pattern* P.

1.4.2 Remarks.

In some definitions there is the further requirement that h is faithful (i.e. h takes the preferred meridian and longitude of T onto the preferred meridian and longitude of T'). This would determine the satellite of C uniquely for a given pattern. However, there is an alternative definition which defines the satellite for a framed companion knot. The framing determines the choice of homeomorphism in the previous definition by identifying a preferred curve on T' to which the longitude of T should be mapped, therefore, it is essential to have an agreed framing on the companion knot. Let C be a framed knot and P be a knot diagram in an annulus. The parallel of C determines an embedding of an annulus in S^3. Let S be the image of P under the embedding. We call S a *satellite knot*. The knot C is called the *companion* knot and P is known as the *pattern*. The knot S will sometimes be denoted as $C * P$.

If the pattern P is framed then S will be framed, inheriting its framing from P. The condition that h is faithful corresponds to the requirement that the knot C has zero framing in this definition. The example in Figure 1.2 is of a satellite of the figure of eight knot. This satellite construction lies at the heart of the

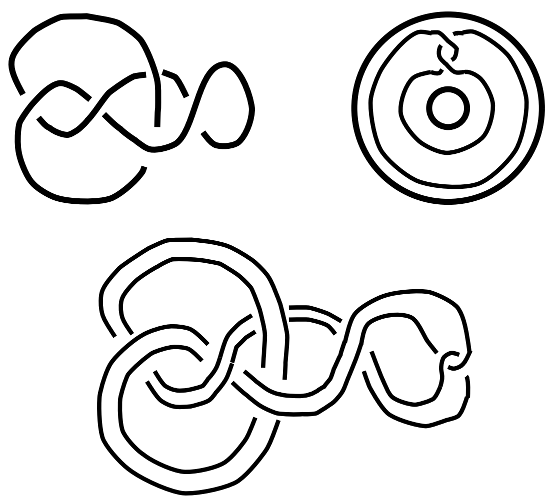


Figure 1.2: A satellite of the figure-eight knot with framing -1.

connection between quantum invariants of coloured links and the framed Homfly polynomial. In Chapter 4, we calculate a pattern, Q, for each colour, V, and show that the quantum invariant of a knot, K coloured by V is equal to the framed Homfly polynomial of the satellite $K * Q$.

Chapter 2

Combinatorial results.

2.1 Introduction.

We first discuss the Homfly polynomial and some basic skein theory. Connections between these and the quantum invariants of links will be established in Chapter 3.

We then look at partitions and Young diagrams. The approach is combinatorial, making use of the connections between Young diagrams and symmetric polynomials. I.G. Macdonald gives a detailed account of the theory of symmetric functions and their connections with Young diagrams in [Mac].

We describe a ring structure on the set of Young diagrams, set up to reflect the representation theory of the Lie algebra $sl(N)$ and its quantum enveloping algebra, which will be discussed in Chapter 3. Connections with the representation theory of the group algebra $\mathbb{C}S_n$ and the Hecke algebras of type A will be established in Chapter 4.

2.2 The Homfly polynomial.

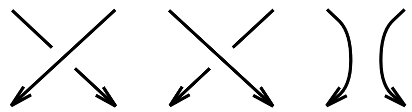
2.2.1 Definition.

The *Homfly polynomial*, $P(L) \in \mathbb{C}[v^{\pm 1}, z^{\pm 1}]$, is an ambient isotopy invariant of an oriented link L which is multiplicative over distant union. It was discovered by several groups; [FYHLMO, PT].

It is determined up to a scalar multiple by the skein relation

$$v^{-1}\,P(L_+) - vP(L_-) = zP(L_0)\,.$$

where L_+, L_- and L_0 are oriented links which differ only in the disc indicated below.



L_+ L_- L_0

We normalise P by setting the value of the Homfly polynomial for the empty link to be 1 (rather than the more usual normalisation, $P(\bigcirc) = 1$.) As a direct consequence of the skein relation

$$P(L \sqcup O) = \frac{v^{-1} - v}{z}\,P(L)\,,$$

where $L \sqcup O$ denotes the link L together with a distant simple closed curve.

We will often write skein relations schematically. For example we can describe the skein relation of the Homfly polynomial pictorially as

$$v^{-1}\,P\left(\quad\right) \quad - \quad v\,P\left(\quad\right) \quad = \quad z\,P\left(\quad\right)\,.$$

The Homfly polynomial is invariant under all three Reidemeister moves, although this can't be proved directly from the skein relation.

If, instead, we require a regular isotopy invariant we must modify the Homfly polynomial to take account of changes in the writhe of a link. We will work with $\mathcal{X}(L) \in \mathbb{C}[x^{\pm 1}, v^{\pm 1}, z^{\pm 1}]$, which is an invariant of framed oriented links. We will call $\mathcal{X}$ the *framed Homfly polynomial.* Discussion of this invariant can be found in sections 18 and 19, Chapter 6 of Kauffman's book [K1] and also in [M2].

It is constructed from the Homfly polynomial by extending the coefficient ring to include an indeterminate x and setting $\mathcal{X}(L) = (xv^{-1})^{\omega(D)}P(L)$, where $\omega(D)$ is the writhe of the knot diagram. The following relation, therefore, holds

$$\mathcal{X}\left(\quad\right) \quad = \quad (xv^{-1})\,\mathcal{X}\left(\quad\right) \quad .$$

The skein relation needs to be adjusted to take account of this curl relation. The framed skein relation is

$$x^{-1}\,\mathcal{X}\left(\quad\right) - x\,\mathcal{X}\left(\quad\right) = z\,\mathcal{X}\left(\quad\right)\,.$$

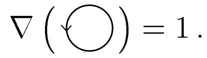
Set $z = s - s^{-1}$ and $\delta = (v^{-1} - v)/z$. Following Morton and Traczyk, [MT], we will set Λ to be a quotient of the ring of polynomials in $x^{\pm 1}$, $v^{\pm 1}$, z and δ,

$$\Lambda = \mathbb{C}[x^{\pm 1}, v^{\pm 1}, z, \delta] / < v^{-1} - v = \delta z > \ .$$

It is easy to see from its construction that the framed Homfly polynomial is an element of Λ. The variable δ is introduced to keep track of the occurrences of z^{-1} and allows for specialisations of Λ in which z is mapped to 0.

2.2.2 Notation.

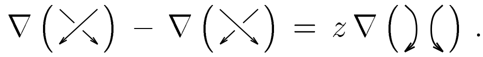
We will denote the evaluation of $P(L)/\delta$ at $v = 1$, by $\nabla(L)$. Note that when we calculate the Homfly polynomial we can always reduce diagrams to a collection of unknots, therefore, $P(L)$ always has a factor of δ. The value of ∇ on the unknot is 1,



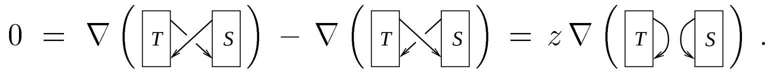
(Note that if we had normalised $P(v, z)$ to be 1 on the unknot, then the value of $\nabla(L)$ would be exactly the Homfly polynomial evaluated at $v = 1$.)

We can calculate ∇ from $\mathcal{X}$ by setting $x = v = 1$.

Since P is an ambient isotopy invariant, ∇ is also an ambient isotopy invariant which is polynomial in z. It satisfies the skein relation



From the skein relation, we have



Therefore, if L is a split link then $\nabla(L) = 0$.

2.2.3 Remarks.

We introduce ∇ here because we use the fact that it vanishes on split links to prove Proposition 4.8.2. In fact the invariant ∇ is a well known invariant called the Conway polynomial. It is closely related to the Alexander polynomial and pre-dates the Homfly polynomial. We define it in terms of the Homfly polynomial here as it will only appear in this context in this thesis. Details of the original definition can be found in J.H. Conway's paper [C].

2.3 Skein theory.

Skein theory was first introduced by J.H. Conway, [C]. Here we are interested in the skein theory associated to the Homfly polynomial. In particular we are interested in the skein theory of the annulus and its connections to satellite knots. The Homfly skein theory is discussed in detail in the work of Lickorish and Millet [LiM, Li2].

2.3.1 Definition.

Let F be a planar surface. If F has boundary, we fix a (possibly empty) set of distinguished points on the boundary.

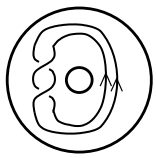
In this context we define a *diagram* in F to be a collection of oriented closed curves and arcs joining the distinguished boundary points, allowing only simple crossings. Two diagrams are equivalent if one can be obtained from the other by a finite number of Reidemeister moves II and III.

Let $D(F)$ be the set of Λ-linear combinations of diagrams in F, up to equivalence.

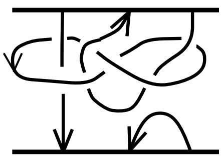
2.3.2 Examples.

If $F = \mathbb{R}^2$ then $D(F)$ is the set of Λ-linear combinations of oriented link diagrams.

Let $F = S^1 \times I$ be an annulus. In this case $D(F)$ is the set of linear combinations of link diagrams such as the one below.



Set $I = [0, 1]$ and fix integers n and m. Let $F = I^2$ with distinguished boundary points at $(i/(m+1), 0)$ for $i = 1 \dots m$ and $(j/(n+1), 1)$ for $j = 1 \dots n$. Then $D(F)$ will be the set of Λ-linear combinations of diagrams such as the one below, in the case where $m = n = 3$

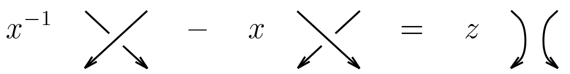


We will call such diagrams *tangles*.

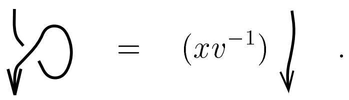
More importantly, we shall be concerned with a subset of these diagrams. Firstly we will insist that there are n boundary points at both the top and the bottom of the rectangle. Secondly, we insist that the points at the top are inputs and those at the bottom are outputs. By this, we mean that the strings which meet the top of the square are oriented into the square and the strings which meet the bottom of the square are oriented out of it. We shall denote the set of such diagrams by $D(R_n^n)$. The set of n-string braids is contained in $D(R_n^n)$, however, the diagram above is not an element of $D(R_3^3)$ since two of its strings turn back on themselves.

2.3.3 Definition.

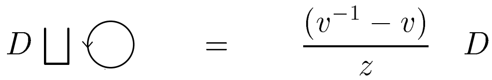
The *framed Homfly skein* $\mathcal{S}(F)$ of a planar surface F with a distinguished set of boundary points is the quotient of $D(F)$ by the relations

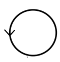


and



Again, it is a consequence that



where  is a null-homotopic loop in F.

The following definition explains how to map the linear skein of one surface, F, into the linear skein of another, F'.

A *wiring* w of F into F' is a choice of inclusion of F into F' and a choice of a fixed diagram of curves and arcs in $F' \backslash F$. The boundary of this fixed diagram is the union of the distinguished sets of F and F'.

Examples of this can be found in Examples 2.3.5.

2.3.4 Theorem.

A wiring of F into F' induces a linear map of the skein of F into the skein of F' defined by

$$\begin{array}{rcl} \mathcal{S}(w) : \mathcal{S}(F) & \to & \mathcal{S}(F') \\ D & \mapsto & w(D) \end{array}$$

where $w(D)$ is the inclusion of the diagram D in the fixed diagram described above.

Proof. Suppose a collection of diagrams in $\mathcal{S}(F)$ satisfy the skein relations. Since the relations are defined locally, the diagrams will continue to satisfy the relations when we extend to $\mathcal{S}(F')$. ■

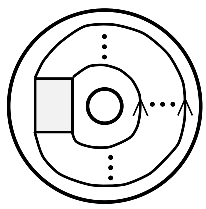
In fact we can extend this idea to wiring several surfaces, F_1, F_2,. . .,F_n into a surface F. This gives us a multi-linear map

$$\mathcal{S}(F_1) \times \mathcal{S}(F_2) \times \cdots \times \mathcal{S}(F_n) \to \mathcal{S}(F)$$

In some of the following examples this idea will be used to turn linear skein modules into algebras.

2.3.5 Examples.

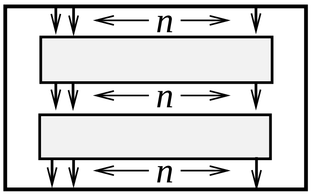
We can wire the rectangle R_n^n into the annulus as indicated below.



This element will be called the *closure* of the tangle. For a tangle T we will denote its closure by $\widehat{T}$.

There is an obvious wiring of the annulus into the plane by "forgetting" the annulus.

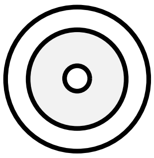
Two copies of the rectangle R_n^n can be wired together as follows,



.

This gives us a bi-linear map $\mathcal{S}(R_n^n) \times \mathcal{S}(R_n^n) \to \mathcal{S}(R_n^n)$ which defines a product on $\mathcal{S}(R_n^n)$. Therefore, $\mathcal{S}(R_n^n)$ is an algebra over Λ.

The next example also defines a product this time in $\mathcal{S}(S^1 \times I)$. A wiring of $\mathcal{S}(S^1 \times I) \times \mathcal{S}(S^1 \times I) \to \mathcal{S}(S^1 \times I)$ is given by the following diagram



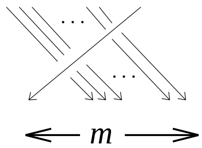
The two annuli are stacked one inside the other. This product is obviously commutative (lift the inner annulus up and stretch it so that the outer one will fit on the inside of it). Hence $\mathcal{S}(S^1 \times I)$ is a commutative Λ-algebra.

2.3.6 Notation.

Let $\mathcal{C} = \mathcal{S}(S^1 \times I)$ as a Λ-algebra. Let $\mathcal{C}^+$ be the sub-algebra generated by the diagrams which have all their strings running anti-clockwise through some meridian.

2.3.7 Theorem.[T2]

The framed Homfly skein $\mathcal{C}^+$ is freely generated as an algebra by φ_m^+, $m \in \mathbb{N}$, where φ_m^+ is the closure of the braid in the picture below.



■

2.3.8 Corollary

The algebra $\mathcal{C}^+$ is graded. Let $\mathcal{C}^{(n)}$ be the linear space spanned by all the terms $(\varphi_{i_1}^+)^{j_1}(\varphi_{i_2}^+)^{j_2}\cdots(\varphi_{i_p}^+)^{j_p}$ where $\sum_{k=1}^{p} i_k j_k = n$. Then

$$\mathcal{C}^+ = \bigoplus_{n=0}^{\infty} \mathcal{C}^{(n)}$$

(Note that $\mathcal{C}^{(n)}$ is the linear space of all diagrams with algebraic intersection equal to n on some meridional arc.)

Proof. By Theorem 2.3.7, every element of $\mathcal{C}^+$ is a Λ-linear combination of such monomials. Since the elements φ_m^+ generate $\mathcal{C}^+$ freely as an algebra, the monomials must generate $\mathcal{C}^+$ as a Λ-module.

Let $c^{(m)} \in \mathcal{C}^{(m)}$ and $c^{(n)} \in \mathcal{C}^{(n)}$. Without loss of generality we can assume that they are monomials. Let

$$c^{(m)} = (\varphi_{i_1}^+)^{j_1}(\varphi_{i_2}^+)^{j_2}\cdots(\varphi_{i_p}^+)^{j_p},$$

with $\sum_{t=1}^{p} i_t j_t = m$, and

$$c^{(n)} = (\varphi_{k_1}^+)^{l_1}(\varphi_{k_2}^+)^{l_2}\cdots(\varphi_{k_r}^+)^{l_r}$$

with $\sum_{t=1}^{r} k_t l_t = n$. Then

$$c^{(m)}c^{(n)} = (\varphi_{i_1}^+)^{j_1}(\varphi_{i_2}^+)^{j_2}\cdots(\varphi_{i_p}^+)^{j_p}(\varphi_{k_1}^+)^{l_1}(\varphi_{k_2}^+)^{l_2}\cdots(\varphi_{k_r}^+)^{l_r}.$$

is a monomial in $\mathcal{C}^{(m+n)}$ as required. ■

2.3.9 Comment

Note that all the elements of $\mathcal{C}^{(n)}$ are linear combinations of terms, each of which has exactly n strings running anti-clockwise through some meridian.

2.3.10 Theorem.[MT]

Let $w : \mathcal{S}(R_n^n) \to \mathcal{S}(S^1 \times I)$ be the wiring as described in the first example in Examples 2.3.5. Define

$$W = \bigcup_{n\in\mathbb{N}} w(\mathcal{S}(R_n^n))$$

Then W is a sub-algebra of $\mathcal{C}$, namely the sub-algebra $\mathcal{C}^+$.

Proof. It is obvious that W is closed under addition, product and scalar multiplication. It is also clear that $W \subseteq \mathcal{C}^+$ since the arcs in the fixed diagram all run round anti-clockwise.

To see that $\mathcal{C}^+ \subseteq W$ consider a element $c \in \mathcal{C}^+$. For each term in the expression c there is a meridian somewhere through which all the strings pass in the same direction. If we cut along each of these meridians and open out the resulting rectangles we obtain an element of $\mathcal{S}(R_n^n)$, for some whole number n, which closes to c as required. ∎

2.3.11 Motivation.

We can think of $\mathcal{C}$ as a collection of patterns for satellite links. In Chapter 3 we introduce quantum link invariants, which depend on the link and a choice of colouring. At the end of Chapter 4 we show that these quantum invariants can be calculated by finding the framed Homfly polynomial of certain satellites of the link, with an appropriate specialisation of the ring Λ. Since we are only concerned with the framed Homfly polynomial of these satellite links and any two patterns which are equivalent in $\mathcal{C}$ will produce satellite links with the same framed Homfly polynomials, we can assume that the patterns are elements of $\mathcal{C}$. In fact, a property of the quantum invariants allows us just to consider patterns in $\mathcal{C}^+$.

2.4 Partitions.

2.4.1 Definitions.

Fix a natural number $n \in \mathbb{N}$. Set $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$, with $\lambda_i \in \mathbb{N}$, $\sum_{i=1}^{k} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. We call λ a *partition* of n.

Often it is stipulated that the λ_i must be non-zero, however, since we wish to compare partitions of different numbers later, we allow our partitions to have a finite number of zeros at the end. We will make no distinction between two partitions which differ only by a collection of zeros.

We shall formally include (0) as a partition of 0.

We can represent a partition of n by a *Young diagram*, a collection of n cells arranged in rows, with λ_1 cells in the first row, λ_2 cells in the second row and so forth. We shall abuse notation by denoting both the partition and its Young diagram by λ. Note that the Young diagram for (0) is the empty diagram.

We define the *size* of a partition, denoted by $|\lambda| = \sum_i \lambda_i$, to be the number of cells in the Young diagram.

The *conjugate* of λ, $\lambda^\vee$, is the partition obtained from λ by setting $\lambda_i^\vee$ to be the number of $\lambda_j \geq i$. In terms of Young diagrams, $\lambda^\vee$ is the diagram whose rows are the columns of λ.

Suppose that λ is a Young diagram with n cells and $T(\lambda)$ is an assignment of the numbers 1 to n to the cells of λ such that they increase from left to right and top to bottom. We call $T(\lambda)$ a *standard tableau.*

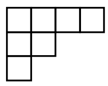
We can define a total ordering on the set of Young diagrams using the lexicographic order. Set $\lambda = (\lambda_1, \cdots, \lambda_k)$ and $\mu = (\mu_1, \cdots, \mu_m)$. Let j be the smallest value of i for which $\lambda_i - \mu_i \neq 0$. If $\lambda_j - \mu_j$ is positive then $\lambda > \mu$.

Note that we do not insist that λ and μ are both partitions of the same natural number or that they have the same number of rows. If $k < m$ we can add a collection of empty rows to λ so that $\lambda_i - \mu_i$ is well defined for $i > k$.

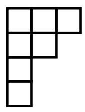
For a given cell in a Young diagram we define the *hook length* to be the number of cells below it in the same column and to the right of it in the same row. The cell itself is included in the count.

2.4.2 Example.

Let $\lambda = (4, 2, 1)$, then $|\lambda| = 7$. The Young diagram for this partition is

$\lambda \quad =$  .

It follows that $\lambda^\vee = (3, 2, 1, 1)$ and has the following Young diagram

$\lambda^\vee \quad =$  .

Below we give two examples of standard tableaux for the partition λ.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

The diagram below shows the hook lengths for each cell in λ.

$$\begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

2.4.3 Definition.

Let each Young diagram determine a vertex of a graph. An edge will join μ to λ if μ can be obtained from λ by removing one cell. Figure 2.1 shows the graph for Young diagrams with up to 7 cells, arranged in increasing size down the page. This graph is usually used to indicate how irreducible representations of the Hecke algebra, H_n (respectively $\mathbb{C}S_n$) decompose when restricted to representations of H_{n-1} (respectively $\mathbb{C}S_{n-1}$). It is known as a Brattelli diagram. We will return to these topics in Chapter 4.

We define two integers, d_λ and σ_λ, recursively using the graph. Let $d_\square = 1$. The integer $\sigma_\square$ is not defined. Instead we set $\sigma_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = -1$ and $\sigma_{\square\square} = 1$. We then define

$$\begin{aligned} d_\lambda &= \sum d_{\lambda'} \\ \sigma_\lambda &= \sum \sigma_{\lambda'} \end{aligned}$$

where the sum is over all those Young diagrams with one fewer cell which are connected to λ by an edge of the graph. In fact there is a closed formula for d_λ.

2.4.4 Proposition.[FRT]

The number of standard tableaux of the Young diagram λ is d_λ and

$$d_\lambda = \frac{n!}{\prod \text{hook lengths}}.$$

where $|\lambda| = n$. ■

2.5 The ring of Young diagrams.

Next we put a ring structure on the set of formal Λ-linear combinations of Young diagrams. It is devised to have the same structure constants as the ring of repre-

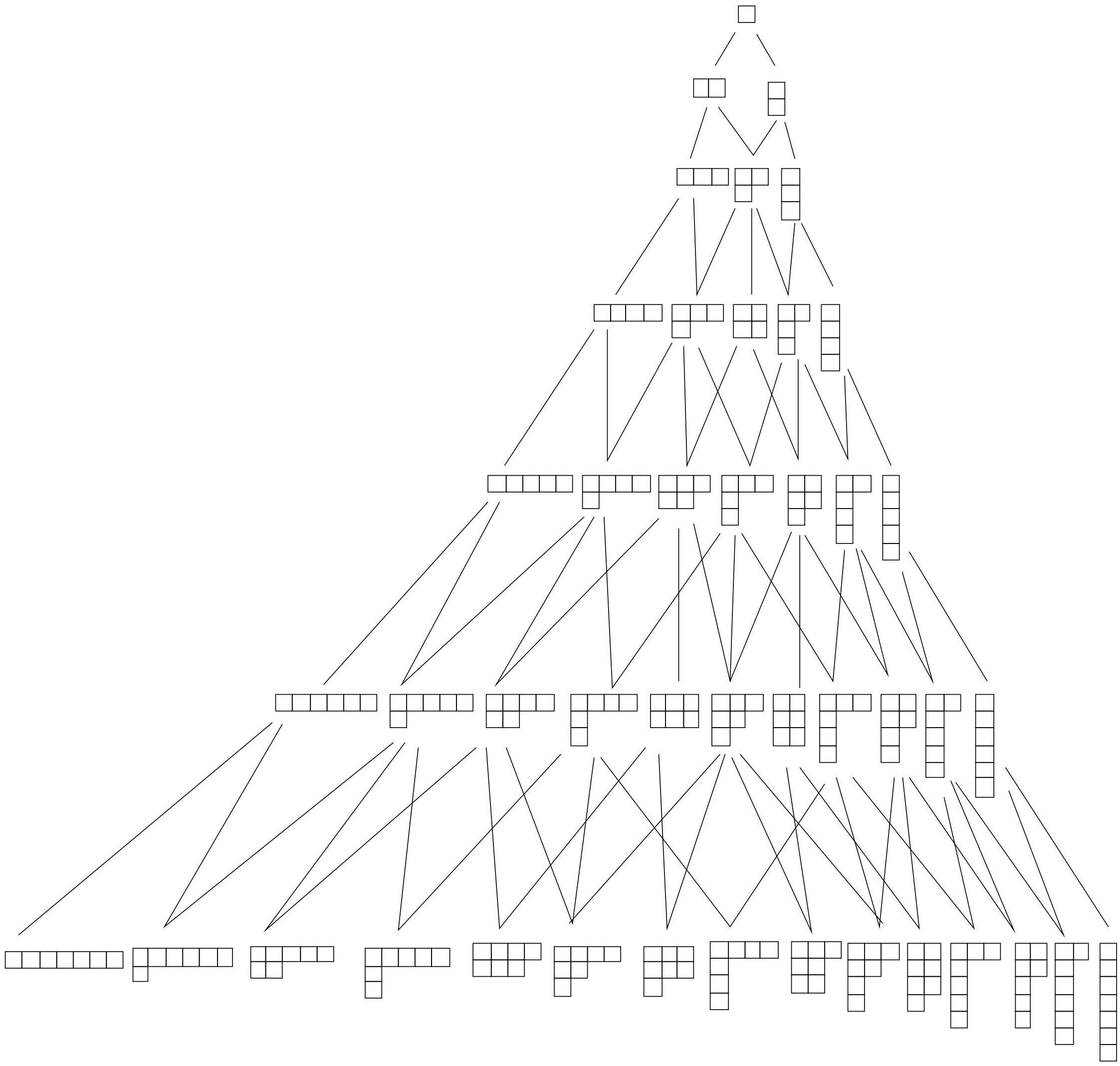


Figure 2.1: The graph of Young diagrams.

sentations of the Lie algebra $sl(N)$. However, it is defined purely combinatorially and completely independently of N.

We have already mentioned that, in Chapter 3, we introduce quantum invariants which depend on a coloured link. In fact a colouring is a choice of an element of this ring for each component of the link. In Chapter 4, therefore, we need only to construct an element, $Q_\lambda \in \mathcal{C}^+$, for each Young diagram λ, for which the framed Homfly polynomial of the satellite $C * Q_\lambda$ is the quantum invariant of C coloured by λ. Since the Young diagrams span the ring we will then be able to calculate the quantum invariant of a link with any colouring via the framed Homfly polynomial.

2.5.1 Definitions.

The *ring of Young diagrams*, Y, is defined to be the set of formal Λ-linear combinations of Young diagrams where addition is given by formal linear combination and the product is defined by the following formula

$$\lambda\mu = \sum_{|\nu|=|\lambda|+|\mu|} a^{\nu}_{\lambda\mu}\nu\,.$$

where the structure constants, $a^{\nu}_{\lambda\mu}$, are the the *Littlewood-Richardson coefficients* This product is associative and commutative with identity the empty partition. Neither of these properties is obvious from the combinatorics; they follow from the representation theory of Lie algebras, described in Proposition 3.4.9.

These coefficients can be calculated combinatorially as the number of ways the diagram ν can be obtained from a strict expansion of λ by μ, as follows.

Let $\lambda = (\lambda_1, \cdots, \lambda_k)$ and $\mu = (\mu_1, , \cdots, \mu_m)$ be two Young diagrams.

A *μ-expansion* of λ is obtained by first adding μ_1 boxes to λ, each labelled with a 1. No two boxes can be placed in the same column and the result must be a legitimate Young diagram. Then add μ_2 boxes labelled 2 (respecting the same rules) and continue until you have added μ_m boxes labelled m. At each stage no two cells with the same label can appear in the same column.

For any given cell, let n_i be the number of cells numbered i above and to the right of it (including the cell itself). The expansion is called *strict* if, for any cell, $i < j$ implies that $n_i \geq n_j$.

A clear description of this method of calculating the Littlewood-Richardson coefficients can be found in [J].

2.5.2 Remarks.

Let $Y^{(n)}$ be the linear space generated by the Young diagrams with exactly n cells, then

$$Y = \bigoplus_{n=0}^{\infty} Y^{(n)}$$

Note that, with respect to this decomposition, the ring is graded, i.e. if $\lambda \in Y^{(m)}$ and $\mu \in Y^{(n)}$ then $\lambda\mu \in Y^{(m+n)}$

2.5.3 Proposition.

The ring Y is generated as a polynomial ring by the Young diagrams with a single column.

Proof. The proof goes by induction on the number of columns and the number of cells in the final column. The empty diagram can be thought of as a single column with no cells. It is obvious that any Young diagram with a single columns can be written as a polynomial in the Young diagrams with one column.

Now, assume that we know the result for all Young diagrams with at most m columns and fewer than k cells in the last column. Write $\lambda^\vee = (\lambda_1^\vee, \cdots, \lambda_m^\vee)$ with $\lambda_m^\vee = k$. Let μ be the Young diagram obtained from λ by removing the last column. Since μ has $m-1$ columns, there is an expression for μ in terms of the diagrams with a single column. Certainly λ is a summand of $\mu\, c_k$. Since the only way to add k cells to μ to obtain λ is to add the ith cell of c_k to the ith row of μ, the scalar $a_{\mu\, c_k}^{\lambda}$ must be 1. Following the rules of strict expansion all the other summands of $\mu\, c_k$ must have at most m columns and at most $k-1$ cells in the mth column. They, therefore, have an expression in terms of the diagrams with a single column. Thus we have an expression for λ in terms of the diagrams with a single column.

■

2.5.4 Remarks.

Let c_i be the diagram with a single column of i cells. There is a formula which will express any Young diagram, λ, as a polynomial in the c_i, known as the Giambelli formula.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a Young diagram with $\lambda_1 = m$ then $\lambda^\vee$ will have m rows and the formula for λ in terms of the c_i is given by

$$\lambda = \begin{vmatrix} c_{\lambda_1^\vee} & c_{\lambda_1^\vee+1} & \cdots & c_{\lambda_1^\vee+m-1} \\ c_{\lambda_2^\vee-1} & c_{\lambda_2^\vee} & \cdots & c_{\lambda_2^\vee+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\lambda_m^\vee-m+1} & c_{\lambda_m^\vee-m+2} & \cdots & c_{\lambda_m^\vee} \end{vmatrix} .$$

As there is an obvious symmetry between the rows and columns, it is not too surprising that there is a similar formula for λ in terms of the Young diagrams with one row. Let d_i denote the Young diagram with one row of i cells, then

$$\lambda = \begin{vmatrix} d_{\lambda_1} & d_{\lambda_1+1} & \cdots & d_{\lambda_1+k-1} \\ d_{\lambda_2-1} & d_{\lambda_2} & \cdots & d_{\lambda_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ d_{\lambda_k-k+1} & d_{\lambda_k-k+2} & \cdots & d_{\lambda_k} \end{vmatrix} .$$

2.5.5 Notation.

We will denote the ring of polynomials in an infinite number of indeterminates over Λ, by $\mathcal{R}_\infty$.

$$\mathcal{R}_\infty = \Lambda[c_1, c_2, \cdots, c_i, \cdots] .$$

Note that since they are both freely generated Λ-algebras on a countably infinite set of generators, $\mathcal{R}_\infty$ is isomorphic to $\mathcal{C}^+$. We shall return to the relationship between these two rings in Chapter 4.

2.5.6 Lemma.

Let c_i be of weight i. The ring $\mathcal{R}_\infty$ is graded by weighted degree. (For definitions of weight and weighted degree see Definition 5.2.12.)

Proof. Define $\mathcal{R}_\infty^{(n)}$ to be the linear space generated by monomials of weighted degree n. Every monomial is in $\mathcal{R}_\infty^{(n)}$ for some n. Every polynomial in $\mathcal{R}_\infty$ is a linear combination of these monomials, therefore,

$$\mathcal{R}_\infty = \bigoplus_{n=0}^{\infty} \mathcal{R}_\infty^{(n)} .$$

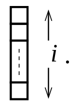
Weighted degree behaves additively under multiplication of monomials and it follows that $\mathcal{R}_\infty$ is graded. ∎

2.5.7 Proposition.

The ring Y is isomorphic to R_∞.

Proof. Define an algebra homomorphism $f : \mathcal{R}_\infty \to Y$ by

$$f \;:\; c_i \quad \longmapsto$$



This is surjective by Proposition 2.5.3.

By definition, $f : \mathcal{R}_\infty^{(n)} \to Y^{(n)}$. The Young diagrams with n cells form a linear basis of $Y^{(n)}$. We know that the monomials of weighted degree n are a linear basis for $\mathcal{R}_\infty^{(n)}$. Since f is surjective they must span $Y^{(n)}$. By Lemma 2.5.8, we see that this spanning set has cardinality equal to the the number of partitions of n. Therefore, the images of the monomials must be a linear basis for $Y^{(n)}$ and hence, f must be injective. ∎

2.5.8 Lemma.

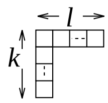
Let c_i have weight i. The number of monomials of weighted degree n in $\mathcal{R}_\infty$ is equal to the number of partitions of n.

Proof. Let $c_{i_1}^{j_1} c_{i_2}^{j_2} \cdots c_{i_m}^{j_m}$ be a monomial of weighted degree n. We can assume, without loss of generality, that $i_1 > i_2 > \cdots > i_m$. The monomial is of weighted degree n if and only if $\sum_{k=1}^{m} j_k i_k = n$. This, however, uniquely describes the partition of n with j_k columns of length i_k. This establishes a one to one relationship between the partitions of size n and the monomials of weighted degree n. ∎

2.5.9 Remarks.

We next examine the product structure of Y in more detail. Consider the product of a Young diagram with a single column and one with a single row. What Young diagrams can be obtained by applying the rules of strict expansion? If we add the cells from the single row to the single column each cell must be placed in a different column. Hence, there are only two possibilities; one cell is added to the first column and all the others start new columns or all the cells from the single row start a new column. The corresponding diagrams have the shape of a hook.

For $k, l > 0$, write $\mu_{k,l} \in Y$ for the Young diagram below, with $k+l-1$ cells,



Then taking $c_0 = d_0 = 1 \in Y$,

$$c_k d_l = \begin{cases} c_k & \text{if } l = 0 \\ d_l & \text{if } k = 0 \\ \mu_{k+1,l} + \mu_{k,l+1} & \text{otherwise.} \end{cases}$$

Note that $\mu_{k,1} = c_k$ and that $\mu_{1,l} = d_l$.

2.5.10 Proposition.[We]

Let

$$C(X) = \sum_{k=0}^{\infty} (-1)^k c_k X^k \qquad \text{and} \qquad D(X) = \sum_{l=0}^{\infty} d_l X^l$$

be formal power series with coefficients in Y. These series satisfy the relation

$$C(X)D(X) = 1.$$

Proof. Let $C(X)D(X) = \sum_{m=0}^{\infty} a_m X^m$ with $a_m = \sum_{k=0}^{m} (-1)^k c_k d_{m-k}$. Then $a_0 = c_0 d_0 = 1$. For $m > 0$,

$$\begin{aligned} a_m &= \sum_{k=0}^{m} (-1)^k c_k d_{m-k} \\ &= d_m + \sum_{k=1}^{m-1} \Big((-1)^k (\mu_{k+1,m-k} + \mu_{k,m-k+1}) \Big) \quad + \quad (-1)^m c_m \end{aligned}$$

$$
\begin{aligned}
&= d_m + \sum_{k=2}^{m}(-1)^{k-1}\mu_{k,m-k+1} \quad + \quad \sum_{k=1}^{m-1}(-1)^{k}\mu_{k,m-k+1} \quad + \quad (-1)^m c_m \\
&= d_m + (-1)^{m-1}\mu_{m,1} + (-1)\mu_{1,m} + (-1)^m c_m \\
&\qquad\qquad + \sum_{k=2}^{m-1}(-1)^{k-1}(\mu_{k,m-k+1} - \mu_{k,m-k+1}) \\
&= d_m - d_m + (-1)^{m-1}(c_m - c_m) - 0 \\
&= 0.
\end{aligned}
$$

■

2.5.11 Definition.

Let $\mathcal{R}_N$ be the quotient ring

$$\mathcal{R}_N = \mathcal{R}_\infty / \langle c_k = 0\,;\, \forall k > N \rangle \ .$$

It is obvious that $\mathcal{R}_N \cong \Lambda[c_1, \cdots, c_N]$.

We will denote the quotient homomorphism by $p_N : \mathcal{R}_\infty \to \mathcal{R}_N$.

To interpret Proposition 2.5.10 in $\mathcal{R}_N$, set $C_N(X)$ to be the image of $C(X)$ in the quotient ring. Then

$$C_N(X) = 1 - c_1 X + \cdots + (-1)^N c_N X^N.$$

This polynomial can be formally factorised;

$$C_N(X) = \Pi_{i=1}^{N}(1 - x_i X)\,.$$

Thus c_k is the kth elementary symmetric function in $\{x_i\}_{i=1}^N$. In particular

$$c_1 = \sum_{i=1}^{N} x_i\,.$$

The ring $\mathcal{R}_N$ can, therefore, be thought of as the ring of symmetric polynomials in $\{x_i\}_{i=1}^N$. In this interpretation, d_l is the lth complete symmetric polynomial, i.e. the sum of all the monomials of degree l.

We shall see C_N again in Section 4.9 and in Proposition 5.2.15.

2.5.12 Proposition.

The ring $\mathcal{R}_N$ is isomorphic to the quotient, Y_N, of the ring Y, given by setting all Young diagrams with more than N rows equal to zero.

Proof. Recall that we have an isomorphism $f : \mathcal{R}_\infty \to Y$. Let I denote the image, in Y, of the ideal in $\mathcal{R}_\infty$ generated by the c_i for $i > N$. Let $\mathcal{I}$ be the ideal given by setting all Young diagrams with more than N rows equal to zero. We need to show that these two ideals are the same. It is clear that $I \subseteq \mathcal{I}$, since c_i has more than N rows if $i > N$. To show that $\mathcal{I} \subseteq I$, consider the Giambelli polynomial of λ, in terms of the c_i. If λ has more than N rows then $\lambda_1^\vee > N$ and so the entries in the top row of the matrix are all elements of I. Expanding by the top row we see that $\lambda \in I$ as required.

■

2.5.13 Remarks.

Proposition 2.5.10 is used extensively at the end of Chapter 4, to find an alternative generating set for $\mathcal{C}^+$ to that of Turaev (see Theorem 2.3.7). The generators are elements of $\mathcal{C}^+$ which correspond to the power sums $\sum_i x_i^m$, in $\mathcal{R}_N$. It is well known that these generate the ring of symmetric functions. The expression for the mth power sum in $\mathcal{C}^+$ will be a linear combination of the closures of m braids on m strings.

Chapter 3

Quantum group invariants.

3.1 Introduction.

Quantum groups were introduced by Drinfel'd [D1] and Jimbo [Ji1]. They are certain types of Hopf algebra obtained from the universal enveloping algebras of semi-simple Lie algebras by a 1-parameter deformation.

The first part of the chapter will discuss Hopf algebras and the construction of quantum groups for generic values of a parameter q. We will then go on to investigate how link invariants are constructed from quantum groups.

We leave discussion of quantum groups at a root of unity and 3-manifold invariants until Chapter 5.

3.2 Hopf algebras.

3.2.1 Definition.

A coalgebra is a triple (C, Δ, ϵ) where C is a vector space over a field k with linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$, such that the diagrams in Figure 3.1 commute. We say that the coalgebra is coassociative and counital. The map Δ is called the *comultiplication* and the map ϵ is called the *counit*

The tensor product $C \otimes C$ is generated by the elements $x \otimes y$, where x and y are elements of C. Therefore, if $z \in C \otimes C$, we can express z in terms of elements

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\Delta \downarrow & & \downarrow 1 \otimes \Delta \\
C \otimes C & \xrightarrow{\Delta \otimes 1} & C \otimes C \otimes C
\end{array}
$$

$$
\begin{array}{ccccc}
 & & C & & \\
 & \swarrow & \downarrow \Delta & \searrow & \\
k \otimes C & \xleftarrow{\epsilon \otimes 1} & C \otimes C & \xrightarrow{1 \otimes \epsilon} & C \otimes k
\end{array}
$$

Figure 3.1: The comultiplication is coassociative and counital.

of this form,

$$z = \sum_i x_i \otimes y_i \,.$$

This is known as Sweedler's notation, [Sw], for elements of $C \otimes C$. We will use it often in this chapter. Let $x \in C$, then $\Delta(x) = \sum_i x_i' \otimes x_i''$.

A *bialgebra* is a vector space, A, which has both algebra and coalgebra structure, i.e. it has a multiplication, unit, comultiplication and counit.

Let A be an algebra, with multiplication μ and unit η, and let C be a coalgebra as above. Let f,$g \in \mathrm{Hom}(C, A)$. We define the convolution product as follows

$$f * g = \mu \circ (f \otimes g) \circ \Delta \,,$$

or in the Sweedler notation

$$f * g(x) = \sum_i f(x_i')g(x_i'') \,.$$

Let $(A, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. An anti-automorphism S of A is called an *antipode* if

$$S * 1 = 1 * S = \eta \circ \epsilon$$

While not all bialgebras have an antipode, if the antipode exists it can be shown to be unique.

A *Hopf algebra* is a bialgebra with antipode.

3.2.2 Example.

Let $\mathbf{g}$ be a Lie algebra with universal enveloping algebra $U(\mathbf{g})$, then, $U(\mathbf{g})$ is a Hopf algebra. The antipode, comultiplication and counit are defined by

$$\left.\begin{array}{l} S(g) = -g \\ \Delta(g) = g \otimes 1 + 1 \otimes g \\ \epsilon(g) = 0 \end{array}\right\} \quad \forall g \in \mathbf{g}\,.$$

3.2.3 Definitions.

A Hopf algebra A is called *quasi-triangular* (or, sometimes, quasi-cocommutative) if there exists an invertible element R of the algebra $A \otimes A$ such that for all $x \in A$, we have

$$\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1}\,.$$

Here $\Delta^{\mathrm{op}} = \tau_{A,A} \circ \Delta$ and $\tau_{A,A}$ is the flip map

$$\begin{array}{rcl} \tau \ : \ A \otimes A & \to & A \otimes A \\ \sum x_i^{'} \otimes x_i^{''} & \mapsto & \sum x_i^{''} \otimes x_i^{'}\,. \end{array}$$

Such an element is called a *universal R-matrix.*

A quasi-triangular Hopf algebra is *braided* if the universal R-matrix satisfies the relations

$$(\Delta \otimes 1)R = R_{13}R_{23} \tag{3.1}$$

$$(1 \otimes \Delta)R = R_{13}R_{12} \tag{3.2}$$

where, if $R = \sum_i s_i \otimes t_i$ in Sweedler notation,

$$R_{13} = \sum_i s_i \otimes 1 \otimes t_i\,,$$

$$R_{12} = \sum_i s_i \otimes t_i \otimes 1$$

and

$$R_{23} = \sum_i 1 \otimes s_i \otimes t_i\,.$$

Set u to be

$$u = \sum_i S(t_i)s_i\,.$$

This element is invertible with

$$u^{-1} = \sum_i t_i S^2(s_i)\,.$$

The element u has the property that for any element $a \in A$,

$$uau^{-1} = S^2(a)\,.$$

A *ribbon Hopf algebra* is a quasi-triangular Hopf algebra, A, with a central element $y \in A$ such that

$$y^2 = uS(u)\,, \qquad S(y) = y\,, \qquad \epsilon(y) = 1$$

and

$$\Delta(y) = (R_{21}R_{12})^{-1}(y \otimes y)\,.$$

The element uy^{-1} also has the following properties;

$$(uy^{-1})a(uy^{-1})^{-1} = S^2(a) \qquad \forall a \in A\,, \tag{3.3}$$

(which follows immediately from the fact that the relation holds for u and that y is central) and

$$\sum_i s_i y u^{-1} t_i = \sum_i t_i u y^{-1} s_i\,. \tag{3.4}$$

Let A be a quasi-triangular Hopf algebra. We denote by Rep$A$ the category of finite dimensional linear representations of A. Its objects are A-modules and its morphisms are module homomorphisms. The comultiplication of A induces a tensor product on Rep$A$. The action of $a \in A$ on $V \otimes W$ is given by

$$a \cdot (v \otimes w) = \sum b_i v \otimes c_i w\,,$$

where $\Delta(a) = \sum b_i \otimes c_i$. The antipode allows us to turn the dual linear spaces into A-modules. Let $V^* = \mathrm{Hom}(V, \Lambda)$. The action of A on V^* is given by

$$a \cdot (f(x)) = f(S(a) \cdot x) \qquad \forall x \in V\,.$$

There is a canonical isomorphism $V \cong V^{**}$, with

$$a \cdot x^{**} = S^2(a) \cdot x = (uau^{-1}) \cdot x\,.$$

3.2.4 Definition.

Let A be a ribbon Hopf algebra. We define the *quantum dimension* of an object V in Rep$A$ to be the trace of the linear operator $f : V \to V$, where $f : x \mapsto uy^{-1} \cdot x$. In general, if $g : V \to V$ we define $tr_q(g)$ to be the trace of $x \mapsto (uy^{-1}) \cdot g(x)$ i.e.

$$tr_q(g) = tr(f \circ g)\,.$$

3.2.5 Theorem.

The universal R-matrix satisfies the equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,.$$

This is known as the quantum Yang-Baxter equation.

Proof.

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes 1)(R) \qquad \text{by equation 3.1} \\ &= (\Delta^{\mathrm{op}} \otimes 1)(R)R_{12} \\ &= (\tau_{AA} \otimes 1)(\Delta \otimes 1)(R)R_{12} \\ &= (\tau_{AA} \otimes 1)(R_{13}R_{23})R_{12} \\ &= (R_{23}R_{13})R_{12}\,. \end{aligned}$$

■

3.2.6 Comment.

More information about Hopf algebras can be found in [Ab] and [Sw].

3.3 Representing Lie algebras

In this section we provide a rough guide to the representation theory of semi-simple complex Lie algebras. Although we will not explicitly calculate the representations of $sl(N)$, the terminology developed will be used in the subsequent sections. The details can be found in several books. The approach we follow here is that of J.E. Humphreys [H].

Let $\mathbf{g}$ be a semi-simple Lie algebra. We can find a maximal abelian sub-algebra, $\mathbf{h}$ which acts diagonally on $\mathbf{g}$, under the adjoint representation. We call such a sub-algebra a *Cartan sub-algebra*. We, therefore, have a decomposition of $\mathbf{g}$ with respect to this action,

$$\mathbf{g} = \mathbf{h} \oplus (\bigoplus_{\alpha} \mathbf{g}_{\alpha}) \,,$$

where $\mathbf{g}_{\alpha} = \{X \in \mathbf{g} : H.X = \alpha(H)X, \ \forall H \in \mathbf{h}\}$. We call α a root of $\mathbf{g}$. Every set of roots has a *base*, a subset of roots $\{\alpha_i\}$ for which every root, β, can be written as a linear combination of the roots in the base, with coefficients either all positive integers or all negative integers. If the coefficients are positive we call β a positive root. Otherwise β is a negative root.

Let V be a finite dimensional $\mathbf{g}$-module. Since $\mathbf{h}$ is abelian, it will act diagonally on V and V will decompose in an analogous way to $\mathbf{g}$,

$$V = \bigsqcup V_{\beta}$$

where $V_{\beta} = \{v \in V : H.v = \beta(H)v \ \forall H \in \mathbf{h}\}$. Now $\mathbf{g}_{\alpha}$ takes V_{β} onto $V_{\beta+\alpha}$. Let $v \in V_{\beta}$. We call v *maximal* if $v \in \mathrm{Ker}\mathbf{g}_{\alpha}$ for all positive roots α. Let $U(\mathbf{G})$ be the universal enveloping algebra of the Lie algebra $\mathbf{g}$. If $V = U(\mathbf{G}).v$ we call v a highest weight vector, with highest weight β. Obviously, in this case V is a simple module. We call β dominant integral if $\beta(H) > 0$ for all $H \in \mathbf{h}$.

3.3.1 Theorem.

For each weight β, there is an irreducible finite dimensional representation V_{β}, with highest weight β if and only if β is dominant integral. ■

The proof of this Theorem can be found in [H].

3.4 The Lie algebra $sl(N)$.

Firstly, we define the Lie algebra $sl(N)$ and its universal enveloping algebra. We then describe the ring of (complex) representations of the universal enveloping algebra of $sl(N)$. We will not calculate explicit representations but, we will identify an index set for the irreducible representations and state how the tensor product of two irreducibles decomposes in terms of this index set.

3.4.1 Definition.

The Lie algebra $sl(N)$ is the complex vector space of $N \times N$ matrices with zero trace. The Lie bracket is given by $[X, Y] = XY - YX$.

3.4.2 Theorem.[Se]

The universal enveloping algebra $U(sl(N))$ is generated by $\{X_i, Y_i, H_i\}_{i=1}^N$, with the relations

$$H_i H_j = H_j H_i \,, \qquad X_i Y_j - Y_j X_i = \delta_{ij} H_i \,,$$

$$H_i X_j - X_j H_i = a_{ij} X_j \,, \quad H_i Y_j - Y_j H_i = -a_{ij} Y_j \,.$$

Also, for $i \neq j$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} X_i^k X_j X_i^{1-a_{ij}-k} = 0$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} Y_i^k Y_j Y_i^{1-a_{ij}-k} = 0$$

where a_{ij} is the (i, j)th entry in the Cartan matrix. For $sl(N)$ the Cartan matrix is given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In this presentation of the universal enveloping algebra, the elements H_i generate the Cartan sub-algebra as defined in section 3.3.

3.4.3 Theorem.[FH]

The irreducible representations of the universal enveloping algebra $U(sl(N))$ are indexed by the Young diagrams with at most N rows. The irreducible representations V_λ and V_μ are isomorphic if and only if $\lambda_i - \mu_i$ is a constant independent of i for $1 \leq i \leq N$. We have complete reducibility of representations. ■

3.4.4 Remarks.

This theorem implies that if two irreducible representations are isomorphic then their Young diagrams must differ by a collection of columns with N cells. The irreducible modules are, therefore, uniquely indexed by Young diagrams with fewer than N rows.

The fundamental representation of $sl(N)$ (i.e. that of dimension N) is indexed by the Young diagram with one cell, $\square$.

The problem of finding the coefficients for the decomposition of tensor products is known as the *Clebsch-Gordan* problem. For $sl(N)$, the decomposition of the tensor product is known. Let V_λ and V_μ be simple modules indexed by the Young diagrams λ and μ respectively. Then

$$V_\lambda \otimes V_\mu = \sum a^\nu_{\lambda\mu} V_\nu$$

where the $a^\nu_{\lambda\mu}$ are the Littlewood-Richardson coefficients described in Chapter 2, with the assumption that $V_\nu = 0$ if ν has more than N rows.

3.4.5 Proposition.

Let V_λ be some irreducible representation of $sl(N)$, with $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$. Setting $a_i = \lambda_i - \lambda_{i+1}$ for $i < N$ and $a_N = \lambda_N$, we have

$$\lambda = (\sum_{i=1}^{N} a_i, \sum_{i=2}^{N} a_i, \cdots, \sum_{i=N}^{N} a_i)\,.$$

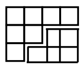
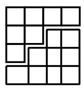
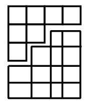
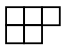
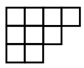
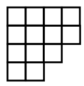
Then $(V_\lambda)^* \cong V_{\lambda^*}$, where

$$\lambda^* = (\sum_{i=1}^{N-1} a_i, \sum_{i=1}^{N-2} a_i, \cdots, a_1)$$

Pictorially, λ^* is the Young diagram which remains if you remove λ from a $\lambda_1 \times N$ grid of cells and rotate the picture through π. ■

3.4.6 Example.

For $\nu = (4, 2, 1)$, we give ν^* for $N = 3$, 4 and 5.

$N = 3$	$N = 4$	$N = 5$
$\nu^* = (3, 2)$	$\nu^* = (4, 3, 2)$	$\nu^* = (4, 4, 3, 2)$
		
$\nu^* \quad =$ 	$\nu^* \quad =$ 	$\nu^* \quad =$ 

3.4.7 Definition.

Let $[V_\lambda]$ denote the isomorphism class of the representation V_λ. Form the free abelian group, $\mathcal{R}_N$, on these classes, quotiented out by the relations $[V] = [V'] \oplus [V'']$ whenever V is isomorphic to $V' \oplus V''$. The complete reducibility of the representations of $sl(N)$ implies that $\mathcal{R}_N$ is free abelian. We give $\mathcal{R}_N$ a ring structure by defining the product to be the tensor product of representations.

3.4.8 Comment.

Note that we have a minor notation problem here. The representation ring $\mathcal{R}_N$ defined in this chapter is isomorphic to a quotient of the ring $\mathcal{R}_N$ defined in Chapter 2, the extra relation being $c_N = 1$. However, this extra condition doesn't make any material difference to any of the results in Chapter 2, we can just substitute $c_N = 1$ into any formula. Therefore, from now on we shall take $\mathcal{R}_N$ to be the quotient of $\mathcal{R}_\infty$ which sets $c_k = 0$, for $k > N$ and $c_N = 1$.

Note, however, that with this adjustment the grading of $\mathcal{R}_N$ by the number of cells is destroyed.

3.4.9 Proposition.

The ring $\mathcal{R}_N$ is isomorphic to the ring $Y_N / < c_N = 1 >$, where Y_N is as defined in Chapter 2.

Proof. This just a restatement of the representation theory in terms of Young diagrams.

$\blacksquare$

3.5 The quantum enveloping algebra.

In this section we shall define $U_q(sl(N))$, the quantum enveloping algebra of $sl(N)$ for generic values of q. We shall consider the situation when q is primitive root of unity in Chapter 5. The construction given here follows that given by Drinfel'd in [D1] and independently by Jimbo in [Ji1] for a general semi-simple Lie algebra.

We then describe its irreducible representations. Results of Rosso [R] and Lusztig [L] showed that for a generic value of q the representations are deformations of those for $U(sl(N))$.

The subject of quantum groups is covered extensively in Kassel's book [Ka].

3.5.1 Definition.

For a parameter h set $q = e^h$. Let $s = e^{h/2}$. Let $n \in \mathbb{Z}$. We define the associated *quantum integer* $[n]$ to be

$$[n] = \frac{s^n - s^{-n}}{s - s^{-1}} .$$

The *quantum binomial* is then given by the formula

$$\left[\begin{array}{c} n \\ k \end{array} \right] = \frac{[n]!}{[k]![n-k]!}$$

where $[n]! = [n][n-1]\cdots[1]$.

Let $G = \{X_i, Y_i, K_i\}_{i=1}^N$. The *quantum enveloping algebra* $U_q(sl(N))$ is the quotient of the free algebra over G by the following relations,

$$K_iK_j = K_jK_i\,, \qquad X_iY_j - Y_jX_i = \delta_{ij}\tfrac{K_i - K_i^{-1}}{s - s^{-1}}\,,$$

$$K_iX_j = s^{a_{ij}}X_jK_i\,, \qquad K_iY_j = s^{-a_{ij}}Y_jK_i\,,$$

and for $i \neq j$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix} X_i^k X_j X_i^{1-a_{ij}-k} = 0\,,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix} Y_i^k Y_j Y_i^{1-a_{ij}-k} = 0$$

where a_{ij} is the (i,j)th entry in the Cartan matrix.

Note that setting $K_i = \exp(hH_i/2)$ and taking $s = e^{h/2}$ then these relations become the relations for the universal enveloping algebra $U(sl(N))$, as given in Definition 3.4.2, when $h \to 0$.

3.5.2 **Proposition.**[D2, Ji2, T1]

The quantum enveloping algebra of $sl(N)$ is a quasi-triangular Hopf algebra. The antipode, comultiplication and counit are defined as follows,

$$\begin{array}{lll} S(X_i) = -K_iX_i\,, & \epsilon(X_i) = 0\,, & \Delta(X_i) = X_i \otimes 1 + K_i^{-1} \otimes X_i\,, \\ S(Y_i) = -Y_iK_i^{-1}\,, & \epsilon(Y_i) = 0\,, & \Delta(Y_i) = Y_i \otimes K_i + 1 \otimes Y_i\,, \\ S(K_i) = K_i^{-1}\,, & \epsilon(K_i) = 1\,, & \Delta(K_i) = K_i \otimes K_i\,. \end{array}$$

The R-matrix for the fundamental representation is given by

$$R = s^{-\frac{1}{N}} \left(\sum_{i \neq j} e_{ii} \otimes e_{jj} + s \sum_i e_{ii} \otimes e_{ii} + (s - s^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji} \right)$$

where e_{ij} is the $N \times N$ elementary matrix with (k,l)th entry $\delta_{ik}\delta_{jl}$ and in which $s^{-1/N}$ is taken to be $e^{-h/2N}$. ■

3.5.3 **Remarks.**

Calculations of the R-matrix for the fundamental representation can be found in [D2, Ji2]. The version given here is that of Drinfel'd.

3.5.4 Theorem.[Ji2, L, R]

Any simple integrable highest weight module V of $U(sl(N))$ admits a quantum deformation: there exists a simple $U_q(sl(N))$ module $\tilde{V}$ such that $\tilde{V}$ specialises to V as q tends to 1; the dimensions of the weight spaces of $\tilde{V}$ (with respect to K_i) are the same as those of the corresponding weight spaces of V (with respect to the H_i). ∎

3.5.5 Remarks.

This result is proved for Lie algebras of type A by Rosso (The $sl(2)$) case is dealt with by Jimbo). The general reslut is the work of Lusztig.

We therefore have an isomorphism between the representation ring of the Lie algebra and that of the quantum group. All the properties of the representation ring of the Lie algebra will, therefore, carry through to the quantum group.

3.6 The quantum group link invariants.

3.6.1 Introduction.

We first give the instructions for building an invariant given a link diagram , some $U_q(sl(N))-$modules and some module homomorphisms. Only then will we deal with the technicalities of determining the homomorphisms and the relations between them.

3.6.2 A basic construction method.

Let $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_k$ be a framed link, with diagram D. The diagram can be regularly isotopped so that it lies in levels. In each level all but two of the strings will run through parallel. One pair of strings will either cross over, form a cap or form a cup.

We colour L by assigning a representation of $U_q(sl(N))$, V_i, to each component L_i. At any point in the diagram where a string is running from top to bottom of a level it is coloured with the representation colouring that component. If the string has the reverse orientation it is coloured by the dual module.

Each of these layers will determine a module homomorphism, which will be built up from the elementary tangles of crossings, cups and caps. The pictorial rules in Figure 3.2 describe how we can compose them. Let T be a coloured tangle, i.e. a tangle with an irreducible module assigned to each of its components. We denote by $J(T)$ the module homomorphism obtained from the composition of the module homomorphism determined by each elementary layer. Note we can only define $J(TS)$ if the colourings are compatible. In Figure 3.2 we require $l = m$ and $V_i' = W_i$ for $i = 1 \dots m$ to be able to compose the module homomorphisms in this way.

$$\boxed{T} \qquad J(T) : V_1 \otimes \cdots \otimes V_k \longrightarrow V_1' \otimes \cdots \otimes V_l'$$

$$\boxed{S} \qquad J(S) : W_1 \otimes \cdots \otimes W_m \longrightarrow W_1' \otimes \cdots \otimes W_n' \,,$$

$$\begin{array}{|c|}\hline S \\ \hline T \\ \hline \end{array} \qquad J(TS) = J(T) \circ J(S) \,,$$

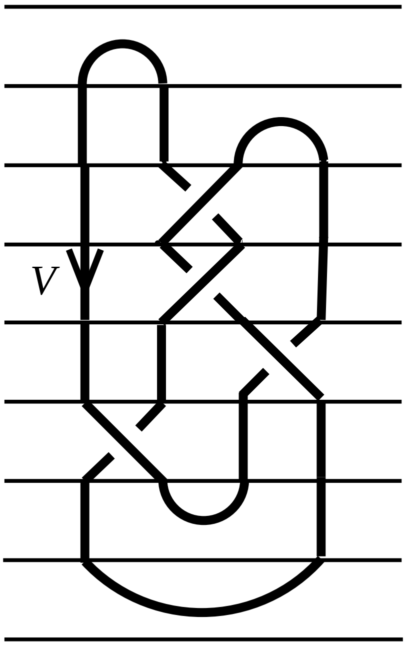
$$\begin{array}{|c|c|}\hline S & T \\ \hline \end{array} \qquad J(T \otimes S) = J(T) \otimes J(S) \,.$$

Figure 3.2: The pictorial composition of the module homomorphisms.

Applying these to a link diagram we can construct a module homomorphism, from the scalars to the scalars, dependent on the diagram and the colouring.

3.6.3 Example.

Figure 3.3 shows a figure-eight knot, K, coloured by the module V and arranged into layers. Each layer determines a module homomorphism from the module at the top of the layer to the module at the bottom, as indicated on the right hand side. Taking the compositions of these, the knot K determines a module homomorphism from the scalars at the top of the picture to the scalars at the bottom.



$$\begin{array}{c}
\mathbb{C}[s^{\pm 1/N}] \\
\downarrow \\
V \otimes V^* \\
\downarrow \\
V \otimes V^* \otimes V \otimes V^* \\
\downarrow \\
V \otimes V \otimes V^* \otimes V^* \\
\downarrow \\
V \otimes V^* \otimes V \otimes V^* \\
\downarrow \\
V \otimes V^* \otimes V^* \otimes V \\
\downarrow \\
V^* \otimes V \otimes V^* \otimes V \\
\downarrow \\
V^* \otimes V \\
\downarrow \\
\mathbb{C}[s^{\pm 1/N}]
\end{array}$$

Figure 3.3: The figure-eight knot arranged in layers.

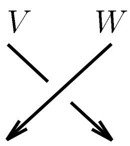
3.6.4 Definitions.

We define the elementary module homomorphisms for each of the elementary tangles. The definitions given here are those of [RT2]. A detailed description of the $U_q(sl(2))$ case can be found in [KM].

The straight string will correspond to the identity homomorphism of the colour.

$$\Big\downarrow V \quad = \quad \mathrm{Id}_V$$

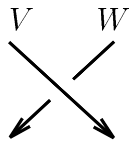
Let $R_{V,W}$ be the map assigned to the positive crossing



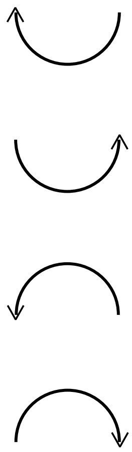
where

$$\begin{aligned} R_{V,W} = \tau_{V,W} R : V \otimes W &\rightarrow W \otimes V\,. \\ x \otimes y &\mapsto \sum t_i y \otimes s_i x \end{aligned}$$

The negative crossing coloured by V and W (as shown below) will correspond to the homomorphism $R^{-1}_{W,V}$.



Although neither τ_{VW} nor R are module homomorphisms, it turns out that their composite is. The cup and cap homomorphisms are given in Figure 3.4, with the assumption that the string oriented downward is coloured by V.



$$\cup : V^* \otimes V \to \mathbb{C}[s^{\pm 1/N}]$$
$$f \otimes v \mapsto f(v)$$

$$\cup : V \otimes V^* \to \mathbb{C}[s^{\pm 1/N}]$$
$$v \otimes f \mapsto f((uy^{-1}).v)$$

$$\cap : \mathbb{C}[s^{\pm 1/N}] \to V \otimes V^*$$
$$1 \mapsto \textstyle\sum_i e_i \otimes e^i$$

$$\cap : \mathbb{C}[s^{\pm 1/N}] \to V^* \otimes V$$
$$1 \mapsto \textstyle\sum_i e^i \otimes (yu^{-1})e_i$$

Figure 3.4: The cup and cap homomorphisms.

3.6.5 Theorem.[RT1]

The maps R_{VW}, R^{-1}_{VW}, $\cup$, $\cup$, $\cap$ and $\cap$ are module homomorphisms. They satisfy the identities described pictorially by the Reidemeister moves RII and $RIII$.

They also satisfy the pictorial identities of Figure 3.5 (and those obtained from Figure 3.5 by changing the sign of the crossing) with all possible colourings and orientations.

The relations of Figure 3.6 also hold for all possible choices of orientation.

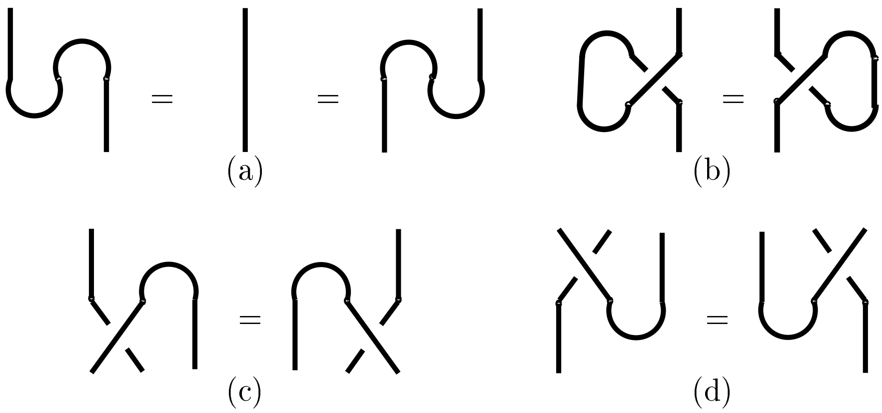


Figure 3.5: The pictorial isotopy relations between the homomorphisms.

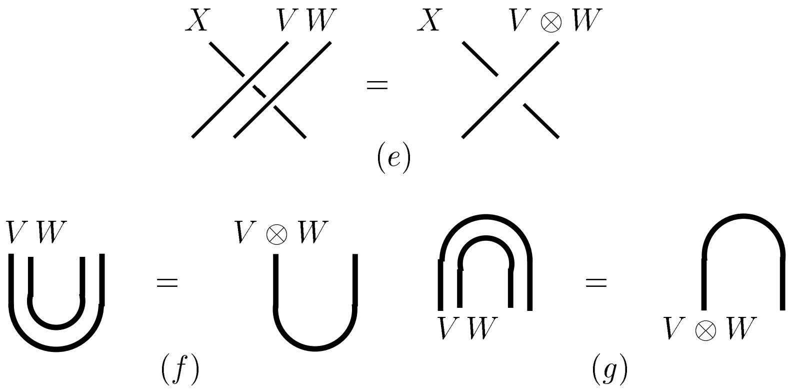
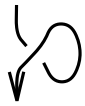


Figure 3.6: Further relations between the module homomorphisms.

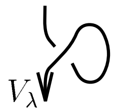
Let $J(L; V_1, V_2, \cdots, V_k)$ denote the module homomorphism determined by the link L, coloured by V_1, ..., V_k. Then $J(L; V_1, V_2, \cdots, V_k)$ is a regular isotopy invariant. ∎

3.6.6 Comment.

If the tangle T below is coloured by a simple module then, by Schur's Lemma, it must represent either the zero homomorphism or a scalar multiple of the identity.

$T \quad =$ 

The Whitney trick (shown in Figure 1.1) implies that the homomorphism determined by T must be a non-zero isomorphism, hence



$=\quad f_\lambda$

for some non-zero scalar f_λ.

3.6.7 Proposition.

Let T be a coloured (n, n) tangle with associated module homomorphism $J(T)$. Let L be a coloured link which is the closure of T. Then

$$J(L) = \mathrm{tr}_q J(T)\,.$$

where tr_q is as defined in Definition 3.2.4. ■

3.6.8 Proposition.

Let L_i be a component of an oriented link L. Let V_i be the colour assigned to L_i. Define $\overline{L}$ to be the link L with the orientation of L_i reversed and the colour V_i replaced by V_i^*.

$$J(L; V_1, \ldots, V_{i-1}, V_i, V_{i+1}, \ldots V_k) = J(\overline{L}; V_1, \ldots, V_{i-1}, V_i^*, V_{i+1}, \ldots V_k)$$

■

3.7 Quantum invariants and the Homfly polynomial.

In this section we give an explicit formula for the $U_q(sl(N))$ invariants in terms of the framed Homfly polynomial $\mathcal{X}$, when all components are coloured by the irreducible representation $V_{\square}$.

3.7.1 Theorem.[T1]

The invariant $J(L; V_{\square}, \cdots, V_{\square})$ is given as a function of s by the framed Homfly polynomial $\mathcal{X}(L)$, evaluated at $x = s^{-1/N}$ and $v = s^{-N}$.

Proof. In this proof, we consider the universal R-matrix constructed by Drinfel'd, given in Proposition 3.5.2. The method of proof follows that of Turaev in [T1]. The notation in [T1] is not consistent with our notation. What we denote by $s = e^{h/2}$ is denoted by $-q$ in [T1]. When we write $s^{1/N}$ we shall mean $e^{h/2N}$ rather than any other Nth root of s.

The invariant $J(L; V_{\square}, \cdots, V_{\square})$ is a function of s, since it is a module endomorphism of the ring $\mathbb{C}[s^{\pm 1/N}]$, as described in Example 3.6.3.

The universal R-matrix in [D2] differs from that given in [T1] by a multiple of $e^{h/2N}$. Therefore, by [T1], the R-matrix for the fundamental representation satisfies the following relation,

$$s^{1/N} R - s^{-1/N} R^{-1} = (s - s^{-1}) \mathrm{Id}^{\otimes 2} .$$

Therefore, with the substitution $x = s^{-1/N}$, $\mathcal{X}$ and $J(L)$, will satisfy the same skein relation. We next consider the substitution for v, by comparing the values of the two invariants on the unknot with zero framing. For the framed Homfly polynomial

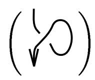
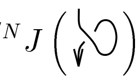
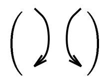
$$\mathcal{X}\left(\bigcirc\right) = \frac{v^{-1} - v}{s - s^{-1}} .$$

The $U_q(sl(N))$ invariant for the unknot is calculated in [T1] to be

$$J\left(\bigcirc\right) = \frac{s^N - s^{-N}}{s - s^{-1}} .$$

These two values agree for $v = s^{-N}$ or $v = -s^N$. We require, however, that $v \to 1$ as $h \to 0$. Since $s = e^{h/2}$, this forces the choice of substitution for v

to be $v = s^{-N}$. The curl factor for $\mathcal{X}$ is given by xv^{-1}. By Schur's Lemma, since the curl on a string coloured by $V_\square$ is an endomorphism of a simple module which commutes with all other endomorphisms, it must be a scalar multiple of the identity. Also, since a positive and a negative curl cancel each other out, the scalar for the negative curl must be the inverse of that for the positive curl. Denote by α, the scalar for the positive curl. From

$s^{1/N} J$  $- \; s^{-1/N} J$  $= \; (s - s^{-1}) J$  ,

we see that α must satisfy the quadratic relation

$$\alpha^2 - s^{-1/N}(s^N - s^{-N})\alpha - s^{-2/N} = 0\,,$$

and therefore, $\alpha = s^{N-\frac{1}{N}}$ or $\alpha = -s^{-N-\frac{1}{N}}$. Since we wish the curl factor to tend to 1 as h tends to 0, we set $\alpha = s^{N-\frac{1}{N}} = xv^{-1}$

With the substitutions $x = s^{-1/N}$ and $v = s^{-N}$, the Homfly polynomial satisfies the same skein relations as the quantum invariant $J(L)$ and the two invariants have the same value on the unknot with zero framing. Thus we have proved the theorem.

■

3.7.2 Remarks and notation.

The obvious question to ask is "Can a similar formula be found for quantum invariants of links coloured by higher dimensional representations?"

The answer is yes. In Chapter 4, we will show how to find patterns, Q_λ, for each Young diagram, λ, such that when $|\lambda| = n$ we can take Q_λ as the closure of some (n,n)-tangle T_λ which represents the projection map from $V_\square^{\otimes n}$ onto V_λ, when coloured by $V_\square$. We can then apply the above theorem to the satellite of a knot, with Q_λ as pattern, to find the quantum invariant of the knot coloured by V_λ. A feature of the construction is that Q_λ is completely independent of N. We state a version of the "Satellite Theorem" here for completeness. The proof can be found on page 89, Theorem 4.6.16.

We will denote the evaluation of the framed Homfly polynomial at $v = s^{-N}$ and $x = s^{-1/N}$ by $\mathcal{X}_N$.

3.7.3 Theorem.

For each Young diagram, λ, there exists $Q_\lambda \in \mathcal{C}^+$ for which, given a link L,

$$J(L; V_{\lambda_1}, \ldots, V_{\lambda_k}) = \mathcal{X}_N(L_1 * Q_{\lambda_1} \sqcup \ldots \sqcup L_k * Q_{\lambda_k})$$

■

Chapter 4

Idempotents of the Hecke algebra.

4.1 Introduction.

In this chapter we construct quasi-idempotent elements of the Hecke algebra following the prescription set out by A. Gyoja in [G]. We will build our idempotents from the positive permutation braids, which are a set of generators for the Hecke algebra, as described in [Jo2]. Positive permutation braids (see Definition 1.3.3) were first defined by Elrifai and Morton [EM]. Details of the relationship between them and the Hecke algebras can be found in [M2].

We will exploit the relationships between the Hecke algebras, the braid groups and the skein theory of Chapter 2 to obtain quasi-idempotent elements of the Hecke algebra expressed as linear combinations of braids. The closures of these elements in the skein of the annulus provide the patterns required for the Satellite Theorem 3.7.3. They provide the key to the translation between the skein theory and the quantum group representations.

At the end of this chapter there is some discussion of recent work by Yokota [Y]. Connections between this thesis and Yokota's work are indicated at appropriate places during this chapter.

4.2 The Hecke algebras.

4.2.1 Definitions.

The nth *Hecke algebra* (of type A), over the ring Λ (defined in Definition 2.2.1), will be denoted H_n. It has the following presentation,

$$H_n \quad = \quad \left\langle \begin{array}{c} \sigma_i \ \ : \ \ i=1,\dots n-1 \end{array} \quad \middle| \quad \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \quad : \quad |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ x^{-1}\sigma_i - x\sigma_i^{-1} = z\,, \end{array} \right\rangle ,$$

where $z = s - s^{-1}$. Notice that v doesn't appear in the presentation. However, we will see later that H_n is isomorphic to $\mathcal{S}(R_n^n)$. The indeterminate v is required to keep track of framing of these diagrams.Elements of $H_n \otimes H_m$ can be written as linear combinations of terms of the form $h_n \otimes h_m$, where $h_n \in H_n$ and $h_m \in H_m$. These terms are represented in $\mathcal{S}(R_{n+m}^{n+m})$ by the juxtaposition of a tangle in $\mathcal{S}(R_n^n)$ with a tangle in $\mathcal{S}(R_m^m)$.

If we set $v = x = s = 1$, we recover a presentation of the symmetric group algebra $\mathbb{C}S_n$. It is obvious from the presentation that H_n is isomorphic to a quotient of the group algebra ΛB_n, where B_n is the n-string braid group as described in Proposition 1.3.2.

4.2.2 Proposition.[MT]

The Hecke algebra, H_n, is isomorphic to $\mathcal{S}(R_n^n)$. As a Λ-module, it is freely generated by the $n!$ positive permutation braids. ■

4.3 Connections with the symmetric group algebras.

In this section we consider the representation theory of the symmetric group algebra, $\mathbb{C}S_n$. We then discuss the relationship between the representations of $\mathbb{C}S_n$ and those of H_n.

It is well known that the number of distinct irreducible representations of $\mathbb{C}S_n$ is equal to the number of conjugacy classes of S_n. The conjugacy classes of the symmetric group are determined by the cycle type of the permutations, therefore, the number of irreducible representations of $\mathbb{C}S_n$ is equal to the number of partitions of n. For $\mathbb{C}S_n$, we can give a direct relation between the conjugacy classes and the irreducible representations.

The following results are well known in the theory of representations of the symmetric group. Details can be found in James' book [Ja] and in the first few chapters of Fulton and Harris' book [FH].

4.3.1 Definitions.

Let D be a standard tableau of the Young diagram λ. Set

$$P_D = \{p \in S_n : p \text{ preserves the rows of } D\}$$

$$Q_D = \{q \in S_n : q \text{ preserves the columns of } D\}\,.$$

We define two elements A_D, $B_D \in \mathbb{C}S_n$ by

$$A_D = \sum_{p \in P_D} p\,, \qquad B_D = \sum_{q \in Q_D} (-1)^{l(q)} q$$

where $l(q)$ is the length of the permutation q. The *Young symmetriser*, C_D, is defined to be

$$C_D = A_D B_D \,\in\, \mathbb{C}S_n\,.$$

4.3.2 Comment.

As with the braid group in Chapter 1, we will take the product to act on the right. Thus, pq is the permutation obtained by performing p and then q.

4.3.3 Example.

Let D be the Young tableau given below,

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}\,.$$

Then P_D is generated by the set of transpositions $\{(12), (23), (34), (56)\}$ and Q_D is generated by the set $\{(15), (57), (26)\}$.

4.3.4 Lemma.

Let D and D' be different Young tableaux for the same Young diagram. We can find some permutation τ for which

$$C_D = \tau C_{D'} \tau^{-1}\,.$$

Proof. Let τ be the permutation which takes the label in a given cell of D to the label in the corresponding cell of D'. Note that τ takes the rows of D to the rows of D' and the columns of D to the columns of D'.

Let $p' \in P_{D'}$. Then $\tau p' \tau^{-1}$ is a permutation which takes the rows of D to the rows of D, i.e. $\tau p' \tau^{-1} \in P_D$. For any $p \in P_D$, $\tau(\tau^{-1} p \tau)\tau^{-1} = p$ and $\tau^{-1} p \tau$ is a permutation of the rows of D'. Hence

$$P_D = \{\tau p' \tau^{-1} : p' \in P_{D'}\}\,,$$

therefore,

$$A_D = \tau A_{D'} \tau^{-1}\,.$$

Similarly, we have that

$$B_D = \tau B_{D'} \tau^{-1}\,.$$

Hence

$$\begin{aligned} C_D &= A_D B_D \\ &= \tau A_{D'} \tau^{-1} \tau B_{D'} \tau^{-1} \\ &= \tau A_{D'} B_{D'} \tau^{-1} \\ &= \tau C_{D'} \tau^{-1}\,. \end{aligned}$$

■

4.3.5 Theorem.

The element C_D is a quasi-idempotent i.e.

$$C_D^2 = a_D C_D\,.$$

for some non-zero scalar a_D. Let $c_D = \frac{1}{a_D} C_D$, then c_D is a genuine idempotent.

The image of $\mathbb{C}S_n$ under right multiplication by c_D is a simple left module of $\mathbb{C}S_n$. Thus $\mathbb{C}S_n c_D$ is a minimal left ideal. Every minimal left ideal of $\mathbb{C}S_n$ is equal to $\mathbb{C}S_n c_D$ for some tableau D, therefore the c_D determine all the simple $\mathbb{C}S_n$ modules

Two ideals $\mathbb{C}S_n c_D$ and $\mathbb{C}S_n c_{D'}$ are isomorphic if and only if D and D' are tableaux of the same Young diagram.

The ideal $\mathbb{C}S_n c_D \mathbb{C}S_n$ is a minimal two-sided ideal. Let $\mathcal{D}_\lambda$ denote the set of tableaux for the Young diagram λ, then

$$\mathbb{C}S_n c_D \mathbb{C}S_n = \bigoplus_{D' \in \mathcal{D}_\lambda} \mathbb{C}S_n c_{D'} \,.$$

If we fix one Young tableau D_λ for each Young diagram λ with n cells we obtain a direct sum decomposition for $\mathbb{C}S_n$,

$$\mathbb{C}S_n \;=\; \bigoplus_{|\lambda|=n} \mathbb{C}S_n c_{D_\lambda} \mathbb{C}S_n \,.$$

Sketch proof. We indicate how to proceed to prove that C_D is a quasi-idempotent. First we must prove the following two statements:

1. $C_D C_{D'} = 0$ when D and D' are tableaux for different Young diagrams.
2. For all $x \in S_n$ the element $C_D x C_D$ is a scalar multiple of C_D.

The proof then proceeds as follows. Let $V_D = \mathbb{C}S_n c_D$. Then, by statement 2, $c_D V_D \subseteq \mathbb{C}S_n c_D$. Suppose that W is a sub-representation of V_D, then $c_D W$ is either 0 or $\mathbb{C}c_D$. However, if $W = \mathbb{C}c_D$ then $V_D \subseteq W$. A similar argument can be used to show that C_D^2 is a non-zero multiple of C_D.

Suppose that $\mathbb{C}S_n C_D \cong \mathbb{C}S_n C_{D'}$ and that D is a tableau for the Young diagram λ. If we multiply by C_D on the left then $\mathbb{C}S_n C_{D'}$ will be non-zero if and only if D' is also a tableau for the Young diagram λ.

The details of this proof can be found in [FH]. ■

4.3.6 Comments and notation.

The details of the proof have been omitted because we will be using an identical method to prove that we have idempotent elements of the Hecke algebra.

Let $T(\lambda)$ be the standard tableau whose cells are numbered 1 to n, from left to right, from top to bottom.

In view of Lemma 4.3.4 and Theorem 4.3.5, we will abuse notation by writing C_λ for the quasi-idempotent C_D constructed using $D = T(\lambda)$. All other idempotents for the same Young diagram will be conjugate and the irreducible representations they index will be isomorphic.

We will denote by a_λ and c_λ, respectively, the scalar and genuine idempotent associated with C_λ.

The decomposition of $\mathbb{C}S_n$ given in Theorem 4.3.5 can be restated using this notation:

$$\mathbb{C}S_n = \bigoplus_{|\lambda|=n} \mathbb{C}S_n c_\lambda \mathbb{C}S_n \,.$$

4.3.7 Proposition.[FRT]

The dimension of the simple $\mathbb{C}S_n$-module $\mathbb{C}S_n c_\lambda$ is equal to the number of standard tableaux, which by Proposition 2.4.4 can be calculated as

$$d_\lambda = \dim \mathbb{C}S_n c_\lambda = \frac{n!}{\prod \text{hook lengths}} \,.$$

Therefore the dimension of the subspace $\mathbb{C}S_n c_\lambda \mathbb{C}S_n$ is d_λ^2. ■

4.3.8 Remarks.

Details of the last result, due to J.S. Frame, G. de B. Robinson and R.M. Thrall, can be found in G.D. James book on the representation theory of the symmetric groups [Ja].

The calculation of the scalar a_λ can be found in [FH, p54]. In fact,

$$a_\lambda = \prod \text{hook lengths} \,.$$

Since a_λ is invertible in $\mathbb{C}S_n$, Theorem 4.3.5 will hold with C_λ in place of c_λ.

4.3.9 Theorem.[B, Go]

Every irreducible representation of H_n is a deformation of some irreducible representation of $\mathbb{C}S_n$. ■

4.4 Construction of the idempotent elements.

By Theorem 4.3.9, we need only to find idempotent elements of the Hecke algebra which specialise to the Young symmetrisers and we have all the irreducible representations of H_n. We first produce building blocks from which we will construct quasi-idempotent elements.

Much of the earlier part of this section is based on the paper by Morton [M2].

4.4.1 Definition.

Let $E_n(\sigma_1, \sigma_2, \cdots, \sigma_{n-1})$ be defined as follows,

$$E_n(\sigma_1, \sigma_2, \cdots, \sigma_{n-1}) = \sum_{\pi \in S_n} \omega_\pi ,$$

where ω_π is the positive permutation braid associated to the permutation π as defined in Definition 1.3.3.

4.4.2 Theorem.[M2]

For each i, we can factorise E_n in H_n, as

$$\begin{aligned} E_n &= E_n^{(i)}(\sigma_i + 1) \\ &= (\sigma_i + 1)E_n^{(i)} \end{aligned}$$

where

$$E_n^{(i)} = \sum_{\pi(i)<\pi(i+1)} \omega_\pi .$$

Proof. We will show this for the case $E_n = (1 + \sigma_i)E_n^{(i)}$; the other case works similarly.

For each permutation π, consider $\pi' = (i, i+1)\pi$. Exactly one of the pair reverses the order of i and $i+1$. Suppose that it is π', so that $\pi(i) < \pi(i+1)$, then the braid $\sigma_i \omega_\pi$ is a positive permutation braid, with permutation π', therefore, we have that $\sigma_i \omega_\pi = \omega_{\pi'}$. Hence,

$$\begin{aligned} E_n &= \sum_{\pi \in S_n} \omega_\pi \\ &= \sum_{\pi(i)<\pi(i+1)} \omega_\pi + \sum_{\pi'(i)>\pi'(i+1)} \omega_{\pi'} \\ &= \sum_{\pi(i)<\pi(i+1)} \omega_\pi + \sum_{\pi(i)<\pi(i+1)} \sigma_i \omega_\pi \\ &= (1+\sigma_i) \sum_{\pi(i)<\pi(i+1)} \omega_\pi \\ &= (1+\sigma_i) E_n^{(i)}, \end{aligned}$$

as required. ∎

Given a scalar γ, we may substitute $\gamma\sigma_i$ for σ_i in ω_π. The element

$$\omega_\pi(\gamma\sigma_1, \cdots, \gamma\sigma_{n-1}) = \gamma^{l(\pi)} \omega_\pi(\sigma_1, \cdots, \sigma_{n-1})$$

where $l(\pi)$ is the length of the permutation. Note that the length of a permutation, π, is equal to the writhe of its positive permutation braid, ω_π.

4.4.3 Definition.

The quadratic relation in H_n can be factorised,

$$(\sigma_i - a)(\sigma_i - b) = 0 \qquad \forall i,$$

where $a = -xs^{-1}$ and $b = xs$.

We define a_n to be

$$a_n = E_n(-a^{-1}\sigma_1, -a^{-1}\sigma_2, \cdots, -a^{-1}\sigma_{n-1})$$

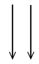
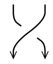
and b_n to be

$$b_n = E_n(-b^{-1}\sigma_1, -b^{-1}\sigma_2, \cdots, -b^{-1}\sigma_{n-1})$$

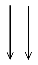
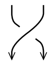
where a and b are the roots of the quadratic equation.

4.4.4 Examples.

Note that $a_1 = b_1$ is just a single string. The expression for a_2 is

$a_2 =$  $+\ x^{-1}s$  ,

and

$b_2 =$  $-\ x^{-1}s^{-1}$  .

Note that if we set $x = s = 1$ then a_n and b_n are just the Young symmetriser and anti-symmetriser for $\mathbb{C}S_n$.

4.4.5 Proposition.

We can factorise a_n and b_n as

$$a_n = (1 - a^{-1}\sigma_i)a_n^{(i)} = a_n^{(i)}(1 - a^{-1}\sigma_i)$$
$$b_n = (1 - b^{-1}\sigma_i)b_n^{(i)} = b_n^{(i)}(1 - b^{-1}\sigma_i)$$

where $a_n^{(i)} = \sum_{\pi(i)<\pi(i+1)}(-a)^{-l(\pi)}\omega_\pi$ and $b_n^{(i)} = \sum_{\pi(i)<\pi(i+1)}(-b)^{-l(\pi)}\omega_\pi$.

Proof. The proof of this result can be deduced directly from the proof of 4.4.2 by substituting $-a^{-1}\sigma_i$ for σ_i.

■

4.4.6 Remark.

Since $1-a^{-1}\sigma_i = -a^{-1}(\sigma_i - a)$, the element a_n has a right (respectively left) factor $(\sigma_i - a)$. Similarly, b_n has a left (respectively right) factor $(\sigma_i - b)$. Therefore, since $(\sigma_i - a)(\sigma_i - b) = 0$, we have that $a_n b_m = 0$ (respectively $b_m a_n = 0$), for every n, $m > 1$.

We shall make repeated use of this factorisation throughout this chapter.

4.4.7 Theorem.[M2]

Let ϕ_a and ϕ_b be linear homomorphisms from the Hecke algebra, H_n, to the ring of scalars Λ defined by $\phi_a(\sigma_i) = a$ and $\phi_b(\sigma_i) = b$ for $i = 1, \ldots, n-1$. Then for all $h \in H_n$,

$$a_n h = \phi_b(h) a_n = h a_n$$

and

$$b_n h = \phi_a(h) b_n = h b_n \, .$$

Proof. It is enough to show this for each σ_i, $1 \leq i < n$. Applying Proposition 4.4.5 and the factorisation of the quadratic relation we have that

$$a_n(\sigma_i - b) = -a^{-1}\, a_n^{(i)}(\sigma_i - a)(\sigma_i - b) = 0$$

and therefore

$$a_n \sigma_i = b a_n \, .$$

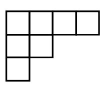
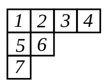
■

4.4.8 Remarks.

In his paper [G], A. Gyoja constructed idempotent elements of H_n which specialise to the Young symmetrisers. We shall construct versions of these idempotents in $\mathcal{S}(R_n^n)$ by appealing to the isomorphism between H_n and $\mathcal{S}(R_n^n)$ established by Morton and Traczyk [MT]. As in our construction of the Young symmetrisers we shall index our idempotents by the Young diagrams with n cells. However, we shall really be referring to the tableau $T(\lambda)$ whose cells are numbered from 1 to n left to right, from top to bottom.

We show below that the elements constructed for other tableaux with the same Young diagram are conjugate to the elements we produce. Since the main goal is to look at the closure of these idempotents, and conjugate elements in H_n will close to the same element of $\mathcal{C}^+$, it is legitimate to work with a preferred tableau.

Throughout we shall illustrate the method of construction using the partition $\nu = (4, 2, 1)$.

ν =  $T(\nu)$ = 

We can consider a 3-dimensional picture for the idempotents, which will be given as linear combinations of braids in a cube rather than their diagrams in a rectangle. Recall that the Young symmetriser is a sum of products of permutations which preserve the rows and permutations which preserve columns. The idea is to replace permutations with positive permutation braids in the rows and columns of the Young diagram. However, in terms of calculation the 3-dimensional viewpoint has certain disadvantages. For example, although the pictures of the idempotents are intuitive in 3-dimensions, we cannot compose them directly with each other. We will, therefore, do calculations with a 2-dimensional version. It will, however, be useful to bear in mind the 3-dimensional picture on occasion. For example, if we take the closure of the diagram in $\mathcal{C}^+$ then the order of the strings in not so tightly constrained. We give the 3-dimensional picture for ν in Figure 4.1. Note how the strings are first grouped in rows and then in columns. Although the braids sit in the vertical plane the boxes are drawn in the horizontal plane to give the impression of how closely the idempotent is related to the shape of the Young diagram. Yokota [Y] doesn't interpret the

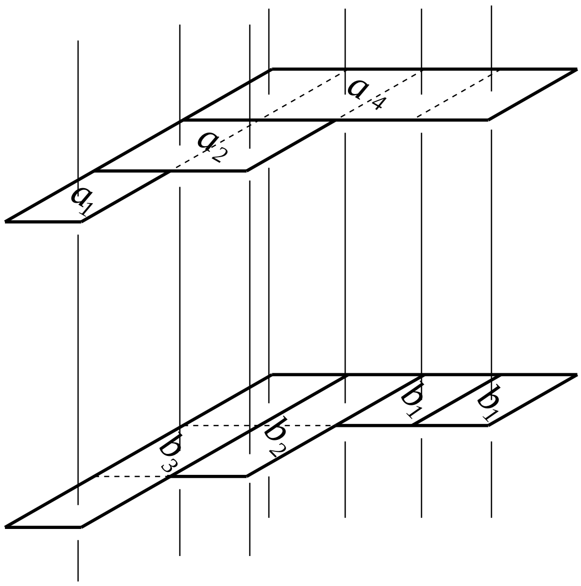


Figure 4.1: The 3-dimensional quasi-idempotent associated to ν.

quasi-idempotents in this 3-dimensional way. However, a 3-dimensional picture of his elements $\dot{e}_\lambda$ would carry an extra copy of the lower Young diagram, labelled with the b_i, on top. Note that Theorem 4.4.7 is stated in [Y, equation 7].

From now on we will work with a 2-dimensional picture. We use the labels in the cells of the tableau to provide an ordering for the strings of the braids upon flattening the picture. With this in mind, it is not difficult to see that the idempotents for two tableaux, D and D', with the same Young diagram will be conjugate. Let π be the permutation which takes the labels of D to those of D'. If e_D is the idempotent associated to D then $\omega_{\pi^{-1}} e_D \omega_\pi^{-1}$ will be the idempotent associated to D'. For example if $D = T(\nu)$ and

$$D' \quad = \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \; ,$$

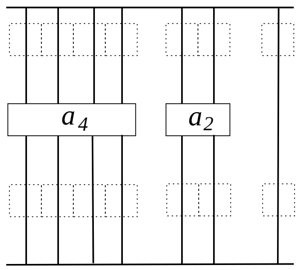
then $\pi = (235)(476)$.

4.4.9 Definitions.

We will draw λ as a line of cells, with cells from the same row of λ grouped together. We will sometimes draw in the rows of cells of the Young diagram (in feint dotted lines) to emphasize this grouping. To each cell we assign a braid string. On the collection of strings which correspond to the ith row of λ, we place a_{λ_i}. We will denote this linear combination of braids by $E_\lambda(a)$.

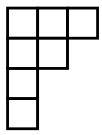
Similarly we define $E_\lambda(b)$ by replacing a_{λ_i} by b_{λ_i} for $1 \le i \le k$, in the above definition.

$E_\nu(a) \quad =$



Recall that we obtain $\lambda^\vee$ from λ by interchanging rows and columns.

$\nu^\vee \quad =$



Note, however, that this doesn't take $T(\lambda)$ to $T(\lambda^\vee)$. Under the exchange of rows and columns the cell labelled i by $T(\lambda)$ is taken to some cell of $\lambda^\vee$. Let j be the number assigned to this cell by $T(\lambda^\vee)$. We define the permutation π_λ by $\pi_\lambda(i) = j$. Let $T(\lambda)^\vee$ be the image of $T(\lambda)$ under the interchange of rows and columns. For the Young diagram ν we have

$$T(\nu)^\vee \quad = \quad \begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \; , \qquad T(\nu^\vee) \quad = \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

and

$$\pi_\nu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 6 & 7 & 2 & 5 & 3 \end{pmatrix}.$$

Let ω_{π_λ} be the positive permutation braid associated with the permutation π_λ. We define $e_\lambda \in H_n$ to be the element

$$e_\lambda = E_\lambda(a)\omega_{\pi_\lambda} E_{\lambda^\vee}(b)\omega_{\pi_{\lambda^\vee}}^{-1} .$$

The element e_ν is shown in Figure 4.2.

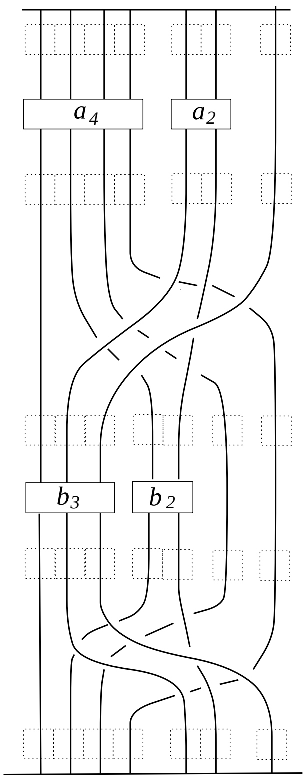


Figure 4.2: The quasi-idempotent e_ν.

4.4.10 Remark.

This picture can be obtained from the 3-dimensional picture by sliding the rows apart at the top of the diagram and sliding the columns apart at the bottom. The permutation braid ω_{π_λ} provides a way of flattening the 3-dimensional picture.

4.4.11 Theorem.

For each Young diagram λ, with n cells, e_λ is a quasi-idempotent element of H_n, i.e.

$$e_\lambda^2 = \alpha_\lambda e_\lambda$$

for some scalar α_λ.

Let $\lambda \neq \mu$ be two Young diagrams both with n cells then e_λ and e_μ are orthogonal in H_n,

$$e_\lambda e_\mu = 0 \quad \text{for } \lambda \neq \mu \, .$$

■

4.5 Proof of Theorem 4.4.11.

To prove this theorem we need some further definitions and results. The proof of Theorem 4.4.11 is deferred until page 77.

4.5.1 Definition.

Let λ and μ be two Young diagrams with $|\lambda| = |\mu| = n$. We will call them *inseparable* if every permutation $\pi \in S_n$ sends some pair of numbers in the same row of $T(\lambda)$ to the same row of $T(\mu)$.

If there is some permutation for which no pair of numbers in the same row of $T(\lambda)$ are mapped to the same row of $T(\mu)$ then we say that λ and μ are *separable.*

4.5.2 Example.

For any Young diagram λ, the pair λ and $\lambda^\vee$ are separable. For example, the permutation π_λ will always separate a Young diagram and its conjugate. Note that if t is a transposition which switches two numbers in the same row of $T(\lambda)$ (respectively $T(\lambda^\vee)$) then $t\pi_\lambda$ (respectively $\pi_\lambda t$) will also separate the two Young diagrams. (Recall that we are multiplication to act on the right.)

4.5.3 Definition.

Let $R(\lambda)$ be the set of permutations of $T(\lambda)$ which preserve the rows of $T(\lambda)$. For example $R(\nu)$ is generated by the set of transpositions $\{(12), (23), (34), (56)\}$.

4.5.4 Lemma.

Let λ and μ be two Young diagrams, both with n cells. If $\lambda > \mu$ then λ and $\mu^\vee$ are inseparable.

Proof. This is shown by induction on the number of cells.

$\underline{n = 1.}$ Here $\lambda = \mu$ and the result is vacuously true.

$\underline{n = 2.}$ If $\lambda > \mu$, then $\lambda = \square\square$ and $\mu = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. Hence $\mu^\vee = \lambda$. Now λ and $\mu^\vee$ both have one row, so both cells in λ are permuted to cells in the same row of $\mu^\vee$ under either of the permutations in S_2, hence λ and $\mu^\vee$ are inseparable.

We now assume the result for all $i < n$, and deduce the result for $|\lambda| = |\mu| = n$. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ and let $\mu = (\mu_1, \mu_2, \cdots, \mu_m)$. We have two cases to consider.

$\underline{\text{Case:}\lambda_1 > \mu_1.}$ Note that μ_1 is the number of rows in $\mu^\vee$. Here the number of cells in the first row of λ is greater than the number of rows in $\mu^\vee$, therefore, any permutation must map at least two numbers from the first row of λ to the same row of $\mu^\vee$, so the two Young diagrams can't be separable.

$\underline{\text{Case:}\lambda_1 = \mu_1.}$ For any Young diagram α, let $r(\alpha)$ be the Young diagram obtained from α by removing the first row. Since $\lambda_1 = \mu_1$, then $|r(\lambda)| = |r(\mu)| < n$, and since $\lambda > \mu$, it follows that $r(\lambda) > r(\mu)$. Hence, by induction, $r(\lambda)$ and $r(\mu)^\vee$ are inseparable.

It remains to prove that if $r(\lambda)$ and $r(\mu)^\vee$ are inseparable, then so are λ and $\mu^\vee$. In fact we show that if λ and $\mu^\vee$ are separable then so are $r(\lambda)$ and $r(\mu)^\vee$.

Let π be the permutation which separates λ and $\mu^\vee$. Since $\lambda_1 = \mu_1$, the number of cells in the first row of λ is equal to the number of rows of $\mu^\vee$. As π separates, it must send exactly one cell from the first row of λ to each row of μ. We can suppose, without loss of generality, that it is the first cell in each row of $\mu^\vee$, for if not, there is a transposition $s \in R(\lambda^\vee)$ which will switch the first cell of the row with the image of the cell in the first row of λ, and πs is a permutation in S_n, which also separates λ and $\mu^\vee$. We can now restrict π to all but the first row of λ, and its image will be exactly the cells of $r(\mu)^\vee$. This restricted permutation must, therefore, separate $r(\lambda)$ and $r(\mu)^\vee$. ■

4.5.5 Corollary.

Given any two Young diagrams, λ and μ, either

1. λ and $\mu^\vee$ are inseparable.
2. $\lambda^\vee$ and μ are inseparable.
3. $\lambda = \mu$.

It therefore follows that if λ and $\mu^\vee$ are separable and $\lambda \neq \mu$, then $\lambda^\vee$ and μ are inseparable.

Proof. Given any two Young diagrams, either case 3 holds or $\lambda > \mu$ or $\lambda < \mu$. By Lemma 4.5.4, if $\lambda > \mu$, then case 1 holds and if $\lambda < \mu$, then case 2 holds. ■

The following lemma is at the crux of proving that the quasi-idempotents are orthogonal. Roughly speaking, the lemma says that if two strings leave a box labelled a_i and arrive at a box labelled b_j then, whatever happens in between, then the braid combination is equal to 0 as an element of the Hecke algebra.

4.5.6 Lemma.

Given any two Young diagrams, $\lambda \neq \mu$ with $|\lambda| = |\mu|$, if λ and μ are inseparable then

$$E_\lambda(a) H_n E_\mu(b) = 0$$

and

$$E_\mu(b) H_n E_\lambda(a) = 0\,.$$

Proof. We will show that $E_\lambda(a)H_nE_\mu(b) = 0$. The proof of the second statement is similar.

Since H_n is spanned by positive permutation braids, it is enough to show that $E_\lambda(a)\omega_\pi E_\mu(b) = 0$ for an arbitrary positive permutation braid, ω_π.

Since λ and μ are inseparable, there are two cells in some row of λ, the lth say, which are sent to two cells in the same row of μ, the pth say, by ω_π.

Suppose that the two cells in the lth row of λ are not adjacent. We can find a transposition, β, for which $\omega_\beta\omega_\pi$ sends two adjacent cells of the lth row of λ to the pth row of μ. Now β preserves the rows of λ i.e. $\beta \in R(\lambda)$. In fact β only permutes strings in the lth row of λ. Hence $a_{\lambda_l}\omega_\beta = \phi(\omega_\beta)a_{\lambda_l}$. Therefore, at the expense of a scalar, we can assume that the two cells are an adjacent pair, i and $i+1$ say.

Similarly, we can suppose the two cells in μ are an adjacent pair j and $j+1$, again at the expense of some scalar. Note that

$$\sigma_i\omega_\pi = \omega_\pi\sigma_j\,. \tag{4.1}$$

By Proposition 4.4.5 we know that

$$a_{\lambda_l} = a^{(i)}_{\lambda_l}(1 - a^{-1}\sigma_i) = -a^{-1}a^{(i)}_{\lambda_l}(\sigma_i - a)\,,$$

and

$$b_{\mu_p} = (1 - b^{-1}\sigma_j)b^{(j)}_{\mu_p} = -b^{-1}(\sigma_j - b)b^{(j)}_{\mu_p}\,.$$

Hence $E_\lambda(a)\omega_\pi E_\mu(b)$ has the term $(\sigma_i - a)\omega_\pi(\sigma_j - b)$ in its expression. This is demonstrated pictorially in Figure 4.3. By equation 4.1 $(\sigma_i - a)\omega_\pi = \omega_\pi(\sigma_j - a)$, therefore, we can rewrite this subexpression as $\omega_\pi(\sigma_j - a)(\sigma_j - b)$.

Now $(\sigma_j - a)(\sigma_j - b) = 0 \in H_n$, therefore, $E_\lambda(a)H_nE_\mu(b) = 0$. ∎

4.5.7 Lemma.

Let τ be a permutation which separates λ and $\lambda^\vee$. We can find permutations $\rho_1 \in R(\lambda)$ and $\rho_2 \in R(\lambda^\vee)$ for which $\rho_1\tau\rho_2 = \pi_\lambda$.

Proof. Recall that λ_j is the number of rows in $\lambda^\vee$ with at least j cells. Since τ separates λ and $\lambda^\vee$, exactly one cell from the jth row of λ must be sent to each of the first λ_j rows of $\lambda^\vee$.

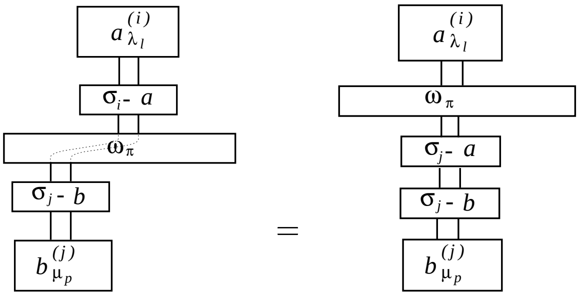


Figure 4.3: A subexpression of $E_\lambda(a)\omega_\pi E_\mu(b)$.

(This can be seen by considering each row of λ in turn. There are exactly λ_1 rows in $\lambda^\vee$. Exactly one cell from the first row of λ must be sent to each row of $\lambda^\vee$ by τ, since τ separates. Only the rows of $\lambda^\vee$ with more than one cell will have spaces left after we have dealt with the image of the first row of λ under τ. The number of such rows is exactly λ_2 and so forth.)

We can find a permutation $\rho_{1,j} \in R(\lambda)$ which reorders the cells in the jth row of λ, so that , for $1 \leq i \leq \lambda_j$, the ith cell of the row is sent to the ith row of $\lambda^\vee$ by $\rho_{1,j}$.

Note that, since $\rho_{1,j}$ only permutes cells in the jth row of λ, all the cycles in $\rho_{1,j}$ are disjoint from those in $\rho_{1,j'}$ if $j \neq j'$ and so they commute. Set $\rho_1 = \prod_{j=1}^{k} \rho_{1,j}$. Then $\rho_1 \in R(\lambda)$.

Similarly, for the jth row of $\lambda^\vee$, we can find a permutation $\rho_{2,j} \in R(\lambda^\vee)$, which reorders the cells in the jth row of $\lambda^\vee$, so that the ith cell is the image of a cell in the ith row of λ, under $\rho_1\tau\rho_{2,j}$. Let $\rho_2 = \prod_{j=1}^{\lambda_1} \rho_{2,j} \in R(\lambda^\vee)$. Then $\rho_1\tau\rho_2$ sends the ith cell of the jth row of λ to the jth cell of the ith row of $\lambda^\vee$, i.e. it interchanges rows and columns. Hence $\pi_\lambda = \rho_1\tau\rho_2$ as required. $\blacksquare$

4.5.8 Example.

In this example, we show how to calculate ρ_1 and ρ_2 for the Young diagram ν.

Recall that $\nu = \yng(4,2,1)$. The permutation $\tau = \binom{1234567}{7635142}$ separates ν and $\nu^\vee$.

$$\young(1234)\ \young(56)\ \young(7) \xrightarrow{\tau} \young(573)\ \young(64)\ \young(2)\ \young(1)$$

Note that τ sends the third cell of the first row of ν to the first row of $\nu^\vee$

whereas π_ν sends the first cell of the first row to the first row of $\nu^\vee$. Therefore $\rho_{1,1}$ must send 1 to 3. By similar arguments we find that $\rho_{1,1} = \binom{1234567}{3421567}$ and $\rho_{1,2} = \rho_{1,3} = 1$. Hence $\rho_1 = \rho_{1,1}$ and $\rho_1\tau$ is as follows,

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}\ \begin{array}{|c|c|}\hline 5&6\\\hline\end{array}\ \begin{array}{|c|}\hline 7\\\hline\end{array} \xrightarrow{\rho_1} \begin{array}{|c|c|c|c|}\hline 4&3&1&2\\\hline\end{array}\ \begin{array}{|c|c|}\hline 5&6\\\hline\end{array}\ \begin{array}{|c|}\hline 7\\\hline\end{array} \xrightarrow{\tau} \begin{array}{|c|c|c|}\hline 5&7&1\\\hline\end{array}\ \begin{array}{|c|c|}\hline 6&2\\\hline\end{array}\ \begin{array}{|c|}\hline 3\\\hline\end{array}\ \begin{array}{|c|}\hline 4\\\hline\end{array}\,.$$

(Note that the permutations refer to the position of the cell in the diagram rather than the labels the cells carry.)

We then find that $\rho_{2,1} = \binom{1234567}{2314567}$ and $\rho_{2,2} = \binom{1234567}{1235467}$.

Noting that $\rho_{2,3} = \rho_{2,4} = 1$, we have that $\rho_2 = \binom{1234567}{2315467}$.

$$\begin{array}{|c|c|c|}\hline 5&7&1\\\hline\end{array}\ \begin{array}{|c|c|}\hline 6&2\\\hline\end{array}\ \begin{array}{|c|}\hline 3\\\hline\end{array}\ \begin{array}{|c|}\hline 4\\\hline\end{array} \xrightarrow{\rho_2} \begin{array}{|c|c|c|}\hline 1&5&7\\\hline\end{array}\ \begin{array}{|c|c|}\hline 2&6\\\hline\end{array}\ \begin{array}{|c|}\hline 3\\\hline\end{array}\ \begin{array}{|c|}\hline 4\\\hline\end{array}$$

Therefore,

$$\begin{aligned} \rho_1\tau\rho_2 &= (1324)(172645)(123)(45) \\ &= (247365) \\ &= \pi_\nu\,. \end{aligned}$$

4.5.9 Corollary.

With λ, τ, ρ_1 and ρ_2 as defined in Lemma 4.5.7, we have that

$$\omega_\tau = \omega_{\rho_1^{-1}}\omega_{\pi_\lambda}\omega_{\rho_2^{-1}}\,.$$

Proof. From Lemma 4.5.7, we have that $\tau = \rho_1^{-1}\pi_\lambda\rho_2^{-1}$. We must prove that $\omega_{\rho_1^{-1}}\omega_{\pi_\lambda}\omega_{\rho_2^{-1}}$ is a positive permutation braid. We know that $\omega_{\rho_1^{-1}} \in R(\lambda)$, $\omega_{\rho_2^{-1}} \in R(\lambda^\vee)$ and they are both positive permutation braids. Note that ω_{π_λ} doesn't cross strings belonging to the same row of λ or the same row of $\lambda^\vee$. Suppose $\omega_{\rho_1^{-1}}$ crosses the ith and the jth string. Since $\rho_1^{-1} \in R(\lambda)$, the ith and jth string must belong to the same row of λ. The braid ω_{π_λ} does not cross these two strings since it doesn't cross any two strings in the same row of λ. Since ω_{π_λ} separates λ and $\lambda^\vee$, the ith and jth strings will end up in different rows of $\lambda^\vee$. Since $\omega_{\rho_2^{-1}} \in R(\lambda^\vee)$ it will not cross over two strings in different rows of $\lambda^\vee$. Similar arguments show that if one of the three braids, $\omega_{\rho_1^{-1}}$, $\omega_{\rho_2^{-1}}$ and ω_{π_λ} crosses a pair of strings, neither of the other two braids will. Therefore, since the three braids are positive permutation braids, $\omega_{\rho_1^{-1}}\omega_{\pi_\lambda}\omega_{\rho_2^{-1}}$ is a braid which crosses no two strings more than once and all the crossings have positive sign. Hence ω_τ and $\omega_{\rho_1^{-1}}\omega_{\pi_\lambda}\omega_{\rho_2^{-1}}$ are two positive permutation braids which represent the same permutation and must, therefore, be the same braid. ∎

4.5.10 Proof of Theorem 4.4.11.

We will first show that e_λ is quasi-idempotent.

We know that $e_\lambda^2 = E_\lambda(a)\, h\, E_{\lambda^\vee}(b)\omega_{\pi_{\lambda^\vee}}^{-1}$ where

$$h = \omega_{\pi_\lambda} E_{\lambda^\vee}(b)\omega_{\pi_{\lambda^\vee}}^{-1} E_\lambda(a)\omega_{\pi_\lambda}\,.$$

We can express h as a linear combination of positive permutation braids,

$$h = \sum_{\tau \in S_n} \gamma_\tau \omega_\tau\,.$$

Then

$$e_\lambda^2 = \sum_{\tau \in S_n} \gamma_\tau\ E_\lambda(a)\,\omega_\tau\, E_{\lambda^\vee}(b)\,\omega_{\pi_\lambda}^{-1}\,.$$

A similar argument to Lemma 4.5.6 shows that if τ doesn't separate λ and $\lambda^\vee$, then $E_\lambda(a)\omega_\tau E_{\lambda^\vee}(b) = 0$. Hence, we need only consider those τ which separate λ and $\lambda^\vee$. In this case we can write $\omega_\tau = \omega_{\rho_1^{-1}}\omega_{\pi_\lambda}\omega_{\rho_2^{-1}}$ for some $\rho_1 \in R(\lambda)$ and $\rho_2 \in R(\lambda^\vee)$, by Lemma 4.5.7.

Now, by Theorem 4.4.7

$$E_\lambda(a)\omega_{\rho_1^{-1}} = \phi_b(\omega_{\rho_1^{-1}})E_\lambda(a)$$

and

$$\omega_{\rho_2^{-1}}E_\lambda(b) = \phi_a(\omega_{\rho_2^{-1}})E_\lambda(b)\,.$$

Therefore, if we set

$$\alpha(\tau) = \phi_b(\omega_{\rho_1^{-1}})\phi_a(\omega_{\rho_2^{-1}})$$

and

$$\alpha_\lambda = \sum_{\tau \in S_n} \alpha(\tau)\gamma_\tau$$

we have that $e_\lambda^2 = \alpha_\lambda e_\lambda$ as required.

Suppose now that $\lambda \neq \mu$. We can see readily that $e_\lambda e_\mu = 0$. For

$$e_\lambda e_\mu = E_\lambda(a)\omega_{\pi_\lambda}E_{\lambda^\vee}(b)\omega_{\pi_{\lambda^\vee}}^{-1}E_\mu(a)\omega_{\pi_\mu}E_{\mu^\vee}(b)\omega_{\pi_{\mu^\vee}}^{-1}\,.$$

Since $\lambda \neq \mu$, either λ and $\mu^\vee$ are inseparable or $\lambda^\vee$ and μ are inseparable by Corollary 4.5.5. If $\lambda^\vee$ and μ are inseparable, then by Lemma 4.5.6,

$$E_{\lambda^\vee}(b)(\omega_{\pi_{\lambda^\vee}}^{-1})E_\mu(a) = 0\,.$$

If λ and $\mu^\vee$ are inseparable, then by Lemma 4.5.6,

$$E_\lambda(a)\left(\omega_{\pi_\lambda}E_{\lambda^\vee}(b)\omega_{\pi_{\lambda^\vee}}^{-1}E_\mu(a)\omega_{\pi_\mu}\right)E_{\mu^\vee}(b) = 0\,.$$

Hence, if $\lambda \neq \mu$, then e_λ and e_μ are orthogonal as required. ∎

4.5.11 Comment.

The orthogonality of the quasi-idempotents is proved in [Y, Proposition 2.9], also using the fact that a symmetriser and and anti-symmetriser must be joined by two strings.

4.6 Specialisation of the Hecke algebra.

We wish to show that the quasi-idempotent elements we have produced specialise to the Young symmetrisers when we set $v = x = s = 1$. From this we can show that the ideals $H_n e_\lambda H_n$ are the minimal two-sided ideals of H_n and so correspond to the simple modules.

4.6.1 Notation.

Let $g : \Lambda \to \mathbb{C}$ denote the ring homomorphism defined by

$$g(x) = g(v) = g(s) = 1\,.$$

Note that $g(z) = g(s - s^{-1}) = 0$.

We can regard $\mathbb{C}$ as a Λ-module, with the action of Λ given by

$$r \cdot w = g(r)w \quad \forall\ r \in \Lambda,\ w \in \mathbb{C}\,.$$

Therefore, $H_n \otimes_\Lambda \mathbb{C}$ is a Λ-module. In fact, we can give $H_n \otimes_\Lambda \mathbb{C}$ a $\mathbb{C}$-algebra structure, the action of $\mathbb{C}$ being given by

$$(x \otimes w) \cdot w' = x \otimes ww' \quad \forall w, w' \in \mathbb{C}, \quad x \in H_n\,.$$

We define the product by

$$(h \otimes w)(h' \otimes w') = hh' \otimes ww' \qquad \forall h,\, h' \in H_n,\ w,\, w' \in \mathbb{C}$$

4.6.2 Lemma

The $\mathbb{C}$-algebra $H_n \otimes_\Lambda \mathbb{C}$ is generated by $\{\sigma_i \otimes 1 \ : \ i = 1 \ldots n-1\}$. ∎

4.6.3 Proposition.

There is an $\mathbb{C}$-algebra isomorphism,

$$f : H_n \otimes_\Lambda \mathbb{C} \to \mathbb{C}S_n\,,$$

defined by

$$f(\sigma_i \otimes 1) = (i\ i+1)\,.$$

Proof. First, note that any relation in $H_n \otimes_\Lambda \mathbb{C}$ must be inherited from one of the two algebras or the definition of the tensor product. Therefore, the following is a complete list of relations in $H_n \otimes_\Lambda \mathbb{C}$.

If $|i-j| \geq 2$, then

$$\begin{aligned}(\sigma_i \otimes 1)(\sigma_j \otimes 1) &= \sigma_i\sigma_j \otimes 1 \\ &= \sigma_j\sigma_i \otimes 1 \\ &= (\sigma_j \otimes 1)(\sigma_i \otimes 1)\,.\end{aligned}$$

For $i = 1 \ldots\ n-2$,

$$\begin{aligned}(\sigma_{i+1} \otimes 1)(\sigma_i \otimes 1)(\sigma_{i+1} \otimes 1) &= \sigma_{i+1}\sigma_i\sigma_{i+1} \otimes 1 \\ &= \sigma_i\sigma_{i+1}\sigma_i \otimes 1 \\ &= (\sigma_i \otimes 1)(\sigma_{i+1} \otimes 1)(\sigma_i \otimes 1)\,.\end{aligned}$$

From the definition of tensor product, $rh \otimes 1 = h \otimes g(r)$, for $r \in \Lambda,\ h \in H_n$. Therefore, for $i = 1 \ldots\ n-1$,

$$\begin{aligned}0 \otimes 1 &= x^{-1}\sigma_i \otimes 1 - x\sigma_i^{-1} \otimes 1 - (s - s^{1})1 \otimes 1 \\ &= \sigma_i \otimes g(x^{-1}) - \sigma_i^{-1} \otimes g(x) - 1 \otimes g(s - s^{-1}) \\ &= \sigma_i \otimes 1 - \sigma_i^{-1} \otimes 1\end{aligned}$$

If we define $f : H_n \otimes \mathbb{C} \to \mathbb{C}S_n$ by $\sigma_i \mapsto (i\ i+1)$ then it is obvious that f is an isomorphism since the two algebras have identical presentations. ■

4.6.4 Notation.

We will write $\Gamma : H_n \to \mathbb{C}S_n$, for the composite map

$$H_n \cong H_n \otimes_\Lambda \Lambda \stackrel{1\otimes g}{\to} H_n \otimes \mathbb{C} \stackrel{f}{\to} \mathbb{C}S_n \,.$$

Then Γ is a ring homomorphism satisfying

$$\Gamma(r\sigma_i) = g(r)(i\ i+1) \qquad \text{for } r \in \Lambda \,.$$

We shall sometimes describe the effect of Γ as "specialising H_n to $\mathbb{C}S_n$".

4.6.5 Proposition.

The image of the quasi-idempotent e_λ under Γ is the Young symmetriser $C_\lambda \in \mathbb{C}S_n$,

$$\Gamma(e_\lambda) = C_\lambda \,.$$

Proof. In order to prevent the notation becoming too clumsy, we will denote the Young tableau $T(\lambda)$ by D and the tableau $T(\lambda)^\vee$ by $D^\vee$.

Recall that $C_\lambda = A_D B_D$, where

$$A_D = \sum_{p \in P_D} p \,, \qquad B_D = \sum_{q \in Q_D} (-1)^{l(q)} q \,,$$

P_D is the set of permutations in S_n which preserve the rows of D and Q_D is the set of permutations which preserve the columns.

Recall, also, that in the construction of e_λ, it was mentioned that the tableau D is implicitly involved and that with respect to this tableau we can define

$$E_\lambda(a) = \sum_{p \in P_D} (xs^{-1})^{-l(p)} \omega_p \,.$$

Now $g((xs^{-1})^{-l(p)}) = 1$ and $\Gamma(\omega_p) = p$. Hence

$$\Gamma(E_\lambda(a)) = A_D \,.$$

It remains to show that $\omega_{\pi_\lambda} E_{\lambda^\vee}(b)\omega^{-1}_{\pi^{-1}_\lambda}$ specialises to B_D.

$$\omega_{\pi_\lambda} E_{\lambda^\vee}(b)\omega^{-1}_{\pi^{-1}_\lambda} = \sum_{p\in P_{D^\vee}} (-xs)^{-l(p)}\omega_{\pi_\lambda}\omega_p\omega^{-1}_{\pi^{-1}_\lambda},$$

which specialises to

$$\sum_{p\in P_{D^\vee}} (-1)^{-l(p)}\pi_\lambda p \pi_\lambda^{-1}.$$

Note that, since π_λ sends columns of D to rows of $D^\vee$, p preserves rows of $D^\vee$ and π_λ^{-1} takes rows of $D^\vee$ to columns of D, each term in this sum is a permutation that preserves the columns of D. We need only show that any permutation which preserves the columns of D can be uniquely written in this form.

Suppose $q \in Q_D$. We can show that $\pi_\lambda^{-1} q \pi_\lambda \in P_{D^\vee}$ by using similar arguments to those above. Obviously $q = \pi_\lambda(\pi_\lambda^{-1} q\pi_\lambda)\pi_\lambda^{-1}$.

Since conjugating by π_λ is an isomorphism, no two elements of Q_D will give rise to the same element of $P_{D^\vee}$ or vice versa. Also note that the sign of a permutation is preserved by conjugation therefore,

$$\begin{aligned}\sum_{p\in P_{D^\vee}} (-1)^{-l(p)}\pi_\lambda p\pi_\lambda^{-1} &= \sum_{q\in Q_D} (-1)^{-l(\pi_\lambda^{-1}q\pi_\lambda)} q \\ &= \sum_{q\in Q_D} (-1)^{-l(q)} q\end{aligned}$$

Noting that $(-1)^{-l(q)} = (-1)^{l(q)}$, we have shown that

$$\Gamma(e_\lambda) = C_\lambda.$$

■

4.6.6 Remark.

So far we have been able to work with Λ as the ring

$$\Lambda = \mathbb{C}[x^{\pm 1}, v^{\pm 1}, s^{\pm 1}, \delta]/ < v^{-1} - v = \delta(s - s^{-1}) > .$$

We now wish to show that, with the substitutions $x = s^{-1/N}$ and $v = s^{-N}$, the closures of our elements e_λ will provide the required patterns to prove the Satellite Theorem 3.7.3.

Note that with these substitutions δ becomes a genuine Laurent polynomial in s,

$$\delta = \frac{s^N - s^{-N}}{s - s^{-1}} = s^{N-1} + s^{N-3} + \cdots + s^{-N+1}\,.$$

Therefore, the framed Homfly polynomial of a knot, evaluated at $x = s^{-1/N}$, $v = s^{-N}$ will be a Laurent polynomial in $s^{1/N}$. However, to ensure the invertibility of the scalar α_λ we must further make the substitution of $s = e^{h/2}$ and write everything as a power series in terms of h. The ring homomorphism g defined in 4.6.1 is now defined by $g(h) = 0$.

From this point, therefore, we will take Λ to be the ring of power series in h, $\mathbb{C}[\![h]\!]$. We will then be able to prove invertibility of certain elements of Λ simply by showing that they have non-zero constant term.

4.6.7 Lemma.

The scalar α_λ is invertible in $\Lambda = \mathbb{C}[\![h]\!]$.

Proof. To show that α_λ is invertible, consider what happens to e_λ^2 upon specialisation. Since e_λ specialises to C_λ and Γ is a ring homomorphism, e_λ^2 specialises to C_λ^2. Now

$$C_\lambda^2 = a_\lambda C_\lambda$$

and since $\alpha_\lambda e_\lambda = e_\lambda^2$, α_λ must specialise to $a_\lambda \in \mathbb{Q}$. Therefore, as a power series in h, the element α_λ must have a non-zero constant term and must be invertible.

$\blacksquare$

4.6.8 Remark.

In the proof of the last Lemma, it was noted that under Γ the scalar α_λ specialised to the scalar a_λ, i.e.

$$\Gamma(\alpha_\lambda) = \prod \text{hook lengths}\,.$$

In particular, note that this is non-zero and corresponds to the evaluation of α_λ at $x = s = v = 1$. This fact will be used in the proof of Theorem 4.8.8.

4.6.9 Theorem.

With the substitutions described above, the Hecke algebra H_n has a direct sum decomposition

$$H_n = \bigoplus_{|\lambda|=n} H_n e_\lambda H_n$$

where the 2-sided ideals $H_n e_\lambda H_n$ have Λ-dimension d_λ^2.

Proof. Recall that $\mathbb{C}S_n$ has a complex vector space basis given by the $n!$ permutations, $\{\pi_i \ : \ i = 1 \ldots n!\}$ in S_n.

By Proposition 4.2.2, H_n is freely generated as a Λ-module by the $n!$ positive permutation braids, $\{\omega_i \ : \ i = 1 \ldots n!\}$. Under Γ, a positive permutation braid is taken to its associated permutation. We may choose the ordering so that $\Gamma(\omega_i) = \pi_i$. By Theorem 4.3.5

$$\mathbb{C}S_n = \bigoplus \mathbb{C}S_n C_\lambda \mathbb{C}S_n \,.$$

It is an immediate consequence of Theorem 4.3.5 and Proposition 4.6.5 that

$$\Gamma\left(\bigoplus_{|\lambda|=n} H_n e_\lambda H_n\right) = \mathbb{C}S_n \,.$$

Since Γ is surjective, we can find $v_i \in \bigoplus_{|\lambda|=n} H_n e_\lambda H_n$ for which $\Gamma(v_i) = \pi_i$. We can write v_i as a linear combination of the positive permutation braids

$$v_i = \sum_{j=1}^{n!} b_{ij}\omega_j \,.$$

Let B be the $n! \times n!$ matrix whose (ij)th entry is b_{ij}. It is enough to show that the determinant of this matrix is non-zero, since then $\omega_i \in \bigoplus H_n e_\lambda H_n$ for each i and thus $H_n = \bigoplus H_n e_\lambda H_n$ as required.

We know that $\Gamma(v_i - \omega_i) = 0$, therefore, in $\mathbb{C}S_n$,

$$(g(b_{ii}) - 1)\pi_i + \sum_{i \neq j} g(b_{ij})\pi_j = 0 \,.$$

However, since the π_i are a vector space basis for $\mathbb{C}S_n$, this implies that

$$g(b_{ij}) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta. Hence b_{ij} is a power series in h which has non-zero constant term if and only if $i = j$. We can write B as

$$B = I + hB'$$

for some matrix B'. It follows that the determinant of B is a power series in h with constant coefficient 1 and hence is invertible in $\Lambda = \mathbb{C}[\![h]\!]$ as required.

A consequence of this result is that $\{\omega_i \ : \ i = 1 \dots n!\}$ is a Λ-basis for $\bigoplus_{|\lambda|=n} H_n e_\lambda H_n$.

Let ∂_λ denote the Λ-dimension of $H_n e_\lambda H_n$ for each λ. From the first part of this proof, we have that

$$\sum_{|\lambda|=n} \partial_\lambda = n!\,. \tag{4.2}$$

Since $\Gamma(e_\lambda) = C_\lambda$ and Γ is surjective,

$$\Gamma(H_n e_\lambda H_n) = \mathbb{C}S_n C_\lambda \mathbb{C}S_n\,. \tag{4.3}$$

From the classical result Proposition 4.3.7, the dimension of $\mathbb{C}S_n C_\lambda \mathbb{C}S_n$ as a complex vector space is d_λ^2. By Theorem 4.3.5 it follows that

$$\sum_{|\lambda|=n} d_\lambda^2 = n!\,. \tag{4.4}$$

Suppose that $H_n e_\lambda H_n$ has Λ-basis $\{w_1, w_2, \dots, w_f\}$. By equation 4.3, the set $\{\Gamma(w_1), \Gamma(w_2), \dots, \Gamma(w_f)\}$ must span $\mathbb{C}S_n C_\lambda \mathbb{C}S_n$. Therefore, for each λ,

$$\partial_\lambda \geq d_\lambda^2\,. \tag{4.5}$$

Combining equations 4.4 and 4.2 we have that

$$\sum_{|\lambda|=n} d_\lambda^2 = \sum_{|\lambda|=n} \partial_\lambda\,. \tag{4.6}$$

Equations 4.5 and 4.6 together imply that $\partial_\lambda = d_\lambda^2$ as required. ■

4.6.10 Comments.

We can now reintroduce the quantum group invariants of Chapter 3 and prove Theorem 3.7.3. First we formalise an idea that was implicit in the construction of the quantum group invariants. Given a diagram in R_n^n, we have a recipe for constructing an element of the endomorphism ring of $V_\Box^{\otimes n}$, where $V_\Box$ denotes the fundamental representation of $U_q(sl(N))$. Theorem 3.7.1 allows us to consider elements of $\mathcal{S}(R_n^n)$ since two tangles which are equivalent in $\mathcal{S}(R_n^n)$ give rise to the same module endomorphism, provided we work with $\Lambda = \mathbb{C}[\![h]\!]$. Thus we obtain a representation of the Hecke algebra H_n on $\mathrm{End}(V_\Box^{\otimes n})$.

4.6.11 Theorem.[Ji2]

Recall that $V_\Box$ denotes the fundamental representation of $U_q(sl(N))$. For each n and N, there is a representation of the Hecke algebra H_n on $\mathrm{End}(V_\Box^{\otimes n})$,

$$\phi : H_n \to \mathrm{End}(V_\Box^{\otimes n})$$

given by the substitutions $x = s^{-1/N}$ and $v = s^{-N}$ and $\sigma_i \mapsto 1 \otimes \cdots \otimes 1 \otimes R \otimes 1 \otimes \cdots \otimes 1$ where the R sits in the $(i, i+1)$ position of the n-fold tensor. This homomorphism is surjective. ∎

We wish to consider the images of the endomorphisms $\phi(e_\lambda)$. First we prove the following Lemma.

4.6.12 Lemma.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ and $\lambda^\vee = (\lambda_1^\vee, \lambda_2^\vee, \cdots, \lambda_m^\vee)$ be a Young diagram and its conjugate diagram. The representation V_λ is the only summand to occur in both $c_{\lambda_1^\vee} \otimes c_{\lambda_2^\vee} \otimes \cdots c_{\lambda_m^\vee}$ and $d_{\lambda_1} \otimes d_{\lambda_2} \otimes \cdots \otimes d_{\lambda_k}$. It occurs with multiplicity 1.

Proof. By considering the leading terms of the two Giambelli formulae for λ, we see that λ is a summand in the decomposition of both the tensor products. The main work is to show that it is the *only* summand common to both products.

Any summand of $c_{\lambda_1^\vee} \otimes c_{\lambda_2^\vee} \otimes \cdots \otimes c_{\lambda_m^\vee}$ must come from a succession of strict expansions of $c_{\lambda_1^\vee}$, by columns labelled from 1 to $\lambda_i^\vee$. The only cells which can be added to the first row must carry the label 1. Therefore, at each stage we can add at most one cell to the first row. It follows that any summand of $c_{\lambda_1^\vee} \otimes c_{\lambda_2^\vee} \otimes \cdots \otimes c_{\lambda_m^\vee}$ can have at most λ_1 cells in the first row. If a summand has λ_1 cells in the first row we can use a similar argument to show that it must have at most λ_2 in the second row and so forth. Continuing in this manner, we must either obtain λ as a summand or at some point the length of the first column must be increased to be longer than $\lambda_1^\vee$. Such a diagram, therefore, cannot be a summand of $d_{\lambda_1} \otimes d_{\lambda_2} \otimes \cdots \otimes d_{\lambda_k}$ since at most $\lambda_1^\vee - 1$ cells can be added to the first column of d_{λ_1}. This follows from the fact that no two cells with the same label can be placed in the same column of a strict expansion.

From this description, it is clear that in either tensor product there is only one way to obtain λ via a strict expansion. The multiplicity of V_λ must, therefore, be 1.

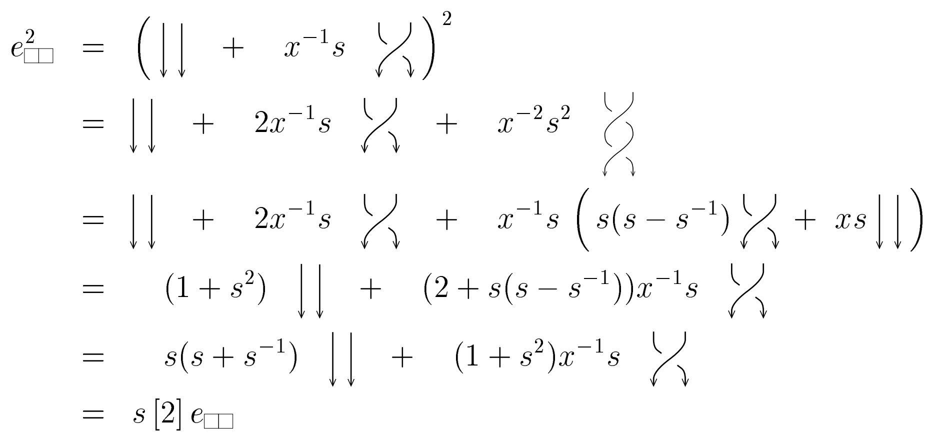
■

4.6.13 Lemma.

The scalars α for the Young diagrams $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ are given by

$$\alpha_{\square\square} = s[2]\,, \quad \alpha_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = s^{-1}[2]\,.$$

Proof. We give the proof for $e_{\square\square}$. The proof for $e_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ is similar.



■

4.6.14 Theorem.

The endomorphism $\phi(e_\lambda)$ of $V_\square^{\otimes n}$ is a scalar multiple of the projection map onto a single copy of the irreducible $U_q(sl(N))$-module V_λ.

Proof. The proof is an induction on n.

$\underline{n=1.}$ Since $e_{\square}$ is the identity in H_1 and ϕ is an algebra homomorphism, $\phi(e_{\square})$ must be the identity map as required.

$\underline{n=2.}$

We have the following decomposition for $V_{\square}^2$,

$$V_{\square}^2 = V_{\square\square} \oplus V_{\substack{\square\\\square}} .$$

The endomorphism ring has dimension 2 and is spanned by, for example, the identity and the R-matrix. Thus since $\phi(e_\lambda)$ is a non-zero linear combination of these two maps, it must be non-zero.

We know that $e_{\square\square} e_{\substack{\square\\\square}} = 0$ as elements of the Hecke algebra, therefore, their images must be disjoint; the image of $\phi(e_{\square\square})$ being either $V_{\square\square}$ or $V_{\substack{\square\\\square}}$. The image of $\phi(e_{\substack{\square\\\square}})$ will then be the complementary summand.

By Schur's Lemma, upon restriction to the appropriate irreducible module, the homomorphisms, being non-zero, must be isomorphisms. Since $\phi(e_{\square\square})^2 = \alpha_\lambda \phi(e_{\square\square})$, it follows that upon restriction to the irreducible summand the map is a scalar multiple of the identity map i.e. up to a scalar, $\phi(e_{\square\square})$ is a projection onto one of the irreducible summands. Similarly, $\phi(e_{\substack{\square\\\square}})$ must be a projection onto the other summand.

We decide which summand is the image by applying Theorem 3.7.1. Evaluating the framed Homfly polynomial at $x = s^{-1/N}$ and $v = s^{-N}$ we find that

$$\frac{1}{\alpha_{\square\square}} \mathcal{X}(\hat{e}_{\square\square}) = [N][N+1]/[2] ,$$

which is the quantum dimension of $V_{\square\square}$. On the other hand

$$\frac{1}{\alpha_{\substack{\square\\\square}}} \mathcal{X}(\hat{e}_{\substack{\square\\\square}}) = [N][N-1]/[2] .$$

which is the quantum dimension of $V_{\substack{\square\\\square}}$.

We now assume that the result is true for all $n < k$.

$\underline{n=k.}$

This is split into two cases. The cases where the Young diagram is a single row or single column require a little more thought and will be dealt with later. For now, assume λ is a Young diagram with k cells and at least two columns and two rows.

Consider the skein diagram for e_λ. By the induction hypothesis, the image of $\phi(e_\lambda)$ is contained in the image of the composition of the projection map to $d_{\lambda_1} \otimes d_{\lambda_2} \otimes \cdots \otimes d_{\lambda_k}$ and the projection map to $c_{\lambda_1^\vee} \otimes c_{\lambda_2^\vee} \otimes \cdots \otimes c_{\lambda_m^\vee}$.

By Lemma 4.6.12, the image of $\phi(e_\lambda)$ must, therefore, be either 0 or the irreducible V_λ. Hence, applying Schur's Lemma, $\phi(e_\lambda)|_{V_\lambda}$ is an isomorphism or the zero homomorphism. Fortunately, $\phi(e_\lambda)$ cannot be the zero homomorphism since Theorem 4.6.11 says that ϕ is surjective and by Lemma 4.6.12, no other ideal $H_n e_\mu H_n$ could possibly contain the preimage of the projection map for V_λ. Therefore, $\phi(e_\lambda)$ must be an isomorphism of V_λ and $\phi(e_\lambda^2) = \phi(e_\lambda)^2 = \alpha_\lambda \phi(e_\lambda)$. We know that α_λ is non-zero, and therefore $\phi(e_\lambda)$ must be a scalar multiple of the identity when restricted to V_λ. Hence $\phi(e_\lambda)$ is α_λ times the projection map as required.

$\underline{\lambda = d_n.}$

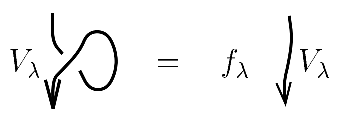
Set $\lambda' = d_{n-1}$. We know that $(e_{\lambda'} \otimes 1)e_\lambda$ is a non-zero scalar multiple of the element e_λ. Hence the image of $\phi(e_\lambda)$ is a summand of the image of $\phi(e_{\lambda'} \otimes 1)$. By the induction hypothesis, the image of $\phi(e_\lambda)$ is therefore a summand of $d_{n-1} \otimes V_\square$. This decomposes into two summands, $d_n + V_\mu$, where $\mu = (n-1, 1)$. But we have already shown that $\phi(e_\mu)$ is the projection map from $V_\square^{\otimes n}$ onto V_μ and since λ and μ are orthogonal, the image of $\phi(e_\lambda)$ must be disjoint from V_μ. Hence it must be d_n. We use similar arguments to those above to show that $\phi(e_\lambda)$ is not the zero map and is in fact the required projection. The proof for the case $\lambda = c_n$ is similar.

■

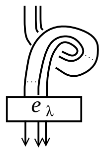
4.6.15 Remarks and notation.

Note that the above proof implies that the endomorphism $\phi(e_\lambda)$ is the zero map if and only if it is projecting onto the zero module in $U_q(sl(N))$. Therefore, $\phi(e_\lambda)$ will be the zero homomorphism if and only if λ has more than N rows.

A consequence of the above result is that e_λ behaves nicely under framing change. In the quantum invariant set up, we know that



for some scalar, f_λ. It follows that



$$= \quad f_\lambda e_\lambda \,.$$

The value of f_λ, calculated in [M1] by evaluating the full twist coloured by $c_1^{\otimes n}$ in two ways, is

$$f_\lambda = x^{|\lambda|^2} v^{-|\lambda|} s^{n_\lambda}$$

where

$$n_\lambda = \frac{|\lambda|(|\lambda|-1)\sigma_\lambda}{d_\lambda}$$

and σ_λ and d_λ are as defined in Definition 2.4.3. It is not difficult to show that n_λ is always an integer. It satisfies the following recursive relation. Let λ' denote the Young diagram obtained from λ by removing the first column (which contains $\lambda_1^\vee$ cells). Then

$$n_\lambda = n_{\lambda'} + 2|\lambda'| + \lambda_1^\vee - (\lambda_1^\vee)^2 \,.$$

We will now prove the Satellite Theorem 3.7.3. Write ε_λ for the genuine idempotent $\frac{1}{\alpha_\lambda} e_\lambda$. Let $Q_\lambda = \widehat{\varepsilon}_\lambda$ be its closure in $\mathcal{C}^+$.

4.6.16 Theorem.

Let C be a framed knot coloured by the irreducible representation V_λ. Let S be the satellite knot $C * Q_\lambda$ with companion C and pattern Q_λ. Then

$$J(C; V_\lambda) = \mathcal{X}_N(S) \,.$$

The result also holds for links where each component coloured by V_λ is decorated by Q_λ.

Proof. If λ has n cells then V_λ is a summand of $V_\square^{\otimes n}$ with multiplicity d_λ. The knot C is the closure of some $(1,1)$ tangle, say T. Let $T^{(n)}$ be the n-string parallel of T. Then

$$\begin{array}{cccc} J(T^{(n)}) & : & V_\square^{\otimes n} & \longrightarrow & V_\square^{\otimes n} \\ & & \sum_{|\mu|=n} d_\mu V_\mu & \mapsto & \sum_{|\mu|=n} t_\mu d_\mu V_\mu \end{array}$$

for scalars

$$t_\mu = J(T; V_\mu) \in \Lambda \,.$$

Now S is the closure of $\varepsilon_\lambda \circ T^{(n)}$. Therefore, applying Proposition 3.6.7 and Theorem 3.7.1

$$\begin{aligned}
\mathcal{X}_N(S) &= J(S; V_\Box, V_\Box, \cdots, V_\Box) \\
&= tr_q(J(\varepsilon_\lambda) \circ J(T^{(n)}; V_\Box, V_\Box, \cdots, V_\Box)) \\
&= tr_q(J(\varepsilon_\lambda) \circ J(T; V_\Box^{\otimes n}))\,.
\end{aligned}$$

By Theorem 4.6.14, we know that $J(\varepsilon_\lambda)$ is the projection onto the irreducible summand V_λ, therefore,

$$tr_q(J(\varepsilon_\lambda) \circ J(T; V_\Box^{\otimes n})) = J(C; V_\lambda)\,.$$

Now, assume that L is a link with k components and that the kth component is coloured by V_λ, where $|\lambda| = n$. We can present L as the closure of a $(1,1)$-tangle, T, (by cutting open the kth component). Let $T^{(n)}$ be the tangle obtained from T by taking the n-string parallel of the kth component of L. Then

$$\begin{aligned}
J(L; V_{\lambda_1}, \ldots, V_{\lambda_{k-1}}, V_{\lambda_k}) &= tr_q(J(T; V_{\lambda_1}, \ldots, V_{\lambda_{k-1}}, V_{\lambda_k})) \\
&= tr_q(J(\varepsilon_\lambda T^{(n)}; (V_{\lambda_1}, \ldots, V_{\lambda_{k-1}}, V_\Box, \ldots, V_\Box)) \\
&= tr_q(J(\varepsilon_\lambda) \circ J(T^{(n)}; V_{\lambda_1}, \ldots, V_{\lambda_{k-1}}, V_\Box, \ldots, V_\Box)) \\
&= J(L'; V_{\lambda_1}, \ldots, V_{\lambda_{k-1}}, V_\Box, \ldots, V_\Box)\,,
\end{aligned}$$

where L' is the link obtained from L by decorating the kth component with Q_{λ_k}. We can repeat this process for each component, to obtain a link with every component coloured by $V_\Box$ and we can then apply Theorem 3.7.1. ■

4.6.17 Comments and notation.

Theorem 4.6.16 holds only if we are very careful about how we index the representations of $U_q(sl(N))$. We must restrict ourselves to those Young diagrams with fewer than N rows. Later we will show that if we instead restrict our set of colours to the representations which correspond to Young diagrams with a single column then if c_k is the zero module (i.e. $k > N$) then the substitutions for x and v will kill $\mathcal{X}_N$. Since the representation ring $\mathcal{R}_N$ is generated as an algebra by these representations, in theory, we can calculate all possible $U_q(sl(N))$-invariants by colouring with polynomials in this restricted set of colours. Hence we can calculate the quantum invariants of any link with any colouring for all N at once.

We will show that the patterns Q_λ are given by the Giambelli polynomials for the Young diagrams λ. From this and the above comment it will follow that we will be able to calculate the quantum invariants for all N at once without having to express every colour as a polynomial in the Young diagrams with a single row.

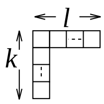
4.7 Genuine idempotents and the skein of the annulus.

The last theorem required genuine idempotent elements of the Hecke algebras. So far, we have only calculated the scalar α_λ for three Young diagrams, namely □, □□ and ⊟.

Here we calculate the scalars α for the Young diagrams c_k, d_l and $\mu_{k,l}$ for every k, $l \in \mathbb{N}$.

4.7.1 Remarks and notation.

To avoid clumsy notation, we shall denote the quasi-idempotent associated to the Young diagram $\mu_{k,l}$ by $e_{k,l}$, where $\mu_{k,l}$ is the hook-shaped diagram as defined in Remarks 2.5.9 and shown below.

$\mu_{k,l} =$ 

Similarly, we will denote the scalar by $\alpha_{k,l}$ rather than $\alpha_{\mu_{k,l}}$. Hence

$$e_{k,l}^2 = \alpha_{k,l} e_{k,l} \, .$$

We will also denote by $Q_{k,l}$ the closure of the genuine idempotent in $\mathcal{C}^+$ related to the Young diagram $\mu_{k,l}$:

$$Q_{k,l} = \frac{1}{\alpha_{k,l}} \, \widehat{e}_{k,l} \, .$$

In the diagrams, we will represent the elements $E_i(a)$ and $E_j(b)$ by boxes. The boxes representing $E_j(b)$, corresponding to columns in the Young diagram, will be shaded to distinguish them from those boxes representing the elements

$E_i(a)$, which correspond to the rows. We shall label the boxes by the number of strings they involve. For example, $e_{k,1}$ will be pictured as a shaded box bearing the label k.

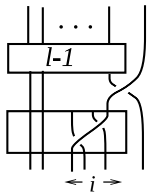
If $k' \geq k$ then $e_{k',1}e_{k,1} = \phi_a(e_{k,1})e_{k',1}$ and if $l' \geq l$ then $e_{1,l'}e_{1,l} = \phi_b(e_{1,l})e_{1,l'}$. Therefore, $\alpha_{k,1} = \phi_a(e_{k,1})$, which we calculate by induction.

First we will prove a Lemma which will be used repeatedly to manipulate the quasi-idempotents.

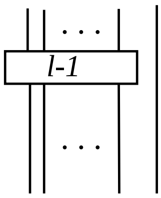
4.7.2 Lemma.

In H_n, we can decompose $e_{1,l}$ into a linear combination of terms which involve $e_{1,l-1}$:

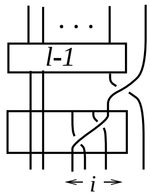
$$e_{1,l} = e_{1,l-1} \otimes e_{1,1} + \sum_{i=0}^{l-2} (x^{-1}s)^{i+1}$$



$$=$$



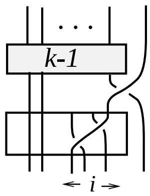
$$+ \sum_{i=0}^{l-2} (x^{-1}s)^{i+1}$$



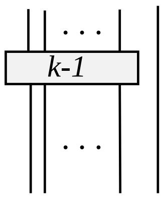
.

Similarly,

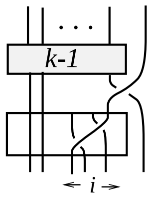
$$e_{k,1} = e_{k-1,1} \otimes e_{1,1} + \sum_{i=0}^{k-2} (-x^{-1}s^{-1})^{i+1}$$



$$=$$



$$+ \sum_{i=0}^{k-2} (-x^{-1}s^{-1})^{i+1}$$



.

Proof. We give the proof for $e_{k,1}$. The proof of the result for $e_{1,l}$ differs only in the weights assigned to the crossings.

Let ω_π be a positive permutation braid on k strings, with $\pi(k) = i$. Define a permutation $\pi' \in S_k$ by

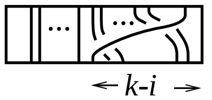
$$\pi'(j) = \begin{cases} \pi(j) & 1 \le \pi(j) < i \\ k & j = k \\ \pi(j) - 1 & i < \pi(j) \le k \end{cases} \tag{4.7}$$

Note that π' is uniquely determined by π. Let π_i denote the permutation

$$\pi_i = (i \;\; i+1 \;\; \ldots \;\; k) \in S_k \,.$$

The positive permutation braid ω_{π_i} has $k - i$ positive crossings,

$$\omega_{\pi_i} \quad = $$



The braid $\omega_{\pi'}\omega_{\pi_i}$ is a positive permutation braid, since the kth string doesn't cross any string in $\omega_{\pi'}$. Hence

$$\omega_{\pi'}\omega_{\pi_i} = \omega_\rho$$

for some permutation $\rho \in S_k$. Now

$$\rho(j) = \pi_i(\pi'(j)) \,,$$

hence,

$$\rho(j) = \begin{cases} \pi'(j) & 1 \le \pi'(j) < i \\ \pi'(j) + 1 & i \le \pi'(j) < k \\ i & \pi'(j) = k \end{cases} .$$

From its definition, $\pi'(j) = \pi(j)$ for $\pi'(j) < i$, and when $i \le \pi'(j) < k$, then $\pi'(j) = \pi(j) - 1$. Finally if $\pi'(j) = k$ then $j = k$, therefore

$$\rho(j) = \begin{cases} \pi(j) & 1 \le \pi'(j) < i \\ \pi(j) - 1 + 1 & i \le \pi'(j) < k \\ i = \pi(k) & j = k \end{cases} .$$

Therefore,

$$\omega_\pi = \omega_{\pi'}\omega_{\pi_i} \,.$$

Given a permutation $\pi \in S_k$ and a fixed i, the permutation π' is uniquely determined by π. Suppose we have two permutations π_1,$\pi_2 \in S_k$ with $\pi_1(k) = \pi_2(k) = i$ and $\pi_1^{'} = \pi_2^{'}$. When $1 \le \pi_1(j) < i$, then

$$\pi_1^{'}(j) = \pi_2^{'}(j) = \pi_1(j) \,.$$

Therefore, $1 \le \pi_2'(j) < i$ and this implies that

$$1 \le \pi_2(j) < i\,.$$

Hence,

$$\pi_2'(j) = \pi_2(j) = \pi_1(j)\,.$$

We can similarly show that for $i < \pi_1(j) \le k$,

$$\pi_1(j) = \pi_2(j)\,,$$

proving uniqueness.

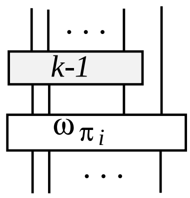
Now, suppose we take $\rho \in S_k$ for which $\rho(k) = k$ and fix i. Define π as follows,

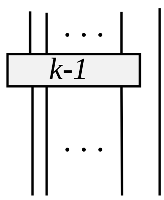
$$\pi(j) = \begin{cases} \rho(j) & 1 \le \rho(j) < i \\ \rho(j) + 1 & i \le \rho(j) < k \\ i & j = k\,. \end{cases}$$

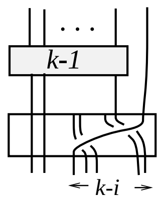
Then π is uniquely determined by ρ and $\rho = \pi'$.

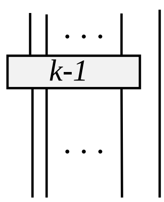
We have established that any positive permutation braid, ω_π, with $\pi(k) = i$ can be written uniquely as a product of a permutation braid in which the kth string passes straight through and ω_{π_i} and that all such products occur.

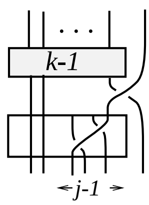
Hence, taking into account the weighting given to the crossings in $e_{k,1}$, we get

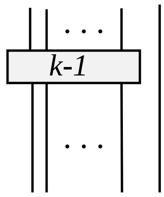
$e_{k,1} = \sum_{i=1}^{k} (-x^{-1}s^{-1})^{k-i}$ 

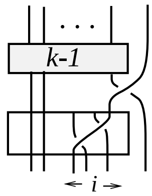
$=$ 

$+ \sum_{i=1}^{k-1} (-x^{-1}s^{-1})^{k-i}$ 

$=$ 

$+ \sum_{j=1}^{k-1} (-x^{-1}s^{-1})^{j}$ 

$=$ 

$+ \sum_{i=0}^{k-2}(-x^{-1}s^{-1})^{i+1}$ 

as required. ∎

4.7.3 Proposition.

The scalar $\alpha_{k,1}$ (associated to the Young diagram c_k) is given by

$$\alpha_{k,1} = \frac{[\,k\,]!}{s^{k(k-1)/2}}\,.$$

The scalar for $\alpha_{1,l}$ (associated to d_l) is given by

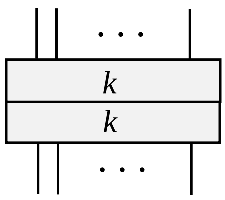
$$\alpha_{1,l} = \frac{[\,l\,]!}{s^{-l(l-1)/2}}\,.$$

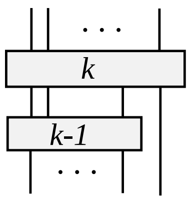
Proof. We will prove the result for the Young diagrams c_k. The proof for d_l is almost identical.

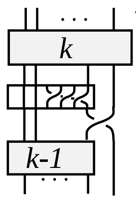
First, recall that $e_{k,1} = b_k$ as defined in Definition 4.4.3, and that b_k could "swallow" any braid on k strings at the expense of a scalar as described in Theorem 4.4.7. The proof goes by induction.

For $k = 1$ we know that $\alpha_{k,1} = 1 = 1/s^0$.

For $k > 1$, note that for any $i \in \mathbb{N}$, $\alpha_{i,1} = \phi_a(e_{i,1})$. By Lemma 4.7.2

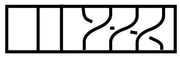
$e_{k,1}^2 \;=$ 

$=$ 

$+ \sum_{i=0}^{k-2} \Bigg((-x^{-1}s^{-1})^{i+1}$ 

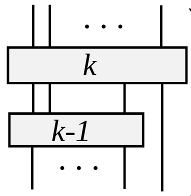
$\Bigg)$

where the braid



in the ith summand has i positive crossings. Therefore, by the induction hypothesis and Theorem 4.4.7

$$e_{k,1}^2 = \frac{[\,k-1\,]!}{s^{(k-1)(k-2)/2}} e_{k,1} + \sum_{i=0}^{k-2} \left((-x^{-1}s^{-1})^{i+1}(-xs^{-1})^{i+1} \right.$$



$$\begin{aligned}
&= \left(\frac{[\,k-1\,]!}{s^{(k^2-3k+2)/2}} + \frac{[\,k-1\,]!}{s^{(k^2-3k+2)/2}} \sum_{i=0}^{k-2} s^{-2i-2} \right) e_{k,1} \\
&= \frac{[\,k-1\,]!}{s^{(k^2-k)/2}} \left(\frac{1 + s^{-k}[\,k-1\,]}{s^{-k+1}} \right) e_{k,1} \\
&= \frac{[\,k-1\,]!}{s^{(k^2-k)/2}} \left(\frac{(s-s^{-1}) + s^{-k}(s^{k-1} - s^{-k+1})}{(s-s^{-1})s^{-k+1}} \right) e_{k,1} \\
&= \frac{[\,k-1\,]!}{s^{(k^2-k)/2}} \left(\frac{s^{k-1}s - s^{-2k+1}s^{k-1}}{(s-s^{-1})} \right) e_{k,1} \\
&= \frac{[\,k\,]!}{s^{k(k-1)/2}} e_{k,1}
\end{aligned}$$

as required. ■

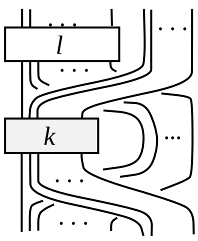
4.7.4 Proposition.

The scalar $\alpha_{k,l}$, defined in Remarks 4.7.1, is

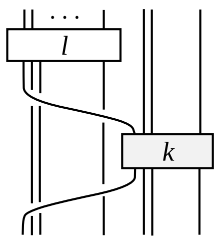
$$\alpha_{k,l} = \frac{[\,k+l-1\,][\,k-1\,]![\,l-1\,]!}{s^{(k(k-1)-l(l-1))/2}}.$$

Proof. First, we give a different presentation of $e_{k,l}$, which is easier to work with,

$e_{k,l}$ =



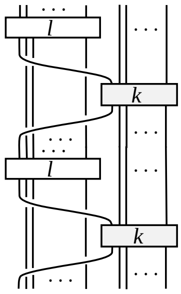
=



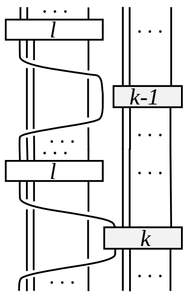
.

Therefore, applying Lemma 4.7.2

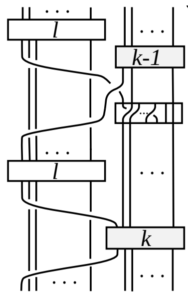
$$e_{k,l}^2 \quad =$$



$$=$$



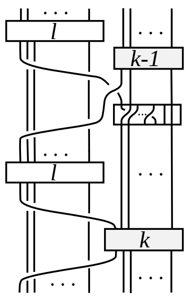
$$+ \quad \sum_{i=0}^{k-2} \left((-x^{-1}s^{-1})^{i+1} \right.$$



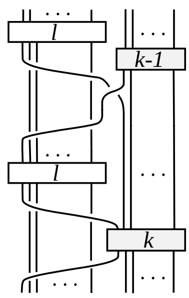
$$\left. \right) .$$

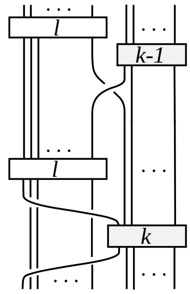
Now, apply Theorem 4.4.7, first to the boxes with i positive crossings and then to the strings which join white and shaded boxes :

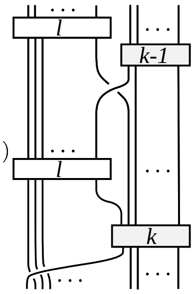
$$\sum_{i=0}^{k-2} (-x^{-1}s^{-1})^{i+1}$$



$$= \quad \left(\sum_{i=0}^{k-2} (-x^{-1}s^{-1})^{i+1} (-xs^{-1})^{i} \right)$$



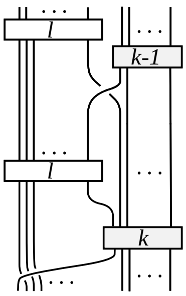
$$= \quad -x^{-1}\left(\sum_{i=0}^{k-2} s^{-2i-1}\right)(xs)^{l-1}(xs)^{-(l-1)}$$ 

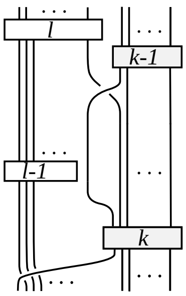
$$= \quad -x^{-1}(s^{-1}+\cdots+s^{-2k+3})(xs)^{-(l-1)}$$  .

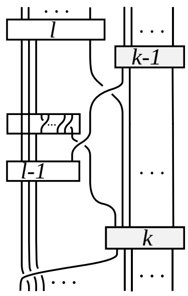
The coefficient of the last diagram simplifies

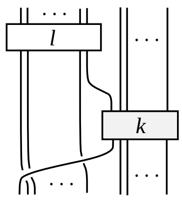
$$\begin{aligned} x^{-1}(s^{-1}+\cdots+s^{-2k+3})(xs)^{-(l-1)} &= x^{-1}s^{-k+1}(s^{k-2}+\cdots+s^{-k+2})(xs)^{-(l-1)} \\ &= x^{-1}s^{-k+1}[k-1](xs)^{-(l-1)}. \end{aligned}$$

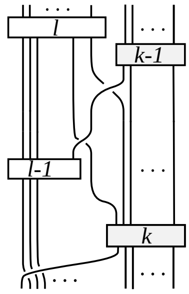
Apply Lemma 4.7.2 to the lower of the two rows of l cells and then use Theorem 4.4.7 to obtain



$=$ 

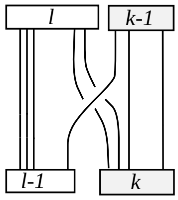
$+ \quad \sum_{j=0}^{l-2}(x^{-1}s)^{j+1}$ 

$$= \quad (-xs^{-1})\alpha_{1,l-1}\alpha_{k-1,1}$$ 

$$+ \quad \left(\sum_{j=0}^{l-2} (x^{-1}s)^{j+1}(xs)^j \right)$$ 

.

Note that

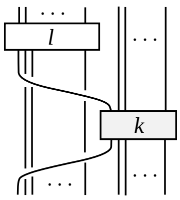


$= 0\,.$

To see this, note that the $l-1$st and lth strings are two strings of $e_{1,l}$ which are also adjacent strings in $e_{k,1}$. We can factorise both $e_{1,l}$ and $e_{k,1}$ by Proposition 4.4.5, to obtain the term

$$(\sigma_{l-1} - a)(\sigma_{l-1} - b) = 0\,.$$

Therefore,

$$e^2_{k,l} \;=\; \alpha_{k-1,1}\alpha_{1,l}e_{k,l} \quad + \quad s^{-k}[k-1](xs)^{-(l-1)}(xs)^{l-1}\alpha_{k-1,1}\alpha_{1,l-1}$$ 

$$= \; (\alpha_{k-1,1}\alpha_{1,l} \,+\, s^{-k}[k-1]\alpha_{k-1,1}\alpha_{1,l-1})e_{k,l}$$

Now, applying Proposition 4.7.3,

$$\begin{aligned}
\alpha_{k-1,1}\alpha_{1,l} \;&+\; s^{-k}[k-1]\alpha_{k-1,1}\alpha_{1,l-1} \\
&= \frac{[k-1]!\,[l-1]!}{s^{(k-1)(k-2)/2}} \left(\frac{[l]}{s^{-l(l-1)/2}} + \frac{s^{-k}[k-1]}{s^{-(l-1)(l-2)/2}} \right) \\
&= \frac{[k-1]!\,[l-1]!}{s^{(k-1)(k-2)/2}} \left(\frac{s^{l-1}[l] \,+\, s^{-k}[k-1]}{s^{-l(l-1)/2}s^{l-1}} \right) \\
&= \frac{[k-1]!\,[l-1]!}{s^{(k-1)(k-2)/2}} \left(\frac{s^{l-k}\big(s^{l-k-1} - s^{-l-k-1} + s^{k-1-l} - s^{-k+1-l}\big)}{s^{-l(l-1)/2}s^{l-1}\big(s - s^{-1}\big)} \right) \\
&= \frac{[k-1]!\,[l-1]!}{s^{(k-1)(k-2)/2}s^{-l(l-1)/2}} \left(s^{-k+1}\frac{\big(s^{l+k-1} - s^{-l-k+1}\big)}{\big(s - s^{-1}\big)} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{[k-1]!\,[l-1]!\,[k+l-1]}{s^{\{(k-1)(k-2)+2(k-1)\}/2}\,s^{-l(l-1)/2}} \\
&= \frac{[k-1]!\,[l-1]!\,[k+l-1]}{s^{k(k-1)/2-l(l-1)/2}}
\end{aligned}
$$

as required. ■

4.7.5 Remarks.

We can easily show that the scalar $\alpha_{k,l}$ specialises to the scalar $a_{\mu_{k,l}}$, defined in Remarks 4.3.8.

For the Young digram $\mu_{k,l}$ the hook length of the corner cell is given by $k+l-1$. Down the column the cells have hook lengths $k-1$, $k-2$, ..., 2, 1. Along the row, the product of the hook lengths is $(l-1)!$. Therefore,

$$\prod \text{hook lengths} = (k+l+1)(k-1)!(l-1)!$$

Now, since $[\,i\,] \to i$ as $s \to 1$, it follows that $a_{k,l}$ specialises to a_λ as expected.

Recall that Q_λ denotes the closure of the genuine idempotent $(1/\alpha_\lambda)\ e_\lambda$.

4.7.6 Proposition.

The elements Q_λ form a free Λ-basis for $\mathcal{C}^+$.

Proof. We will show that the set $\{Q_\lambda \ : \ |\lambda| = n\}$ is a Λ- basis for $\mathcal{C}^{(n)}$. By Theorem 2.3.10 we have a surjective linear map

$$w : \mathcal{S}(R_n^n) \to \mathcal{C}^{(n)}\,.$$

By Proposition 4.2.2, since $\mathcal{S}(R_n^n)$ is isomorphic to H_n, every element of $\mathcal{C}^{(n)}$ can be expressed as the closure of some element $h \in H_n$.

Let $c \in \mathcal{C}^{(n)}$. Choose $h \in H_n$ such that $\widehat{h} = c$. By Theorem 4.6.9, we can write

$$
\begin{aligned}
h &= \sum_{|\lambda|=n} h'_\lambda e_\lambda h''_\lambda \\
&= \sum_{|\lambda|=n} (1/\alpha_\lambda)\ h'_\lambda e_\lambda^2 h''_\lambda\,.
\end{aligned}
$$

Therefore,

$$\begin{aligned} c &= \sum_{|\lambda|=n} (1/\alpha_\lambda) \;\; h'_\lambda \widehat{e_\lambda^2} h''_\lambda \\ &= \sum_{|\lambda|=n} (1/\alpha_\lambda) \;\; e_\lambda \widehat{h_\lambda} e_\lambda \end{aligned}$$

where

$$h_\lambda = h''_\lambda h'_\lambda \,.$$

The remainder of the proof applies the techniques used in the proof of Theorem 4.4.11 (the proof can be found on p.77). From the definition of e_λ,

$$e_\lambda h_\lambda e_\lambda = E_\lambda(a) \left(\omega_{\pi_\lambda} E_{\lambda^\vee}(b) \omega_{\pi_{\lambda^\vee}}^{-1} h_\lambda E_\lambda(a) \omega_{\pi_\lambda} \right) E_{\lambda^\vee}(b) \omega_{\pi_{\lambda^\vee}}^{-1} \,.$$

We can write $\omega_{\pi_\lambda} E_{\lambda^\vee}(b) \omega_{\pi_{\lambda^\vee}}^{-1} h_\lambda E_\lambda(a) \omega_{\pi_\lambda}$ as a linear combination of positive permutation braids, say $\sum_\tau \gamma_\tau \omega_\tau$. Therefore

$$e_\lambda h_\lambda e_\lambda = \sum_\tau \gamma_\tau E_\lambda(a) \omega_\tau E_{\lambda^\vee}(b) \omega_{\pi_{\lambda^\vee}}^{-1} \,.$$

If τ is a permutation which doesn't separate λ and $\lambda^\vee$ then

$$E_\lambda(a) \omega_\tau E_{\lambda^\vee}(b) \omega_{\pi_{\lambda^\vee}}^{-1} = 0 \,.$$

When τ does separate λ and $\lambda^\vee$, then by Lemma 4.5.7 we can find permutations $\rho_1 \in R(\lambda)$ and $\rho_2 \in R(\lambda^\vee)$ for which

$$\omega_\tau = \omega_{\rho_1} \omega_{\pi_\lambda} \omega_{\rho_2} \,.$$

Substituting this into our expression for $e_\lambda h_\lambda e_\lambda$, we can remove ω_{ρ_1} and ω_{ρ_2} at the expense of some scalar, $\beta(\tau) \in \Lambda$. Therefore,

$$e_\lambda h_\lambda e_\lambda = \sum_\tau \beta(\tau) \gamma_\tau e_\lambda \,.$$

Thus

$$\begin{aligned} c &= \sum_{|\lambda|=n} (1/\alpha_\lambda) \left(\sum_\tau \gamma_\tau \beta(\tau) \right) \widehat{e}_\lambda \\ &= \sum_{|\lambda|=n} \zeta_\lambda Q_\lambda \end{aligned}$$

where

$$\zeta_\lambda = \sum_\tau \gamma_\tau \beta(\tau) \in \Lambda \,.$$

We have, therefore, shown that given any element of $\mathcal{C}^{(n)}$ we can express it as a Λ-linear combination of the Q_λ for which λ has n cells. Therefore, $\{Q_\lambda \ : \ |\lambda| = n\}$ spans $\mathcal{C}^{(n)}$. In fact, this set is a basis. By Corollary 2.3.8 and Lemma 2.5.8 the Λ-dimension of $\mathcal{C}^{(n)}$ is equal to the number of partitions of n. We have a set of this cardinality which spans $\mathcal{C}^{(n)}$, hence, since Λ is a commutative ring, $\{Q_\lambda \ : \ \lambda \in Y\}$ is a Λ-basis for $\mathcal{C}^+$.

$\blacksquare$

We next wish to calculate the value of the framed Homfly polynomial $\mathcal{X}$ (defined in Definition 2.2.1) for the unknot decorated by $Q_{k,1}$. From this we derive the value of the unknot decorated by $Q_{1,l}$.

4.7.7 Proposition.

$$\mathcal{X}(Q_{k,1}) = \prod_{i=1}^{k} \frac{v^{-1}s^{-(i-1)} - vs^{i-1}}{s^i - s^{-i}}.$$

Proof. The result follows by an induction on k, starting at $k = 1$,

$$\mathcal{X}(Q_{1,1}) = \mathcal{X}\left(\bigcirc\right) = \frac{v^{-1} - v}{s - s^{-1}}.$$

Assume that we have the result for all $i < k$. It is enough to show that

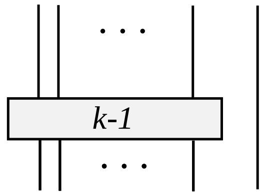
$$\mathcal{X}(Q_{k,1}) = \left(\frac{v^{-1}s^{-k+1} - vs^{k-1}}{s^k - s^{-k}}\right) \mathcal{X}(Q_{k-1,1}) \tag{4.8}$$

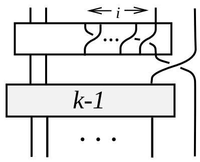
as we can then use the induction hypothesis to obtain

$$\begin{aligned} \mathcal{X}(Q_{k,1}) &= \left(\frac{v^{-1}s^{-k+1} - vs^{k-1}}{s^k - s^{-k}}\right) \prod_{i=1}^{k-1} \frac{v^{-1}s^{-i+1} - vs^{i-1}}{s^i - s^{-i}} \\ &= \prod_{i=1}^{k} \frac{v^{-1}s^{-i+1} - vs^{i-1}}{s^i - s^{-i}}. \end{aligned}$$

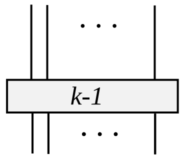
We now establish equation 4.8. Recall that since we are working in $\mathcal{C}^+$, we can slide pieces of tangle off the top of a diagram and reintroduce them at the bottom.

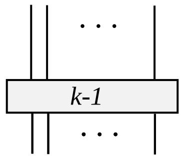
$$\mathcal{X}(Q_{k,1}) \;=\; \frac{1}{\alpha_{k,1}}\mathcal{X}(\hat{e}_{k,1})$$

$$= \frac{1}{\alpha_{k,1}} \mathcal{X} \Bigg($$


$$+ \sum_{i=0}^{k-2} (-x^{-1}s^{-1})^{i+1}$$


$$\Bigg)$$

$$= \frac{1}{\alpha_{k,1}} \mathcal{X} \Bigg(\frac{v^{-1} - v}{s - s^{-1}}$$


$$+ \sum_{i=0}^{k-2} (-x^{-1}s^{-1})^{i+1} (xs^{-1})^{i} (xv^{-1})$$


$$\Bigg)$$

$$\begin{aligned}
&= \frac{\alpha_{k-1,1}}{\alpha_{k,1}} \left(\frac{v^{-1} - v}{s - s^{-1}} - v \left(\sum_{i=0}^{k-2} s^{-2i-1} \right) \right) \mathcal{X}(Q_{k-1,1}) \\
&= \frac{[k-1]!}{[k]!} \frac{s^{k(k-1)/2}}{s^{(k-1)(k-2)/2}} \left(\frac{v^{-1} - v}{s - s^{-1}} - v^{-1}s^{-k+1}[k-1] \right) \mathcal{X}(Q_{k-1,1}) \\
&= \frac{s^{k-1}}{[k]} \left(\frac{v^{-1} - v - v^{-1}s^{-k+1}(s^{k-1} - s^{-k+1})}{s - s^{-1}} \right) \mathcal{X}(Q_{k-1,1}) \\
&= \left(\frac{s^{k-1}(v^{-1} - v - v^{-1} + v^{-1}s^{-2k+2})}{s^{k} - s^{-k}} \right) \mathcal{X}(Q_{k-1,1}) \\
&= \left(\frac{v^{-1}s^{-k+1} - vs^{k-1}}{s^{k} - s^{-k}} \right) \mathcal{X}(Q_{k-1,1}) .
\end{aligned}$$

Therefore,

$$\mathcal{X}(Q_{k,1}) = \prod_{i=1}^{k} \frac{v^{-1}s^{-i+1} - vs^{i-1}}{s^{i} - s^{-i}}$$

as required. ■

4.7.8 Corollary.

Evaluating the framed Homfly polynomial at $v = s^{-N}$ and $x = s^{-1/N}$,

$$\mathcal{X}_N(Q_{k,1}) = \begin{cases} 1 & \text{for } k = N \\ 0 & \text{for } k > N. \end{cases}$$

Proof. By Proposition 4.7.7

$$\mathcal{X}(Q_{k,1}) = \prod_{i=1}^{k} \frac{v^{-1}s^{-(i-1)} - vs^{i-1}}{s^i - s^{-i}}\,.$$

When $i = N+1$ and $v = s^{-N}$

$$\begin{aligned}
\frac{v^{-1}s^{-(i-1)} - vs^{i-1}}{s^i - s^{-i}} &= \frac{s^N s^{-N-1+1} - s^{-N}s^{N+1-1}}{s^{N+1} - s^{-(N+1)}} \\
&= \frac{1-1}{s^{N+1} - s^{-(N+1)}} \\
&= 0\,.
\end{aligned}$$

Therefore, since this term is a factor of $\mathcal{X}_N(Q_{k,1})$, for $k > N$,

$$\mathcal{X}_N(Q_{k,1}) = 0 \qquad \text{for } k > N\,.$$

By Proposition 4.7.7

$$\begin{aligned}
\mathcal{X}_N(Q_{N,1}) &= \frac{(v^{-1} - v)}{(s - s^{-1})}\,\frac{(v^{-1}s^{-1} - vs)}{(s^2 - s^{-2})} \cdots \frac{(v^{-1}s^{-N+1} - vs^{N-1})}{(s^N - s^{-N})} \\
&= \frac{(s^N - s^{-N})}{(s - s^{-1})}\,\frac{(s^{N-1} - s^{-N+1})}{(s^2 - s^{-2})} \cdots \frac{(s - s^{-1})}{(s^N - s^{-N})} \\
&= 1\,.
\end{aligned}$$

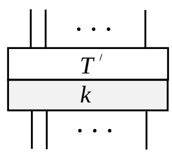
■

4.7.9 Proposition.

Let L be a link with n components. Let L' be the satellite of L, obtained from L by decorating the ith component by the pattern Q_{λ_i}, for $i < n$ and the nth component of L by $Q_{k,1}$. If $k > N$ then

$$\mathcal{X}_N(L') = 0\,.$$

Proof. The link L can be presented as the closure of a $(1,1)$-tangle, T, on the nth component. Therefore, we can present L' as the closure of $T' \circ \varepsilon_{k,1}$ which is a (k,k) tangle. (Here $\varepsilon_{k,1} = (1/\alpha_{k,1})\, e_{k,1}$.)

$L' \quad = \quad 1/\alpha_{k,1}$ 

Since T' is an element of $\mathcal{S}(R_k^k)$, it can be swallowed up by $e_{k,1}$ at the expense of a scalar, $\gamma(T')$. Therefore,

$$\mathcal{X}_N(\widehat{T' e_{k,1}}) = \gamma(T')\mathcal{X}_N(\hat{e}_{k,1})\,.$$

Now by Proposition 4.7.7, since $k > N$,

$$\mathcal{X}_N(\hat{e}_{k,1}) = 0\,,$$

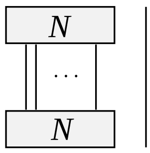
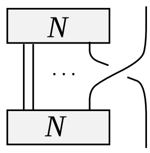
therefore,

$$\mathcal{X}_N(L') = 0\,.$$

■

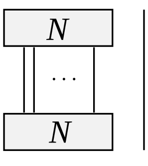
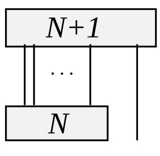
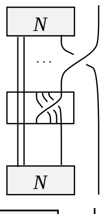
4.7.10 Lemma.

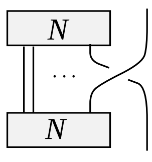
The following relation holds, at $v = s^{-N}$ and $x = s^{-1/N}$:

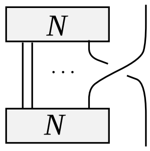
$\mathcal{X}_N\Bigg($  $\Bigg)$ $= x^{-1}s^{-N}[N]\;\mathcal{X}_N\Bigg($  $\Bigg)$

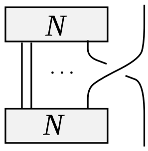
where the relationship is local, i.e. if two link diagrams are identical except where shown in the above relation, then $\mathcal{X}_N$ will evaluate to the same value on either diagram.

Proof. For readability, we express the equalities as equalities between diagrams, despite the fact it is the values of $\mathcal{X}_N$ for each diagram which are equal.

 $=$  $- \sum_{i=0}^{N_1}(-x^{-1}s^{-1})^{i+1}$  by Lemma 4.7.2

$= -\left(\sum_{i=0}^{N-1}(-x^{-1}s^{-1})^{i+1}(-xs^{-1})^{i}\right)$  by Prop. 4.7.9

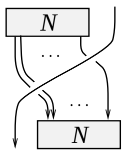
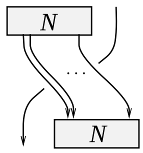
$$= \quad x^{-1}\left(\sum_{i=0}^{N-1} s^{-2i-1}\right)$$ 

$$= \quad x^{-1}s^{-N}[N]$$  .

∎

4.7.11 Lemma.

Let L_t and L_b be two link diagrams which differ only where shown.

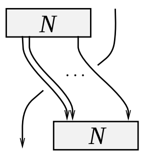
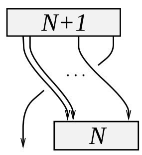
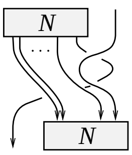
 

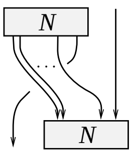
L_t $\qquad$ L_b

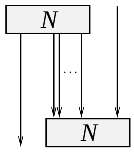
The framed Homfly polynomial of L_t is equal to that of L_b,

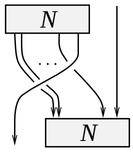
$$\mathcal{X}_N(L_t) = \mathcal{X}_N(L_b)\,.$$

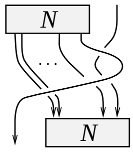
Proof. To prove this, we consider the value of the framed Homfly polynomial, $\mathcal{X}$. To make the proof easier to read, we will write down equalities as if they held for the diagrams, although the equalities will hold only for their quantum invariants.

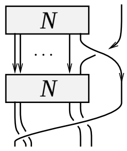
 $=$  $+ \quad x^{-1}s^{-N}$  by Lemma 4.7.10

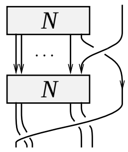
$= \quad x^{-1}s^{-N}[N]$ 

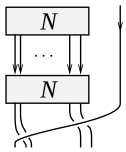
$$= \quad x^{-1}s^{-N}[N](-xs^{-1})^{-N+1}$$  by Theorem 4.4.7

$$= \quad x^{-1}s^{-N}[N](-xs^{-1})^{-2(N-1)}$$ 

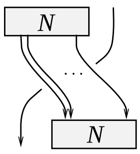
$$= \quad x^{-1}s^{-N}[N](-xs^{-1})^{-2(N-1)}$$ 

$$= \quad x^{-1}s^{-N}[N](-xs^{-1})^{-2(N-1)}$$ 

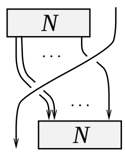
$$= \quad x^{-1}s^{-N}[N](-xs^{-1})^{-2(N-1)} \Bigg(x^{-2}$$ 

$$- x^{-1}(s - s^{-1})$$  $\Bigg)$.

We can apply Lemma 4.7.10 to the first term of the expression on the righthand side, to obtain



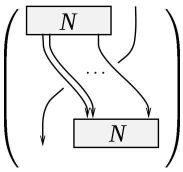
$$= \quad \Bigg(\frac{x^{-3}s^{-N}[N](-xs^{-1})^{-2(N-1)}}{x^{-1}s^{-N}[N]}$$

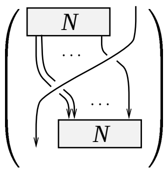
$$- \; x^{-2}s^{-N}[N](s - s^{-1})(-xs^{-1})^{-2(N-1)} \Bigg)$$ 

We now simplify the coefficient of L_t in the above expression.

$$
\begin{aligned}
\frac{x^{-3}s^{-N}[N](-xs^{-1})^{-2(N-1)}}{x^{-1}s^{-N}[N]} & - \ x^{-2}s^{-N}[N](s-s^{-1})(-xs^{-1})^{-2(N-1)} \\
& = \ x^{-2N}s^{2N-2} - x^{-2N}s^{-N}(s^{N}-s^{-N})s^{2N-2} \\
& = \ x^{-2N}s^{2N-2} - x^{-2N}s^{N-2}(s^{N}-s^{-N}) \\
& = \ x^{-2N}(s^{2N-2} - s^{2N-2} + s^{-2}) \\
& = \ x^{-2N}s^{-2}
\end{aligned}
$$

Upon setting $x = s^{-1/N}$, we have that

$\mathcal{X}_N$ 

$=$ $\mathcal{X}_N$ 

as required. ∎

Lemma 4.7.11 is a specific case of [Y, Lemma 1.3], namely [Y, equation 20]. The more general result in not needed in this thesis.

4.7.12 Proposition.

Let $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_n$ be an n-component link. Denote the link obtained from L by removing the nth component by L'. Suppose the ith component is coloured by V_{λ_i}, for $i \leq n$ and that $V_{\lambda_n} = c_N$. Then

$$\mathcal{X}_N(L) = \mathcal{X}_N(L')\,.$$

Proof. By Lemma 4.7.11, we can switch the sign of any crossing which involves the nth component of L without altering the value of the invariant. Therefore, we can assume the nth component of L is unlinked from all the other components. Further, we can unknot the component to obtain a distant union of L' and an unknot decorated by $Q_{N,1}$. Therefore, evaluating $\mathcal{X}_N$, we get

$$
\begin{aligned}
\mathcal{X}_N(L) & = \mathcal{X}_N(L' \sqcup Q_{N,1}) \\
& = \mathcal{X}_N(L')\mathcal{X}_N(Q_{N,1}) \\
& = \mathcal{X}_N(L') \qquad \text{by Proposition 4.7.8.}
\end{aligned}
$$

∎

4.7.13 Remarks.

A consequence of Propositions 4.7.9 and 4.7.12 is that we can calculate the $U_q(sl(N))$ invariants for all N at once, so long as we stick to colouring links with the c_i. However, since the c_i generate $\mathcal{R}_\infty$ as an algebra, we can calculate every quantum invariant, as a linear combination of links coloured by the c_i.

We next demonstrate that a relationship which we know to hold in $\mathcal{R}_\infty$ also holds in $\mathcal{C}^+$ and go on to calculate the value of $\mathcal{X}$ for the unknot decorated by $Q_{1,l}$. It is not difficult to establish the value of $\mathcal{X}(Q_{1,l})$ directly (as we did for $\mathcal{X}(Q_{k,1})$, but the indirect approach taken here is used in the proof of Proposition 4.9.8. We first establish a skein theoretic result for $\mathcal{C}^+$.

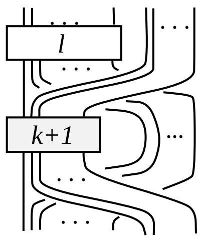
4.7.14 Lemma.

Consider the closures of the $e_{k,l}$ in the skein of the annulus, i.e. $\widehat{e}_{k,l} \in \mathcal{C}^+$. The following relation holds :

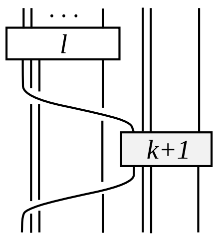
$$s^l[\,l\,]\widehat{e}_{k+1,l} + s^{-k}[\,k\,]\widehat{e}_{k,l+1} = s^{l-k}[\,l+k\,]\,\widehat{e}_{1,l}\,\widehat{e}_{k,1}\,.$$

Proof. For the sake of simplicity, the pictures only show the tangle which we close to form an element $\mathcal{S}(S^1 \times I)$. However, since we are working with the closures, we can slide pieces of the diagram off the top and place them on the bottom of the picture without changing the diagram as an element in the skein of the annulus.

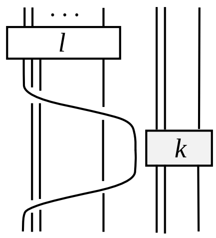
$\widehat{e}_{k+1,l} \quad =$



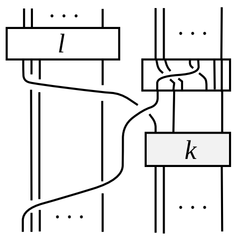
$=$



$=$

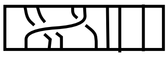


$+ \quad \sum_{i=0}^{k-1} \left((-x^{-1}s^{-1})^{i+1} \right.$

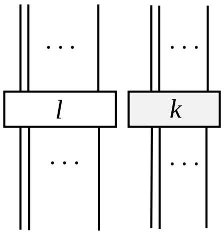
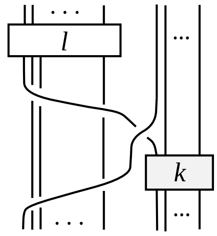


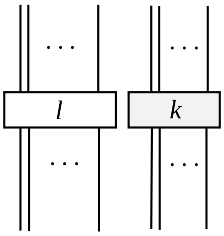
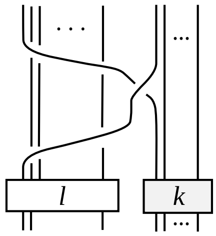
$\left. \right) ,$

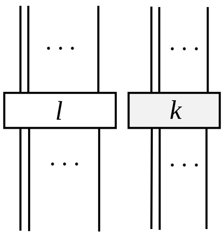
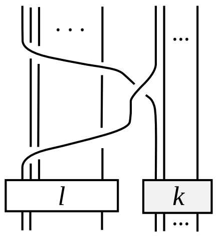
where there are i crossings in the braid

.

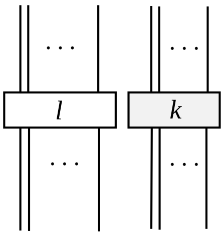
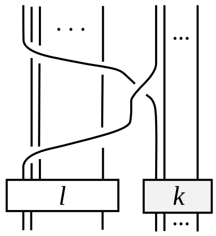
Since we are working in the skein of the annulus, we can slide this braid around the annulus so that it appears under $e_{k,1}$. The properties of $e_{k,1}$ mean that we can remove each of these braids at the expense of a scalar, $(-xs^{-1})^i$. We obtain

$\hat{e}_{k+1,l} \quad =$  $+ \sum_{i=0}^{k-1} \Big((-x^{-1}s^{-1})^{i+1}(-xs^{-1})^i$  $\Big)$

$=$  $- \quad x^{-1}\left(\sum_{i=0}^{k-1} s^{-2i-1}\right)$ 

$=$  $- \quad x^{-1}s^{-k}[\,k\,]$  .

Using exactly the same technique on $\widehat{e}_{k,l+1}$ we obtain

$\widehat{e}_{k,l+1} \quad =$  $+ \quad x^{-1}s^{l}[\,l\,]$  .

Hence

$$\begin{aligned} s^l[\,l\,]\widehat{e}_{k+1,l} + s^{-k}[\,k\,]\,\widehat{e}_{k,l+1} &= (s^l[\,l\,] + s^{-k}[\,k\,])\,\widehat{e}_{1,l}\,\widehat{e}_{k,1} \\ &= s^{l-k}\,[\,l+k\,]\,\widehat{e}_{1,l}\,\widehat{e}_{k,1} \end{aligned}$$

■

4.7.15 Comment.

A version of Lemma 4.7.14 appears as [Y, Lemma 1.2]. However, since Yokota's quasi-idempotents contain an extra row of antisymmetrisers, the expression involves a term which is not one of the quasi-idempotents. Thus the idea that this is the "Giambelli polynomial" for the decomposition of $e_{k,1}e_{1,l}$ is not emphasised.

4.7.16 Proposition.

The idempotents $Q_{k,l}$ satisfy

$$Q_{k+1,l} + Q_{k,l+1} = Q_{k,1}Q_{1,l}\,.$$

Note that this is the decomposition of the product of c_k and d_l in the ring of Young diagrams.

Proof. From Lemma 4.7.14 we know that

$$s^l[l]\alpha_{k+1,l}Q_{k+1,l} + s^{-k}[k]\alpha_{k,l+1}Q_{k,l+1} = s^{l-k}[l+k]\alpha_{k,1}\alpha_{1,l}Q_{k,1}Q_{1,l}$$

Now

$$\begin{aligned}
\frac{s^l[l]\alpha_{k+1,l}}{s^{l-k}[l+k]\alpha_{k,1}\alpha_{1,l}} &= \frac{s^l[l][k+l][k]![l-1]!s^{k(k-1)/2}s^{-l(l-1)/2}}{s^{l-k}[l+k]s^{k(k+1)/2}s^{-l(l-1)/2}[k]![l]!} \\
&= \frac{s^l}{s^{l-k}s^k} \\
&= 1\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{s^{-k}[k]\alpha_{k,l+1}}{s^{l-k}[l+k]\alpha_{k,1}\alpha_{1,l}} &= \frac{s^{-k}[k][k+l][k-1]![l]!s^{k(k-1)/2}s^{-l(l-1)/2}}{s^{l-k}[l+k]s^{k(k-1)/2}s^{-l(l+1)/2}[k]![l]!} \\
&= \frac{s^{-k}}{s^{l-k}s^l} \\
&= 1\,.
\end{aligned}$$

Therefore

$$Q_{k+1,l} + Q_{k,l+1} = Q_{k,1}Q_{1,l}\,.$$

■

4.7.17 Proposition.

Let us denote by $Q_C(X)$ and $Q_D(X)$ the power series

$$Q_C(X) = \sum_{k=0}^{\infty} (-1)^k Q_{k,1} X^k$$

and

$$Q_D(X) = \sum_{l=0}^{\infty} Q_{1,l} X^l \,.$$

(These formal power series are can be thought of as $\mathcal{C}^+$ versions of the formal power series, $C(X)$ and $D(X)$, defined in Proposition 2.5.10.) Then,

$$Q_C(X) Q_D(X) = \left(\sum_{k=0}^{\infty} (-1)^k Q_{k,1}\, X^k \right) \left(\sum_{l=0}^{\infty} Q_{1,l}\, X^l \right) = 1 \,.$$

Proof. The coefficient of X^0 in $Q_C(X)Q_D(X)$ is given by $Q_{0,1}Q_{1,0}$ which is the empty knot, which acts as the identity in $\mathcal{C}^+$. Therefore, the coefficient of X^0 in $Q_C(X)Q_D(X)$ is 1. It remains to show that the coefficient of X^m is zero for $m > 0$ i.e.

$$\sum_{k=0}^{m} (-1)^k Q_{k,1} Q_{1,m-k} = 0 \,.$$

To do this we use the relation established in Proposition 4.7.16.

$$\begin{aligned}
\sum_{k=0}^{m} (-1)^k Q_{k,1} Q_{1,m-k} &= Q_{1,m} + (-1)^m \, Q_{m,1} \\
&\quad + \sum_{k=1}^{m-1} (-1)^k \left(Q_{k+1,m-k} + Q_{k,m-k+1} \right) \\
&= Q_{1,m} + (-1)^m Q_{m,1} + \sum_{k=1}^{m-1} (-1)^k Q_{k+1,m-k} \\
&\quad + \sum_{k=0}^{m-2} (-1)^{k+1} Q_{k+1,m-k} \\
&= Q_{1,m} + (-1)^m Q_{m,1} + (-1)^{m-1} Q_{m,1} + (-1) Q_{1,m} \\
&\quad + \sum_{k=1}^{m-2} (-1)^k \left(Q_{k+1,m-k} - Q_{k+1,m-k} \right) \\
&= 0 \,.
\end{aligned}$$

■

The following Lemma, which provides a factorisation for $\mathcal{X}(Q_C(X))$, was suggested, in it's present form, by Morton.

4.7.18 Lemma.

We can express $\mathcal{X}(Q_C(X))$ as an infinite product of rational functions in X with coefficients in the ring Λ.

$$\mathcal{X}(Q_C(X)) = \prod_{k=0}^{\infty} \frac{1 - vs^{2k+1}X}{1 - v^{-1}s^{2k+1}X}$$

Proof. Let us denote the right hand side by $P(X)$. As a formal power series,

$$P(X) = \sum_{r=0}^{\infty} p_r X^r \,.$$

Note that

$$P(X) = \frac{1 - vsX}{1 - v^{-1}sX} P(s^2 X)$$

and so

$$(1 - v^{-1}sX)P(X) = (1 - vsX)P(s^2X)\,.$$

Expand both sides as power series in X and compare coefficients of X^{r+1}. Then

$$p_{r+1} - v^{-1}sp_r = s^{2r+2}p_{r+1} - vs^{2r+1}p_r\,,$$

and so

$$\begin{aligned} p_{r+1} &= \frac{(v^{-1}s - vs^{2r+1})}{(1 - s^{2r+2})} p_r \\ &= \frac{s^{r+1}(vs^r - v^{-1}s^{-r})}{s^{r+1}(s^{r+1} - s^{-r-1})} p_r \\ &= \frac{(vs^r - v^{-1}s^{-r})}{(s^{r+1} - s^{-(r+1)})} p_r\,. \end{aligned}$$

The value of p_0 is given by setting $X = 0$,

$$p_0 = P(0) = 1\,.$$

Therefore, for $r > 0$,

$$\begin{aligned} p_r &= \prod_{i=0}^{r} \frac{vs^{i-1} - v^{-1}s^{-(i-1)}}{s^i - s^{-i}} \\ &= \prod_{i=0}^{r} (-1) \frac{v^{-1}s^{-(i-1)} - vs^{i-1}}{s^i - s^{-i}} \\ &= (-1)^r \mathcal{X}(Q_{r,1}) \quad \text{by Proposition 4.7.7}\,. \end{aligned}$$

Thus,

$$P(X) = \mathcal{X}(Q_C(X)) .$$

■

4.7.19 Proposition.

The value of the framed Homfly polynomial on the unknot decorated by $Q_{1,l}$ is

$$\mathcal{X}(Q_{1,l}) = \prod_{i=1}^{l} \frac{v^{-1}s^{i-1} - vs^{-(i-1)}}{s^i - s^{-i}} .$$

Proof. This result can be proved directly, as for $\mathcal{X}(Q_{k,1})$, but in light of Lemma 4.7.18 we give the following, more elegant proof. By Proposition 4.7.17,

$$\mathcal{X}(Q_C Q_D) = \mathcal{X}(Q_C)\mathcal{X}(Q_D) = 1 .$$

Therefore, by Lemma 4.7.18

$$\mathcal{X}(Q_D) = \prod_{k=0}^{\infty} \frac{1 - v^{-1}s^{2k+1}X}{1 - vs^{2k+1}X} .$$

The power series expansion of $\mathcal{X}(Q_D)$ is therefore given by writing v for v^{-1} in the power series $\mathcal{X}(Q_C)$. Therefore,

$$\begin{aligned} \mathcal{X}(Q_D) &= \sum_{l=0}^{\infty} \left((-1)^l \prod_{i=1}^{l} \frac{vs^{-(i-1)} - v^{-1}s^{i-1}}{s^i - s^{-i}} \right) X^l \\ &= \sum_{l=0}^{\infty} \left(\prod_{i=1}^{l} \frac{v^{-1}s^{i-1} - vs^{-(i-1)}}{s^i - s^{-i}} \right) X^l \end{aligned}$$

and hence,

$$\mathcal{X}(Q_{1,l}) = \prod_{i=1}^{l} \frac{v^{-1}s^{i-1} - vs^{-(i-1)}}{s^i - s^{-i}} .$$

■

4.7.20 Corollary.

The denominator of $\mathcal{X}_N(Q_{1,l})$ is "no worse than" $[N-1]!$. By this we mean that there are no poles at any roots of unity of order N or larger. This will be important in Chapter 5.

Proof. By Proposition 4.7.19

$$\mathcal{X}(Q_{1,l}) = \frac{(v^{-1}-v)}{(s-s^{-1})}\frac{(v^{-1}s-vs^{-1})}{(s^2-s^{-2})}\cdots\frac{(v^{-1}s^{l-1}-vs^{-l+1})}{(s^l-s^{-l})}.$$

Therefore,

$$\begin{aligned}\mathcal{X}_N(Q_{1,l}) &= \frac{(s^N-s^{-N})}{(s-s^{-1})}\frac{(s^Ns-s^{-N}s^{-1})}{(s^2-s^{-2})}\cdots\frac{(s^Ns^{l-1}-s^{-N}s^{-l+1})}{(s^l-s^{-l})}\\ &= \frac{[N]}{[1]}\frac{[N+1]}{[2]}\cdots\frac{[N+l-1]}{[l]}.\end{aligned}$$

It is clear that if $l < N$, then the result is true. Suppose $l \geq N$. Then

$$\begin{aligned}\mathcal{X}_N(Q_{1,l}) &= \frac{[N][N+1]\cdots[2N-2][2N-1][2N]\cdots[N+l-1]}{[1][2]\cdots[N-1][N][N+1]\cdots[N+l-N]}\\ &= \frac{[l+1][l+2]\cdots[l+N-1]}{[N-1]!}.\end{aligned}$$

■

We next prove a result concerning the dual Young diagrams for the c_i.

4.7.21 Proposition.

$$\mathcal{X}_N(Q_{k,1}) = \mathcal{X}_N(Q_{N-k,1}).$$

Proof. By Proposition 4.7.7 and Theorem 4.6.16

$$\begin{aligned}\mathcal{X}_N(Q_{N-k,1}) &= \prod_{j=1}^{N-k}\frac{s^Ns^{-j+1}-s^{-N}s^{j-1}}{s^j-s^{-j}}\\ &= \prod_{j=1}^{N-k}\frac{s^{N-j+1}-s^{-N+j-1}}{s^j-s^{-j}}\\ &= \frac{[N]}{[1]}\frac{[N-1]}{[2]}\cdots\frac{[k+1]}{[N-k]}\\ &= \frac{[N]!}{[N-k]!\,[k]!}\\ &= \frac{[N]}{[1]}\cdots\frac{[N-k+1]}{[k]}\\ &= \mathcal{X}_N(Q_{k,1}).\end{aligned}$$

■

4.7.22 Comment.

This result follows automatically from the properties of the quantum group invariants since the Young diagram c_{N-k} is the index for the dual representation of c_k in $\mathcal{R}_N$.

Later, we will show that this holds for all Young diagrams and their duals. This will be important when we come to define a 3-manifold invariant in Chapter 5.

4.8 A homomorphism from $\mathcal{R}_\infty$ to $\mathcal{C}^+$.

We next define an algebra homomorphism from $\mathcal{R}_\infty$ to $\mathcal{C}^+$. Later, we will demonstrate a set of generators for $\mathcal{C}^+$ by considering the image of a generating set of $\mathcal{R}_\infty$ under this homomorphism.

4.8.1 Definition.

We define the algebra homomorphism $\theta : \mathcal{R}_\infty \to \mathcal{C}^+$ by

$$\theta(c_i) = Q_{c_i} \quad \forall i \in \mathbb{N}$$

4.8.2 Proposition.

The homomorphism θ is an algebra isomorphism.

Proof. For this proof, we denote Q_{c_i} by Q_i.

The ring $\mathcal{R}_\infty$ is defined to be the free algebra generated by c_i, $i \in \mathbb{N}$. Hence, the homomorphism θ is an isomorphism if we can show that the elements Q_i generate $\mathcal{C}^+$ as a free algebra.

By 2.3.7, the elements φ_m^+, $m \in \mathbb{N}$, generate $\mathcal{C}^+$ freely, as an algebra, where φ_m^+ is the closure of the braid in Figure 4.4.

By induction on n, we will show that φ_n^+ can be expressed in terms of the Q_i, for $i \leq n$. Thus, $\{Q_i \ : \ i \in \mathbb{N}\}$ generates $\mathcal{C}^+$ as an algebra.

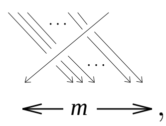


Figure 4.4: The braid which closes to $\varphi_m^+ \in \mathcal{C}^+$.

For $n = 1$, we have that $Q_1 = \varphi_1^+$, and we are done.

Now, assume that we have an expression for φ_m^+ in terms of the Q_i for all $m < n$. It follows from the definition of $C^{(n)}$ (see Corollary 2.3.8) that $Q_n \in \mathcal{C}^{(n)}$. Therefore, there is an expression for Q_n in terms of monomials of weighted degree n in the φ_m^+. The monomials of weighted degree n can be indexed by the partitions of n, hence

$$Q_n = \beta_n \varphi_n^+ + \sum_{|\lambda|=n} \beta_\lambda \varphi_{\lambda_1}^+ \varphi_{\lambda_2}^+ \cdots \varphi_{\lambda_k}^+$$

for some scalars $\beta_\lambda \in \Lambda$.

The aim is to prove that β_n is invertible in Λ. Then, by the induction hypothesis, we can write φ_n^+ in terms of the Q_i, with $i \leq n$. This then proves that the elements Q_i generate $\mathcal{C}^+$ as an algebra.

Recall from Notation 2.2.2, that the Conway polynomial can be calculated from the framed Homfly polynomial by setting $x = v = 1$.

Note that all the terms on the right hand side, except for φ_n^+, are split links. Since the Conway polynomial is 0 on any split link and φ_n^+ is ambient isotopic to the unknot,

$$\nabla(Q_n) = \nabla(\beta_n \varphi_n^+) = \beta_n^{'}$$

where $\beta_n^{'}$ is the evaluation of β_n at $x = v = 1$.

Since we are working in $\Lambda = \mathbb{C}[\![h]\!]$, β_n is invertible if and only if it has non-zero constant term when written in terms of h. Recall that the power series expressions for x and v, in terms of h, have constant term 1. Therefore the constant term of β_n is equal to that of $\beta_n^{'}$. If we can show that the constant term of $\nabla(Q_n)$ is non-zero, we have shown that β_n is invertible.

Consider calculating the constant term of ∇ for each positive permutation braid in Q_n.

For a link, L, with k components the Conway polynomial, written as a polynomial in z has the form

$$\nabla(L) = m_L z^{k-1} + \text{ higher order terms in } z\,,$$

where m_L is a scalar depending on the linking numbers of the components of L. For a knot, K,

$$\nabla(K) = 1 + \text{ higher order terms in } z.$$

Since $z = s - s^{-1}$, as a power series in h

$$z = h + \text{ higher order terms in } h.$$

Therefore, no power of z contributes to the constant term of $\nabla(Q_n)$. Hence the only terms of Q_n which contribute to the constant term of ∇, as a power series in h, are those positive permutation braids which close to knots.

An n-string positive permutation braid closes to a knot if and only if the permutation has order n. Let π be such a permutation. The coefficient of ω_π in Q_n is $(-x^{-1}s^{-1})^{l(\pi)}$. Since π is of order n,

$$l(\pi) \equiv n - 1 \mod 2\,.$$

The constant term of $(-x^{-1}s^{-1})^{l(\pi)}$, as a power series in h, is therefore, $(-1)^{n-1}$.

There are $(n-1)!$ such positive permutation braids and therefore the constant term of $\nabla(Q_n)$ is $(-1)^{n-1}(n-1)!$. Hence β_n is invertible in $\Lambda = \mathbb{C}[\![h]\!]$.

By induction, we can write φ_n^+ in terms of Q_i for any $n \in \mathbb{N}$. Since the elements φ_n^+ generate $\mathcal{C}^+$ as an algebra it follows that the elements Q_i will also generate $\mathcal{C}^+$ as an algebra. Thus, θ is surjective.

It remains to show that $\mathcal{C}^+$ is generated freely by Q_i, $i \in \mathbb{N}$.

Since $\{Q_i \ : \ i \in \mathbb{N}\}$ generates $\mathcal{C}^+$ as an algebra, the set of monomials in the Q_i must span $\mathcal{C}^+$ as a Λ-module. The set of monomials $Q_{i_1}^{j_1} Q_{i_2}^{j_2} \cdots Q_{i_m}^{j_m}$ for which $\sum_{k=1}^{m} i_k j_k = n$, generate $\mathcal{C}^{(n)}$ as a Λ-module. We already know that $\mathcal{C}^{(n)}$ is freely generated by a set of elements with cardinality the number of partitions of n. A similar count to that of Lemma 2.5.8 will show that we have exactly this number of monomials. Therefore, since Λ is commutative, the monomials form a free Λ-basis for $\mathcal{C}^{(n)}$. We know that $\mathcal{C}^+ = \bigoplus_{n \in \mathbb{N}} \mathcal{C}^{(n)}$, therefore the set of all the monomials forms a free Λ-basis for $\mathcal{C}^+$. It follows that $\{Q_i \ : \ i \in \mathbb{N}\}$ generates $\mathcal{C}^+$ as a free algebra.

$\blacksquare$

We next demonstrate that $\theta(d_l) = Q_{1,l}$ and that $\theta(\mu_{k,l}) = Q_{k,l}$. The aim is to prove that $\theta(\lambda) = Q_\lambda$ for any Young diagram λ.

4.8.3 Proposition.

Under θ, the Young diagram with a single row of l cells is mapped to the closure of the genuine idempotent $(1/\alpha_{1,l})\, e_{1,l}$,

$$\theta(d_l) = Q_{1,l}\,.$$

Proof. We already know that $C(X)D(X) = 1$. Therefore, for $m \geq 1$,

$$\sum_{k=0}^{m} (-1)^k c_k d_{m-k} = 0\,.$$

Since θ is an algebra isomorphism,

$$\sum_{k=0}^{m} (-1)^k \theta(c_k)\theta(d_{m-k}) = 0\,.$$

The proof is an induction on the number of cells.

We know that $d_1 = c_1$, therefore

$$\theta(d_1) = Q_{1,1}$$

as required.

Assume we have the result for d_i, for $i < m$. We consider the sum

$$\sum_{k=0}^{m} (-1)^k Q_{k,1} Q_{1,m-k}\,.$$

It follows from the induction hypothesis that

$$\sum_{k=0}^{m} (-1)^k Q_{k,1} Q_{1,m-k} = Q_m + \sum_{k=1}^{m} (-1)^k \theta(c_k)\theta(d_{m-k})\,.$$

By Proposition 4.7.17, we know that

$$\sum_{k=0}^{m} (-1)^k Q_{k,1} Q_{1,m-k} = 0\,.$$

Therefore, by the induction hypothesis

$$\begin{aligned} 0 &= Q_m + \sum_{k=1}^{m}(-1)^k\theta(c_k)\theta(d_{m-k}) - \sum_{k=0}^{m}(-1)^k\theta(c_k)\theta(d_{m-k}) \\ &= Q_m - \theta(d_m) \end{aligned}$$

Since θ is an algebra isomorphism, it follows that

$$\theta(d_m) = Q_m$$

as required. ■

From this result, we can find the images of the hook shaped diagrams $\mu_{k,l}$ under θ, by induction. The proof is a direct application of the skein relation deduced in Lemma 4.7.14.

4.8.4 Proposition.

The images of $\mu_{k,l}$ under θ, for $k \geq 1$, $l \geq 1$ are given by

$$\theta(\mu_{k,l}) = (1/\alpha_{k,l})\ \widehat{e}_{k,l} = Q_{k,l}\,.$$

The scalar $\alpha_{k,l}$ is given in Proposition 4.7.4 as

$$\alpha_{k,l} = \frac{[\,k+l-1\,][\,k-1\,]![\,l-1\,]!}{s^{(k(k-1)-l(l-1))/2}}\,.$$

Proof. We go by induction on the number of cells, and the length of the first column, noting that $\theta(\mu_{1,l}) = \theta(d_l)$ for every l, by Proposition 4.8.3. It follows that we have the result for $m = 1$.

The induction step works as follows. We assume the result for all diagrams with n cells where $n < m$. We also assume we know the result for all hook shaped diagrams with m cells and at most k cells in the first column. Since θ is an algebra isomorphism,

$$\theta(\mu_{k+1,m-k}) = \theta(c_k d_{m-k}) - \theta(\mu_{k,m-k+1})\,.$$

Now, by the induction hypothesis and Lemma 4.7.14

$$
\begin{aligned}
\theta(c_k d_{m-k}) \quad - \quad & \theta(\mu_{k,m-k+1}) \\
= \quad & \frac{s^{k(k-1)/2} s^{-(m-k)(m-k-1)/2}}{[k]!\,[m-k]!} \\
& \qquad \left(\frac{s^k [m-k]}{[m]}\, \widehat{e}_{k+1,m-k} + \frac{s^{-m+k}[k]}{[m]}\, \widehat{e}_{k,m-k+1} \right) \\
& - \quad \frac{s^{(k(k-1)-(m-k+1)(m-k))/2}}{[k+(m-k+1)-1]\,[k-1]!\,[(m-k+1)-1]!}\, \widehat{e}_{k,m-k+1} \\
& \\
= \quad & \frac{s^{(k^2-k-m^2+2mk+m-k^2-k)/2} s^k}{[k]!\,[m-k]![m]}\, \widehat{e}_{k+1,m-k} \\
& \qquad + \left(\frac{s^{(k^2-k-m^2+2mk+m-k^2-k)/2} s^{-m+k}}{[k-1]!\,[m-k]!\,[m]} \right. \\
& \qquad\qquad \left. - \frac{s^{(k^2-k-m^2+2mk-k^2-m+k)/2}}{[m]\,[k-1]![m-k]!} \right) \widehat{e}_{k,m-k+1} \\
& \\
= \quad & \frac{s^{(-m^2+2mk+m)/2}}{[k]!\,[m-k-1]![m]}\, \widehat{e}_{k+1,m-k} \\
& \quad + \left(\frac{s^{-m^2+2mk-m)/2} - s^{-m^2+2mk-m)/2}}{[m]\,[k-1]!\,[m-k]!} \right) \widehat{e}_{k,m-k+1} \\
& \\
= \quad & \frac{s^{(k+1)k/2-(m-k)(m-k-1)/2}}{[(k+1)-1)!\,[(m-k)-1]!\,[(k+1)+(m-k)-1]}\, \widehat{e}_{k+1,m-k}\,, \\
= \quad & Q_{k+1,m-k}
\end{aligned}
$$

as required. ■

To show that $\theta(\lambda) = Q_\lambda$ for every Young diagram λ, we require the following Lemma.

4.8.5 Lemma.

Let $Q(y_1, y_2, y_3)$ be a polynomial in 3 variables.

$$Q(y_1, y_2, y_3) = \sum_{c_{\mathbf{i}} \in \mathbb{C} \quad \mathbf{i} \in \mathbb{N}^3} c_{\mathbf{i}} \; y_1^{i_1} \, y_2^{i_2} \, y_3^{i_3}$$

Suppose that $Q(y_3^{N^2}, y_3^N, y_3)$ is identically equal to zero for every value of $N \in \mathbb{N}$. Then

$$Q(y_1, y_2, y_3) \equiv 0 \,.$$

Proof. We suppose that $Q(y_1, y_2, y_3)$ is not identically zero and show that this leads to a contradiction. Let $\mathbf{j}$ be the largest of the indexes $\mathbf{i}$ (in the lexicographic ordering) for which $c_{\mathbf{i}} \neq 0$. We claim that, for N large enough, this is the only term that contributes to the highest power of y_3 in $Q(y_3^{N^2}, y_3^N, y_3)$. Now

$$Q(y_3^{N^2}, y_3^N, y_3) = \sum_{\mathbf{i}} c_{\mathbf{i}} \; y_3^{i_1 N^2 + i_2 N + i_3} \,.$$

We wish to show that if we choose N large enough then

$$j_1 N^2 + j_2 N + j_3 > i_1 N^2 + i_2 N + i_3 \,.$$

for all $\mathbf{i} < \mathbf{j}$ with $c_{\mathbf{i}} \neq 0$. Suppose that $j_1 > i_1$. Then

$$(j_1 - i_1) N^2 + (j_2 - i_2) N + (j_3 - i_3) > 0 \quad \Longleftrightarrow \quad N^2 > \frac{(i_2 - j_2)}{(j_1 - i_1)} N - \frac{(i_3 - j_3)}{(j_1 - i_1)} \,.$$

Now for any a, $b \in \mathbb{Q}$

$$\frac{aN + b}{N^2} \to 0 \;\; \text{as } N \to \infty \,,$$

Therefore, we can find $n_{\mathbf{i}} \in \mathbb{N}$ for which

$$N^2 > \frac{(i_2 - j_2)}{(j_1 - i_1)} N + \frac{(i_3 - j_3)}{(j_1 - i_1)} \quad \forall \, N > n_{\mathbf{i}} \,.$$

If $j_1 = i_1$ and $j_2 > i_2$ then

$$(j_2 - i_2) N + (j_3 - i_3) > 0 \quad \Longleftrightarrow \quad N > \frac{i_3 - j_3}{j_2 - i_2} \,.$$

Therefore

$$n_{\mathbf{i}} = \frac{i_3 - j_3}{j_2 - i_2} \,.$$

Finally, if $j_1 = i_1$ and $j_2 = i_2$ then $j_3 > i_3$ by the assumption that $\mathbf{i} < \mathbf{j}$. Therefore

$$j_3 - i_3 > 0 \qquad \forall\, N \in \mathbb{N}$$

as required. We can take $n_{\mathbf{i}} = 1$.

Since Q is a polynomial the number of $\mathbf{i}$ for which $c_{\mathbf{i}}$ is non-zero is finite. Hence, we can set n to be the maximum of the $n_{\mathbf{i}}$. Then for every $N > n$

$$j_1 N^2 + j_2 N + j_3 > i_1 N^2 + i_2 N + i_3 \qquad \forall\, \mathbf{i} < \mathbf{j}\,.$$

Therefore, for $N > n$, $c_{\mathbf{j}}$ is the coefficient of the largest power of y_3 in $Q(y_3^{N^2}, y_3^N, y_3)$. However, for any $N \in \mathbb{N}$, $Q(y_3^{N^2}, y_3^N, y_3)$ is identically zero. Therefore, $c_{\mathbf{j}} = 0$ which contradicts the assumption that it was non-zero. Therefore $Q(y_1, y_2, y_3)$ must also be identically zero. ∎

4.8.6 Remark.

The above result will hold, even if we know only that $Q(y_3^{N^2}, y_3^N, y_3)$ if identically zero only for all $N > m$ say. Taking $n' = \max\{n, m\}$ we can still derive the contradiction, since $c_{\mathbf{j}}$ will be forced to be zero for $N > n'$.

4.8.7 Lemma.[TW]

Let s be a primitive root of unity and λ and μ be two q-admissible Young diagrams (see Definition 5.2.1). Recall that $J(H; V_\lambda, V_\mu)$ denotes the quantum invariant for the Hopf link (with positive linking) with one component coloured by V_λ and the other by V_μ. Set M to be the matrix $(M_{\lambda,\mu})$ where

$$M_{\lambda,\mu} = J(H; V_\lambda, V_\mu)\,,$$

where the indexing set runs over all q-admissible diagrams. Then M is invertible. ∎

4.8.8 Theorem.

The image of the Young diagram λ under the algebra isomorphism θ is Q_λ

$$\theta(\lambda) = Q_\lambda\,.$$

The element Q_λ, therefore, is given by the Giambelli formula for λ.

Proof. Let G_λ denote the Giambelli formula for λ in terms of the c_i. Since θ is an algebra isomorphism

$$\theta(\lambda) = \theta(G_\lambda)\,.$$

Hence, $\theta(\lambda) \in \mathcal{C}^{(n)}$, where n is the number of cells in λ.

By Proposition 4.7.6 we can express $\theta(\lambda) - Q_\lambda$ as a linear combination of the Q_μ where μ is a Young diagram with n cells;

$$\theta(\lambda) - Q_\lambda = \sum_{|\mu|=n} b_{\lambda\mu} Q_\mu\,. \tag{4.9}$$

We will show that $b_{\lambda\mu} = 0$ for every μ with n cells. Then, since Proposition 4.8.2 proves that θ is an algebra isomorphism

$$\theta(\lambda) = Q_\lambda$$

as required.

Suppose $b_{\lambda\mu} \neq 0$ for some μ. We will show that this assumption leads to a contradiction. From the proof of Proposition 4.7.6, we know that $b_{\lambda\mu}$ is a Laurent polynomial in x, v and s divided by some power of $(s - s^{-1})$. Therefore, when we substitute $s = x^{-N}$ and $v = x^{N^2}$, we have a Laurent polynomial in x (with a finite number of isolated poles and zeros) or the zero polynomial. Let $\beta_{\lambda\mu}$ denote the scalar obtained from $b_{\lambda\mu}$ by making these substitutions.

We will show that $\beta_{\lambda\mu}$ is identically zero, for any $N > n$. Assuming that we have shown that $\beta_{\lambda\mu}$ is zero we can apply Lemma 4.8.5, as follows, to show that $b_{\lambda\mu} = 0$, giving us the required contradiction. If we remove a factor of the largest power of s, and the smallest powers of x and v from $b_{\lambda\mu}$, we see that $b_{\lambda\mu}$ is the product of a Laurent monomial and a "genuine" polynomial in x, v and $t = s^{-1}$ divided by a power of $s - s^{-1}$. We can then apply Lemma 4.8.5 to the polynomial (with $y_1 = v$, $y_2 = t = s^{-1}$ and $y_3 = x$) to determine that $b_{\lambda\mu} \equiv 0$. This contradicts the assumption that $b_{\lambda\mu}$ was non-zero. Therefore, $b_{\lambda\mu} = 0$ for every Young diagram μ with n cells.

To show that $\beta_{\lambda\mu} \equiv 0$ we consider how it behaves in a neighbourhood of $x = 1$. It is possible that $x = 1$ is a pole, or a zero of $\beta_{\lambda\mu}$. However, since all poles and zeros are isolated there is some neighbourhood of $x = 1$ for which $\beta_{\lambda\mu}$ is well defined and non-zero. Note also that we know that $1/\alpha_\mu$ is well defined and non-zero at $x = s = 1$ from Remark 4.6.8. We can therefore choose a neighbourhood of $x = 1$ for which both $\beta_{\lambda\mu}$ and $1/\alpha_\mu$ are both well defined and non-zero.

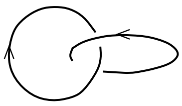
In particular, we can pick $p \in \mathbb{N}$ for which $x = e^{-2\pi i/pN}$ lies in this neighbourhood. Therefore, the scalars β_λ and $1/\alpha_\lambda$ are well defined and non-zero at the primitive pth root of unity $s = e^{2\pi i/p}$. Set r_μ to be minimum value of p for which this is true.

We can find such an r_μ for every $b_{\lambda\mu}$ which is not identically zero. Set r to be the maximum of all the r_μ and $2N$. (We require $r \geq 2N$ since all the Young diagrams of size n must fit into a rectangle of length $r - N$ and height $N - 1$.)

Order the Young diagrams with fewer than N rows and at most $r - N$ columns (for example by size). We can extend the expression for $\theta(\lambda) - Q_\lambda$ to a linear combination of these Young diagrams by setting $b_{\lambda\mu_j} = 0$ if $|\mu_j| \neq n$.

Given any link, if we decorate any component by $\theta(\lambda)$ and calculate $\mathcal{X}_N$, for any $N > n$, this will evaluate to the same Laurent polynomial in x as $\mathcal{X}_N$ evaluated on the link with the component decorated by Q_λ. This follows from the definition of the quantum invariants, Theorem 4.6.16 and the fact that $\theta(\lambda) = \theta(G_\lambda)$ where G_λ is the Giambelli polynomial for λ in terms of the c_k.

In particular if H denotes the Hopf link

$H \quad = $  .

then for each i

$$J(H; \mu_i, \theta(\lambda) - Q_\lambda) = \sum \beta_{\lambda,\mu_j} J(H; \mu_i, \mu_j) = 0\,. \tag{4.10}$$

Define the matrix $J(H)$ by setting the (i,j)th entry to be the value of the $U_q(sl(N))$ quantum invariant for the Hopf link with one component coloured by μ_i and the other by μ_j.

We can rewrite the set of equations 4.10 in matrix form :

$$\begin{pmatrix} & & \\ & J(H; \mu_i, \mu_j) & \\ & & \end{pmatrix} \begin{pmatrix} \\ \beta_{\lambda\mu_j} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By Lemma 4.8.7 the matrix $J(H)$ is invertible and this implies that $\beta_{\lambda\mu_j}$ is zero for all μ_j. However, since we assumed that $\beta_{\lambda\mu}$ was non-zero this is a contradiction.

Therefore, $\beta_{\lambda\mu} = 0$ for all μ and hence we have proved that

$$\theta(\lambda) = Q_\lambda \,.$$

■

4.8.9 Comments.

It is worth noting that for many sizes of Young diagram there is a more direct proof of this result. Recall from Remarks 4.6.15 that the elements Q_λ behave nicely under change of framing. Let U_n denote the unknot with framing $n \in \mathbb{Z}$. Then

$$\mathcal{X}(U_n * Q_\lambda) = f_\lambda^n \, \delta_\lambda$$

where δ_λ denotes the value of $\mathcal{X}$ evaluated on the unknot decorated by Q_λ. Therefore, if we set

$$\theta(\lambda) - Q_\lambda = \sum_\mu b_\mu Q_\mu$$

by Lemma 4.8.5

$$\mathcal{X}\left(U_n * (\theta(\lambda) - Q_\lambda)\right) = 0\,.$$

This gives rise to a set of linear equations namely

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ f_{\mu_1} & f_{\mu_2} & \cdots & f_{\mu_m} \\ f_{\mu_1}^2 & f_{\mu_2}^2 & \cdots & f_{\mu_m}^2 \\ \vdots & \vdots & \vdots & \vdots \\ f_{\mu_1}^{m-1} & f_{\mu_2}^{m-1} & \cdots & f_{\mu_m}^{m-1} \end{pmatrix} \begin{pmatrix} b_{\mu_1}\delta_{\mu_1} \\ b_{\mu_2}\delta_{\mu_2} \\ b_{\mu_3}\delta_{\mu_3} \\ \vdots \\ b_{\mu_m}\delta_{\mu_m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where m is the number of Young diagrams with $|\lambda|$ cells. The matrix is a Vandermonde matrix and is invertible if and only if $f_{\mu_i} \neq f_{\mu_j}$ for $i \neq j$. If the matrix is invertible then $b_\mu = 0$ for every μ and $\theta(\lambda) = Q_\lambda$.

For $|\lambda| = 1$, 2, 3, 4 and 5 this is not a problem. Unfortunately there are some pairs of Young diagrams with the same number of cells which have the same curl factor. For example, when $|\lambda| = 6$. The two diagrams in Figure 4.5(a) have the same curl factor, $x^{36}v^{-6}s^6$. Other cases arise when there is more than one Young diagram with a fixed number of cells which is self conjugate (i.e. has reflectional symmetry along the leading diagonal). For these Young diagrams $\sigma_\mu = 0$, therefore, the curl factor for such Young diagrams is $x^{|\mu|^2}v^{-|\mu|}$. For example, when $|\lambda| = 8$, there are two distinct Young diagrams which are both self conjugate. These are shown in Figure 4.5(b). Hence, the matrix is not

$f_\lambda = x^{36}v^{-6}s^6$ $\qquad$ $f_\lambda = x^{64}v^{-8}$

(a) $\qquad$ (b)

Figure 4.5: Young diagrams with the same curl factors.

invertible when $n = 6$ or $n = 8$. These are not the only cases, although I can see no way of predicting when pairs of Young diagrams with this property will occur.

4.8.10 Corollary to Theorem 4.8.8.

Given N and any Young diagram λ

$$\mathcal{X}_N(Q_\lambda) = \mathcal{X}_N(Q_{\lambda^*}).$$

Proof. Let G_λ denote the Giambelli polynomial for λ with $c_i = 0$ for $i > N$ and $c_N = 1$. This is an identity in $\mathcal{R}_N$. The Giambelli polynomial G_{λ^*}, for λ^*, in $\mathcal{R}_N$ can be described by taking that for λ and replacing c_i by c_{N-i} for $i < N$. Now, we know from Proposition 4.7.21 that the invariant of the unknot decorated by c_i is equal to that of the unknot decorated by c_{N-i}. Also, because we are considering the unknot, the invariant of the unknot decorated by $\mu\nu$ is equal to the product of the invariant of the unknot decorated by μ and the invariant of the unknot decorated by ν. Hence, by Theorem 4.8.8

$$\begin{aligned} \mathcal{X}_N(Q_{\lambda^*}) &= \mathcal{X}_N(\theta(G_{\lambda^*})) \\ &= \mathcal{X}_N(\theta(G_\lambda)) \\ &= \mathcal{X}_N(Q_\lambda). \end{aligned}$$

■

4.8.11 Proposition.

Let T be an (n,n) tangle which closes to $c \in \mathcal{C}^{(n)}$. Suppose that for all λ with $|\lambda| = n$ there exist $\eta_\lambda \in \Lambda$ such that

$$J(T; V_\square, \cdots, V_\square) = \sum_{|\lambda|=n} \eta'_\lambda V_\lambda$$

where η'_λ is obtained from η_λ by setting $v = s^{-N}$ and $x = s^{-1/N}$. Then

$$\theta(\sum_{|\lambda|=n} \eta_\lambda \lambda) = c\,.$$

Proof. We can write any element of $\mathcal{C}^{(n)}$ as a linear combination of elements Q_λ where λ has n cells. In particular

$$c = \sum_{|\lambda|=n} \gamma_\lambda Q_\lambda\,,$$

for some coefficients $\gamma_\lambda \in \Lambda$, where c is the closure of T. Since θ is an isomorphism

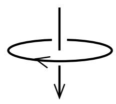
$$\theta(\sum_{|\lambda|=n} \gamma_\lambda \lambda) = c\,.$$

Now

$$\theta(\sum_{|\lambda|=n} \eta_\lambda \lambda) = \sum_{|\lambda|=n} \eta_\lambda Q_\lambda\,.$$

We will denote the evaluation of γ_λ at $x = s^{-1/N}$ and $v = s^{-N}$ by γ'_λ.

Let $H = H_1 \sqcup H_2$ denote the Hopf link (as in the proof of Theorem 4.8.8). We can think of H as the closure of the $(1,1)$ tangle S shown below.

$S =$ 

For any Young diagram μ with less than N rows

$$\begin{aligned}
\sum_{|\lambda|=n} \gamma'_\lambda J(H; V_\mu, V_\lambda) &= \sum_{|\lambda|=n} \gamma'_\lambda \mathcal{X}_N(H_1 * Q_\mu \sqcup H_2 * Q_\lambda) \\
&= \mathcal{X}_N(H_1 * Q_\mu \sqcup H_2 * c) \\
&= \mathcal{X}_N(H_1 * Q_\mu \sqcup H_2 * \widehat{T}) \\
&= tr_q(J(S; V_\mu, \sum \eta'_\lambda V_\lambda)) \quad \text{by Prop. 3.6.7} \\
&= \sum_{|\lambda|=n} \eta'_\lambda tr_q(J(S; V_\mu, V_\lambda)) \\
&= \sum_{|\lambda|=n} \eta'_\lambda J(H; V_\mu, V_\lambda)\,.
\end{aligned}$$

Thus, for any Young diagram, μ, with fewer than N rows

$$\sum_{|\lambda|=n} (\gamma_\lambda^{'} - \eta_\lambda^{'}) J(H; V_\mu, V_\lambda) = 0\,.$$

We can use exactly the same techniques as were used in the proof of Theorem 4.8.8 to show that $\gamma_\lambda^{'} - \eta_\lambda^{'} = 0$ and therefore, that

$$\gamma_\lambda - \eta_\lambda = 0\,.$$

Hence,

$$\sum_{|\lambda|=n} \gamma_\lambda Q_\lambda = \sum_{|\lambda|=n} \eta_\lambda Q_\lambda$$

as required. ∎

4.8.12 Remarks.

The proof of the Theorem 4.8.8 relies upon our knowledge of the representation ring $\mathcal{R}_N$, quantum invariants and their relation with the framed Homfly polynomial. However, the result is a relationship which holds in $\mathcal{C}^+$ and is purely skein theoretic. It would be pleasing to have a direct proof, in terms of the skein theory.

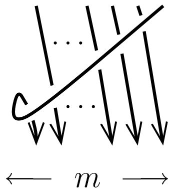
We consider the preimage of φ_m^+ next. By Proposition 4.8.11, the preimage of φ_m^+ is given by the endomorphism of $V_\Box^{\otimes m}$ represented by the braid in Figure 4.4. Note that φ_m^+ is a torus knot. The quantum invariants of torus knots have been calculated by Rosso and Jones [JR] and Strickland [S]. We give a simplified version of Strickland's result below.

4.8.13 Theorem. [S]

Let m and p be coprime integers. Let $K^{(m,p)}$ denote the (m,p) cable of the knot K. Then

$$J(K^{(m,p)}; c_1) = \sum_{\tau \in \psi_m(c_1)} f_\tau^{p/m} J(K; \tau)\,.$$

where the (m,p)-cable of K is the satellite of K with pattern the pth power of the tangle



and $\psi_m(c_1)$ is the mth Adams operation (see Definition 4.9.1). ■

Applying Proposition 4.8.11, Theorem 4.8.13 with $p = 1$ and Proposition 4.9.2 we obtain the following expression for $\theta^{-1}(\varphi_m^+)$:

$$\theta^{-1}(\varphi_m^+) = (xv^{-1})^{-1}\sum_{k=1}^{m}(-1)^{k-1} f_{\mu_{k,m-k+1}}^{1/m}\mu_{k,m-k+1}$$

where, for a Young diagram λ, f_λ is the framing factor given in Remarks 4.6.15 as

$$f_\lambda = x^{|\lambda|^2}v^{-|\lambda|}s^{n_\lambda}$$

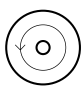
Recall that σ_λ and d_λ are defined in Definition 2.4.3 and a recursive definition for n_λ is given in Remarks 4.6.15.

For $\lambda = \mu_{k,m-k+1}$, we have $n_\lambda = m^2 - 2mk + m$. Therefore $f_\lambda^{\frac{1}{m}}$ contains only integer powers of x, v and s.

Let φ_m^- denote the m-string braid obtained from φ_m^+ by switching the $(m-1)$ crossings from positive to negative. By applying Theorem 4.8.13 with $p = -1$ we see that

$$\theta^{-1}(\varphi_m^-) = (xv^{-1})\sum_{k=1}^{m}(-1)^{k-1} f_{\mu_{k,m-k+1}}^{-1/m}\mu_{k,m-k+1}$$

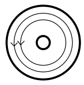
and hence, the value of $\theta^{-1}(\varphi_m^-)$ can be obtained by replacing s, x and v by s^{-1}, x^{-1} and v^{-1} respectively in the preimage of φ_m^+. This can be demonstrated easily for the case where $m = 2$, using the skein relations. Note that

 $= \quad \theta(c_1)$

and

 $= \quad xs\,\theta(\square\square) - xs^{-1}\,\theta\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right).$

Thus

 $= \quad x^{-2}$  $- \quad x^{-1}(s-s^{-1})$ 

$$\begin{aligned}
&= x^{-1}s\theta(\mu_{1,2}) - x^{-1}s^{-1}\theta(\mu_{2,1}) - x^{-1}(s-s^{-1})\theta(c_1^2) \\
&= x^{-1}(s-s+s^{-1})\theta(\mu_{1,2}) - x^{-1}(s^{-1}+s-s^{-1})\theta(\mu_{2,1}) \\
&= x^{-1}s^{-1}\theta(\mu_{1,2}) - x^{-1}s\theta(\mu_{2,1}) \\
&= x^{-1}s^{-1}\,\theta\left(\,\square\square\,\right) - x^{-1}s\,\theta\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right).
\end{aligned}$$

The relationship between $\theta^{-1}(\varphi_m^+)$ and $\theta^{-1}(\varphi_m^+)$ is not so easy to see in general.

4.8.14 Proposition.

The following equalities hold :

$$\begin{aligned}
\varphi_m^+ &= x^{m-1}\sum_{k=1}^{m}(-1)^{k-1}s^{m-k}[k]\theta(c_k d_{m-k}) \\
\varphi_m^- &= x^{-(m-1)}\sum_{k=0}^{m-1}(-1)^{k}s^{k}[m-k]\theta(c_k d_{m-k})
\end{aligned}$$

Proof.

$$\begin{aligned}
\varphi_m^+ &= x^{-1}v\sum_{k=1}^{m}(-1)^{k-1}x^m v^{-1}s^{m-2k+1}\theta(\mu_{k,m-k+1}) \\
&= \sum_{k=1}^{m}(-1)^{k-1}x^{m-1}s^{m-2k+1}\theta(\mu_{k,m-k+1}) \\
&= x^{m-1}\sum_{k=1}^{m}(-1)^{k-1}s^{m-2k+1}\sum_{i=k}^{m}(-1)^{i-k}\theta(c_i d_{m-i}) \\
&= x^{m-1}\sum_{i=1}^{m}\sum_{k=1}^{i}(-1)^{i-1}s^{m-2k+1}\theta(c_i d_{m-i}) \\
&= x^{m-1}\sum_{i=1}^{m}(-1)^{i-1}s^{m+1}\left(\sum_{k=1}^{i}s^{-2k}\right)\theta(c_i d_{m-i}) \\
&= x^{m-1}\sum_{i=1}^{m}(-1)^{i-1}s^{m+1}s^{-2}\frac{(1-s^{-2i})}{(1-s^{-2})}\theta(c_i d_{m-i}) \\
&= x^{m-1}\sum_{i=1}^{m}(-1)^{i-1}s^{m-i}[i]\theta(c_i d_{m-i}).
\end{aligned}$$

Since, by Proposition 4.8.3, $\sum_{i=0}^{m}(-1)^i\theta(c_i)\theta(d_{m-i}) = 0$ for $m > 0$, we can add multiples of it to φ_m^+.

$$\begin{aligned}
\varphi_m^+ &= \varphi_m^+ + \theta(x^{m-1}[m]\sum_{i=0}^{m}(-1)^i c_i d_{m-i}) \\
&= [m]x^{m-1}\theta(c_0 d_m) + x^{m-1}\sum_{i=1}^{m}(-1)^{i-1}(s^{m-i}[i] - [m])\theta(c_i d_{m-i}) \\
&= [m]x^{m-1}\theta(c_0 d_m) + x^{m-1}\sum_{i=1}^{m}(-1)^i\left(\sum_{k=1}^{m-i} s^{-m-1+2k}\right)\theta(c_i d_{m-i})\,.
\end{aligned}$$

Then

$$\begin{aligned}
\varphi_m^- &= x^{-(m-1)}\sum_{i=0}^{m}(-1)^i\left(\sum_{k=1}^{m-i} s^{m+1-2k}\right)\theta(c_i d_{m-i}) \\
&= x^{-(m-1)}\sum_{i=0}^{m}(-1)^i s^i[m-i]\theta(c_i d_{m-i}) \\
&= x^{-(m-1)}\sum_{i=0}^{m-1}(-1)^i s^i[m-i]\theta(c_i d_{m-i}).
\end{aligned}$$

■

4.8.15 Definition.

We define $\Phi^+(X)$ and $\Phi^-(X)$ to be the following two power series :

$$\Phi^+(X) = \sum_{m=1}^{\infty}\varphi_m^+ X^{m-1} \tag{4.11}$$

and

$$\Phi^-(X) = \sum_{m=1}^{\infty}\varphi_m^- X^{m-1} \quad . \tag{4.12}$$

We also define "quantum derivatives" of $C(X)$ and $D(X)$,

$$C'_q(X) = \sum_{k=1}^{\infty}(-1)^k[k]c_k X^{k-1} \quad \text{and} \quad D'_q(X) = \sum_{l=1}^{\infty}[l]d_l X^{l-1}\ .$$

4.8.16 Proposition.

The following three identities hold :

$$\Phi^+(X) = -\theta\left({C'}_q(xX)D(xsX)\right) ,$$

$$\Phi^-(X) = \theta\left(C(x^{-1}sX){D'}_q(x^{-1}X)\right)$$

and

$$\Phi^-(X) = -\theta\left({C'}_q(x^{-1}X)D(x^{-1}s^{-1}X)\right) .$$

Proof. The first two identities follow automatically from the definitions. To show the third identity, note that

$$\sum_{m=1}^{\infty} x^{-m+1}[m]\left(\sum_{k=0}^{m}(-1)^k c_k d_{m-k}\right) X^{m-1} \quad = \quad 0\,.$$

We can, therefore, subtract any multiple of the image of this sum under θ from $\Phi^-(X)$ by applying Proposition 2.5.10. Hence

$$\begin{aligned}
\Phi^-(X) &= \Phi^-(X) - \sum_{m=1}^{\infty} x^{-m+1}\sum_{k=0}^{m}(-1)^k[m]\theta(c_k d_{m-k})X^{m-1} \\
&= \sum_{m=1}^{\infty} x^{-m+1}\sum_{k=0}^{m}(-1)^k(s^k[m-k]-[m])\theta(c_k d_{m-k})X^{m-1} \\
&= \sum_{m=1}^{\infty} x^{-m+1}\sum_{k=0}^{m}(-1)^k\left(\frac{s^m-s^{-m+2k}-s^m+s^{-m}}{s-s^{-1}}\right)\theta(c_k d_{m-k})X^{m-1} \\
&= \sum_{m=1}^{\infty} x^{-m+1}\sum_{k=0}^{m}(-1)^k s^{-m+k}\left(\frac{-s^k+s^-k}{s-s^{-1}}\right)\theta(c_k d_{m-k})X^{m-1} \\
&= \sum_{m-1}^{\infty} x^{-m+1}\sum_{k=0}^{m}(-1)^{k-1}s^{-m+k}[k]\theta(c_k d_{m-k})X^{m-1}\,.
\end{aligned}$$

The third equation now follows by comparing the coefficient of X^{m-1} on the right with that of $\theta\left(C(x^{-1}sX){D'}_q(x^{-1}X)\right)$. ∎

4.9 Adams operations.

Here we consider another generating set for the ring $\mathcal{R}_N$. Recall that we can express the elements of $\mathcal{R}_N$ as symmetric functions in variables x_i, for $i = 1 \ldots N$.

It is well known that the power sums $\sum_i x_i^m$, $m \in \mathbb{N}$, generate $\mathcal{R}_N$ as an algebra. We will give an expression for the image of the mth power sum, under θ, as a sum of m elements in $\mathcal{C}^+$. The advantage of these over the elements Q_λ is that the number of terms in each expression is linear rather than factorial in the number of cells and all the terms are reasonably simple braids. For the definitions of C_N and p_N the reader is referred back to Definition 2.5.11.

4.9.1 Definition.

The *Adams operations*, $\{\psi_m\}_{m\in\mathbb{N}}$, are a family of $\mathcal{R}_N$-endomorphisms,

$$\psi_m : \mathcal{R}_N \to \mathcal{R}_N \,,$$

defined by their images on the x_i :

$$\psi_m(x_i) = x_i^m.$$

Hence $\psi_m(c_1) = \sum x_i^m$, the mth Newton power sum. This is well known to be a polynomial in the c_i which is independent of N, i.e. there is a polynomial $b_m(\mathbf{c}) \in \mathcal{R}_\infty$ with $p_N(b_m) = \psi_m(c_1)$ for all N.

4.9.2 Proposition.

The following identities hold for $\psi_m(c_1)$,

$$\psi_m(c_1) \;=\; p_N\left(\sum_{k=1}^{m}(-1)^{k-1}kc_kd_{m-k}\right) \;=\; p_N\left(\sum_{k=1}^{m}(-1)^{k-1}\mu_{k,m-k+1}\right)\,.$$

Proof. The function $\ln(C(X))$ has a formal power series expansion

$$\ln(C(X)) = \sum_{m=1}^{\infty} b_m(\mathbf{c})X^m$$

where $b_m(\mathbf{c})$ is a polynomial in $\{c_k\}_{k=1}^{\infty}$. Differentiating $\ln(C(X))$ with respect to X, we get

$$\frac{C'(X)}{C(X)} = C'(X)D(X) = \sum_{m=1}^{\infty} mb_m(\mathbf{c})X^{m-1}\,.$$

By comparing coefficients it follows that

$$\begin{aligned}
mb_m(\mathbf{c}) &= \sum_{k=1}^{m}(-1)^k k c_k d_{m-k} \\
&= \sum_{k=1}^{m-1}\Big((-1)^k k(\mu_{k+1,m-k}+\mu_{k,m-k+1})\Big) \quad + (-1)^m m\mu_{m,1} \\
&= (-1)^m m\mu_{m,1} + \sum_{k=1}^{m-1}(-1)^k k\mu_{k,m-k+1} + \sum_{k=2}^{m}(-1)^{k-1}(k-1)\mu_{k,m-k+1} \\
&= -\mu_{1,m} + (-1)^m(m-m+1)\mu_{m,1} + \sum_{k=2}^{m-1}(-1)^k(k-k+1)\mu_{k,m-k+1} \\
&= \sum_{k=1}^{m}(-1)^k\mu_{k,m-k+1} \quad .
\end{aligned}$$

For each N we have that

$$\begin{aligned}
\ln(C_N(X)) &= \sum_{i=1}^{N}\ln(1-x_iX) \\
&= -\sum_{m=1}^{\infty}\sum_{i=1}^{N}\frac{x_i^m}{m}X^m \\
&= -\sum_{m=1}^{\infty}\frac{\psi_m(c_1)}{m}X^m\,.
\end{aligned}$$

Now $\ln(C_N(X)) = p_N(\ln(C(X)))$ and so

$$\psi_m(c_1) = -p_N(mb_m(\mathbf{c}))\,.$$

The result follows directly from this equation. ∎

4.9.3 Remarks.

These formulae are independent of N. For a given N, we find $\psi_m(c_1)$ by applying p_N. For most purposes we can treat $\psi_m(c_1)$ as if it were an element of $\mathcal{R}_\infty$.

4.9.4 Lemma.

Let $\Psi(X) = \sum_{m=1}^{\infty}\psi_m(c_1)X^{m-1}$. Then

$$\Psi(X) = -C'(X)D(X) = D'(X)C(X)\,.$$

Proof. From the proof of 4.9.2 we know that

$$\Psi(X) = -\frac{d}{dX}\ln(C(X)) = -C'(X)D(X)\,.$$

It remains to note that

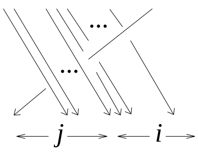
$$D'(X)C(X) = \left(\frac{1}{C(X)}\right)' C(X) = \frac{-C(X)'}{C(X)^2}C(X) = -C'(X)D(X)\,.$$

$\blacksquare$

The objective now is to interpret the Adams operations, $\psi_m(c_1)$, as the image under θ of a simple sum of m braids on m-strings.

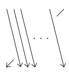
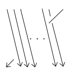
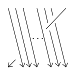
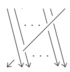
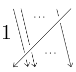
4.9.5 Definition.

Let $\varphi_{i,j} \in \mathcal{C}^+$ be the closure of the diagram given below, with i positive crossings and j negative crossings (so $\varphi_m^+ = \varphi_{m-1,0}$).



4.9.6 Theorem.

The element $\psi_m(c_1)$ is a scalar multiple of the image of the sum of braids

$P_m = x^{m-1}$  $+\ x^{m-3}$  $+\ x^{m-5}$  $+ \cdots +\ x^{-m+3}$  $+\ x^{-m+1}$ 

under the homomorphism θ. In fact,

$$P_m = [m]\theta(\psi_m(c_1)) \quad .$$

Proof. The proof goes by induction on m.
For $m = 1$ we have $P_1 = c_1$ and $\psi_1(c_1) = 1(c_1)$. Note that $[1] = 1$.

Now, assuming the result holds for all $k < m$, we can apply Lemma 4.9.7 to obtain the following,

$$P_m = \sum_{k=1}^{m-1}(s - s^{-1})x^{m-1-k}P_k\varphi_{0,m-1-k} \quad + mx^{m-1}\varphi_{0,m-1}\,.$$

Therefore, by the induction hypothesis,

$$P_m = \sum_{k=1}^{m-1} (s^k - s^{-k}) x^{m-1-k} \theta(\psi_k(c_1)) \varphi_{0,m-1-k} \quad + m x^{m-1} \varphi_{0,m-1} \, .$$

Using the definition in Lemma 4.9.4 and equation 4.12 it can be shown that

$$\sum_{k=1}^{m-1} x^{m-1-k} (s^k - s^{-k}) \theta(\psi_k(c_1)) \varphi_{0,m-k}$$

is the coefficient of X^{m-2} in

$$s\,\theta\,(\,\Psi(sX)\,)\,\Phi^-(xX) - s^{-1}\,\theta\left(\,\Psi(s^{-1}X)\,\right)\Phi^-(xX)\,.$$

Applying Lemma 4.9.4 and Proposition 4.8.16

$$\begin{aligned} s\theta\,(\,\Psi(sX)\,)\,\Phi^-(xX) \quad - &\quad s^{-1}\theta\left(\,\Psi(s^{-1}X)\,\right)\Phi^-(xX) \\ = &\quad s\,\theta\left(-C'(sX)D(sX)C(sx^{-1}xX){D'}_q(x^{-1}xX)\,\right) \\ &\quad + s^{-1}\theta\left(D'(s^{-1}X)C(s^{-1}X)D(x^{-1}s^{-1}xX){C'}_q(x^{-1}xX)\right) \\ = &\quad \theta\left(\,s^{-1}D'(s^{-1}X){C'}_q(X) - s{D'}_q(X)C'(sX)\,\right) \end{aligned}$$

by Proposition 2.5.10. The coefficient of X^{m-2} in the expression above is given by

$$\theta(b_{m-2}) = \theta\left(\sum_{k=1}^{m-1} (-1)^k [k](m-k) s^{-m+k} c_k d_{m-k} \; - \sum_{k=1}^{m-1} (-1)^k [m-k] k s^k c_k d_{m-k}\right).$$

The coefficient of $c_k d_{m-k}$ in b_{m-2} is

$$\begin{aligned} (-1)^k \left((m-k)\frac{(s^{-m+2k} - s^{-m})}{s - s^{-1}}\right. \quad & - \quad \left. k\frac{(s^m - s^{-m+2k})}{s - s^{-1}}\right) \\ & = \quad (-1)^k \left(m s^{-m+k}[k] - k\frac{(s^m - s^{-m})}{s - s^{-1}}\right) \\ & = \quad (-1)^k (m s^{-m+k}[k] - k[m]) \\ & = \quad (-1)^k m s^{-m+k}[k] + (-1)^{k-1} k[m]\,. \end{aligned}$$

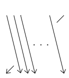
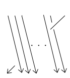
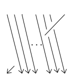
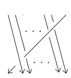
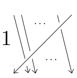
Recall that $\varphi_{0,m-1} = \varphi_m^-$ and add back on the term $m x^{m-1}\theta(\varphi_{0,m-1})$. By Proposition 4.8.14,

$$
\begin{aligned}
P_m &= \theta\left(\sum_{k=1}^{m-1}(-1)^k(ms^{-m+k}[k]-k[m])c_kd_{m-k}\right. \\
&\qquad\qquad\qquad \left. + m\sum_{k=0}^{m-1}(-1)^k s^k[m-k]c_kd_{m-k}\right) \\
&= \theta\left(\sum_{k=1}^{m-1}(-1)^k\left(ms^k\frac{(s^{k-m}-s^{-k-m}+s^{m-k}-s^{k-m})}{s-s^{-1}}-k[m]\right)c_kd_{m-k}\right. \\
&\qquad\qquad\qquad \left. + m[m]c_0d_m\right) \\
&= \theta\left(\sum_{k=1}^{m-1}(-1)^k\left(ms^k\frac{(s^{m-k}-s^{-m-k})}{s-s^{-1}}-k[m]\right)c_kd_{m-k} \quad + m[m]c_0d_m\right) \\
&= \theta\left(\sum_{k=1}^{m-1}(-1)^k(m[m]-k[m])c_kd_{m-k} \quad + \quad m[m]c_0d_m\right) \\
&= \theta\left([m]\sum_{k=1}^{m}(-1)^{k-1}kc_kd_{m-k} \quad - \quad (-1)^{m-1}m[m]c_md_0\right. \\
&\qquad\qquad\qquad \left. + m[m]c_0d_m \quad + m[m]\sum_{k=1}^{m-1}(-1)^kc_kd_{m-k}\right) \\
&= \theta\left([m]\psi_m(c_1)+m[m]\sum_{k=0}^{m}(-1)^kc_kd_{m-k}\right) \qquad \text{by Proposition 4.9.2} \\
&= [m]\theta\left(\psi_m(c_1)\right)+m[m]0 \qquad \text{by Proposition 2.5.10} \\
&= [m]\theta\left(\psi_m(c_1)\right)
\end{aligned}
$$

■

4.9.7 Lemma.

Let

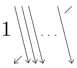
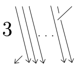
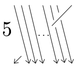
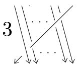
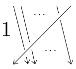
$P_m \;=\; x^{m-1}$  $+\; x^{m-3}$  $+\; x^{m-5}$  $+ \cdots +\; x^{-m+3}$  $+\; x^{-m+1}$ 

$$= \sum_{i=0}^{m-1} x^{m-1-2i} \varphi_{i,m-1-i} \, .$$

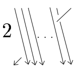
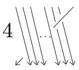
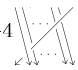
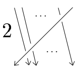
Then

$$P_m = \sum_{k=1}^{m-1} (s - s^{-1}) x^{m-1-k} P_k \varphi_{0,m-1-k} + m x^{m-1} \varphi_{0,m-1} \, .$$

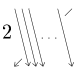
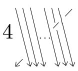
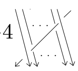
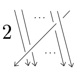
Proof. The method here is to switch and smooth the positive crossings in the braids using the skein relation. Numbering the crossings from bottom left to top right on the diagrams, we start with the $(m-1)$st (positive) crossing.

$P_m = x^{m-1}$  $+ \, x^{m-3}$  $+ \, x^{m-5}$  $+ \cdots + x^{-m+3}$  $+ \, x^{-m+1}$ 

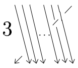
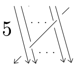
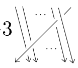
$= x^{m-1} \varphi_{0,m-1}$

$+ \, x^{-1} \Big(x^{m-2}$  $+ \, x^{m-4}$  $+ \cdots + x^{-m+4}$  $+ \, x^{-m+2}$  $\Big)$

$= x^{m-1} \varphi_{0,m-1} + (s - s^{-1}) P_{m-1} \varphi_{0,0}$

$+ \, x \Big(x^{m-2}$  $+ \, x^{m-4}$  $+ \cdots + x^{-m+4}$  $+ \, x^{-m+2}$  $\Big)$

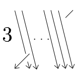
$= 2x^{m-1} \varphi_{0,m-1} + (s - s^{-1}) P_{m-1} \varphi_{0,0}$

$+ \, x^{m-3}$  $+ \cdots + x^{-m+5}$  $+ \, x^{-m+3}$  .

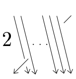
Applying the skein relation to the $(m-2)$nd crossing we then see that

$$\begin{aligned} P_m = {} & 2x^{m-1} \varphi_{0,m-1} + (s - s^{-1})(P_{m-1} \varphi_{0,0} + x P_{m-2} \varphi_{0,1}) + x^{m-1} \varphi_{0,m-1} \\ & + \text{ weighted sum of diagrams with} \\ & \quad \text{the } (m-1)\text{st and } (m-2)\text{nd crossings negative.} \end{aligned}$$

Applying the skein relation to all the positive $(m-3)$rd, $(m-4)$th, …, 2nd crossings we finally arrive at

$P_m = (m-1) x^{m-1} \varphi_{0,m-1} + (s - s^{-1}) \sum_{k=2}^{m-1} x^{m-1-i} P_k \varphi_{0,m-k-1} + x^{m-3}$ 

$= (m-1) x^{m-1} \varphi_{0,m-1} + (s - s^{-1}) \sum_{k=2}^{m-1} x^{m-1-i} P_k \varphi_{0,m-k-1}$

$+ \, x^{-1} \Big(x^{m-2}$  $\Big)$

$$
\begin{aligned}
&= (m-1)x^{m-1}\varphi_{0,m-1} + (s-s^{-1})\sum_{k=2}^{m-1} x^{m-1-k}P_k\varphi_{0,m-k-1} \\
&\quad +(s-s^{-1})P_1\varphi_{o,m-2} + xx^{m-2}\varphi_{o,m-1} \\
&= mx^{m-1}\varphi_{0,m-1} + (s-s^{-1})\sum_{k=1}^{m-1} x^{m-1-k}P_k\varphi_{0,m-k-1}
\end{aligned}
$$

as required. ■

We now calculate the framed Homfly polynomial for the Adams operations, using the formula for ψ_m in terms of the logarithm of $C(X)$. We then show that this calculation agrees with the calculation of the framed Homfly polynomial for P_m.

4.9.8 Proposition.

$$
\mathcal{X}(\theta(\psi_m(c_1))) = \frac{v^m - v^{-m}}{s^m - s^{-m}} .
$$

Proof. From Proposition 4.9.2 we know that

$$
\ln(C_N(X)) = \sum_{m=1}^{\infty} b_m(\mathbf{c})X^m
$$

and that for each N,

$$
\psi_m(c_1) = -p_N(mb_m(\mathbf{c})) .
$$

Consider

$$
\begin{aligned}
\mathcal{X}(\theta(\ln(C(X)))) &= \mathcal{X}(\ln(Q_C(X))) \\
&= \ln\left(\prod_{k=0}^{\infty} \frac{1-vs^{2k+1}X}{1-v^{-1}s^{2k+1}X}\right) \\
&= \sum_{k=0}^{\infty} \ln(1-vs^{2k+1}X) - \ln(1-v^{-1}s^{2k+1}X) \\
&= \sum_{k=0}^{\infty}\left(\sum_{m=1}^{\infty} -\frac{(vs^{2k+1}X)^m}{m} \quad - \quad \sum_{m=1}^{\infty} -\frac{(v^{-1}s^{2k+1}X)^m}{m}\right) \\
&= \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} \frac{(v^{-m}-v^m)}{m} s^{2km+m}X^m
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \frac{(v^{-m} - v^{m})}{m} s^m X^m \left(\sum_{k=0}^{\infty} (s^{2m})^k \right) \\
&= \sum_{m=1}^{\infty} \frac{(v^{-m} - v^{m})}{m} \frac{s^m}{1 - s^{2m}} X^m .
\end{aligned}
$$

Therefore,

$$\mathcal{X}(\theta(b_m(\mathbf{c})) = \frac{v^{-m} - v^m}{m\,(s^m - s^{-m})}$$

and hence

$$\mathcal{X}(\theta(\psi_m(c_1))) = \frac{v^m - v^{-m}}{s^m - s^{-m}} .$$

■

4.9.9 Remark.

Note that this calculation agrees with the evaluation of $\mathcal{X}$ on P_m.

$$
\begin{aligned}
\mathcal{X}(P_m) &= \sum_{i=0}^{m-1} x^{m-1-2i} \mathcal{X}(\varphi_{i,m-1-i}) \\
&= \sum_{i=0}^{m-1} x^{m-1-2i} (xv^{-1})^{i-m+1+i} \frac{(v^{-1} - v)}{s - s^{-1}} \\
&= \sum_{i=0}^{m-1} x^{m-1-2i} (xv^{-1})^{-m+2i+1} \frac{(v^{-1} - v)}{s - s^{-1}} \\
&= \sum_{i=0}^{m-1} v^{m-2i-1} \frac{(v^{-1} - v)}{s - s^{-1}} \\
&= \frac{(v^{-1} - v)}{s - s^{-1}} v^{m-1} \sum_{i=0}^{m-1} v^{-2i} \\
&= \frac{(v^{-1} - v)}{s - s^{-1}} v^{m-1} (1 + v^{-2} + \cdots + v^{-2m+2}) \\
&= \frac{(v^{-1} - v)}{s - s^{-1}} v^{m-1} v^{-m+1} (v^{m-1} + v^{m-3} + \cdots + v^{-m+1}) \\
&= \frac{v^m - v^{-m}}{s - s^{-1}} .
\end{aligned}
$$

By Theorem 4.9.6

$$P_m = [m]\theta(\psi_m(c_1)) ,$$

hence,

$$\mathcal{X}(\theta(\psi_m(c_1))) = \frac{v^m - v^{-m}}{s^m - s^{-m}}$$

which agrees with the calculation in Proposition 4.9.8.

4.10 A discussion of the work of Y. Yokota.

While writing up this thesis, I was given a copy of a preprint by Yokota entitled "Skeins and quantum $SU(N)$ invariants of 3- manifolds." [Y]. Some of the results in this and the subsequent chapter of this thesis also appear in this preprint. Here I highlight some of the similarities and differences between the two bodies of work.

Yokota works with the substitutions for x and v in terms of s already in place, giving a 1-parameter family of skein relations. The idempotent building blocks $f^{(l)}$ and $g^{(k)}$ are, respectively, equal to $1/\alpha_{1,l}\, e_{1,l}$ and $1/\alpha_{k,1}\, e_{k,1}$. They are defined inductively rather than directly as in Definition 4.4.3. However, it is easily verified that the $1/\alpha_{1,l}\, e_{1,l}$ and $1/\alpha_{k,1}\, e_{k,1}$ satisfy the inductive definitions [Y, equations 5, 6], by applying Lemma 4.7.2. In [Y] the quasi-idempotents are made by sandwiching a row of symmetrisers $f^{\lambda_k} \otimes \cdots \otimes f^{\lambda_1}$ between two rows of anti-symmetrisers $g^{\lambda_1^\vee} \otimes \cdots \otimes g^{\lambda_m^\vee}$ to produce the element $\dot{e}_\lambda$. Note, that upon closure, since the building blocks $g^{(k)}$ are idempotent, the two rows of anti-symmetrisers can be replaced by one, hence,

$$\widehat{\dot{e}}_\lambda = \frac{1}{\alpha_{\lambda_1} \cdots \alpha_{\lambda_k}\ \alpha_{\lambda_1^\vee} \cdots \alpha_{\lambda_m^\vee}}\ \ \widehat{e}_\lambda\,.$$

The description of Young diagrams in [Y, §2] leads to the row of symmetrisers appearing in the reverse order and the crossings are all negative (compare Figure 4.2 with [Y, Figure 3.]). However, there is no material difference between e_λ and $\dot{e}_\lambda$. Our elements e_λ can easily be rearranged to look like those of Yokota. For example, when working with the Young diagrams $\mu_{k,l}$ in Proposition 4.7.4 we rearranged our diagram for $e_{k,l}$ to present it in the same form as Yokota's quasi-idempotent for $\mu_{k,l}$, to simplify the calculation of $\alpha_{k,l}$.

It is worth noting that, even working with idempotent building blocks, the resulting $\dot{e}_\lambda$ is only quasi-idempotent and not an idempotent. Yokota scales this element to obtain a genuine idempotent.

The decomposition of the Hecke algebra H_n into minimal 2-sided ideals of Theorem 4.6.9 relied upon the fact that the idempotents ε_λ specialised to the Young symmetrisers of the symmetric group algebra $\mathbb{C}S_n$. Yokota makes no use of this fact. Instead he works purely in the Hecke algebra with the idempotents associated to the rectangular Young diagrams with l columns and n rows, where $1 \leq n \leq N-1$.

In [Y, Proposition 2.4], the framed Homfly polynomial is calculated for the unknot decorated by Q_λ for any Young diagram λ. It is reasonably quick to check that this agrees with our calculations in Propositions 4.7.7 and 4.7.19 for $Q_{k,1}$ and $Q_{1,l}$. To show that we can obtain the $U_q(sl(N))$ invariants from $\mathcal{X}$ we note, in Corollary 4.7.8, that $\mathcal{X}_N(Q_{k,1}) = 0$ for $k > N$ and that $\mathcal{X}_N(Q_{N,1}) = 1$. The case for $N+1$ is noted in [Y, equation 19]. The cases where $k > N+1$ follow from the inductive definition of $g^{(k)}$. In [Y, equation 20] Yokota notes that a string can be unlinked from a component carrying $g^{(N)}$ without altering the value of $\mathcal{X}_N$, as we observed in Lemma 4.7.11. However we don't need to introduce the identities of [Y, Figure 8], rather they arise naturally from the properties of $\mathcal{X}_N$, for each N, upon making the substitutions for x and v.

In Chapter 5 we extend $\mathcal{X}_N$ to a 3-manifold invariant. We show that we have invariance under the Rourke-Fenn version of the Kirby moves whereas Yokota works with Kirby's original handle slides. However, the 3-manifold invariants arrived at are equal. To see this consider the correction terms. Let L be a k component link and set

$$\mathrm{sig}(L) = \sigma_+ - \sigma_- \,.$$

Since $\sigma_+ + \sigma_- = k$, we have

$$\begin{aligned} c_+^{-\sigma_+} c_-^{-\sigma_-} &= \rho(r)^{-(\sigma_+ + \sigma_-)} c(r)^{-\sigma_+ + \sigma_-)} \\ &= \rho(r)^{-k} c(r)^{-\mathrm{sig}(L)} \,. \end{aligned}$$

(For the definition of sig(L) see Definition 5.5.3, for c_+, c_-, $\rho(r)$ and $c(r)$ see Notation 5.5.5.)

Chapter 5

The 3-manifold invariants.

5.1 Introduction.

In this chapter we demonstrate that the link invariant $\mathcal{X}_N$, defined in Chapter 4, gives rise to a 3-manifold invariant when evaluated at primitive roots of unity.

We will show that $\mathcal{X}_N$ vanishes on any link if any component is coloured by certain Young diagrams with a single row, when evaluated at a primitive rth root of unity. Let I denote the ideal generated by these Young diagrams. It follows that, at any given root of unity, we can calculate the invariant using colours in the quotient ring $\mathcal{R}_N/I$.

Note that, when q is a root of unity, our scalar ring Λ is the field of complex numbers $\mathbb{C}$. The ring $\mathcal{R}_N/I$ can be described as a complex vector space (rather than a free module) and we will construct a basis for this space.

We give a brief description of the construction of 3-manifolds via surgery on framed links and describe the requirements for a link invariant to determine a 3-manifold invariant.

Finally we provide an example of a 3-manifold invariant by fixing a colour Ω_r in $\mathcal{R}_N/I$, and showing how $\mathcal{X}_N(L)$ behaves under Kirby moves when every component of L is coloured by Ω_r.

Note that throughout this chapter we treat $\mathcal{R}_N$ as an abstract polynomial ring with indeterminates c_1, c_2, $\ldots$, c_{N-1}. The connection with the representation ring of $U_q(sl(N))$ is all but forgotten.

5.2 Two ideals and their connection.

In this section we fix q to be a primitive rth root of unity. We describe two generating sets (which depend on r) and show that they generate the same ideal of $\mathcal{R}_N$. We will demonstrate a vector space basis for the resulting quotient ring.

5.2.1 Definition.

Fix N and a primitive rth root of unity q with $r > N$. We define the *level* l of q to be

$$l = r - N\,.$$

We call a Young diagram *q-admissible* if it has fewer than N rows and at most l columns. We will denote the set of q-admissible diagrams by $Diag_{N,r}$.

5.2.2 Notation.

Let $\mathcal{I}$ be the ideal in $\mathcal{R}_N$ generated by the Young diagrams with exactly $l+1$ columns.

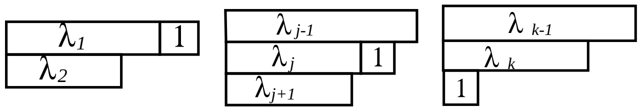
We now show that $Diag_{N,r}$ is a spanning set for the vector space $\mathcal{R}_N/\mathcal{I}$.

5.2.3 Lemma.

Let $\lambda \in \mathcal{R}_N$ be a Young diagram with n columns. Then $\lambda \times c_i$ is a linear combination of diagrams with either $n-1$, n or $n+1$ columns.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $k < N$ and $\lambda_1 = n$. Label the cells of c_i from 1 to i, top to bottom.

The first cell can be added to λ in any position that results in the formation of a legitimate Young diagram. From the combinatorial rules, any cell labelled with j, $j > 1$, must be placed in a position below and to the left of the first cell, hence, the only possible way of increasing the number of columns is shown in Figure 5.1 (a). In this case, the number of columns can't increase further because of the restrictions on the positions of the other cells.



(a) (b) (c)

Figure 5.1: Legitimate strict expansions of λ.

We have, therefore, shown that any summand of $\lambda \times c_i$ must have at most $n+1$ columns.

To show the second inequality, we need to consider the possible ways that λ can have cells added to give a column with N cells.

Let $k = N - m$. To reduce the number of columns, we must add m cells to the first column of λ and no more. Any cell to the right of or above these m cells must have a smaller label. Therefore, these m cells must be labelled $(i-m+1)$,$(i-m+2)$,$\ldots$,i, and the remaining cells must all have been placed in the first k rows, so no other column can have been filled.

This tells us that any summand can have at most one fewer columns than λ.

■

5.2.4 Theorem.

Let λ be a Young diagram in $\mathcal{R}_N$ with more than $l+1$ columns. Then λ can be expressed as a linear combination of Young diagrams with at most l columns modulo $\mathcal{I}$. Furthermore, if λ has $l+1+i$ columns where $0 \leq i \leq l$ then λ can be expressed as a linear combination of Young diagrams with at least $l+1-i$ columns and at most l columns.

Proof. The proof is an induction on the number of columns and the number of cells in the final column of the Young diagrams. For any Young diagram λ with $l+1$ columns, $\lambda \in \mathcal{I}$. Therefore, any linear combination of Young diagrams with all the coefficients equal to 0 will suffice.

Assume that the result is known for all Young diagrams $\mu < \lambda$. Let λ be the Young diagram with $l+1+i$ columns and k cells in the last column and λ' be the Young diagram obtained from λ by removing the last column. Then

$$\lambda' \times c_k = \lambda + \sum_j \mu_j \tag{5.1}$$

where $\mu_j < \lambda$ for every j. By the induction hypothesis we know that each of the μ_j can be expressed as a linear combination of Young diagrams with at most l columns. It remains to show that $\lambda' \times c_k$ can be expressed as a linear combination of diagrams with at most l columns. The quotient map $\rho : \mathcal{R}_N \to \mathcal{R}_N/I$ is a ring homomorphism, hence, $\rho(\lambda' \times c_k) = \rho(\lambda') \times \rho(c_k)$. Now $\rho(c_k) = c_k$ modulo $\mathcal{I}$. By the induction hypothesis we have an expression for $\lambda' \in \mathcal{R}_N/I$ as a linear combination of Young diagrams with at most l columns. By Lemma 5.2.3, the product of any of these terms with c_k contributes terms with at worst one extra or one fewer column. Therefore, the number of columns for the summands of $\rho(\lambda' \times c_k)$ is bounded by 0 below and $l+1$ above. Any term with $l+1$ columns is equivalent to 0 modulo $\mathcal{I}$. We have thus proved the first part of the Theorem.

We now examine the cases where λ has between $l+1$ columns and $2l+1$ columns in more detail to establish the second part.

By Lemma 5.2.3, the μ_j in equation 5.1 will have at most $l+1+i$ columns and at least $l+1+i-2$ columns. Note that $l+1-(i-2) > l+1-i$. Recall that $\mu_j < \lambda$ for every j. Therefore, by the induction hypothesis, we can assume that the expressions for the μ_j modulo $\mathcal{I}$ only involve Young diagrams with at least $l+1-i$ columns as required. It remains to show that this also holds for the product $\lambda' c_k$. By the induction hypothesis λ' can be expressed as a linear combination of Young diagrams with at least $l+1-(i-1)$. If we multiply any of these Young diagrams by c_k the number of columns in each summand is bounded below by $l+1-i$ and above by $l+1$, by Lemma 5.2.3. Any term with $l+1$ columns is equivalent to 0 modulo $\mathcal{I}$ and this proves the second part of the Theorem. Note that for $i > l$ this says no more than the first part of the Theorem.

■

This result implies that the set of Young diagrams with fewer than N rows and at most l columns is a spanning set for the quotient space $\mathcal{R}/\mathcal{I}$. In Corollary 5.2.20 we establish that this set is a vector space basis.

5.2.5 Proposition.

The number of q-admissible Young diagrams $V(r, N)$ is the binomial coefficient

$$V(r, N) = \binom{r-1}{N-1} .$$

Proof. There is a one to one correspondence between the q admissible Young diagrams and the monomials of actual degree l in N variables. (Note that here we are working with actual degree, not weighted degree as elsewhere in the thesis.) Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$. Add $l - \lambda_1$ columns with N cells to λ. The Young diagram corresponds to the monomial

$$c_{\lambda_1^\vee} c_{\lambda_2^\vee} \ldots c_{\lambda_k^\vee} c_N^{l-\lambda_1} .$$

The number of monomials in N variables of degree l is equal to

$$\binom{r-1}{N-1}$$

as required. ∎

5.2.6 Definition.

Recall from Definition 4.9.1 that the rth Adams operator of c_1 is given by the formula

$$\psi_r(c_1) = x_1^r + x_2^r + \cdots + x_N^r ,$$

where the c_i are the elementary symmetric functions in the x_j. We define the ideal I to be

$$I = \langle\, \partial\psi_r/\partial c_i \,:\, 1 \leq i \leq N-1 \,\rangle .$$

5.2.7 Proposition.

The partial derivative of $\psi_r(c_1)$ with respect to c_k is given by

$$\partial\psi_r/\partial c_i = (-1)^{k-1} r d_{r-k} .$$

The ideal I is therefore generated by $\{d_{r-i}\}_{1 \leq i \leq N-1}$

$$I = \langle d_{r-1}, d_{r-2}, \ldots, d_{r-N+1} \rangle .$$

Proof. Recall from the proof of Proposition 4.9.2 that

$$\ln(C(z)) = \sum_{m=1}^{\infty} \frac{-\psi_m(c_1)}{m} z^m .$$

Now

$$\frac{\partial}{\partial c_k} \ln(C(z)) = \frac{\partial \ln(C(z))}{\partial C(z)} \frac{\partial C(z)}{\partial c_k} = (-1)^k z^k D(z) .$$

Therefore, equating the coefficients of z^r, we get

$$(-1)\frac{\partial}{\partial c_k}\frac{\psi_r(c_1)}{r} = (-1)^k d_{r-k} .$$

Now, since $(-1)^{k-1}r$ is invertible and I is generated by the partial derivatives of $\psi_r(c_1)$, we have shown that

$$I = \langle d_{r-1}, d_{r-2}, \ldots, d_{r-N+1} \rangle ,$$

as required.

■

5.2.8 Proposition.

Let q be a primitive rth root of unity. Then for p where $r - N < p < r$

$$\mathcal{X}_N(Q_{1,p}) = 0 .$$

Proof. By Proposition 4.7.19

$$\begin{aligned} \mathcal{X}_N(Q_{1,p}) &= \prod_{i=1}^{p} \frac{s^N s^{i-1} - s^{-N} s^{-(i-1)}}{s^i - s^{-i}} \\ &= s^{-(N+1)p} \prod_{i=1}^{p} \frac{q^{N+i-1} - 1}{q^i - 1} \end{aligned}$$

Now $q^{N+i-1} = 1$ whenever $i = kr - N + 1$ for some $k \in \mathbb{N}$. In particular if $k = 1$

$$q^{N+r-N+1-1} = q^r = 1 .$$

Hence $q^r - 1$ appears in the numerator for $p > r - N$. To ensure that this is not a factor of the denominator, we require p to be less than r. Therefore, for $r - N < p < r$, the quantum invariant evaluates to 0 for q a primitive rth root of unity.

■

5.2.9 Corollary.

Let L be a link with a component coloured by d_p, where $l < p < r$. Then

$$\mathcal{X}_N(L) = 0\,.$$

Proof. The link L can be written as the closure of a $(1,1)$-tangle T, obtained from L by cutting the link at some point along the component coloured by d_p. Hence, for some scalar $\gamma_T \in \Lambda$

$$\mathcal{X}_N(L) = \gamma_T\, \mathcal{X}_N(Q_{1,p})\,.$$

Suppose that the other components of L are coloured by $\lambda_1, \ldots, \lambda_k$. We can write each λ_i as a polynomial in the c_j where $j < N$. The denominator of γ_T is the product of the denominators of the scalars $1/\alpha_{j,1}$. These scalars were calculated in Proposition 4.7.3 and the denominator of $1/\alpha_{j,1}$ is $[j]!$. Since $j < N$, the term $q^r - 1$ is not a factor of the denominator of γ_T. Hence, $\mathcal{X}_N(L) = 0$, by Proposition 5.2.8.

■

5.2.10 Remarks.

In view of Corollary 5.2.9, it may seem peculiar to define I in terms of the partial derivatives of the rth Adams operator. After all, we were looking for an ideal generated by colours which make $\mathcal{X}_N$ vanish and we have shown that the d_i, $l < i < r$, have this property.

However, we will use the fact that I is generated by the partial derivatives of a polynomial in Proposition 5.2.15 on the way to proving that $I = \mathcal{I}$.

We have shown that if we think of $d_{l+1}, \ldots, d_{r-1}$ as polynomials in the c_i then when q is a primitive rth root of unity $(\mathcal{X}_N(c_i))_{1\le i<N} \in \mathbb{C}^{N-1}$ is a solution of

$$(d_{l+1}, \ldots, d_{r-1}) = (0, \ldots, 0)\,.$$

These are not necessarily the only solutions. However, there is no obvious way to interpret other solutions in terms of knot invariants.

5.2.11 Proposition.

The ideal I contains $\mathcal{I}$.

Proof. For any given Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, there is a polynomial expression in $\{d_i\}_{i\geq 0}$, given by

$$\lambda = \begin{vmatrix} d_{\lambda_1} & d_{\lambda_1+1} & d_{\lambda_1+2} & \cdots & d_{\lambda_1+k-1} \\ d_{\lambda_2-1} & d_{\lambda_2} & d_{\lambda_2+1} & \cdots & d_{\lambda_2+k-2} \\ d_{\lambda_3-2} & d_{\lambda_3-1} & d_{\lambda_3} & \cdots & d_{\lambda_3+k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{\lambda_k-k+1} & d_{\lambda_k-k+2} & d_{\lambda_k-k+3} & \cdots & d_{\lambda_k} \end{vmatrix} .$$

If λ is a generator for $\mathcal{I}$, then $\lambda_1 = l+1$ and the top row of the matrix will be

$$(d_{l+1}, d_{l+2}, \ldots, d_{l+k-1}) .$$

Since $k < N$, we have that $l + k - 1 < l + N - 1 = r - 1$, so all the terms in the top row are generators of I. Expanding the determinant by the top row, therefore, gives us $\lambda \in I$. All the generators of $\mathcal{I}$ being in I, we have shown that $\mathcal{I} \subseteq I$.

∎

We now show that these two ideals are the same ideal and calculate its codimension. A corollary of this calculation will be that the spanning set of q-admissible diagrams is a basis of $\mathcal{R}_N/I$ (see Corollary 5.2.20).

5.2.12 Definition.

Let $P(y_1, y_2, \ldots, y_n)$ be a polynomial. We say that P is *weighted homogeneous* of *weighted degree* d with *weights* k_1, k_2, …, k_n if for any scalar t

$$P(t^{k_1}y_1, t^{k_2}y_2, \ldots, t^{k_n}y_n) = t^d P(y_1, y_2, \ldots, y_n) .$$

From the relation $C(z)D(z) = 1$, we know that we can write d_j as a weighted homogeneous polynomial of weighted degree j in the c_i, $1 \leq i \leq N$, where c_i has weight i. Let $F : \mathbb{C}^{N-1} \times \mathbb{R} \to \mathbb{C}^{N-1}$ be defined by

$$F(\mathbf{c}, t) = (d_{r-1}(\mathbf{c}, c_N = t), \ldots, d_{r-N+1}(\mathbf{c}, c_N = t)) .$$

5.2.13 Remarks.

Note that when $t = 1$ the components of $F(\mathbf{c}, t)$ are exactly the polynomials which generate I. However, when $t = 0$ we can apply Lemma 5.2.14 and so we can easily count the number of solutions to $F(\mathbf{c},0)=0$ (with multiplicity). Showing that this number is preserved under a small perturbation from $t=0$ will allow us to calculate the codimension of I and show that the q-admissible Young diagrams form a basis of $\mathcal{R}_N/I$. Note that since we have a spanning set for $\mathcal{R}_N/\mathcal{I}$ and $\mathcal{I}\subseteq I$, we know that I is of finite codimension in $\mathcal{R}_N$.

5.2.14 Lemma.[MO, §2 Theorem 1]

Let $f(y_1,\ldots,y_n)$ be a weighted homogeneous polynomial of weighted degree d with weights w_1, …, w_n with an isolated critical point at the origin. Let

$$f' : (\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$$

be defined by

$$f'(y_1,y_2,\ldots,y_n)=\left(\partial f/\partial y_1,\, \partial f/\partial y_2,\ldots,\, \partial f/\partial y_n\right).$$

Then the local degree of f' at 0 is given by the formula

$$\deg f'=\prod_{i=1}^{n}\frac{(d-w_i)}{w_i}.$$

■

5.2.15 Proposition.

The equation $F(\mathbf{c},0)=0$ has only one solution, namely $c=0$, and this solution occurs with multiplicity $\binom{r-1}{N-1}$.

Proof. Let $\mathbf{c}=(c_1,c_2,\ldots,c_{N-1})$ be a solution of $F(\mathbf{c},0)=0$. At $t=0$, we have that $c_N=0$, hence,

$$C(z)=1-c_1z+c_2z^2-\cdots+(-1)^{N-1}c_{N-1}z^{N-1}$$

$$D(z)=1+d_1z+d_2z^2+\cdots+d_lz^l+d_{l+N}z^{l+N}+\cdots$$

If we work modulo z^{N+l}, we can write

$$D(z)=D'(z)+z^{l+N}D''(z)$$

where D' is a polynomial of degree l. Now at the point $\mathbf{c}$,

$$1 = C(z)D(z) \equiv C(z)D'(z) \qquad \mod z^{l+N}\,.$$

The degree of CD' is at most $l + N - 1$, and so

$$C(z)D'(z) = 1\,,$$

but both C and D' are polynomials and therefore must both be equal to 1. Hence

$$(c_1, c_2, \ldots, c_{N-1}) = (0, 0, \ldots, 0)\,.$$

Note that the weighted degree of $\psi_r(c_1)$ is r. By Lemma 5.2.14 the multiplicity of the root 0 is given by

$$\begin{aligned}
\prod_{i=1}^{N-1} \frac{r-i}{i} &= \frac{(r-1)(r-2)\cdots(r-N+1)}{(N-1)(N-2)\cdots 1} \\
&= \frac{(r-1)!}{(r-N)!(N-1)!} \\
&= \binom{r-1}{N-1}
\end{aligned}$$

■

The following three standard results will be instrumental in showing that the two ideals are equal and that the set of Young diagrams with fewer than N rows and at most $l+1$ columns is actually a vector basis for the quotient space rather than just a spanning set. The proof that the number of critical points (counted with multiplicity) is preserved for our 1-parameter family of maps $F(\mathbf{c}, t)$ was explained to me by J.W. Bruce.

5.2.16 Proposition.[AGV, p. 92]

Suppose that a map, g, has no zeros on the boundary of a bounded domain $U \subset \mathbb{C}^n$ and that the degree of the map $g/|g|$ from the boundary of U to the n dimensional unit sphere is equal to k. Then the system $g = 0$ has a finite number of roots in U and the sum of their indices is equal to k. (For our purposes, the index of a root is equal to its multiplicity.) ■

5.2.17 Theorem.[F]

Let J be an ideal of the polynomial ring $\mathbb{C}[c_1, \ldots, c_n]$ and assume that the variety $V(J) = \{P_1, \ldots, P_m\}$ is a finite set of points. Let $\mathcal{O}_i$ be the local ring associated with the point P_i. Then there is a natural isomorphism

$$\mathbb{C}[c_1, \ldots, c_n]/J \cong \prod_{i=1}^{m} \mathcal{O}_i/J.\mathcal{O}_i$$

■

5.2.18 Corollary.

Using the same notation as in Theorem 5.2.17, let m_{P_i} denote the multiplicity of the point P_i. The dimension of $\mathbb{C}[c_1, \ldots, c_n]/J$ is equal to the sum of the multiplicities of the points P_i,

$$\dim \mathbb{C}[c_1, \ldots, c_n]/J = \sum_{i=1}^{i} m_{P_i} \, .$$

Proof. From Theorem 5.2.17 we have that the dimension of $\mathbb{C}[c_1, \ldots, c_n]/J$ is equal to the sum of the dimensions of $\mathcal{O}_i/J.\mathcal{O}_i$, for each i. This dimension is exactly the definition of the multiplicity of the point P_i and we are done. ■

Now we return to our specific example. Each of the primitive rth roots of unity provide us with a point P_i in the variety $V(I)$. It is not clear all the points in $V(I)$ arise in this way, however, other solutions of $F(\mathbf{c}, 1) = 0$ have no obvious interpretation in terms of knot invariants.

5.2.19 Theorem.

The codimension of the ideal I is $\binom{r-1}{N-1}$.

Proof. For any given scalar α,

$$C(\alpha z) D(\alpha z) = 1 \, ,$$

so the polynomial $d_j(c_1, \ldots, c_{N-1})$ is weighted homogeneous of weighted degree j with weights $(1, 2, \ldots, N)$.

Choose $(\mathbf{c}, t) \in \mathbb{C}^{N-1} \times \mathbb{R}$ such that $F(\mathbf{c}, t) = 0$. Since d_j is weighted homogeneous, then for a given α,

$$\begin{aligned} F(\alpha c_1, \alpha^2 c_2, \ldots, \alpha^{N-1} c^{N-1}, \alpha^N t^N) &= (\alpha^{r-1} d_{r-1}(c,t), \ldots, \alpha^{r-N+1} d_{r-N+1}(c,t)) \\ &= 0\,. \end{aligned}$$

Setting $\alpha = t^{-1/n}$ and $y = (\alpha c_1, \alpha^2 c_2, \ldots, \alpha^{N-1} c_{N-1})$ then $F(y, 1) = 0$. Note also, that given any solution y of $F(y, 1) = 0$, then for each t we can find an x such that $F(x, t) = 0$.

Since $0 < t \leq 1$, we have that $\alpha \geq 1$ and so

$$|y| \geq |c|$$

If we choose $R \in \mathbb{R}$ such that

$$R > \max_{\{y : F(y,1)=0\}} |y|\,,$$

then the ball B_R will contain all the preimages of 0 for $0 < t \leq 1$ and none of these points will lie on $\partial B_R = S_R$. Since the only preimage of 0 for $t = 0$, is 0, in fact the inequality can be extended to $0 \leq t \leq 1$.

By the choice of R, the map

$$\frac{F(\mathbf{c}, t)}{|F(\mathbf{c}, t)|} : S_R \times [0, 1] \to S_1$$

is well defined and is a homotopy from $t = 0$ to $t = 1$, so it has fixed degree for any value of t. By Proposition 5.2.16 the degree, for a given t, is equal to the sum of the multiplicities of the roots of $F(\mathbf{c}, t)$ in the interior of B_R. For $t = 0$, we have that the only root of $F(\mathbf{c}, 0)$ is 0 and the multiplicity of this root is $V(r, N)$, by Proposition 5.2.15. At $t = 1$ we therefore have that the sum of the multiplicities of the roots of $F(\mathbf{c}, 1)$ in the interior of B_R must also be equal to this number, but by the choice of R, all the roots are inside S_R, so this is the sum of the multiplicities of all the roots. Therefore by Theorem 5.2.17 the codimension of the ideal I is $V(r, N)$.

■

5.2.20 Corollary.

The two ideals I and $\mathcal{I}$ are equal :

$$I = \mathcal{I}\,.$$

The set of Young diagrams with at most l columns and fewer than N rows is a vector space basis for the quotient space $\mathbb{C}[c_1, \ldots, c_{N-1}]/I$.

Proof. By combining Theorems 5.2.19, 5.2.4 and Proposition 5.2.11 we have the following set of inequalities from the codimension calculations:

$$\binom{r-1}{N-1} = \dim \mathbb{C}[c_1, \ldots, c_{N-1}]/I \leq \dim \mathbb{C}[c_1, \ldots, c_{N-1}]/\mathcal{I} \leq \binom{r-1}{N-1}\,.$$

Hence $\mathcal{I} = I$ and by Theorem 5.2.4 we have a spanning set which has exactly the correct number of elements by Theorem 5.2.19, namely the Young diagrams with at most l columns and fewer than N rows.

■

5.3 The ring $\mathcal{R}_N/I$.

In this section we consider a property of the structure constants for $\mathcal{R}_N/I$ which is inherited from $\mathcal{R}_N$.

5.3.1 Notation.

We will denote the coefficient of ν in the product $\lambda\mu \in \mathcal{R}_N/I$ by $b^{\nu}_{\lambda\mu}$ and the coefficient of ν^* in the product $\lambda\mu$ by $b_{\lambda\mu\nu}$. Therefore,

$$\lambda\mu = \sum_{\nu} b^{\nu}_{\lambda\mu}\nu = \sum_{\nu^*} b_{\lambda\mu\nu}\nu^*$$

and consequently

$$b_{\lambda\mu\nu} = b^{\nu^*}_{\lambda\mu}\,.$$

This is consistent with the notation for the Littlewood-Richardson coefficients as defined in Definition 2.5.1.

5.3.2 Proposition.

Let $\lambda,\mu \in \mathcal{R}_N$, then the empty Young diagram $\emptyset \subseteq \lambda\,\mu$ if and only if $\mu = \lambda^*$. The coefficient of $\emptyset$ in $\lambda \times \lambda^*$ is 1. ∎

5.3.3 Theorem.

Let $a_{\lambda\mu\nu}$ denote the coefficient of ν^* in $\lambda\mu \in \mathcal{R}_N$, then

$$a_{\lambda\mu\nu} = a_{\mu\nu\lambda} = a_{\nu\lambda\mu}\,.$$

Proof. Let the decomposition of $\lambda\mu \in \mathcal{R}_N$ be

$$\lambda\mu = \sum_\eta a_{\lambda\mu\eta}\eta^*\,.$$

Consider the decomposition of the tensor product $\lambda\mu\nu$.

$$\lambda\mu\nu = \sum_\eta a_{\lambda\mu\eta}\eta^*\nu$$

By Proposition 5.3.2, the empty Young diagram will occur in only one term, namely when $\eta = \nu$, with coefficient $a_{\lambda\mu\nu}$. Similarly, we have that $a_{\mu\nu\lambda}$ is the coefficient of $\emptyset$ in the decomposition of $\mu\nu\lambda$ and $a_{\nu\lambda\mu}$ is the coefficient of $\emptyset$ in $\nu\lambda\mu$. Noting that

$$\lambda\mu\nu = \mu\nu\lambda = \nu\lambda\mu$$

we are done. ∎

5.3.4 Theorem.

The relation in Theorem 5.3.3 holds for the structure constants of $\mathcal{R}_N/\mathcal{I}$. Recall that $b_{\lambda\mu\nu}$ denotes the coefficient of $\nu^* \in Diag_{N,r}$ in the expression for $\lambda\mu$, as an element of $\mathcal{R}_N/I$. Then

$$b_{\lambda\mu\nu} = b_{\mu\nu\lambda} = b_{\nu\lambda\mu}\,.$$

Proof. Let $\lambda,\mu,\nu \in \mathcal{R}_N/\mathcal{I}$ be Young diagrams in $Diag_{N,r}$. The decomposition of $\lambda\mu$, in $\mathcal{R}_N/\mathcal{I}$, is given by

$$\lambda\mu = \sum_{\eta\in Diag_{N,r}} b_{\lambda\mu\eta}\eta^*$$

Note that as η has at most l columns, then so has η^*. Therefore $\eta^* \in Diag_{N,r}$. Hence, $\eta\eta^*$ can be written as a linear combination of Young diagrams with at most $2l$ columns, as an element of $\mathcal{R}_N$. The trivial representation will occur with coefficient 1 from Proposition 5.3.2. Those terms with at least $l+2$ columns can be expressed as linear combinations of Young diagrams with at least 1 row, by the Theorem 5.2.4. Since $Diag_{N,r}$ is a basis and the trivial module doesn't appear in the expressions for diagrams with more than $l+1$ columns, the only contribution to the scalar term is 1. We can then use the same method as for Theorem 5.3.3 to show that

$$b_{\lambda\mu\nu} = b_{\mu\nu\lambda} = b_{\nu\lambda\mu} \ ; \qquad \forall\, \lambda,\ \mu,\ \nu \in Diag_{N,r}\,.$$

■

5.4 Surgery on a link.

5.4.1 Definition.

Let $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_m$ be a framed link in S^3. *Surgery* is a method of producing a 3-manifold by cutting up S^3 using the link as a pattern.

Remove a solid torus neighbourhood V_i of each component L_i from S^3. This gives us a compact 3-manifold with boundary called the *exterior of the link* which we denote by $\mathrm{ext}L$. The manifold $M(L)$ is obtained by gluing a solid torus to each of the boundary components of $\mathrm{ext}L$. The gluing specifies which curves on the boundary of $\mathrm{ext}L$ are to be glued to the meridian of the solid torus. This is determined by the framing of the link. The framing of a link component identifies a choice of parallel curves. It is these curves which are glued to the meridians of the solid tori.

More formally, let U be a regular closed neighbourhood of L in S^3, consisting of solid tori $U_1, \ldots, U_m$. For $i = 1 \ldots m$ identify U_i with $S^1 \times B^2$ so that L_i is identified with $S^1 \times 0$ where 0 is the centre of B^2. The framing of L_i is identified with a constant normal vector field on $S^1 \times 0 \subset S^1 \times B^2$.

Let B^4 be a closed 4-ball bounded by S^3. Glue m copies of the 2-handle $B^2 \times B^2$ to B^4 along the identities $U_i = S^1 \times B^2 = \partial B^2 \times B^2$ for each i.

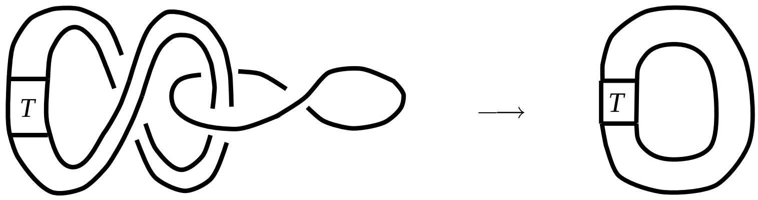
5.4.2 Comments.

Lickorish [Li1] proved that every 3-manifold can be obtained in this way. Of course, it is possible that the same manifold can be obtained by surgery on two different links. The next result, determines exactly when this can happen. Details of these results can be found in Rolfsen's book [Ro].

5.4.3 Theorem.[FR, Ki, Li1]

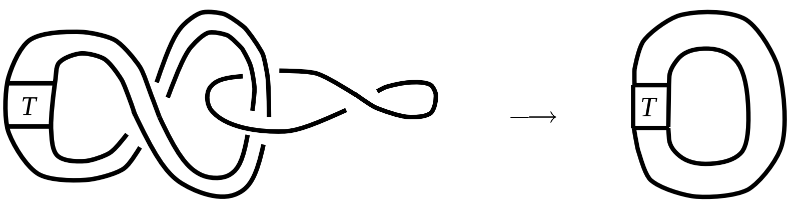
Every closed oriented 3-manifold, M, can be obtained by surgery on a framed link, i.e. given any 3-manifold M there is some link L for which $M = M(L)$.

There is an orientation preserving homeomorphism between $M(L)$ and $M(L')$ if and only if L and L' are related by a finite sequence of Kirby moves. The positive and negative Kirby moves are shown in Figures 5.2 and 5.3.



$L \longrightarrow \varphi_+(L)$

Figure 5.2: The positive Kirby move.



$L \longrightarrow \varphi_-(L)$

Figure 5.3: The negative Kirby move.

$\blacksquare$

5.4.4 Remarks.

Ideally we wish to find an element of the representation ring $\mathcal{R}_N$, Ω say, for which the quantum invariant

$$J(L; \Omega, \ldots, \Omega) = J(L'; \Omega, \ldots, \Omega)$$

where L' and L are related by a Kirby move, then we have a 3-manifold invariant.

In fact, we describe an element $\Omega_r \in \mathcal{R}_N/I$ for which

$$J(L; \Omega_r, \ldots, \Omega_r) = cJ(L'; \Omega_r, \ldots, \Omega_r)$$

for some scalar c when we evaluate at q a primitive root of unity. We need to introduce a correction term to obtain a 3-manifold invariant. Working in $\mathcal{R}_N/I$, which is a finite dimensional space, we know that Ω_r will be a finite linear combination of $\lambda \in Diag_{N,r}$.

5.5 A 3-manifold invariant.

In this section, we use the techniques of Morton and Strickland [MS] to obtain a 3-manifold invariant by evaluating $\mathcal{X}_N$ at a primitive root of unity.

Throughout this section we will assume that q is a fixed primitive rth root of unity. All evaluations of $\mathcal{X}_N$ will be at this root of unity.

5.5.1 Notation.

Let Ω_r denote the following linear combination of elements in $\mathcal{R}_N/I$.

$$\Omega_r = \sum_{\lambda \in Diag_{N,r}} \delta_{\lambda^*} \ \lambda \, .$$

where

$$\delta_{\lambda^*} = \mathcal{X}_N(Q_{\lambda^*}) \, .$$

Note that Corollary 4.8.10 implies that

$$\delta_\lambda = \delta_{\lambda^*} \, .$$

5.5.2 Proposition.

Let L_1 be an oriented link with k components and let L_2 be the link obtained from L_1 by reversing the orientation of the first component. Then

$$\mathcal{X}_N(L_1; \Omega_r, \lambda_2, \ldots, \lambda_k) = \mathcal{X}_N(L_2; \Omega_r, \lambda_2, \ldots, \lambda_k)\,.$$

Proof.

$$\begin{aligned}
\mathcal{X}_N(L_1; \Omega_r, \lambda_2, \ldots, \lambda_k) &= \sum_{\mu \in Diag_{N,r}} \delta_{\mu^*} \mathcal{X}_N(L_1; \mu, \lambda_2, \ldots, \lambda_k) \\
&= \sum_{\mu \in Diag_{N,r}} \delta_{\mu^*} \mathcal{X}_N(L_2; \mu^*, \lambda_2, \ldots, \lambda_k)
\end{aligned}$$

This follows from the properties of the quantum invariants. Now, μ^* has at most the same number of columns as μ, hence, $\mu \in Diag_{N,r}$ if and only if $\mu^* \in Diag_{N,r}$. Therefore, since $\delta_\mu = \delta_{\mu^*}$,

$$\begin{aligned}
\sum_{\mu \in Diag_{N,r}} \delta_{\mu^*} \mathcal{X}_N(L_2; \mu^*, \lambda_2, \ldots, \lambda_k) &= \sum_{\mu \in Diag_{N,r}} \delta_{\mu^*} \mathcal{X}_N(L_2; \mu, \lambda_2, \ldots, \lambda_k) \\
&= \mathcal{X}_N(L_2; \Omega_r, \lambda_2, \ldots, \lambda_k)
\end{aligned}$$

∎

5.5.3 Definition.

Given a framed link L with k components we can define the $k \times k$ matrix $(l_{i,j})$ by

$$l_{i,j} = \begin{cases} \mathrm{lk}(L_i, L_j) & \text{for } i \neq j \\ \text{framing on } L_i & \text{for } i = j \end{cases}.$$

The matrix is symmetric and hence can be thought of as a quadratic form. We define $\mathrm{sig}(L)$ to be the signature of this quadratic form. (Note this is not necessarily equal to the standard definition of the signature of a link.)

5.5.4 Lemma.

If $\overline{L}$ is the link obtained from L by reversing the orientation of every component then

$$\mathrm{sig}(\overline{L}) = \mathrm{sig}(L)\,.$$

Under a Kirby move the signature is altered as follows :

$$\mathrm{sig}(L) = \mathrm{sig}(\varphi_{\pm}(L)) \pm 1\,.$$

■

5.5.5 Notation.

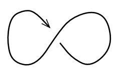
Let c_+ denote the scalar

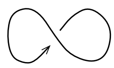
$$c_+ = \sum_{\nu \in Diag_{N,r}} f_\nu \delta_\nu^2$$

and c_- the scalar

$$c_- = \sum_{\nu \in Diag_{N,r}} f_\nu^{-1} \delta_\nu^2\,.$$

The scalar c_+ (respectively c_-) is the values of $\mathcal{X}_N$ on the unknot with framing $+1$ (respectively framing -1) coloured by Ω_r.

$$c_+ = \mathcal{X}_N \left(\Omega_r \right.$$  $$\left. \right)\,.$$

$$c_- = \mathcal{X}_N \left(\Omega_r \right.$$  $$\left. \right)\,.$$

Since $c_+ \in \mathbb{C}$, we can write c_+ as the product of a positive real number $\rho(r)$ and a complex number of unit length $c(r)$

$$c_+ = \rho(r) c(r)\,.$$

5.5.6 Theorem.

The element

$$T(L) = \rho(r)^{-k} c(r)^{-\mathrm{sig}(L)} \mathcal{X}_N(L; \Omega_r, \ldots, \Omega_r)$$

depends only on the manifold given from the k component link L by surgery where $\mathrm{sig}(L)$ is the signature of the quadratic form described in Definition 5.5.3.

■

The proof relies, almost entirely, on the symmetry of the coefficients $b_{\lambda\mu\nu}$ demonstrated in Theorem 5.3.4. First we establish some properties of Ω_r as an element of the ring $\mathcal{R}_N/I$ and look at the behaviour of $\mathcal{X}_N$ under the Kirby moves. The proof of Theorem 5.5.6 can be found on page 169.

5.5.7 Lemma.

The element Ω_r is an eigenvector for multiplication by λ, for any $\lambda \in Diag_{N,r}$.

Proof.

$$\begin{aligned}
\lambda\Omega_r &= \sum_{\mu \in Diag_{N,r}} \delta_{\mu^*}\ \lambda\mu \\
&= \sum_{\mu \in Diag_{N,r}} \sum_{\nu \in Diag_{N,r}} \delta_{\mu^*} b_{\lambda\mu\nu} \nu^* \\
&= \sum_{\nu \in Diag_{N,r}} \sum_{\mu \in Diag_{N,r}} \delta_{\mu^*} b_{\lambda\nu\mu} \nu^* \quad \text{by Theorem 5.3.4.}
\end{aligned}$$

Now,

$$\lambda\nu = \sum_{\mu \in Diag_{N,r}} b_{\lambda\nu\mu}\mu^* \, .$$

Therefore,

$$\delta_{\lambda\nu} = \delta_\lambda \delta_\nu = \sum_{\mu \in Diag_{N,r}} b_{\lambda\nu\mu}\delta_{\mu^*} \, .$$

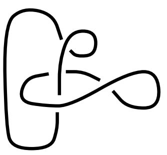
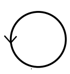
Hence,

$$\begin{aligned}
\lambda\Omega_r &= \sum_{\nu \in Diag_{N,r}} \delta_\lambda \delta_\nu \nu^* \\
&= \delta_\lambda \Omega_r \, .
\end{aligned}$$

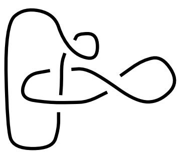
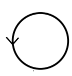
$\blacksquare$

5.5.8 Proposition.

Let λ be a basis element of $\mathcal{R}_N/I$. Then

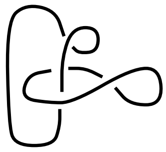
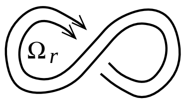
$\mathcal{X}_N$ (λ  Ω_r) $= c_+ \ \mathcal{X}_N$ ( λ)

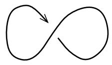
Similarly

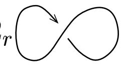
$\mathcal{X}_N$ (λ  Ω_r) $= c_- \ \mathcal{X}_N$ ( λ)

where c_- is the complex conjugate of c_+.

Proof. By Proposition 5.5.2, a component coloured by Ω_r can be oriented in either direction, without changing the value of $\mathcal{X}_N$. Therefore, we can choose the orientation so that the linking number of the two components is 1 rather than -1. Therefore, omitting to write $\mathcal{X}_N$,

λ  Ω_r $=$ λ 

$=$ $\lambda\Omega_r$ 

$=$ $\delta_\lambda\ \Omega_r$  by Lemma 5.5.7.

$=$ $c_+\delta_\lambda\,.$

To show the result for the negative Kirby move change all the framings in the above proof from $+1$ to -1.

To see that $c_- = \overline{c}_+$, note that

$$|f_\lambda| = 1$$

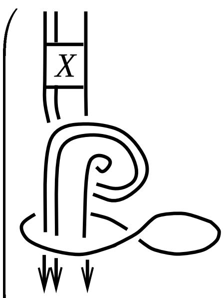
and therefore f_λ^{-1} is the complex conjugate of f_λ. We can calculate δ_λ by writing λ as a polynomial in the c_i. Therefore, δ_λ is an integer linear combination of rationals of quantum integers. Now for any given $n \in \mathbb{Z}$

$$[n] = \frac{s^n - s^{-n}}{s - s^{-1}}\,.$$

Since s has unit length s^{-n} is the conjugate of s^n and so $[n] \in \mathbb{R}$. Hence δ_λ^2 is a real number and therefore self conjugate. ∎

5.5.9 Corollary.

Let L be a link which is the closure $\widehat{X}$ of some (n,n)-tangle X. Then

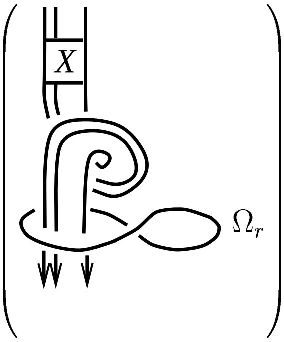
$\mathcal{X}_N$ ( Ω_r) $=$ $c_+\ \ \mathcal{X}_N\,(L)$.

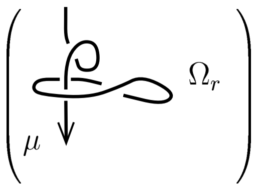
Although the diagram on the lefthand side is drawn as an open tangle, we will evaluate $\mathcal{X}_N$ on its closure.

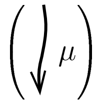
Proof. The tangle X determines an endomorphism of c_1^n. As an element of $\mathcal{R}_N/I$, we can write c_1^n as a linear combination of q-admissible diagrams

$$c_1^n = \sum_{\mu \in Diag_{N,r}} \beta_\mu \mu \, .$$

By Schur's Lemma, as an $\mathcal{R}_N$-module endomorphism the tangle X is a scalar multiple, γ_μ say, of the identity map when restricted to each irreducible piece. Hence we can replace X by $\sum_{\mu \in Diag_{N,r}} \gamma_\mu \beta_\mu \mu$. The result then follows from Proposition 5.5.8.

$\mathcal{X}_N$ 

$= \sum_{\mu \in Diag_{N,r}} \gamma_\mu \beta_\mu \mathcal{X}_N$ 

$= \sum_{\mu \in Diag_{N,r}} c_+ \gamma_\mu \beta_\mu \mathcal{X}_N$ 

$= c_+ \mathcal{X}_N (L) \quad .$

∎

5.5.10 Lemma.

Let L and K be two links and let λ denote an element of the basis of $\mathcal{R}_N/I$. Then

$$J(L; \lambda, \mu_2, \ldots, \mu_m) J(K; \lambda, \eta_2, \ldots, \eta_n) = \delta_\lambda J(L\#K; \lambda, \mu_2, \ldots, \mu_m, \eta_2, \ldots, \eta_n)$$

where $L\#K$ denotes the connected sum of the links L and K.

Proof. Consider the link L as the closure of a $(1,1)$-tangle T on the first component and K as the closure of S.

The tangle T determines a module endomorphism of λ which by Schur's Lemma is a scalar multiple of the identity map. Let ρ_T denote the scalar. Let ρ_S denote the scalar associated to S. Then

$$\begin{aligned} J(L;\lambda,\mu_2,\ldots,\mu_m)J(K;\lambda,\eta_2,\ldots,\eta_n) &= \delta_\lambda\rho_T\delta_\lambda\rho_s \\ &= \delta_\lambda\delta_\lambda\rho_T\rho_S \\ &= \delta_\lambda J(L\#K;\lambda,\mu_2,\ldots,\mu_m,\eta_2,\ldots,\eta_n) \end{aligned}$$

∎

5.5.11 Notation.

Let H (respectively H^*) denote the matrix (over the basis $Diag_{N,r}$) for the Hopf link where the linking number of the two components is $+1$ (respectively -1) and each component has zero framing. Therefore, H is the matrix defined in Lemma 4.8.7 and

$$H_{\lambda\mu} = \mathcal{X}_N(H;\lambda,\mu)\,.$$

Note that, by reversing the orientation of one component we obtain the Hopf link with linking number -1 (respectively 1), therefore,

$$H^*_{\lambda\mu} = H_{\lambda\mu^*}$$

By Lemma 4.8.7 the matrix H is invertible. This is essential to prove that $T(L)$ is a 3-manifold invariant as stated in Theorem 5.5.6.

We define the matrix F to be the matrix whose entries are the framing coefficients f_λ (see Comment 3.6.6) on the diagonal and 0 off the diagonal.

The next Proposition demonstrates how we can use the Kirby moves and the connected sum of links to calculate $\mathcal{X}_N$ is two ways and so derive relationships between H and H^*. The next Proposition does the spade work required to show that H^* is the inverse of H up to a scalar,

$$H^*H = \rho(r)^2 I\,.$$

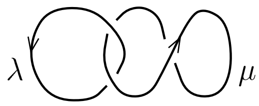
5.5.12 Proposition.

The following hold

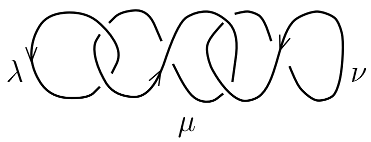
$$HFHF = c_+F^{-1}H^*\,. \tag{5.2}$$

$$H^*F^{-1}H^*F^{-1} = c_-FH\,. \tag{5.3}$$

Proof. Consider the matrix HF, where H and F are as defined in Notation 5.5.11. The (λ,μ)th entry of HF is $f_\mu H_{\lambda,\mu}$ which corresponds to calculating the $\mathcal{X}_N$ of the link below.



Let $T_{\lambda,\mu,\nu}$ denote the value of $\mathcal{X}_N$ for the link below

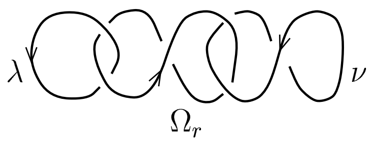


$$\begin{aligned} T_{\lambda,\mu,\nu} &= \delta_\mu^{-1} f_\mu H_{\lambda,\mu} f_\nu H_{\mu,\nu} \quad \text{by Lemma 5.5.10} \\ &= \delta_\mu^{-1}(HF)_{\lambda,\mu}(HF)_{\mu,\nu}\,. \end{aligned}$$

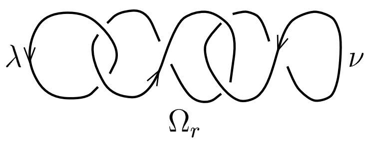
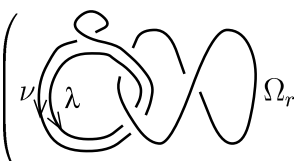
Therefore,

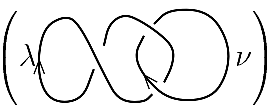
$$((HF)^2)_{\lambda,\nu} = \sum_{\mu\in Diag_{N,r}} \delta_\mu T_{\lambda,\mu,\nu}\,.$$

Alternatively we can think of this as the value of $\mathcal{X}_N$ on the link below.



Now

$\mathcal{X}_N$ () $=$ $\mathcal{X}_N$ ()

$=$ $c_+\mathcal{X}_N$ ()

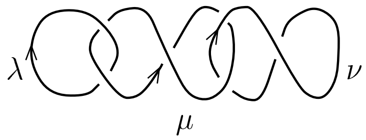
$= c_+(F^{-1}H^*)_{\lambda,\nu}\,.$

Therefore,

$$HFHF = c_+F^{-1}H^*\,.$$

The proof of the second relation is similar. We use Hopf links with negative linking number and negative framing and apply Lemma 5.5.10, providing us with two ways to calculate the same invariant.

Let $T^*_{\lambda\mu\nu}$ denote the value of $\mathcal{X}_N$ for the link below

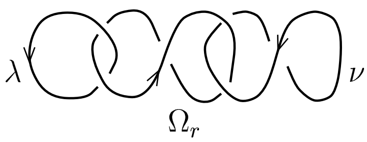


$$\begin{aligned} T^*_{\lambda,\mu,\nu} &= \delta_\mu^{-1} f_\mu^{-1} H^*_{\lambda,\mu} f_\nu^{-1} H^*_{\mu,\nu} \quad \text{by Lemma 5.5.10} \\ &= \delta_\mu^{-1} (H^* F^{-1})_{\lambda,\mu} (H^* F^{-1})_{\mu,\nu} \,. \end{aligned}$$

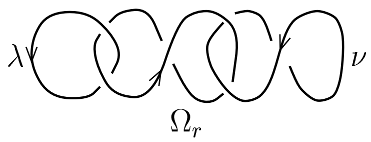
Therefore,

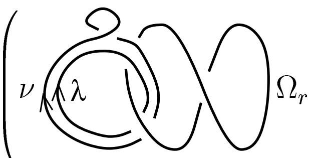
$$((H^* F^{-1})^2)_{\lambda,\nu} = \sum_{\mu \in Diag_{N,r}} \delta_\mu T_{\lambda,\mu,\nu} \,.$$

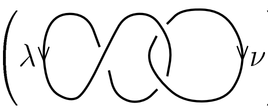
Alternatively we can think of this as the value of $\mathcal{X}_N$ on the link below.



Now

$\mathcal{X}_N \Big($  $\Big)$

$= \mathcal{X}_N \Big($  $\Big)$

$= c_- \mathcal{X}_N \Big($  $\Big)$

$= c_- (FH)_{\lambda,\nu} \,.$

Therefore,

$$H^* F^{-1} H^* F^{-1} = c_- F H \,.$$

■

5.5.13 Corollary.

The scalar c_+ (and therefore c_-) is non-zero and the two Hopf matrices H and H^* are almost inverses of each other

$$H^* H = c_+ c_- I$$

and

$$\rho(r)^2 = c_+ c_- = \sum_{\mu \in Diag_{N,r}} \delta_\mu^2 \,.$$

This equation can be found in [Y, Propsition 4.3].

Proof.

$$\begin{aligned} H^*HFH &= H^*HFHFF^{-1} \\ &= c_+ H^* F^{-1} H^* F^{-1} \qquad \text{by equation 5.2} \\ &= c_+ c_- FH \qquad \text{by equation 5.3.} \end{aligned}$$

Therefore, since F and H are invertible,

$$H^*H = c_+ c_- I \,.$$

Calculating the first entry of H^*H we see that

$$\begin{aligned} \sum_{\lambda \in Diag_{N,r}} H_{\emptyset,\lambda^*} H_{\lambda,\emptyset} &= \sum_{\lambda \in Diag_{N,r}} \delta_\lambda \delta_{\lambda^*} \\ &= \sum_{\lambda \in Diag_{N,r}} \delta_\lambda^2 \qquad \text{by Corollary 4.8.10.} \end{aligned}$$

Since δ_λ^2 is a positive real number, c_+ and c_- can't be zero. ∎

We can now prove that $T(L)$ is a 3-manifold invariant as stated in Theorem 5.5.6

5.5.14 Proof of Theorem 5.5.6

If L has $k+1$ components, $\varphi_+(L)$ will have k. By Lemma 5.5.4 we know that

$$\mathrm{sig}(L) = \mathrm{sig}(\varphi_+(L)) + 1 \,.$$

Therefore,

$$\begin{aligned} T(L) &= \rho(r)^{-k-1} c(r)^{-\mathrm{sig}(L)} \mathcal{X}_N(L; \Omega_r, \ldots, \Omega_r) \\ &= \rho(r)^{-1} \rho(r)^{-k} c(r)^{-(\mathrm{sig}(\varphi_+(L))+1)} c_+ \mathcal{X}_N(\varphi_+(L); \Omega_r, \ldots, \Omega_r) \\ &= \rho(r)^{-1} c(r)^{-1} c_+ T(\varphi_+(L)) \\ &= T(\varphi_+(L)) \,. \end{aligned}$$

For the negative Kirby move, note that $c_- = \rho(r)\overline{c(r)}$ and since $c(r)$ has unit length, $\overline{c(r)} = c(r)^{-1}$. Therefore,

$$\begin{aligned} T(L) &= \rho(r)^{-1}c(r)c_- T(\varphi_-(L)) \\ &= \rho(r)^{-1}c(r)\rho(r)\overline{c(r)}T(\varphi_-(L)) \\ &= c(r)c(r)^{-1}T(\varphi_-(L)) \\ &= T(\varphi_-(L))\,. \end{aligned}$$

This completes the proof. ■

5.5.15 Comments.

In [RT2], Turaev and Reshetikhin gave a method for constructing 3-manifold invariants from modular Hopf algebras. Turaev and Wenzl [TW] and Andersen [An] proved that the representation theory of $U_q(sl(N))$ at q a root of unity gives rise to a modular Hopf algebra. (The case where $N = 2$ had been treated earlier in [RT2].) Hence we can define a 3-manifold invariant from quantum group invariants at q a root of unity. A detailed account of the theory can be found in Turaev's book [T3].

5.5.16 Theorem.

For a fixed N and r the 3-manifold invariant $T(L)$ defined in Theorem 5.5.6 is equal to the Turaev-Reshetikhin invariant for $U_q(sl(N))$ up to normalisation. ■

Discussion.

Theorem 5.5.16 follows directly from the work of Turaev and Wenzl [TW]. Morton and Strickland [MS] proved the result directly for $N = 2$.

We can't identify the ring $\mathcal{R}_N/I$ with a ring generated by the irreducible representations of $U_q(sl(N))$ at a root of unity. Wenzl [Wz2] proved that the q-admissible diagrams index the irreducible representations of the Hecke algebras of type A when q is a root of unity. We can show that the Littlewood-Richardson coefficients for Hecke algebras at a root of unity calculated by Goodman and Wenzl [GW] are equal to our coefficients $b^{\nu}_{\lambda\mu}$. In fact Goodman and Wenzl demonstrated that they can be calculated by considering any Young diagram to represent an element of the ring of symmetric polynomials and reducing modulo the ideal of q-admissible diagrams. (Recall that in Chapter 4 that we described the c_i as the elementary symmetric polynomials in $N-1$ variables.) They comment that these coefficients *should* be related to the decomposition of the irreducible representations of $U_q(sl(N))$ when q is a root of unity.

The representation theory of $U_q(sl(N))$ at a root of unity has been investigated by many people. Andersen [An] demonstrated the existence of a simple $U_q(sl(N))$-module for each q-admissible diagram. He defines a reduced tensor product which gives rise to a ring structure for the set of semi-simple $U_q(sl(N))$-modules. Various properties of this ring were established by Turaev and Wenzl [TW] in their study of 3-manifold invariants. The question of whether it is possible to realise the abstract ring $\mathcal{R}_N/I$ by these modules of $U_q(sl(N))$ is discussed in [Wz3]. It is not clear that the structure coefficients for the decomposition of tensor products tally with the Littlewood-Richardson coefficients of Goodman and Wenzl.

Concluding remarks.

Theorem 4.8.8 states the equivalence of two elements of the Homfly skein of the annulus. However, we had to appeal to the relationship with quantum group representations to prove it. It would be pleasing to have a completely skein theoretic proof of this result.

To this end, it is worth pursuing the possibility of reasonably simple skein relations between the elements Q_λ. This would also facilitate calculation of the scalars α_λ and calculation of invariants for knots by using the skein relations to simplify the colouring. Such a skein relation should be enough to show that the scalar α_λ is a Laurent polynomial in s only. Yokota achieves a form of relationship between the idempotents as elements of the Hecke algebra in [Y, Lemma 1.3]. However, it requires use of the product in the Hecke algebra which is not possible upon taking closure.

What information do the 3-manifold invariants carry for specific values of N and q? Do well known 3-manifold invariants appear when we evaluate at specific roots of unity?

Many of the results in Chapters 4 and 5 rely upon the invertibility of the Hopf matrix. Although this result is proved in [TW], it would be more satisfying to prove it directly from the theory of Chapter 5 and so take a step nearer to divorcing the theory from the representation theory of quantum groups.

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