Sides of polygons

Peter Giblin

1 Triangles

Consider the triangle in which the sides are 1, 1.6 and 2.5. This is a genuine triangle since the longest side, 2.5, is less than the sum of the other two, 2.6. Now consider the ratios p/q where p, q are sides and $p \geq q$, namely 1.6, 2.5 and $\frac{2.5}{1.6} = 1.56$ approximately. All these ratios are > 1.5. Nothing very remarkable in that, but now consider the following result:

Proposition 1.1 *Let* $\phi = \frac{1}{2}(1 + \sqrt{5}) = 1.62$ *approximately (the 'golden ratio'). Then for any triangle, there is a pair of sides whose ratio* r *satisfies* $1 \leq r \leq \phi$.

Note that later I shall also write k_3 for this ϕ since it is a number specially attached to triangles (3-gons). Of course it follows that the same result will hold if we replace ϕ in this inequality with any larger number. After proving the proposition I shall show that ϕ in this proposition cannot be replaced by any *smaller* number. Evidently we cannot replace ϕ by the smaller number 1.5, from the above example.

Proof of the proposition Let the sides of the triangle be $a \geq b \geq c$, where $a \leq b + c$. Assume that $k > 0$ is such that there is no pair of sides whose ratio r satisfies $1 \leq r \leq k$. Thus every ratio $r \geq 1$ is in fact $> k$: $\frac{a}{b} > k$, $\frac{b}{c} > k$, $\frac{a}{c} > k$. It follows that

$$b < \frac{a}{k},\ c < \frac{b}{k} < \frac{a}{k^2};\ a \leq b + c < a\left(\frac{k+1}{k^2}\right).$$

This is a *contradiction* if $(k+1)/k^2 \leq 1$, that is $k^2 - k - 1 \geq 0$. The zeros of this quadratic are $(1 \pm \sqrt{5})/2$ and it is ≥ 0 for positive k if and only if $k \geq \phi$. That is, for any $k \geq \phi$ there is a contradiction and there must be a pair of sides whose ratio satisfies $1 \leq r \leq k$. □

It is clear that some variant of this argument should work for polygons with more than three sides: I shall say something about this below. But first it looks likely that ϕ cannot be replaced by any smaller number in the proposition. To prove this let ε be any small strictly positive number; it is enough to show that ϕ cannot be replaced by $\phi - \varepsilon$. There is a more general argument in §2 but here I shall explicitly construct a scalene triangle for which all ratios $r > 1$ of sides are in fact $> \phi - \varepsilon$.

Let $\varepsilon > 0$ be small and choose a such that

$$(\phi - \varepsilon)(\phi - \tfrac{1}{2}\varepsilon) < a < \phi - \tfrac{1}{2}\varepsilon + 1. \tag{1}$$

First to check such an a exists, we need to show

$$\phi^2 - \tfrac{3}{2}\phi\varepsilon + \tfrac{1}{2}\varepsilon^2 < \phi - \tfrac{1}{2}\varepsilon + 1,$$

which amounts to

$$\varepsilon^2 + \varepsilon(1 - 3\phi) + 2(\phi^2 - \phi - 1) < 0$$

for small $\varepsilon > 0$. Note that when $\varepsilon = 0$ the left-hand side is 0, by definition of ϕ. Its derivative at $\varepsilon = 0$ is $1 - 3\phi$ which is < 0. Hence for small ε the left-hand side is < 0 and there will indeed be a suitable value of a, such as the arithmetic mean of the two expressions in (1). Indeed the quadratic in ε will be < 0 for $0 < \varepsilon < 3\phi - 1$, since this is the other root.

Now consider the triangle with sides $a, b = \phi - \frac{1}{2}\varepsilon, c = 1$, so that $a > b > c$. Then by choice of a we have

$$\frac{a}{c} = a > \phi - \varepsilon,\ \frac{b}{c} = b > \phi - \varepsilon,\ \frac{a}{b} > \phi - \varepsilon,\ \text{and } a < b + c.$$

This triangle therefore exists and has every ratio $r > 1$ being in fact $> \phi - \varepsilon$, as required.

As a numerical example, let $\varepsilon = 0.05$ and choose a as the arithmetic mean of the expressions in (1), which works out as about 2.5455. Then the ratios $r > 1$ are approximately 1.5979, 2.5455 and 1.5930, all of which are greater than $\phi - \varepsilon = 1.5680$.

In the limiting case $\varepsilon = 0$, the triangle with sides $\phi^2, \phi, 1$ is 'flat', since $\phi^2 = \phi + 1$.

2 Other polygons

For a quadrilateral (which need not be planar) with sides $a \geq b \geq c \geq d$ and satisfying the 'existence' criterion $a \leq b + c + d$ the analogous argument to that given in Proposition 1.1 shows that, if all ratios $r \geq 1$ of sides are in fact $> k$, that is $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}$ are all $> k$ then

$$a \leq b + c + d \leq a\left(\frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3}\right),$$

which is a contradiction if the right-hand side is $< a$, that is if $k^3 - k^2 - k - 1 > 0$. This cubic has two turning points at $k = -\frac{1}{3}$ and $k = 1$, at both of which the cubic is negative, and therefore the cubic has a single real root (its discriminant is -44 as an alternative argument for this). This root, which I shall call k_4, is approximately 1.84, bigger than $k_3 = \phi$, but still < 2, and the cubic is > 0 precisely when $k > k_4$. So we can conclude that

Proposition 2.1 *For any quadrilateral there is a pair of sides whose ratio satisfies* $1 \leq r \leq k_4 =$ 1.84 *approximately.*

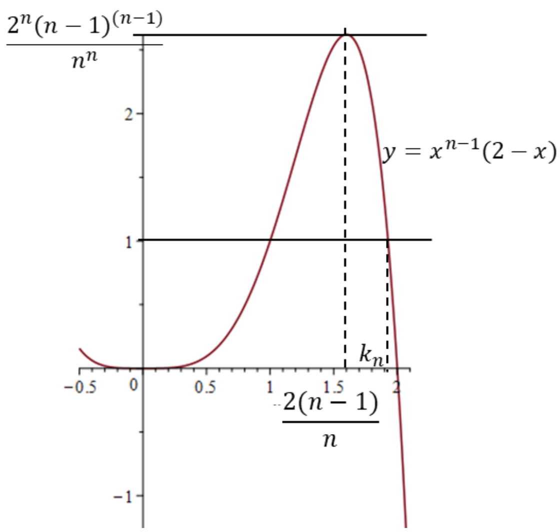
Note that the cubic equation $k^3 - k^2 - k - 1 = 0$ can also be written $k^3(k - 1) = k^3 - 1$ if we introduce the additional solution $k = 1$. For a 5-sided polygon this will become $k^4(k-1) = k^4 - 1$, which has one positive root, k_5 say, besides 1. This k_5 is approximately equal to 1.93, an increase on the value for quadrilaterals, but still < 2. For pentagons there always exists a pair of sides whose ratio $r \geq 1$ satisfies $r < k_5 = 1.93$ approximately. In general for an n-gon we have that $k_n > 1$ satisfies $k_n^{n-1}(k - 1) = k_n^{n-1} - 1$. Hence:

Proposition 2.2 *Any n-sided polygon ($n \geq 3$) has a pair of sides whose ratio r satisfies* $1 \leq r \leq k_n$*, where k_n is the positive solution other than 1 of the equation* $x^{n-1}(x - 1) = x^{n-1} - 1$. □

(In a sense this proposition is true for $n = 2$ too, though here $x = 1$!) It is natural to ask what is the limit of the sequence k_n introduced in this way. Writing $x^{n-1}(x-1) = x^{n-1} - 1$ as $x^{n-1}(2-x) = 1$ we are interested in the solution $x = k_n$ of this which is > 1.

The figure shows the graph of $y = x^{n-1}(2-x)$ for $n > 2$, in fact for $n = 5$. For $x > 0$ the maximum occurs at $x = \frac{2(n-1)}{n} > 1$ and $y = 1$ at $x = 1$ while the derivative $y' = 2(n-1)x^{n-2} - (n+1)x^{n-1}$ is positive, so y is increasing at $x = 1$ and the maximum of y is > 1. Thus the larger value of x for which $y = 1$ is sandwiched between $x = \frac{2(n-1)}{n}$ and 2. This shows that k_n as above has limit 2.

(Incidentally writing the maximum value $N = \dfrac{2^n(n-1)^{n-1}}{n^n}$ as $\left(1 - \dfrac{1}{n}\right)^{n-1} \times \dfrac{2^n}{n}$ it is evident that $N \to \infty$ as $n \to \infty$ since the first term in this product tends to $1/e$.)



In fact we have the following:

Proposition 2.3 *(i) Given $\varepsilon > 0$ there exist an integer $n > 1$ and numbers $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n$, with $a_1 < a_2 + a_3 + \ldots a_n$ (hence the sides of an n-gon) and every ratio $r \geq 1$ of two of the a_i being $> 2 - \varepsilon$.*
(ii) Given n and $\varepsilon > 0$ there exists an n-sided polygon such that every ratio $r \geq 1$ of its sides satisfies $r > k_n - \varepsilon$, where k_n is as in Proposition 2.2.

Proof (i) Let $\alpha = 2 - \frac{1}{2}\varepsilon$ and choose n large enough that α lies between $\frac{2(n-1)}{n}$ and k_n (see the figure, remembering $k_n \to 2$); this guarantees that $\alpha^{n-1}(2-\alpha) > 1$. Let the a_i be $a_1 = \alpha^{n-1}, a_2 = \alpha^{n-2}, \ldots, a_n = 1$. Then $a_1 > a_2 > \ldots > a_n$ since $\alpha > 1$ and all ratios > 1 of pairs of the a_i are $\geq \alpha$ and therefore $> 2 - \varepsilon$. Furthermore

$$a_2 + a_3 + \ldots a_n = 1 + \alpha + \alpha^2 + \ldots \alpha^{n-2} = \frac{\alpha^{n-1} - 1}{\alpha - 1} > \alpha^{n-1} = a_1$$

since $\alpha^{n-1}(2-\alpha) > 1$. This proves the result.
(ii) Take $\alpha = \frac{1}{2}(2 + k_n)$ in the proof of (i) above. Then the resulting sides $a_1, \ldots, a_n$ have the property that every ratio $r \geq 1$ is in fact $\geq \alpha$, and $\alpha > k_n - \varepsilon$ since this amounts to $\varepsilon > \frac{1}{2}k_n - 1$ and the right-hand side of this inequality is < 0 as $k_n < 2$. □

3 What about higher dimensions?

It is known that, given four positive numbers $a_1 \geq a_2 \geq a_3 \geq a_4$, a necessary and sufficient condition for the existence of a proper tetrahedron with these numbers as areas of its faces is $a_1 < a_2 + a_3 + a_4$. This is proved in detail in [1, Ch.3]. It follows by the same argument as in the quadrilateral case, Proposition 2.1, that there is always a pair of faces the ratio r of whose areas satisfies $1 \leq r \leq k_4$ where k_4 is the positive solution other than 1 of $x^4(x-1) = 1$ and is 1.84 approximately. It also follows from the argument of Proposition 2.3(ii) that k_4 this is the smallest possible number for which this result holds.

I do not know whether the result extends to polyhedra with more than four faces!

References

[1] O.A.Ivanov, *Journey to advanced thinking*, trans. Robert G.Burns, Mathematical Association of America Press 2017.

Peter Giblin, Department of Mathematical Sciences,
The University of Liverpool, Liverpool L69 7ZL
pjgiblin@liv.ac.uk