Ancient approximation to the sine function

Peter Giblin

There is a remarkable approximation to the sine function over the interval $[0, \pi]$ which is credited to Aryabhata I (about 500 CE) and which is mentioned in [1], namely (in modern notation)

$$\sin x \approx \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}.$$

A casual glance at Figure 1 tells you that the approximation is a good one, in fact it is nowhere out by more than about 0.00163.

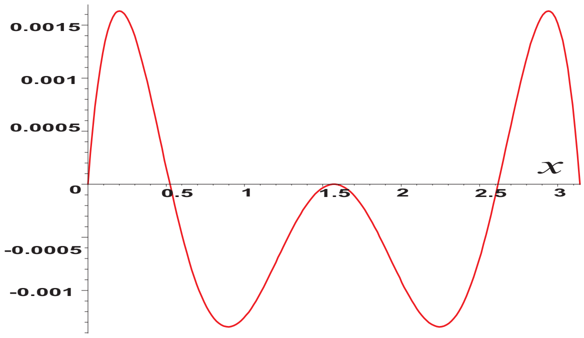

Figure 1: The difference between Aryabhata's approximation and the sine function over the interval $[0, \pi]$.

To discuss this approximation it is better to convert it to an approximation for $\cos x$ by replacing x by $\frac{1}{2}\pi - x$, yielding

$$\cos x \approx \frac{\pi^2 - 4x^2}{\pi^2 + x^2}, \quad -\tfrac{1}{2}\pi \leq x \leq \tfrac{1}{2}\pi.$$

This is an even function of x, taking the correct values 1, 0 at $x = 0, \frac{1}{2}\pi$ and indeed the original approximation for $\sin x$ is symmetric about $\frac{1}{2}\pi$ and takes the correct values 0, 1 at $x = 0, \frac{1}{2}\pi$. Is this the best approximation to $\cos x$ by a rational function with numerator and denominator both of degree 2? Keeping to even functions, they will be functions of x^2 and hence we can assume the form $f(x) = (a + bx^2)/(c + dx^2)$, say. However imposing $f(0) = 1$ gives $a = c$ and scaling we may assume $a = c = \pi^2$. Imposing $f(\frac{1}{2}\pi) = 0$ we can restrict attention (changing notation) to

$$f(x) = \frac{\pi^2 - 4x^2}{\pi^2 + kx^2}.$$

What is the best value of k? For a start, what is "best"? The value $k = 1$ seems pretty good, from Figure 1 where you shift the vertical axis to the middle of the graph to get the explicit graph of $f(x) - \cos x$. A measure of goodness might be the (absolute, unsigned) area between the two curves $y = \cos x$ and $y = f(x)$ over the interval $[0, \frac{1}{2}\pi]$ (since the fuctions are even), that is the integral of $|f(x) - \cos x|$ over this interval. In a more general context we would probably use the integral of

$(f(x)-\cos x)^2$ but in the present case $f(x)-\cos x$ can be integrated explicitly and it does not change sign very often over the given interval. A fairly standard integration exercise is to show

$$\int (f(x) - \cos x)dx = \frac{1}{k^3}\left(4\pi \arctan(kx/\pi) - 4kx + \pi k^2 \arctan(kx/\pi) - k^3 \sin x\right).$$

Remarkably, there is only a tiny range of values of $k > 0$ over which the sign of $f(x)-\cos x$ changes at all. It is quite easy to check that this function

- has a "degenerate minimum" (a minimum where the second derivative vanishes) at $x = 0$ for $k = \frac{1}{2}\sqrt{2\pi^2 - 16} \approx 0.96685$, giving an area under the graph between 0 and $\frac{1}{2}\pi$ of about 0.0025988,
- has a maximum at $x = \frac{1}{2}\pi$ for $k = \frac{2}{\pi}\sqrt{\pi(4-\pi)} \approx 1.04545$, giving an (absolute) area of about 0.0041965.

These extremes are shown for $0 \le x \le \frac{1}{2}\pi$ in Figure 2. Between these extreme values of k, the corresponding graph looks like that of the right-hand half of Figure 1, and outside this range of k the sign of $f(x) - \cos x$ does not change. For the value $k = 1$ in Aryabhata's formula the absolute area, measured by the integral of $|f(x)-\cos x|$, is about 0.0013137, which is certainly better (smaller) than either of the extremes.

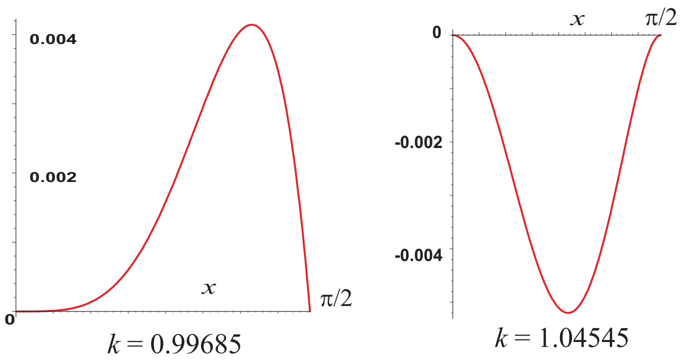

Figure 2: The graphs of $f(x) - \cos x$ for two values of k; only for k between these values does this function change sign over the interval $[0, \frac{1}{2}\pi]$.

So the optimum value of k, minimizing the absolute area, is *very* close to 1. The best estimate I have been able to obtain is 0.99522, with the aboslute area being 0.0012780. Maybe someone else can do better, or calculate the optimum value explicitly, but at any rate it shows that Aryabhata's formula is extremely well chosen!

References

[1] G.G.Joseph, *The Crest of the Peacock: Non-European Roots of Mathematics*, Princeton University Press, Third edition 2010 (Second edition Penguin Books 2000).

Peter Giblin, Department of Mathematical Sciences, The University of Liverpool, Liverpool L69 7ZL.