SATELLITE AND QUANTUM INVARIANTS

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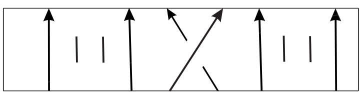
1. Algebras and knot polynomials

One of the most exciting and fruitful features of the burgeoning of ideas released by the discovery of the invariant known as the Jones polynomial is the extent of the different strands which have come to be drawn together in the course of explorations. These include algebraic, combinatoric and geometric aspects, and the involvement of quantum groups originating from theoretical physics.

A key strand from early times has been the appearance of the Hecke algebras (initially those of type A), and their role in both defining the Jones, and subsequently the Homfly polynomials [7], while at the same time their algebraic involvement with the quantum groups of the A series, and their very satisfactory modelling by pieces of knot diagrams [18].

From this standpoint the Hecke algebras H_n of type A are best defined as linear combinations of n-braids, in the sense of Artin, in which the elementary Artin braids $\{\sigma_i\}$ with

$\sigma_i \quad =$ 

$i \quad i+1$

each satisfy a quadratic relation in addition to the braid relations.

Up to isomorphism the quadratic relation can take the form

$$\sigma^2 - z\sigma - 1 = 0$$

for a parameter z, or equally

$$(\sigma - s)(\sigma + s^{-1}) = 0,$$

where $z = s - s^{-1}$.

Replacing σ by $x^{-1}\sigma$ gives an isomorphic algebra with the relation $x^{-1}\sigma - x\sigma^{-1} = z$, so that any other quadratic relation can be used instead. The 2-parameter version lies at the heart of the Homfly polynomial.

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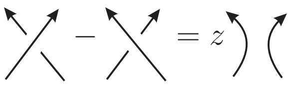
In algebraic literature the letters g_i or T_i often replace σ_i, and the quadratic relation $(g_i - q)(g_i + 1) = 0$ translates to $x = s, q = s^2$ above, while the notation $x = \alpha$ or $x = v$ occurs at an early stage in the work of Kauffman or Morton [11].

As an aside, the cases $v = s^N, z = s - s^{-1}$ showed up very early in the Hecke algebra explorations of Jones, [7], and relate closely to the basic invariants from the quantum groups $sl_q(N)$, with $q = s^2$. The original Jones polynomial fits readily here with the case $N = 2$.

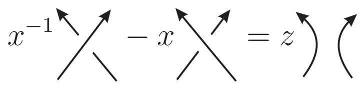
From the knot-theoretic viewpoint the most useful approach is to work with linear combinations of knot diagrams, using a ground ring Λ which is often $\mathbb{Z}[v^{\pm 1}, s^{\pm 1}]$ or some variant with Laurent polynomials in two independent parameters, and possibly some allowed denominators, such as $s^r - s^{-r}$.

At a crossing in the diagram impose the linear relation



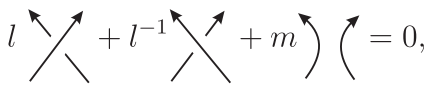
between diagrams which only differ as shown. This is a direct counterpart to the quadratic relation $\sigma_i - \sigma_i^{-1} = z$ Id and gives the basic version of the Hecke algebra H_n, when using braids on n strings as diagrams.

The more general case with



corresponds to using the quadratic with roots $xs, -xs^{-1}$ and $z = s - s^{-1}$. This approach is developed in [12, 1] and [10].

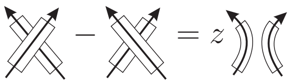
The relation



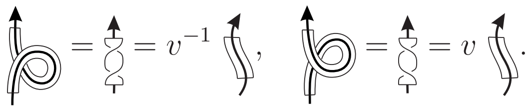
which is the form adopted by Lickorish and Millett for their initial diagram-based construction of the Homfly polynomial, corresponds to $l = ix^{-1}, m = iz$.

Remark 1.1. The version using x and s adapts more readily to quantum group invariants, and also to finite type invariants, and is now generally adopted, up to the letters chosen for the parameters, in favour of the equivalent l, m form.

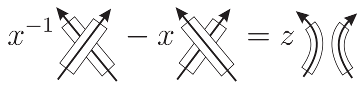
The parameter x can be sidelined quite a bit in calculations by working with framed (or 'banded') diagrams, which carry a specified ribbon neighbourhood for every curve. This is often given implicitly as the 'blackboard framing' in which the band lies parallel to the curve as drawn in the diagram. Then use of Reidemeister moves R_{II} and R_{III} on diagrams preserve the implicit bands, and the basic relation



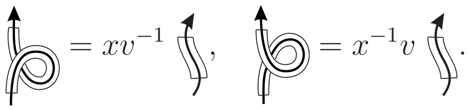
can be used, along with a parameter v to handle the effect of R_I on a diagram, which introduces a twist in the implicit band. This shows up as the additional relation



Banded diagrams can be adapted to incorporate a parameter x by simply multiplying any diagram D by $x^{\mathrm{wr}(D)}$, where $\mathrm{wr}(D)$ is the 'writhe' of the diagram, in other words the number of positive crossings minus the number of negative crossings. In this adaptation the relations become



and



This allows for any adjustment of the quadratic relation that may be wanted when comparing invariants.

Early Hecke algebra calculations of the Homfly polynomial [16] made use of the case $x = 1$ to simplify the work with braids, so that only the parameter z was needed for a great part of the calculations, and the second parameter v was only incorporated in the closing stages. The general method of working with framed diagrams followed Kauffman's similar approach to the second variable in his construction of the 2-variable Kauffman polynomial.

2. Satellite invariants

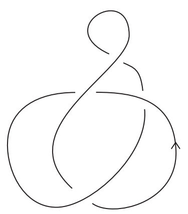
Framed (banded) knots and links form the natural setting for the use of satellites in developing extra invariants based on the Homfly polynomial.

Calculations in [16] showed at an early stage that, unlike the special case of the Alexander polynomial, there were potential further invariants available for a knot K by the simple device of calculating the Homfly polynomial of chosen satellites of K. For example, to compare K and K' we could look at the 2-cable of each (with the same number of twists). If K and K' are equivalent knots then so are the corresponding 2-cables, and hence the cables will have the same Homfly polynomials. However it can happen that cables have different polynomials even when the original knots K, K' have the same polynomial, showing that the polynomial of the chosen cable is an invariant which can carry information that is independent from the polynomial of the knot itself [16, 19].

To give a systematic account of satellites we start with a banded knot K, and 'decorate' K with a 'pattern' Q.

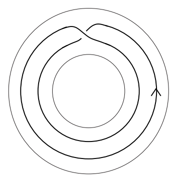
For example, when

$$K \quad =$$



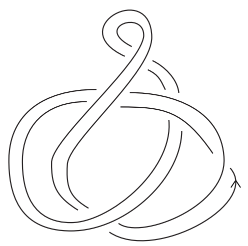
and Q is the simple 2-cable pattern

$$Q \quad =$$



then the resulting satellite is

$$K * Q \quad =$$

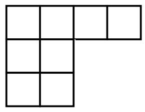


In general a pattern is a framed diagram $Q \subset A \times I$ lying in the standard thickened annulus $A \times I$. Placing the annulus around the chosen band neighbourhood of K carries Q to the satellite $K * Q$.

For each choice of Q we can then regard the Homfly polynomial $P(K * Q)$ as an invariant of the banded knot K, written $P(K\colon Q)$.

While very many choices of Q are available, giving potentially a large number of different invariants for K, it is possible to spot relations between these, where the choices for pattern are closely related as diagrams within $A \times I$. If for example we restrict attention to patterns formed from closed m-string braids in $A \times I$ then these will result in at most $p(m)$ linearly independent resulting invariants, where $p(m)$ is the number of partitions of the positive integer m.

A partition of m is often displayed as a Young diagram with m cells. For example the partition of 8 into three parts $8 = 4 + 2 + 2$ can be visualised as a diagram



with 3 rows.

Organising satellite invariants. There are a number of ways to organise satellite invariants in terms of linear combinations of framed diagrams in the annulus, modulo skein relations. Among these are some particularly useful combinations Q_λ, one for each partition of m.

The resulting invariants $P(K\colon Q_\lambda)$ span all the $p(m)$ invariants arising from closed m-braid patterns. They have good integrality properties in the variables v, s, and change by a scalar multiple depending only on the partition λ when the twist of the band around K is changed.

The elements Q_λ can be constructed in a number of equivalent ways. The original method was as the closure of suitably chosen idempotents in the Hecke algebra H_m, [4, 1, 2, 22]. Later they appeared as the analogue of the Schur functions s_λ when the possible decorating elements in the annulus are interpreted as symmetric polynomials in a large number of variables, [9]. More recently they have been identified with eigenvectors of a simple operation on linear combinations of diagrams in the annulus, [10]. An extension of this approach can be found in [5] and the consequent integrality results in [13], while an account of all the different approaches can be found in [15, 14].

3. Quantum group invariants

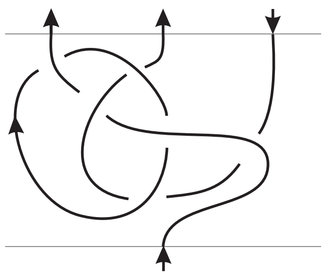
One unexpected development, which emerged closely after the knot polynomial discoveries, was the use of quantum groups in constructing 1-parameter invariants of framed knots and links. The initial work by Kirillov and Reshetikhin [8] on invariants derived from the quantum group $sl_q(2)$ was followed by a more systematic general approach by Reshetikhin and Turaev [20] based on the properties of the universal R-matrix in the algebraic formulations of quantum groups arising from work of Jimbo [6] and Drinfeld [3].

Connections between these invariants and knot polynomials were established at a very early stage [17, 12], although the exact details and general consequences were the subject of much subsequent work.

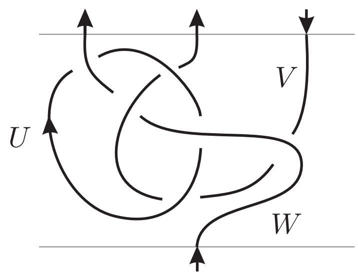
Quantum groups. Quantum groups are algebras $\mathcal{G}$ over the formal power series ring $\mathbf{Q}[[h]] = \Lambda$, They are typically a deformed version of a classical semi-simple Lie algebra, which shows up in the limit when $h \to 0$.

Knot invariants are constructed from finite dimensional $\mathcal{G}$-modules, making use of crucial features of the algebra $\mathcal{G}$. The major properties are that the tensor product $V \otimes W$ of two $\mathcal{G}$-modules is again a $\mathcal{G}$-module, and that there is an invertible $\mathcal{G}$-module homomorphism $R_{V,W} \colon V \otimes W \to W \otimes V$, determined by the universal R-matrix in $\mathcal{G}$. In addition there is a dual module V^* for each V, and homomorphisms $V \otimes V^* \to \Lambda$ and $\Lambda \to V \otimes V^*$, where the ground ring Λ acts as the trivial $\mathcal{G}$-module.

Colouring tangles. Any oriented diagram consisting of closed curves and arcs connecting points at the top and bottom, as shown,



can be *coloured* by making a choice of $\mathcal{G}$-module for each component.

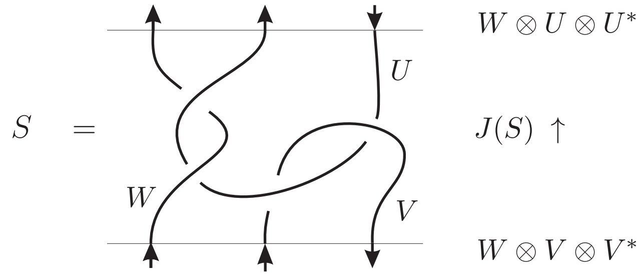


The resulting *coloured tangle* T determines a module homomorphism

$$J(T) \colon T_{\text{bottom}} \to T_{\text{top}},$$

where T_{bottom} is the tensor product of the outgoing string colours at the bottom of T, using the dual colour for any incoming string, and T_{top} is the tensor product of the incoming strings at the top of T, with again the dual colour for any outgoing strings.

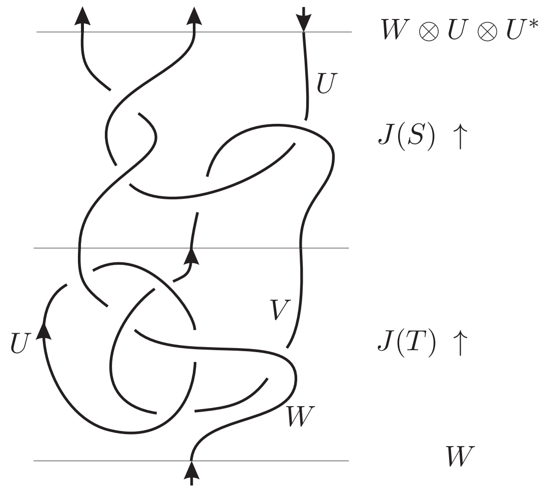
So the example T above yields $J(T) \colon W \to W \otimes V \otimes V^*$, while the coloured tangle



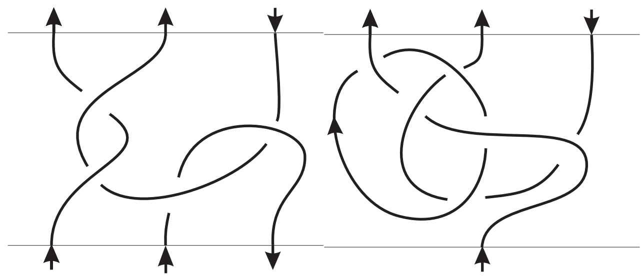
gives $J(S)\colon W \otimes V \otimes V^* \to W \otimes U \otimes U^*$.

Set $T_{\mathrm{top}} = \Lambda$ when no strings of T meet the top, and equally $T_{\mathrm{bottom}} = \Lambda$ when there are no strings at the bottom.

Placing consistently coloured tangles S and T one above the other results in the composite $J(S)J(T)$ of the homomorphisms $J(S)$ and $J(T)$,

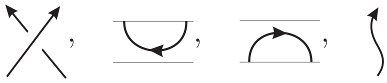


while placing them alongside each other

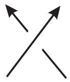
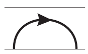


represents their tensor product $J(S) \otimes J(T)$.

Construction of homomorphisms. To construct $J(T)$ for a general coloured tangle T dissect the tangle into elementary pieces



and build up the homomorphism $J(T)$ from a definition on these pieces. Combine the homomorphisms for the elementary pieces, by taking their tensor product when they lie side by side, and composing them when consistently coloured pieces are placed one on top of the other.

Use $R_{V,W}$ or its inverse on a simple crossing of strands  coloured V and W, alongside the homomorphisms $\Lambda \to V \otimes V^*$ and $V \otimes V^* \to \Lambda$ for the cup  and cap  and the identity 1_V for a single strand  coloured by V. Where a string orientation is reversed use the dual module in the homomorphism.

The key feature of this construction is that the resulting homomorphism $J(T)$ can be shown, using algebraic properties of a quantum group, to be unaltered when the tangle T is changed by Reidemeister moves R_{II} and R_{III} on the strings inside it.

Knot invariants. Any knot diagram K when coloured by a module W can be regarded as a coloured tangle with no strings at the bottom or top.

The resulting homomorphism $J(K)$ is then a map from the trivial module $\Lambda = \mathbf{Q}[[h]]$ to itself. This is simply multiplication by a scalar, which we write as $J(K\colon W) \in \mathbf{Q}[[h]]$.

The scalar $J(K\colon W)$ depends only on W and the banded knot K, and gives the 1-parameter quantum group invariant of K for the $\mathcal{G}$-module W. The construction similarly provides an invariant of a banded link depending on a choice of module for each oriented component.

The invariant $J(K)$ is additive under direct sum of modules,

$$J(K\colon W) = J(K\colon W_1) + J(K\colon W_2)$$

when $W = W_1 \oplus W_2$. It is usual to look at the case where $\mathcal{G}$ is semi-simple, so that W decomposes as the sum of irreducible modules. Then we need only consider colourings by *irreducible* modules. These have the added property that introducing a twist in the band around K multiplies $J(K\colon W)$ by a scalar f_W which depends only on the irreducible module W.

Dependence on h. While the quantum invariant $J(K\colon W)$ lies in the power series ring $\mathbf{Q}[[h]]$ it can generally be written as a multiple, depending only on W and the amount of twisting in the band around K, of a Laurent polynomial in $\mathbb{Z}[q^{\pm 1}]$, where $q = e^h$.

The Lie algebras $\{sl(N)\}$ of the A series have corresponding quantum groups $\{sl(N)_q\}$. The knot invariants derived from these have a close relationship with the Homfly satellite invariants discussed above.

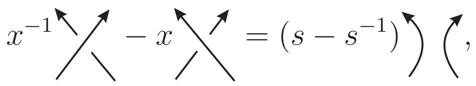
For each N there is a 'fundamental' N-dimensional irreducible $sl(N)_q$-module $V_\square^{(N)}$. Formulae derived from the universal R-matrix show [9] that the homomorphism

$$R = R_{V_\square^{(N)},V_\square^{(N)}} \colon V_\square^{(N)} \otimes V_\square^{(N)} \to V_\square^{(N)} \otimes V_\square^{(N)}$$

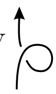
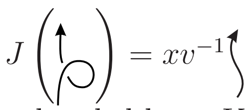
satisfies the quadratic equation

$$e^{\frac{h}{2N}} R - e^{-\frac{h}{2N}} R^{-1} = (e^{\frac{h}{2}} - e^{-\frac{h}{2}})\mathrm{Id}.$$

The invariant where all components of a link are coloured by $V_\square^{(N)}$ then satisfies the equation



with $s = e^{\frac{h}{2}}$ and $x = e^{-\frac{h}{2N}} = s^{-\frac{1}{N}}$.

Calculations for the module endomorphism of $V_\square^{(N)}$ given by  show [9] that  with $v = s^{-N}, x = s^{-\frac{1}{N}}$. The consequence is that for a banded knot K we have

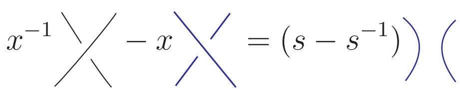
$$J(K \colon V_\square^{(N)}) = x^{wr(K)} P(K)$$

where $P(K)$ is the framed Homfly polynomial of K in v and s, normalised to take the value 1 on the empty knot, and v, s, x are given as above in terms of h.

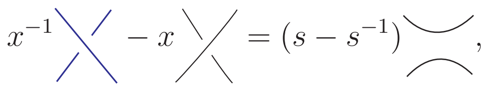
Remark 3.1. The classical Jones polynomial of a link is given from the Homfly polynomial by setting $v = t = s^2$. With $v = s^{-2}$ we get the Jones polynomial up to a sign depending on the number of components of the link.

In the quantum group case with $N = 2$ the modules are all self-dual, and so string orientation in the tangles can be omitted.

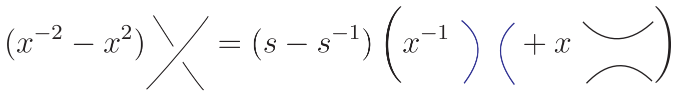
For the fundamental irreducible 2-dimensional module $V_\square^{(2)}$ we then have the relation



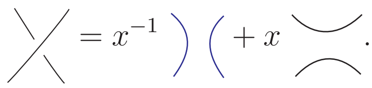
and also



with $s = e^{\frac{h}{2}}, x = e^{-\frac{h}{4}}$. These give



so, since $x^2 = s^{-1}$ we get



The relations here are very close to the classical Kauffman bracket relations with $A = x^{-\frac{1}{2}}$. Sikora [21] looks at the skein theory $J(K\colon V_{\Box}^{(N)})$ and its extension to links, and notes that when $N = 2$ it coincides with the normalised Kauffman bracket up to a sign depending on the number of crossings and link components.

Dependence among invariants. In the light of the huge array of available knot invariants arising from the use of satellites and quantum groups the questions of their interrelations and independence become important. There is, for example, no point in trying to distinguish pairs of knots by use of an invariant which depends on invariants that are already known to agree on the knots in question.

We have noted above that the 2-variable Homfly polynomial $P(K)$ of a knot specialises to the family of 1-variable quantum invariants $\{J(K\colon V_{\Box}^{(N)}\}$. Conversely if we know enough of these quantum invariants then we can recover the whole Homfly polynomial.

There is a much wider result about the whole collection of invariants arising from the A-series of quantum groups $\{sl(N)_q\}$, showing that these quantum invariants of a knot are collectively equivalent to its Homfly satellite invariants.

The relations between the invariants come in a very attractive form, linking up the irreducible quantum group modules in a nice way with the natural set of satellite invariants $\{P(K\colon Q_\lambda)\}$ discussed earlier.

Quantum and satellite invariants. The most striking correspondence comes when we use the irreducible $\{sl(N)_q\}$-modules to colour a knot or link. As for the classical case of $sl(N)$ there is an irreducible $sl(N)_q$-module $V_\lambda^{(N)}$ for every partition λ of an integer into at most $N-1$ parts. The 1-variable quantum invariants $J(K\colon V_\lambda^{(N)})$ are all special cases of the 2-variable Homfly satellite invariant $P(K\colon Q_\lambda)$ corresponding to the same partition λ. The following explicit result, due essentially to Wenzl [22], is carefully discussed in chapter 11 of [9].

Theorem 3.2. *For a partition λ of m into at most $N-1$ parts*

$$J(K\colon V_\lambda^{(N)}) = x^{m^2 \mathrm{wr}(K)} P(K\colon Q_\lambda)$$

with $s = e^{\frac{h}{2}}, v = s^{-N} = e^{-\frac{Nh}{2}}, x = s^{-\frac{1}{N}} = e^{-\frac{h}{2N}}$ replacing the variables x, s and v on the right-hand side, and $\mathrm{wr}(K)$ *is the sum of the crossings in K, counted with sign.*

From this theorem, and its natural extension to links, we can recover any of the quantum invariants once we know sufficiently many satellite invariants. Conversely we can find the satellite invariants from a knowledge of the quantum invariants for sufficiently many N. More detailed comments about the exact requirements can be found in [14].

Other quantum invariants. Wenzl goes on in [23] to study the quantum invariants arising from the quantum groups of the B, C and D series of classical Lie algebras, and relates them to satellite invariants based on the Kauffman 2-variable polynomial for unoriented banded knots and links. The connection in these cases comes from the Birman-Wenzl-Murakami algebras, in place of the Hecke algebras.

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