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by CHUNG-MAN HUI

TITLE: Plane Quartic Curves

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- PREFACE -

I would like to thank my Supervisor Dr. P. J. Giblin, without whose precious supervision I could not have finished this Thesis. I am also grateful to his patience in reading through the first version of this manuscript and correcting many mistakes. Also I want to thank Liverpool University for their partial financial assistance.

Some of the results which I use in this Thesis are due to J.W. Bruce. I would like to thank him for the access of these results, which were first obtained by him in his Liverpool Ph. D. Thesis.

Also, I would like to express my gratitude to Miss Diana Ho for her expertise in typing this thesis.

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Corrections

p. 2 The vertex O removed.

p. 28 Bottom third line " $x^2z^2 + 2xy^2z + ...$"

p.107 Seventh line "$x^2z^2 + 2xy^2z + ...$" ; Gens. for A_1 are $x^4, y^4, xy^3, y^3z, xy^2z, y^2z^2$

p.111 Twelfth line " $\mathcal{E}(n)$ "

p.115 Fourth line "quotient"

Fifth line "represented"

Bottom second line "$g \in \mathcal{E}(n)$"

Bottom third line "$f \in \mathcal{E}(n)$"

p.116 First line "(versal= k-transversal)"

p.118 Bottom fifth line ".... in $M \times V_f$ the singularity $f' = f +$ terms..."

p.119 Bottom line ".... ζxy^2z "

p.120 Third line "....+ ζxy^2 " $(f_{Z,x}, f_{Z,y})'$

Sixth line "....+ ζxy^2 "

Eleventh Line "......, xy^2 form aabase ....."

p.122 Tenth line " y, y^2, y^3, xy^2 form a base...."

Nineth line "$2x + 2y^2 - 2xy^2 \in J_{f_z}$ by $f_{z,x}$ "

p.123 Bottom line "... $3\varepsilon yz^2 + 2\zeta y^2z = 0$ "

p.124 Third line " $yz \in J_{f_x} + m^3$......"

p.132 Bottom third line "..... $k_i(y-1)(z+1)^3$ "

Bottom line "...$+r_i(z+1)^2 + s_i(y-1)^2(z+1)^2$ "

p.133 Third line " $\sum_i [a_i + ...$ "

Fourth line " $+(-4b_i + ...$"

Fifth line " ..... $+2s_i)z]$ "

Eleventh line " $-4b_i + ...$"

p.134 Eleventh line "......$+ (\beta(\alpha_0+1) + \alpha(\beta_0+1) + \alpha\beta)x^2z^2$ "

p.146 Fourth line "Proof : We can write (M^m, O)....."

p.150 Foruth line " ......$j_1^k(x_0, u, v) \not\pitchfork \mathcal{R}$-orbit of h "

p.156 Fifteenth line " ....iff $\mathcal{T}$ "

p.159 Third line " $\gamma = \delta = \varepsilon = \zeta = 0$ "

p.160 Bottom third line ".....$+ \beta_0'' x^3 z$ where "

163 Fourteenth line "p.118, when $\mathcal{T}$ is ...."

p.163 Bottom second line, omit " Let $N \times U \times V \subset \mathcal{G}$, and "

put in " Working within neighborhood $\mathcal{G}$, let $\eta : N \times U \to \mathbb{C}$ "

p.164 First line "$F: (N \times U \times V, 0) \to \mathbb{C}$ "

p.167 Bottom fifth line "$\mathcal{D}$: (UxV...... "

Bottom third line " (u,v,w) $S_\chi(G)$ iff $\mathcal{D}$ (UxV........... $\in S_\chi(F)$ "

p.170 Fourth line " As a versal...."

Eleventh line " $+ \gamma_0 xy^3 + \varepsilon y + \zeta y^2$ "

Corrections (cont.)

p.173 Nineth line " $\left.\begin{array}{l} y_i \to x \\ x_i \to x \end{array}\right\}$ where $x \in X$ "

p.174 Bottom fourth line, omit "(Cross Cap) "

p.178 Bottom Fourth line " ....: $u_2 = w_2 = 0$ "

p.179 Eighth line "argument on p. 166, we have "

p.180 First line " c_1 : $(U_1 \times U_2 \times M$ ....."

Second line " c_2 : ($U_1 \times U_2 \times M$ ......"

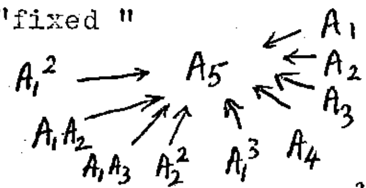
p.183 Bottom third line " On p. 167 in Section 6.3 "

p.188 Sixth line "..... $+ 3y^2$ "

p.195 Seventh line " (see p.166) "

p.198& p.199 Wrong order. (Binding mistake)

p.201 Fourth line "fixed "

p.205 Diagram " 

"

p.207 Thirteenth line "(**) $\frac{1}{3}x^3 - \frac{1}{3}\left(\frac{x^2+\beta}{\delta}\right)^3 + \dots$.."

Eleventh line " $\frac{(x^2+\beta)^2}{\delta^2} + \gamma + \delta x = 0$ "

p.213 Third line " This implies δ big, a contradiction (δ small)"

p.223 Bottom second line " Consider $\mathcal{X}_1^0$ case first "

p.225 Third line " a_2: $U_1 \times U_2 \times M \times W_1 \times W_2 \to$ ...."

Fourth line " $a_2: \pi_2' \circ \vartheta_2^{-1} \circ \pi_2 \circ \vartheta_2$ "

Eleventh line " $a_1 \times a_2$: $U_1 \times U_2 \times M \times W_1 \times W_2 \to$ ...."

Bottom third line "..... hence a submersion at "

p.238 Thirteenth line " (iv).....$-3\delta y_3^2 \quad -4\varepsilon y_3^3$"

p.251 Eighth line " So the condition for A_4 is $\left[\delta\left(-\frac{\alpha}{2\beta}\right) - \frac{\alpha^2}{4\beta^2}\right] \neq 0$"

p.259 Fourth line " Similarly $n_3 - 3\alpha \geqslant 0$ and ...."

p.260 Bottom second line "...... $A_1^3A_3$, $A_1^2 D_4$, A_1^6 "

Chapter 1

Introduction

Our interest in this thesis is in Complex Quartic Curves (not necessary irreducible) in the Complex projective plane $\mathbb{C}P^2$. Such a curve is the set of zeroes of a homogenous quartic polynomial f in three variables,

$$a_1x^4 + a_2y^4 + a_3z^4 + \dots = 0.$$

That is, a quartic curve is given by a nontrivial linear combination of the 15 basis monomials

$$\begin{array}{ccccc} & & x^4 & & \\ & x^3y & & x^3z & \\ x^2y^2 & & x^2yz & & x^2z^2 \\ xy^3 & xy^2z & & xyz^2 & xz^3 \\ y^4 \quad y^3z & & y^2z^2 & & yz^3 \quad z^4 \end{array}$$

We then can represent the space of quartic curves as $M = \mathbb{C}^{15} - \{0\}$ or $\mathbb{C}P^{14}$ which are manifolds. We shall stick to $\mathbb{C}^{15} - \{0\}$.

Our aim is to stratify M according to the singularity types of quartic curves, so that each stratum is a manifold and there will be nice properties such as Whitney regularity between the strata.

We start with a smooth group action on M, corresponding to projective equivalence of curves. Actually our strata will be unions of orbits under this group action on M. The group is $GL(3,\mathbb{C})$. The action is on the left by substitution.

$$GL(3,\mathbb{C}) \times M \to M$$

$$(\theta,f) \to \theta.f = f\circ\theta^{-1}$$

Here we regard $f \in M$ as a map $\mathbb{C}^3 \to \mathbb{C}$ and $\theta \in GL(3,\mathbb{C})$ as a linear map $\mathbb{C}^3 \to \mathbb{C}^3$. The substitution is in the sense as in the example

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . f(x,y,z) \to f(x-y,y,z)$$

The orbit containing f is the image of the function

$$\phi f : GL(3,\mathbb{C}) \to M$$

$$\theta \to \theta.f$$

Then the orbit are cones with vertex O under the above action. We say, two quartic curves are projectively equivalent if they are in the same orbit.

The classification of quartic curves by analytic type of singularity was known for a long time (Hilton 1920). In Chapter 2, the classification is presented in detail and normal forms are obtained for all singularity types. In some cases we have to try very hard to obtain normal forms which will suit our purpose in the later chapters. Note that all the singularities involved are simple (in the sense of Arnol'd) except $\tilde{E}_7$, which is the singularity of four concurrent lines. All our general arguments only apply to simple cases.

The actual stratification of M is as follows. Suppose that C_1 is a curve with isolated simple singularities. Then C_2 is in the same stratum as C_1 if and only if (i) the singularities of C_1 and C_2 correspond one-to-one, and corresponding singularities are analytically equivalent (i.e. equivalent by an analytic change of local coordinates in $\mathbb{C}P^2$, i.e. right-equivalent), (ii) the components of C_2 have the same degrees as those of C_1.

Suppose that C_1 has a non-simple isolated singularity so that in fact C_1 is 4 concurrent lines (see classification in Chapter 2). Then C_2 is in the same stratum as C_1 if and only if C_2 is 4 concurrent lines. Note that in this case C_1 and C_2 have analytically equivalent singularities if and only if cross ratios of the two sets of 4 linea agree.

Suppose that C_1 has a non-isolated singularity (i.e. repeated component) then the stratum of C_1 is its orbit under the above group action (it will turn out that this leads to finitely many strata).

Chapter 3, 4 and 5 are devoted to the proof that the strata are manifolds. In Chapter 3, we have shown that the normal forms themselves are manifolds and they are all "good" normal forms. The notion of Transversal is introduced in Chapter 4 and the list of Transversals for the normal forms is given on Table 4.2. In Chapter 5, we discuss properties of transversals and use them to prove that the strata are manifolds. In Section 5.3 we deal with the case in which we have to choose very special transversals to suit our purpose and at the end of the Chapter we have shown that the stratum

$\tilde{E}_7$ is also a manifold.

In Chapter 6, we prove Whitney (A) and (B) regularity between the strata involving simple (or non-isolated) singularities. In fact a stronger local triviality result is proved. The only cases left are those cases in which the "smaller" or "lower" stratum is $\tilde{E}_7$. (Recent work of Bruce and Giblin (to appear) shows that all possible pairs are A-regular (see also Chapter 8)). Thus the strata generally fit together very nicely. In Section 6.2 we have shown some examples in which we can also prove easy cases of regularity without using the general argument. The general argument for curves with a single singularity is given in Section 6.3. The argument for the cases for two singularities takes up the whole of Section 6.4 and an example is given. Section 6.5 deals with the cases when the curves have more than two singularities.

We are also interested in the problems of specialization: when can a sequence of curves all in one stratum have as limit a curve in another stratum? In Chapter 7, we are able to establish rigorously the specialization diagram for all curves with isolated singularities. But some very tricky cases arise from the more degenerate curves having non-isolated singularities. Some of them are still left unsolved. In Section 7.1, we use elementary methods to find some of the despecializations. Despecialization means the "breaking up" of singularities by looking at the unfolding.

Finally in Chapter 8 we show, using an argument of J.W. Bruce, that certain strata are (A) and (B) regular over $\tilde{E}_7$.

CHAPTER 2

CLASSIFICATION OF QUARTIC CURVES

2.1 The Genus Formula (for irreducible quartics)

Let Γ be an algebraic curve in P^2, given as the set of zeros of a homogeneous polynomial f of degree d say. We assume that Γ has no repeated component i.e. that f is square-free. Then the singularities of Γ are isolated, and associated with each singular point P there are various numbers:

r = number of analytic branches at P

δ = number of double points at P

μ = Milnor number of singularity at P.

By the theorem of Milnor (Milnor 1968, p.85) we have

(2.1.1) $$2\delta = \mu + r - 1$$

and the genus formula asserts that, if Γ is irreducible, then the genus g of Γ is given by

(2.1.2) $$g = \tfrac{1}{2}(d-1)(d-2) - \Sigma\delta_P$$

to be summed over singular points P of Γ. (Serre 1959, p.74) (Milnor 1968, p.85) (also see p.8). An indication of the proof is given at the end of this section.

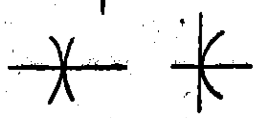
We are of course mainly interested in the case when $d = 4$. Then the genus formula becomes

(2.1.3) $$2g = 6 - \Sigma(\mu_p + r_p - 1)$$

In particular, since $r_p \geq 1$ and $g \geq 0$ we have, for any singular point P,

$$\mu_p \leqslant 6$$

Arnold has already given the analytic classification of singularities up to $\mu \leqslant 7$. (See Arnol'd 1972, Functional Anal. Appl. 6 254-292). In fact by an analytic change of coordinates any such singularity (with $\mu \leqslant 7$) can be turned into exactly one of the following list:

(2.1.4)	Standard Form	r	$2\delta=\mu+r-1$	Picture
A_1	$x^2 + y^2$	2	2	
A_2	$x^3 + y^2$	1	2	
A_3	$x^4 + y^2$	2	4	
A_4	$x^5 + y^2$	1	4	
A_5	$x^6 + y^2$	2	6	
A_6	$x^7 + y^2$	1	6	
D_4	$x^2y + y^3$	3	6	
D_5	$x^2y + y^4$	2	6	
D_6	$x^2y + y^5$	3	8	
E_6	$x^3 + y^4$	1	6	

If we use the formula (2.1.3) more precisely, we can also see that

(2.1.5) $$\mu_p + r_p - 1 \leqslant 6.$$

Now for D_6 in (2.1.4) $\mu + r - 1 = 8$, therefore it is ruled out, and cannot occur on an irreducible quartic. Actually, we shall see in §2.3 that D_6 and also certain other singularities with $\mu \geqslant 7$ can occur on reducible quartics.

Now making use of (2.1.2) again, we can deduce that since $\delta \geqslant 0$, $0 \leqslant g \leqslant 3$. That is, we have only four possible genus 0, 1, 2, 3 in the irreducible quartics. And therefore we can consider the possible existence of singularities on the irreducible quartics according to these four genuses as in table (2.1.6)

(2.1.6)

g	$\Sigma\mu_p+r_p-1$	number of singularities	Types
0	6	3	A_1^3, A_2^3, $A_1^2A_2$, $A_1A_2^2$
0	6	2	A_1A_3 A_2A_3 A_1A_4 A_2A_4
0	6	1	A_5 A_6 D_4 D_5 E_6
1	4	2	A_1^2 A_1A_2 A_2^2
1	4	1	A_3 A_4
2	2	1	A_1 A_2
3	0	0	non-singular

In §2.2, it will be shown that all those can be realized, i.e. the above is a complete list of singularity types of irreducible plane quartic curves.

<u>Proof of Genus Formula for Γ (irreducible)</u>

Γ is topologically a surface with isolated singularities. We want to find the genus of this surface.

First, we would look at the genus for non-singular curve of degree d. The set of coefficients which give non-singular curves form a (open dense) connected set because it is the complement of an algebraic variety. For example, consider $d = 2$, the equation

$$ax^2 + by^2 + ca^2 + 2fyz + 2gxz + 2hxy = 0$$

is singular iff $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ an algebraic variety.

Hence, the connectedness of this set implies that the genus is constant for every non-singular curve of degree d. So it is enough to find the genus for one curve, namely

$$\Gamma_d : x^d + y^d + z^d = 0 \text{ which is non-singular.}$$

Consider the map $\rho_d : \Gamma_d \to \Gamma_1$

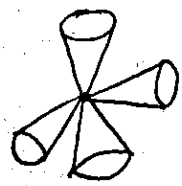
$$(x,y,z) \mapsto (x^d,y^d,z^d) = (\xi,\eta,\rho)$$

where Γ_1 is the line $\xi + \eta + \rho = 0$ and hence homeomorphic to S^2 with Euler characteristic $\chi = 2$. Now, if $\xi\eta\rho \neq 0$, then there are d choices for each of x,y,z. Therefore by homogeneity there are d^2 preimages of each such point in Γ_1. This is true for all points in Γ_1 except (1,0,-1), (0,1,-1), (1,-1,0) where there are only d points in the preimage. Hence, we have the equation of Euler characteristic as

$$\begin{aligned}\chi(\Gamma_d) &= d^2\chi(\Gamma_1) - 3(d^2-d)\\ &= 3d-d^2\\ &= d(3-d)\end{aligned}$$

But, for a connected surface, $\chi = 2-2p$ where p is the genus of the surface. Now since the complement of singular set on a curve is connected (Shafarevich), i.e., non-singular curve is connected, the genus p_0 for Γ_d is $p_0 = \frac{1}{2}(d-1)(d-2)$.

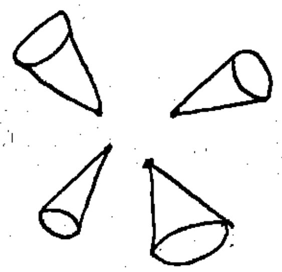
Now let us return to our singular curve . Suppose it is irreducible and it has Euler characteristic $\chi(\Gamma)$. For each singular point P of Γ, we have the local picture (not as embedded in $\mathbb{C}P^2$)



r cones, r = no. of branches at P.

The curve can be deformed in two ways:

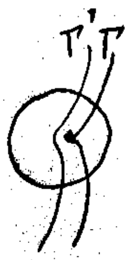
(1) Pull apart the vertices of these cones at each singular point P.



In doing this we get a non-singular surface (connected, see above) whose genus is (by definition) the genus of the curve Γ. The equation of Euler characteristic becomes

$$\chi(\Gamma) + \Sigma(r-1) = 2-2p \qquad \text{(i)}$$

where p is the genus of Γ and the summation is over the singular points of Γ.



(2) Arbitrary close to Γ, there exist non-singular curves of degree d. So deform Γ by making small changes in the coefficients so that Γ becomes a non-singular curve Γ' of degree d. Choosing small discs around the singular points of Γ, and provided Γ' is close enough to Γ, the part of Γ' inside the discs is diffeomorphic to the Milnor fibre of the singularities. Outside the discs the curve Γ' is diffeomorphic to Γ, so $\chi(\Gamma)$ can be calculated from Γ' by removing cones and adding Milnor fibre in the discs. Hence the Euler characteristic equation becomes

$$\chi(\Gamma) - 1 + \Sigma(1-\mu) = 3d-d^2 \qquad \text{(ii)}$$

Again the summation is over the singular points of Γ.

(The $\Sigma 1$ comes from removing the cone and the $\Sigma(1-\mu)$ from adding the Milnor fibre, which is a connected surface with boundary having first homology group free of rank μ). Subtracting equation (ii) from (i), we have

$$\Sigma(\mu+r-1) = 2-2p-3d+d^2$$

Hence $$p = \tfrac{1}{2}(d-1)(d-2) - \tfrac{1}{2}\Sigma(\mu+r-1) \quad \text{Q.E.D.}$$

Note that we don't in general have a genus formula for reducible curves. But if we defined the genus for a reducible curve as the sum of the genera of the non-singular surfaces obtained by pulling apart the singularities, we can also get a genus formula, for reducible curves. Let us proceed as following. If Γ is <u>reducible</u> and has c algebraic components, then removing the singularities by deforming the curve would leave us with c disjoint surfaces, (since all points of intersections of components will be singular). Let us go through the process of deforming the curve again.

(1)' Separating the vertices of the cones as in (1), we get c disjoint non-singular surfaces (each of them connected). Let $p_1, \ldots p_c$ be the genera of the c surfaces. Then the Euler characteristic equation becomes

$$\chi(\Gamma) + \Sigma(r-1) = \sum_{i=1}^{c}(2-2p_i)$$

Define the genus p of Γ (reducible) as

$$p = p_1 + \dots + p_c$$

then $\chi(\Gamma) + \Sigma(r-1) = 2c-2p$ (iii)

where Σ is over singular points of Γ.

(2)' The process is the same as (2). And the Euler Characteristic equation is unchanged as (ii). Subtracting the equation (iii) from (ii), we have

$$2p = (d-1)(d-2) + 2(c-1) - \Sigma(\mu+r-1)$$

where Σ is over singular points of Γ.

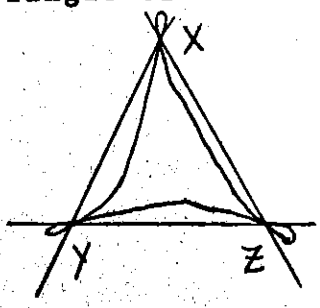
Hence $p = \frac{1}{2}d(d-3) + c - \frac{1}{2}\Sigma(\mu+r-1)$

This is the genus formula for any reducible curve of degree d.

2.2 Enumeration (Normal Forms) for Irreducible Quartics

In this section, we shall find the normal form of each of the possible singularity type mentioned in table (2.1.6). We start from A_1^3.

Suppose the three nodes are at the vertices X, Y, Z of the triangle of reference. Then the curve Γ goes through



X,Y,Z of the homogeneous coordinate. Considering the fifteen basis monomials, then x^4, y^4, z^4 must be absent from

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$$\begin{array}{ccccccccc} & & & & x^4 & & & & \\ & & & x^3y & & x^3z & & & \\ & & x^2y^2 & & x^2yz & & x^2z^2 & & \\ & xy^3 & & xy^2z & & xyz^2 & & xz^3 & \\ y^4 & & y^3z & & y^2z^2 & & yz^3 & & z^4 \end{array}$$

the equation of the curve. Also since X is singular on the curve Γ, near (1,0,0) in the non-homogeneous coordinates the linear terms Y and Z must be absent, i.e. x^3y and x^3z terms in the homogeneous case must be absent. Similarly, for singularity Y and Z, we have respectively xy^3, y^3z and xz^3, yz^3 terms disappearing too. Hence the remaining equation is

$$ax^2y^2 + bx^2z^2 + cy^2z^2 = xyz\ (dx + ey + fz)$$

To ensure that the curve is irreducible, we should have $a \neq 0$, $b \neq 0$, $c \neq 0$. Then choosing U (unit point) by the transformation

$$x \to \lambda x$$

$$y \to uy$$

$$z \to vz$$

we can make $a = b = c = 1$.

The equation becomes

$$x^2y^2 + x^2z^2 + y^2z^2 = xyz\ (\alpha x+\beta y+\gamma z)$$

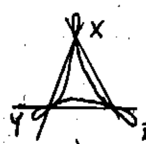
Let us now look at the point (1,0,0) more carefully. We can see that in the non-homogeneous coordinate, the lowest terms are $\alpha YZ + Y^2 + Z^2$, which is two lines (tangents). It is known that when these two tangents are distinct, the singularity at X is a node and when they coincide the singularity becomes a cusp. And the condition for the two tangents to coincide is $\alpha^2 - 4 = 0$ or $\alpha = \pm 2$. Similar results occur if we consider (0,1,0) and (0,0,1) i.e.

cusp at (0,1,0) for $\beta^2 - 4 = 0$ or $= \pm 2$

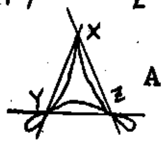
cusp at (0,0,1) for $\gamma^2 - 4 = 0$ or $= \pm 2$.

Therefore we can now write down the normal form not only for A_1^3 but also for $A_1^2A_2$, $A_1A_2^2$ and A_2^3.

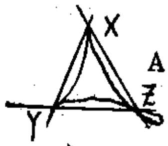
(2.2.1)



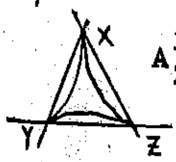
A_1^3 $$x^2y^2 + x^2z^2 + y^2z^2 = xyz\,(\alpha x+\beta y+\gamma z)$$ $\alpha \neq \pm 2,\ \beta \neq \pm 2,\ \gamma \neq \pm 2$



$A_1^2A_2$ $$x^2y^2 + x^2z^2 + y^2z^2 = xyz\,(\alpha x+\beta y+\gamma z)$$ $\beta \pm \gamma \neq 0,\ \beta \neq \pm 2,\ \gamma \neq \pm 2$



$A_1A_2^2$ $$x^2y^2 + x^2z^2 + y^2z^2 = xyz\,(2x+2y+\gamma z)$$ $\gamma \neq \pm 2$



A_2^3 $$x^2y^2 + x^2z^2 + y^2z^2 = xyz\,(2x+2y+2z)$$

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It can be shown that the above formulae give curves of the types indicated for all values of the parameters not excluded by the conditions stated.

Now we consider the values of the parameters which are excluded. In the case $A_1^2A_2$, when $\beta-\gamma = 0$ we have

$$x^2y^2 + x^2z^2 + y^2z^2 = xyz\,(2x+\beta y+\beta z).$$

i.e. $$x^2(y+z)^2 - \beta xyz\,(y+z) + y^2z^2 = 0.$$

Regarding $x(y+z) = A$ and $yz = B$, then we have the quartic equation

$$A^2 - \beta AB + B^2 = 0$$

The actual factorization is

$$\left[A - \frac{-\beta+\sqrt{\beta^2-4}}{2}B\right]\left[A - \frac{-\beta-\sqrt{\beta^2-4}}{2}B\right] = 0.$$

when $B \neq \pm 2$.

That is,

$$\left[xy + xy - \frac{\beta+\sqrt{\beta^2-4}}{2}yz\right]\left[xy + xz - \frac{\beta-\sqrt{\beta^2-4}}{2}yz\right] = 0.$$

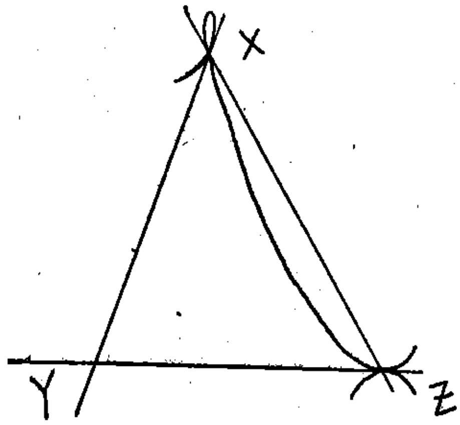
which are two genuine conics. It is easy to check that the conics touch at X and intersect transversally at Y and Z, so we have $A_1^2A_3$ (see classification of reducible quartics). Therefore for $A_1^2A_2$ $\beta - \gamma \neq 0$. Similarly $\beta + \gamma \neq 0$.

Note that, in the above formula we can replace the positive sign before any coefficients by a negative one. This is because by a transformation of the following type we can always change the signs of the coefficients

$$\begin{array}{lll} x \to x & x \to -x & x \to x \\ y \to -y & y \to y & y \to y \\ z \to z, & z \to z, & z \to -z \end{array}$$

This shows that the two different signs of the coefficients are projectively equivalent. We can have either of the signs in the normal form.

A_1A_3



Suppose A_1 at X and A_3 at Z

Take $z = 0$ be a tangent to the node at X

$x = 0$ be the tangent to the tacnode at Z.

x^4, z^4 terms absent (because curve pass through X,Z)

X singular x^3y and x^3y absent

Z singular xz^3 and xz^3 absent

Z is A_3 xyz^2, y^2z^2, y^3z absent

(The technique in calculating μ is described in the Appendix).

If we consider non-homogeneous coordinates at X(1,0,0), the lowest terms are $ky^2 + \ell yz + mz^2 = 0$ - two tangent lines. Since $z = 0$ is chosen to be the tangent to the node at X, we must have y^2 term absent. Hence the remaining equation is

$$ay^4 + bx^2z^2 + cx^2yz + dxy^2z + exy^3 = 0$$

If $a = 0$, x a factor. Hence $a \neq 0$.

If $b = 0$, y a factor. Hence $b \neq 0$.

We want $c \neq 0$, to ensure a node at X.

Then we can choose the unit point to obtain the normal form

A_1A_3 $\quad y^4 + x^2z^2 + x^2yz + \alpha xy^2z + \beta xy^3 = 0$

$\alpha \neq 2$, $\alpha \neq -2$, $\beta^2 - \alpha\beta + 1 \neq 0$ N & S

Observe that to ensure A_3 at Z, α should not be equal to ± 2. For in non-homogeneous coordinates at (0,0,1) the leading terms

$$y^4 \pm 2xy^2 + x^2$$

will then be a perfect square. This implies the Milnor number of the singularity at Z will be higher than 3. Actually when $\alpha = \pm 2$, the curve will become A_1A_4.

Now, let us also check other conditions on α, β such that the curve will remain to be A_1A_3. Consider the normal form $f(x,y,z) = y^4 + x^2z^2 + x^2yz + \alpha xy^2z + \beta xy^3 = 0$.

$$\frac{\partial f}{\partial x} = 2xz^2 + 2xyz + \alpha y^2 z + \beta y^3 = 0 \qquad \text{(i)}$$

$$\frac{\partial f}{\partial y} = 4y^3 + x^2 z + 2\alpha xyz + 3\beta xy^2 = 0 \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial z} = 2x^2 z + x^2 y + \alpha xy^2 = 0 \qquad \text{(iii)}$$

By (iii), we know the singularities must be on $x = 0$ or $2xz + xy + \alpha y^2 = 0$. In the case $x = 0$, we have by (ii) $y = 0$, and $x = y = 0$ satisfying (i) too. Hence one of the singularities is at (0,0,1) as we expect. On the other hand, in case $2xz + xy + \alpha y^2 = 0$, (eq.(iv)) and $x \neq 0$ note that (1,0,0) is obviously a solution which also satisfies (i) and (ii). Hence (1,0,0) is the other singularity as expected.

Now if $y = 0$, by (i) or (ii) $x = 0$ or $z = 0$, i.e. singularity at (0,0,1) or (1,0,0)

if $z = 0$, by (i) or (ii) $y = 0$ i.e. singularity at (1,0,0).

This results in singularities which we have already known. Therefore, we can assume $y \neq 0$, $z \neq 0$ to find other singularities that is not at X or Z.
Multiply (iv) by z

$$2xz^2 + xyz + \alpha y^2 z = 0 \qquad \text{(v)}$$

Then, (i)-(v) we have

$$xyz + \beta y^3 = 0$$

$$\text{i.e. } xz + \beta y^2 = 0 \qquad \text{(vi)}$$

Substitute (vi) into (iv)

$$xy = (2\beta-\alpha)y^2$$

i.e. $x = (2\beta-\alpha)y$. Hence $2\beta \neq \alpha$ (for $x \neq 0$)

Let $y = 1$, by (vi), we have $z = \frac{-\beta}{(2\beta-\alpha)}$.
So the other singularity is at

$$\left((2\beta-\alpha)^2,\ (2\beta-\alpha),\ -\beta\right)$$

We want this to satisfy all (i), (ii)and (iii) equations. But we have already made use of equation (i) and (iii). The condition for this to satisfy (ii) is

$$4(2\beta-\alpha)^3 + (2\beta-\alpha)^4(-\beta) + 2\alpha(2\beta-\alpha)^2\,(2\beta-\alpha)(-\beta)$$
$$+ 3\beta(2\beta-\alpha)^2\,(2\beta-\alpha)^2 = 0.$$

$(2\beta-\alpha) \neq 0$, Since we assume $y \neq 0$.

Hence

$$4 + (2\beta-\alpha)(-\beta) + 2\alpha(-\beta) + 3\beta(2\beta-\alpha) = 0.$$

$$4 - 2\beta^2 + \alpha\beta - 2\alpha\beta + 6\beta^2 - 3\alpha\beta = 0.$$

$$4\beta^2 - 4\alpha\beta + 4 = 0.$$

i.e. $\beta^2 - \alpha\beta + 1 = 0.$

This is the condition on α,β such that the curve have singularity other than X and Z. Therefore for A_1A_3, we must have

$$\beta^2 - \alpha\beta + 1 \neq 0.$$

and, conversely, granted this condition the curve has no other singularities.

It is worth remarking that actually when

$$\beta^2 - \alpha\beta + 1 = 0$$

and $\alpha \neq \pm 2$ i.e. $\beta \neq \pm 1$, the curve becomes two conics

$$(y^2 + \beta xz + \beta xy)(y^2 + \frac{1}{\beta}xz) = 0$$

with the singularity type $A_1^2A_3$  (see reducible classification) i.e. the condition $\beta^2 - \alpha\beta + 1 = 0, \alpha \neq \pm 2$ is for another node added to the curve. Also, when $\beta^2 - \alpha\beta + 1 = 0$ and $\alpha = \pm 2$ i.e. $\beta = \pm 1$, the curve becomes

$$(y^2 \pm xz \pm xy)(y^2 \pm xz) = 0.$$

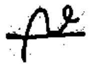
This is A_1A_5  (see reducible classification).

Note: Further specializations can be obtained by considering equation

$$ay^4 + bx^2z^2 + cx^2yz + dxy^2z + exy^3 = 0.$$

For example if we let $a \to 0$, the equation becomes

$$x(bxz^2 + cxyz + dy^2z + ey^3) = 0$$

The vertice Z still remains an A_3, X an A_1, but the line $x = 0$ will cut the cubic at $y^2\ (dz + ey) = 0$, i.e. at (0,0,1) twice, and (0,-d,e) once. Hence curve becomes $A_1^2A_3$ .

If we let $c = e = 0$, the equation is

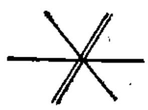
$$(y^2 + mxz)(y^2 + nxz) = 0$$

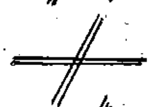
which is two genuine conics with two points contact  A_3^2

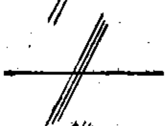
If we let $a = c = e = 0$, we have

$$xz(bxz + dy^2) = 0$$

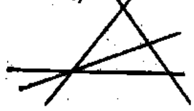
which is two tangent to a conic, hence  $A_1A_3^2$

Also, $a = b = d = e = 0$ is 

$a = c = d = e = 0$ 

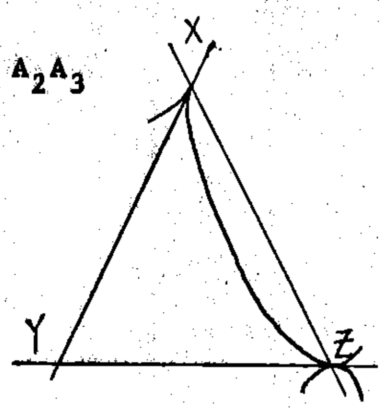
$a = b = c = d = 0$ 

$b = c = d = e = 0$ 

$a = b = e = 0$  etc.

We would expect corresponding result by considering general formula for each singularity type. Therefore for the rest of the cases, we would only give less obvious specializations. Notice that these are just some of the specializations, not all. And also all we have achieved here is just a sequence of curves of a particular type of singularity specializing into another type. It does not necessary show that the whole of the other singularity type is in the closure of the one we start with. We shall deal with specializations specifically in Chapter 7.

A_2A_3



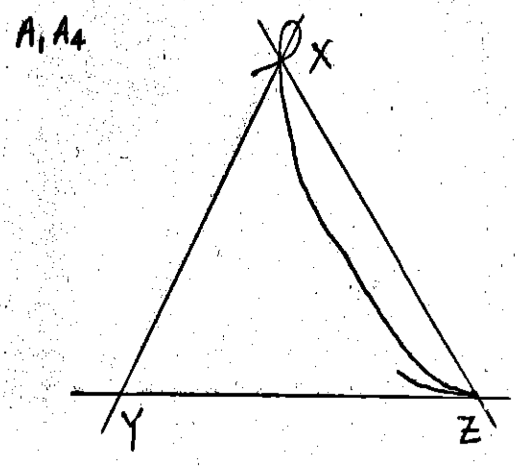
Similar to A_1A_3, except with cusp at X and z = 0 be tangent to the cusp at X. This implies the two tangent lines formed by the lowest terms at (1,0,0) in the non-homogeneous coordinates must coincide. But we want the tangent to be z = 0. Therefore the YZ and Y^2 term in the lowest terms must be absent. That is, x^2yz and x^2y^2 terms are absent.

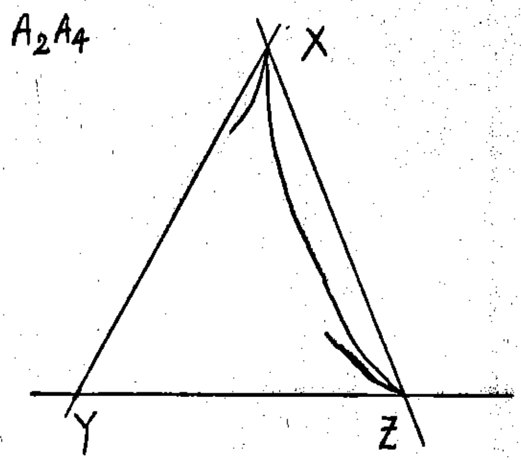
Hence the equation is $ay^4 + bx^2z^2 + dxy^2z + exy^3 = 0$.

To ensure cusp at X, b and e must not be zero. And $a \neq 0$, since cubic will not cut x = 0 again. Therefore by choosing unit point we have normal form.

A_2A_3 $\quad y^4 + x^2z^2 + \alpha xy^2z + xy^3 = 0$ $\quad \alpha \neq \pm 2$ N & S

When $\alpha = \pm 2$, the curve becomes A_2A_4. We can also check that the normal form gives A_2A_3 for all other values of α.





Let the node and the ramphoid cusp be at X and Z respectively. Then the curve passes through X, Z - x^4, z^4 terms disappear; also being singular at those points x^3y, x^3z, xy^3, yz^3 vanish too.

Choosing $x = 0$ be the tangent to the ramphoid cusp at Z y^2z^2 term disappears.

For the singularity at Z to be a ramphoid cusp, we should have y^2z^2, y^3z, xyz^2 terms disappeared and also the leading terms in the non-homogeneous coordinate at Z must form a perfect square. (see appendix).

i.e. we have the equation

$$ax^2z^2 + bxy^2z + cy^4 = dx^2y^2 + ex^2yz + fxy^3$$

where $b^2-4ac = 0$, $a \neq 0$, $c \neq 0$ for irreducibility and $b \neq 0$ because if $b = 0$ we have an A_3.

Now take the transformation

$$x \to x$$

$$y \to y$$

$$z \to z + \lambda y$$

which just amounts to choosing the tangent line to node at X. Substituting Z in the equation by $z+\lambda y$, we have

$$ax^2z^2 + bxy^2z + cy^4 = (d-a\lambda^2 + e\lambda)x^2y^2 + (e-a\lambda)x^2yz + (f-b\lambda)xy^3$$

Now choose $\lambda = \frac{e}{a}$, we have equation

$$ax^2z^2 + bxy^2z + cy^4 = dx^2y^2 + (f - \frac{be}{a})\, xy^3$$

Let $f - \frac{be}{a} = k$.

The equation becomes

$$ax^2z^2 + bxy^2z + cy^4 = dx^2y^2 + kxy^3$$

$$a \neq 0,\ b \neq 0,\ c \neq 0$$

$$b^2-4ac = 0$$

We now claim that $k \neq 0$ for A_4 at Z.

Consider non-homogeneous coordinate at (0,0,1), we have

$$aX^2 + bXY^2 + cY^4 = dX^2Y^2 + kXY^3$$

i.e. $$(X + \frac{b}{2a}\,Y^2)^2 = dX^2Y^2 + kXY^3$$

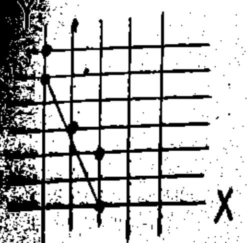
To calculate μ, we can use the following transformation

$$X \to X - \frac{b}{2a}\,Y^2$$

$$Y \to Y$$

$$Z \to Z$$

Hence equation becomes



$$X^2 = d(X - \frac{b}{2a} Y^2)^2 \; Y^2 + k(X - \frac{b}{2a} Y^2) \; Y^3$$

$$X^2 = dX^2Y^2 - \frac{db}{a} XY^3 + \frac{db^2}{4a} Y^6$$

$$+ kXY^3 - \frac{kb}{2a} Y^5$$

The leading terms are

$$X^2 + \frac{kb}{2a} Y^5 + \ldots . = 0$$

Now if $k \neq 0$ $\quad \mu = (5-1)(2-1) = 4$ (see appendix)

if $k = 0$ $\quad \mu = (6-1)(2-1) = 5$

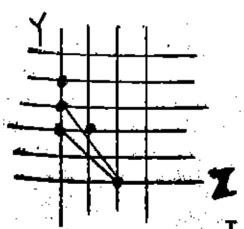
Hence for A_4 at Z, we must have $k \neq 0$.

We can now choose unit point, and the normal form is

$$x^2z^2 + \theta xy^2z + y^4 = \alpha x^2y^2 + xy^3$$

i.e. $(xz + y^2)^2 = \alpha x^2y^2 + xy^3$.

Observe also that $\alpha \neq 0$, because at X the non-homogeneous leading terms are $Z^2 + \alpha Y^2 + \ldots . = 0$, and



$$\mu = (2-1)(2-1) = 1 \qquad A_1$$

If $\alpha = 0$, the leading terms will then be $Z^2 + Y^3 + \ldots . = 0$

therefore $\mu = (3-1)(2-1) = 2$ A_2

Hence normal forms

A_1A_4 $(xz + y^2)^2 = \alpha x^2y^2 + xy^3$ $\alpha \neq 0$

A_2A_4 $(xz + y^2)^2 = xy^3$ NbS

Hence, $\alpha \to 0$, $A_1A_4 \to A_2A_4$

It is checked that for all other values of α, the normal form for A_1A_4 still gives the singularity type indicated. Note that we can also derive some other specializations from the process of classification. For example in the above case, consider the equation before choosing the unit point

$$ax^2y^2 + bxy^2z + cy^4 = dx^2y^2 + kxy^3$$

Let $d \neq 0$ $k \to 0$, the curve comes two proper conics

$$(zx + y^2 + \sqrt{d}xy)(zx + y^2 - \sqrt{d}xy) = 0$$

We can easily check that it has only two singularities, one at (1,0,0) and the other at (0,0,1). Since $d \neq 0$, we know that at (1,0,0) there is still a node, $\left[(z+\sqrt{d}y)(z-\sqrt{d}y) = 0\right.$ distinct tangents$\left.\right]$, whereas at (0,0,1), the non-homogeneous leading terms are (see p. 24)

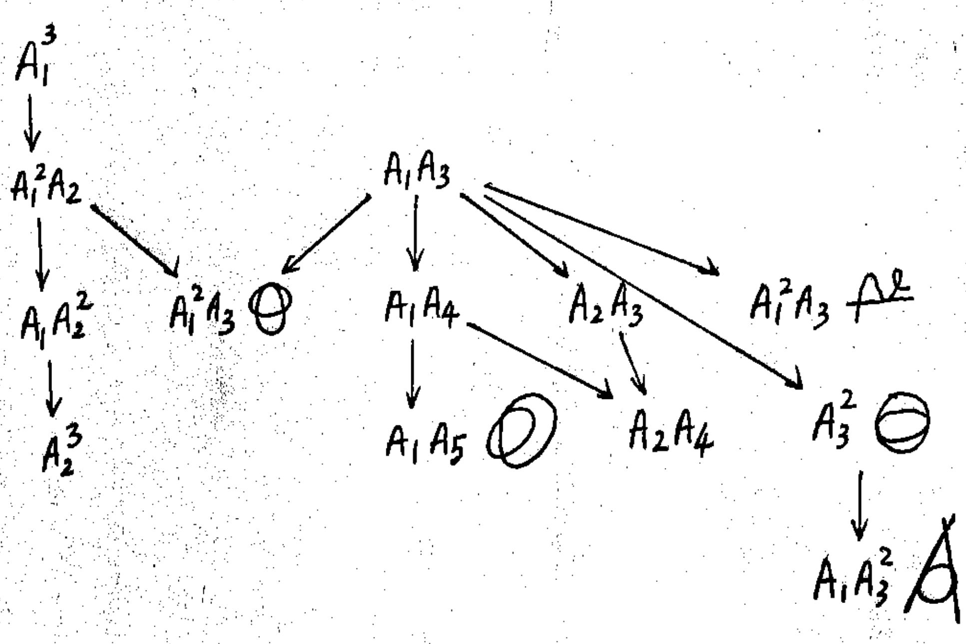
$$x^2 + \frac{db^2}{4a^2}\, y^6 + \ldots\ldots = 0$$

Hence $\quad \mu = (6-1)(2-1) = 5 \qquad\qquad A_5$

In other words, the two conics has a node and a 3-point contact i.e. an A_1A_5 

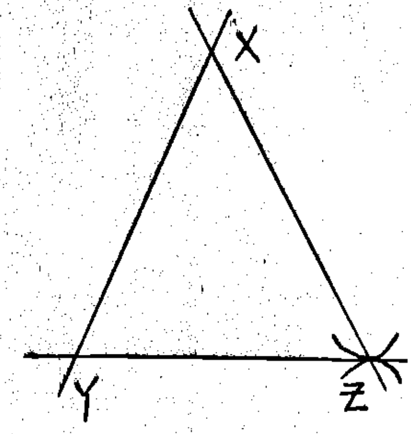
Therefore, we can say A_1A_4 specializes into A_1A_5.

Up till now, our knowledge enable us to draw a small picture of several specializations



As we have said before, the arrows only indicate there is a sequence of curves of the upper singularity type specializing into the lower singularity type. We have not yet shown that lower type is actually in the closure of the upper.

A_5 Osnode $\quad$ A_6 tacnode cusp



Suppose A_5 be at Z, then z^4, xz^3, yz^3 terms disappear.

Choose $x = 0$ to be the tangent to the osnode at Z, y^2z^2 absent. For singularity at Z to be an osnode we should have at least the following two conditions:

(i) y^2z^2, y^3z, xyz^2 terms absent.

(ii) leading terms in the non-homogeneous coordinate at Z must form a perfect square (see A_1A_4).

Hence then we have equation

$$ax^2z^2 + bxy^2z + cy^4 = dx^2y^2 + ex^3y + fxy^3 + gx^4 + hx^2yz + \ell x^3z$$

where $a \neq 0$, $b \neq 0$, $c \neq 0$, $b^2-4ac = 0$ as in previous cases.
Taking the transformation

$$x \to x$$
$$y \to y + \lambda_1 x$$
$$z \to z + \lambda_2 + \lambda_3 y$$

and choosing λ_1, λ_2 and λ_3, we can reduce the equation to

$$ax^2z^2 + bxy^3z + cy^4 = d'x^2y^2 + f'xy^3 + e'x^3y$$

Claim that $f' = 0$, for A_5 at Z (see p. 24). Take transformation

$$X \to X - \frac{b}{2a} Y^2$$

$$Y \to Y$$

$$Z \to Z$$

in non-homogeneous coordinates at Z and consider equation

$$X^2 = d'(X - \frac{b}{2a} Y^2)^2 Y^2 + f'(X - \frac{b}{2a} Y^2) Y^3 + e'(X - \frac{b}{2a} Y^2) Y$$

if $f' \neq 0$ leading terms are $X^2 + \frac{b}{2a} Y^5 + \ldots = 0$, $\mu = 4$

if $f' = 0$, $d' \neq 0$.

leading terms are $X^2 + \frac{b^2}{4a} Y^6 + \ldots = 0$.

Hence $\mu = (6-1)(2-1) = 5$ A_5

Furthermore, if $f' = 0$, $d' = 0$, $e' \neq 0$ leading terms are

$$X^2 - \frac{e'b^3}{8a^3} Y^7 + \ldots = 0$$

Hence $\mu = (7-1)(2-1) = 6$ A_6

Choosing unit point, we can write down the normal forms for A_5 and A_6

A_5 $$x^2z^2 + xy^2z + y^4 = x^2y^2 + \alpha x^3 y$$

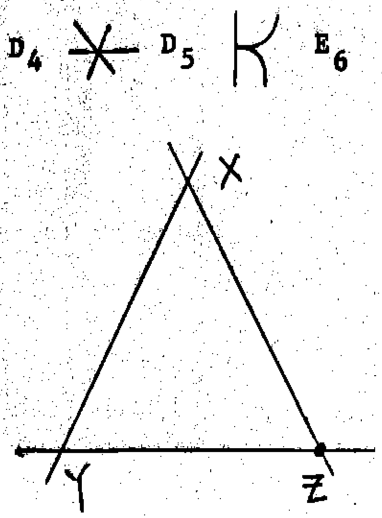
i.e. $(xz + y^2)^2 = x^2(y^2 + \alpha xy)$ $\alpha \neq 0$ (see p. 25)

If $\alpha = 0$ $A_5 \to A_1 A_5$

A_6 $$(xz + y^2)^2 = x^3 y.$$

N k S.

Now, we come up to the types of triple points.



Let the triple point be at Z. Then the terms z^4, xz^3, yz^3, xyz^2, x^2y^2 and y^2z^2 must be absent (condition for triple root). So the equation is $zC(x,y) = A(x,y)$ where C and A are polynomials of degrees 3 and 4 respectively.

(i) First consider D_4. We can see that the tangents at the triple point i.e. at Z are distinct. This implies that C has distinct factors. We can assume that tangents are

$$xy(x + y) = 0$$

Therefore the equation becomes

$$xyz(x + y) = A(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$$

Now, take the transformation

$$x \to x$$

$$y \to y$$

$$z \to z + \lambda x + \mu y$$

We have the equation

$$xyz(x+y) + \lambda x^2 y(x+y) + \mu xy^2(x+y)$$

$$= ax^4 + bx^3 y + cx^2 y^2 + dxy^3 + ey^4$$

i.e. $$xyz(x+y) = ax^4 + (b-\lambda)\ x^3 y + (c-\lambda-\mu)x^2 y^2 + (d-\mu)x^3 y + ey^4$$

Choosing $\lambda = b$, $\mu = d$, we can reduce the equation to

$$xyz(x+y) = ax^4 + (c-b-d)\ x^2 y^2 + ey^4$$

or $$xyz(x+y) = ax^4 + b x^2 y^2 + ey^4 \qquad \text{where } b = (c-b-d)$$

To stop the curve from being reducible, we must have $a, e \neq 0$.
Now, taking unit point, we can have the normal form

D_4 $$xyz(x+y) = x^4 + \alpha x^2 y^2 + \beta y^4 \qquad \beta \neq 0,\ \beta \neq -\alpha-1$$

Observe that when $\beta = -1-\alpha$, we have

$$xyz(x+y) = x^4 + \alpha x^2 y^2 - y^4 - \alpha y^4$$

$$= (x^4 - y^4) + (\alpha x^2 y^2 - \alpha y^4)$$

$$= (x+y)\ (x-y)\ (x^2+y^2) + \alpha y^2 (x+y)\ (x-y).$$

It becomes a reducible quartic. Therefore $\beta \neq -1-\alpha$

Actually we get this condition by considering the normal form

$$f(x,y,z) = xyz\ (x+y) - x^4 - \alpha x^2 y^2 - \beta y^4$$

$$\frac{\partial f}{\partial x} = 2xyz + y^2z - 4x^3 - 2\alpha xy^2 = 0 \qquad \text{(i)}$$

$$\frac{\partial f}{\partial y} = x^2z + 2xyz - 2\alpha x^2y - 4\beta y^3 = 0 \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial z} = x^2y + xy^2 = 0 \qquad \text{(iii)}$$

By (iii), singularities lies on $x = 0$ or $y = 0$ or $x+y = 0$.

If $x = 0$ by (i), $y = 0$ or $z = 0$.

If $x = 0 = y$, (ii) holds $\Longrightarrow (0,0,1)$ is a singularity as expected.

If $x = 0 = z$, (ii) holds if $\beta = 0$.

This means that $(0,1,0)$ is a singularity if $\beta = 0$.

If $\beta \neq 0$, (ii) $\Longrightarrow y = 0$ contradiction. Hence $(0,1,0)$ not singular.

If $y = 0$, by (ii) $x = 0$ or $z = 0$. If $x = y = 0$, $(0,0,1)$ known.

If $y = z = 0$ by (i) $\Longrightarrow x = 0$ contradiction.

$\Longrightarrow (1,0,0)$ non-singular.

If $x+y = 0$, we have $y = -x$

Let $x = 1$, by (i) we have $z = -(2\alpha+4)$.

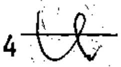
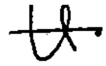
Therefore singularity is at $(1, -1, -(2\alpha+4))$.

Substitute this into (ii) we have

$$4\alpha + 4\beta + 4 = 0$$

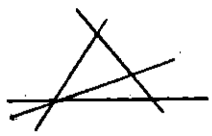
$$\text{i.e.} \quad \alpha + \beta + 1 = 0.$$

This is the condition for a further singular at $(1,-1,-(2\alpha+4))$.

We can easily see that for specialization, when $\beta=0$, the curve becomes A_1D_4  (see reducible cases). Similarly, when $\beta+\alpha+1 = 0$, $\beta\neq 0$ the curve is also A_1D_4 .
But if both $\beta+\alpha+1 = 0$, $\beta=0$ i.e. $\alpha+1 = 0$, then the curve becomes $A_1^2D_4$  .
Also, we can get a further specialization by considering the equation of D_4 before taking the unit point,

$$\text{i.e.} \quad xyz(x+y) = ax^4 + \delta x^2y^2 + ey^4$$

If all a, δ and e = 0, then the curve becomes

$A_1^3D_4$ 

Therefore, we have the specializations

$$D_4$$
$$\downarrow$$
$$A_1D_4$$
$$\downarrow$$
$$A_1^2D_4$$
$$\downarrow$$
$$A_1^3D_4$$

(ii) For D_5, it has a repeated tangent. Then C must have a repeated factor. We can assume C to be x^2y, and equation is

$$x^2yz = A(x,y)$$
$$= ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$$

Take transformation

$$x \to x$$

$$y \to y$$

$$z \to z + \lambda x + \mu y \quad \text{and let } \lambda = b$$

$$\mu = c$$

Equation reduces to

$$x^2yz = ax^4 + dxy^3 + ey^4$$

$a, e \neq 0$ for irreducible case.

Choose unit point by taking transformation $x \to mx$

$$y \to ny$$

$$z \to z$$

we have

$$m^2n\, x^2yz = am^4x^4 + dmn^3xy^3 + en^4y^4$$

$$x^2yz = \frac{am^2}{n} x^4 + \frac{dn^2}{m} xy^3 + \frac{en^3}{m^2} y^4$$

Let $\frac{am^2}{n} = 1$, $\frac{en^3}{m^2} = 1$

then $am^2 = n$, $en^3 = m^2 = \frac{n}{a}$

Therefore $n^2 = \frac{1}{ae}$ i.e. $n = \pm\sqrt{\frac{1}{ae}}$

and $m = \pm\sqrt{\frac{n}{a}}$ i.e. $m = \pm\sqrt{\sqrt{\frac{1}{ae}} \times \frac{1}{a}}$ or $\pm\sqrt{\sqrt{\frac{1}{ae}} \times \frac{1}{a}\, i}$

This means that there are two choices for n and four choices for m.

Let $\alpha = \frac{dn^2}{m}$, we have the normal form

$$D_5 \qquad x^2yz = x^4 + \alpha xy^3 + y^4 \text{ (no restriction on } \alpha)$$

NbS.

where we can replace α by $-\alpha$, $i\alpha$ or $-i\alpha$. But we have observed that the replacement of α by $-\alpha$, $i\alpha$ or $-i\alpha$ in the normal form is equivalent to taking the transformations

$$\begin{matrix} x \to -x & & x \to -ix & & x \to +ix \\ y \to y & & y \to -y & & y \to -y \\ z \to z & , & z \to z & , & z \to z \end{matrix}$$

respectively. That is $\pm\alpha$, $\pm i\alpha$ all give the same curve up to projective equivalence.

We can also check that this singularity at $Z(0,0,1)$ does have $\mu = 5$. The nonhomogeneous leading terms are

$$X^2Y - Y^4$$

Hence $\mu = (2-1)4 + 1 = 5$ (see appendix). Also the singularity is a triple point, so must be D_5 from the list on p. 29. Further, it is easy to check that the curve has no other singularities.

(iii) E_6, the three tangents coincide. Then C is a cube, say $C = x^3$. The equation is

$$\begin{aligned} x^3z &= A(x,y) \\ &= ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \end{aligned}$$

If $e = 0$, curve reducible. Hence $e \neq 0$.

Apply transformation

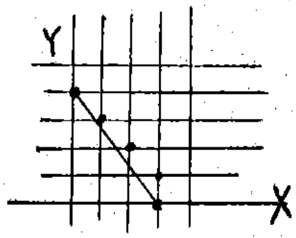
$$x \to x$$
$$y \to y + \lambda_1 x$$
$$z \to z + \lambda_2 x + \lambda_3 y,$$

we can get rid of xy^3, x^4 and x^3y terms. The equation becomes

$$x^3z = cx^2y^2 + ey^4$$

or $\quad x^3z = kx^2y^2 + y^4$ where $k = \frac{c}{e}$, since $e \neq 0$.

Now let us consider the condition for $\mu = 6$. At $Z(0,0,1)$, the leading terms in the non-homogeneous coordinate are



$$X^3 - Y^4$$

Hence $\mu = 6$.

Therefore, we can see that even when $k = 0$, the equation can still have an E_6. But then the question is whether this is projectively equivalent to the case when $k \neq 0$. The answer to this is no. These two cases are actually projectively distinct. When $k \neq 0$ the curve has two inflexions whereas when $k = 0$, it has only one. This is shown as following:

The curve is $x^3z - kx^2y^2 - y^4 = 0$.

The tangent at (x_0, y_0, z_0) is

$$x(3x_0^2z_0 - 2kx_0y_0^2) + y(-2kx_0^2y_0 - 4y_0^3) + z(x_0^3) = 0$$

parametrized by $(1, t, t^2+t^4)$, the tangent equation becomes

$$x\left[3(t^2+t^4) - 2kt^2\right] + y\left[-2kt-4t^3\right] + z = 0$$

Suppose this tangent meets the curve again at $u(1,u,ku^2+u^4)$, then

$$kt^2 + 3t^4 + u\left[-2kt-4t^3\right] + (ku^2+u^4) = 0$$

This factorize $(u-t)^2\ (k+u^2+2ut+3t^2) = 0.$

The condition for $u = t$ to be a triple root is

$$6t^2 + k = 0$$

which has two solutions for $k \neq 0$ and one solution for $k = 0$. Each solution gives an inflexion. Hence the two cases are projective distinct.

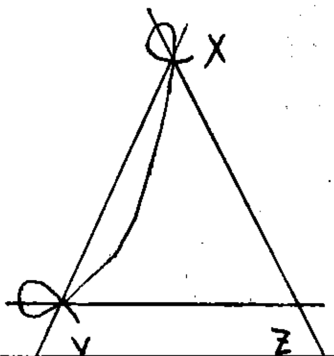
Now for the case, when $k \neq 0$, we can choose unit point to have normal form

E_6 $\quad x^3z = x^2y^2 + y^4$

and when $k = 0$ $\quad x^3z = y^4$

This implies that we have two orbits in E_6, corresponding to $k \neq 0$, $k = 0$.

A_1^2



Let the two nodes be at vertices X and Y of the triangle of reference. Then x^4, y^4, x^3y, x^3z, xy^3, y^3z terms must be absent. We are left with the equation

$$ax^2y^2 + bx^2z^2 + cy^2z^2 + dz^4 + ex^2yz + fxy^2z + gxyz^2 + hxz^3 + kyz^3 = 0$$

Consider the lowest degree terms at (1,0,0) in the non-homogeneous coordinate

$$aY^2 + bZ^2 + eYZ = 0$$

If we choose the line $y = \sqrt{\frac{b}{a}}\, i\, z$ to be the tangent to the node at X, we can have x^2yz term vanished. Similarly, choose line $x = \sqrt{\frac{c}{a}}\, i\, z$ to be the tangent to the node at Y, we can have xy^2z term disappeared too.
Then equation becomes

$$ax^2y^2 + bx^2z^2 + cy^2z^2 + dz^4 + gxyz^2 + hxz^3 + kyz^3 = 0.$$

To ensure nodes at X and Y $a \neq 0$, $b \neq 0$, $c \neq 0$.
Also not all d, h and k equal to zero to keep Z non-singular i.e. to ensure no other singular point.
Now, choosing unit point, we can reduce the equation to

$$A_1^2 \quad x^2y^2 + x^2z^2 + y^2z^2 + \lambda z^4 + \alpha xyz^2 + \beta xz^3 + \gamma yz^3 = 0$$

(not all $\lambda, \beta, \gamma = 0$).

or let $\lambda = \delta + 1$

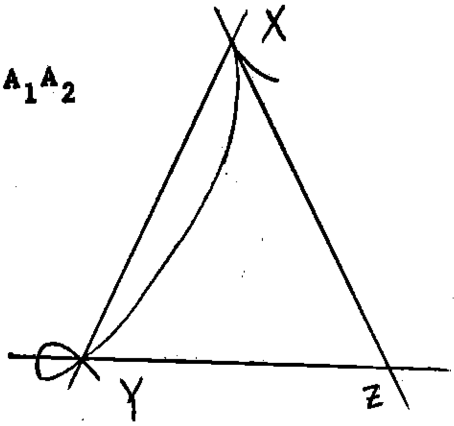
$$x^2y^2 + x^2z^2 + y^2z^2 + z^4 + \alpha xyz^2 + \beta xz^3 + \gamma yz^3 + \delta z^4 = 0$$

i.e. $(x^2+z^2)(y^2+z^2) + \alpha xyz^2 + (\beta x+\gamma y+\delta z)\, z^3 = 0$

N

Note: It is not always easy to check the conditions on the moduli so that the normal form will remain as the singularity type indicated, espcially when there is more than 2 moduli. We shall omit this and produce only reasonably obvious restrictions.

A_1A_2

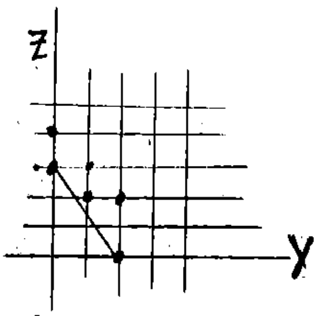


Similarly to A_1^2, except now we have a cusp at X and taking $y = 0$ to be the tangent to the cusp. This implies the terms x^2z^2 and x^2yz are absent.

Therefore the equation is

$$ax^2y^2 + by^2z^2 + dz^4 + gxyz^2 + hxz^3 + kyz^3 = 0.$$

The leading terms in the non-homogeneous coordinate at X are



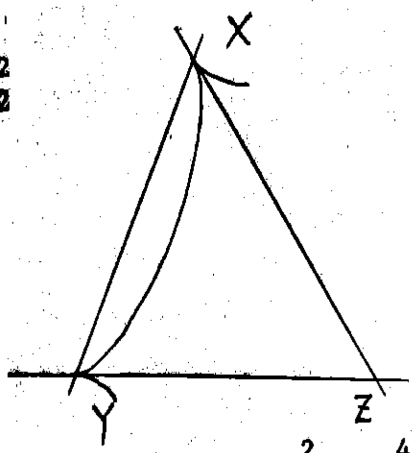
$aY^2 + hZ^3 \qquad a \neq 0$ (irreducible)

$\mu = (3-1)(2-1) = 2$, if $h \neq 0$.

Therefore, for $A_1A_2, h \neq 0$. Then we can choose unit point and obtain the normal form

$$A_1A_2 \qquad x^2y^2 + g^2z^2 + \alpha xyz^2 + xy^3 + \gamma yz^3 + \delta z^4 = 0$$

N



If the singular points at X and Y are both cusps, then we have x^2z^2, y^2z^2, x^2yz and xy^2z terms disappearing as well. We are left with the equation

$$ax^2y+dz^4+gxyz^2+hxz^3+kyz^3 = 0$$

$a \neq 0$ (irreducible)

To ensure cusp at X, Y, we must have $h \neq 0$, $k \neq 0$. Hence, choosing unit point, we have normal form

$A^2_2 \qquad x^2y^2 + \alpha xyz^2 + xz^3 + yz^3 + \delta z^4 = 0$

N

Consider the normal form of A^2_2

$$f = x^2y^2 + \alpha xyz^2 + xz^3 + yz^3 + \delta z^4 = 0$$

$$\frac{\partial f}{\partial x} = 2xy^2 + yz^2 + z^3 = 0 \qquad \text{(i)}$$

$$\frac{\partial f}{\partial y} = 2x^2y + \alpha xz^2 + z^3 = 0 \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial z} = 2\alpha xyz + 3xz^2 + 3yz^2 \; 4\delta z^3 = 0 \qquad \text{(iii)}$$

By (iii), we have singularities on $z = 0$

or $\qquad 2\alpha xy + 3xz + 3yz + 4\delta z^2 = 0 \qquad$ (iv)

If $z = 0$, by (i) $x = 0$ or $y = 0$.

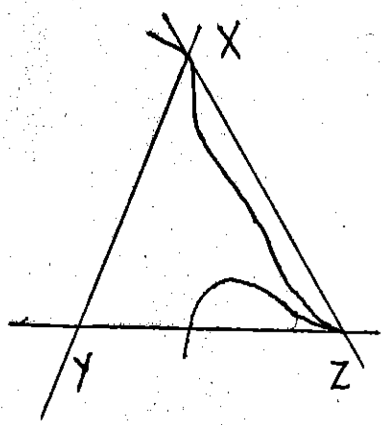
If $x = z = 0$, (ii) satisfied $\Longrightarrow (0,1,0)$ singular

$z = y = 0$, (ii) also satisfied $\Longrightarrow (1,0,0)$ singular.

This is what we would expect.

But what is the condition for other singularities? This is found by solving equation (i), (ii) and (iv), assuming $x \neq 0$, $y \neq 0$ and $z \neq 0$. In doing so, very lengthy calculation is needed. And the condition found will probably be very complicated. Hence this will be of little use to us. Therefore we might as well omit this calculation. Same situation would happen to any normal form with more than two moduli.

A_3



Let the tacnode be at Z. Then z^4, xz^3, yz^3, xyz^2, y^2z^2 and y^3z terms are absent. Let the curve pass through X. Hence x^4 is absent. Consider the lowest terms in non-homogeneous coordinate at (1,0,0). The tangent is

$$\mu Y + \gamma Z = 0 \qquad \mu, \gamma \text{ arbitrary constant.}$$

Choose $y = 0$ to be the tangent to the curve at X, then we must have x^3z term absent.

Therefore, the remaining equation is

$$ay^4 + bx^2z^2 + cx^3yz + dx^3y + exy^2z + fxy^3 + gx^2y^2 = 0$$

Take transformation

$$x \to x$$

$$y \to y$$

$$z \to z + \lambda y$$

The equation becomes

$$ay^4+bx^2(z^2+2\lambda yz+\lambda^2y^2) + cx^2y(z+\lambda y) + dx^3y$$

$$+ exy^2(z+\lambda y) + fxy^3 + gx^2y^2 = 0$$

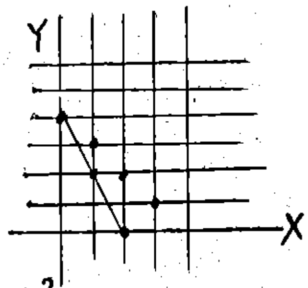
$$ay^4+bx^2z^2 + (2\lambda b+c)\ x^2yz + (b\lambda^2+c\lambda+g)\ x^2y^2$$

$$+ dx^3y + exy^2z + (\lambda e+f)\ xy^3 = 0.$$

Choose $\lambda = -\frac{c}{2b}$, we can get rid of x^2yz term

$$ay^4 + bx^2z^2 + d'x^3y + e'xy^2z + f'xy^3 + g'x^2y^2 = 0$$

Observe that $a = 0$, the curve has a factor x. Hence $a \neq 0$.
Also $a \neq 0$ to ensure non-singularity at X.
At Z, the non homogeneous leading terms are



$$aY^4 + e'XY^2 + bX^2$$

If $e'^2-4ab \neq 0$, $\mu = (4-1)(2-1) = 3$.

Then, choose unit point we can make $a=b=1=d$, i.e., $e' \neq \pm 2$, the normal form is

$$A_3 \qquad x^2z^2 + y^4 + x^3y + \alpha xy^2z + \beta x^2y^2 + \gamma xy^3 = 0$$

N, $\alpha \neq \pm 2$

A_4 Similar to A_3. Consider the normal form of A_3. At Z, in the non-homogeneous coordinate, the equation becomes

$$X^2 + \alpha XY^2 + Y^4 + X^3Y + \beta X^2Y^2 + \gamma XY^3 = 0$$

If $\alpha = +2$, then

$$(X + Y^2)^2 + X^3Y + \beta X^2Y^2 + \gamma XY^3 = 0$$

Take transformation $X \to X-Y^2$

$Y \to Y$

then

$$X^2 + (X-Y^2)^3Y + \beta(X-Y^2)^2Y^2 + \gamma(X-Y^2)Y^3 = 0$$

Therefore, leading terms are

$$X^2 - \gamma Y^5$$

Hence if $\gamma \neq 0$, $\mu = (5-1)(2-1) = 4$.

The normal form is

$$A_4 \quad x^2z^2 + 2xy^2z + y^4 + x^3y + \beta x^2y^2 + \gamma xy^3 = 0$$

$$\text{or } (xz + y^2)^2 + x^3y + \beta x^2y^2 + \gamma xy^3 = 0 \qquad \gamma \neq 0$$

$\beta^2 \neq 4\gamma$ (see p.44)

Consider the normal form of A_4

$$f = x^2z^2 + 2xy^2z + y^4 + x^3y + \beta x^2y^2 + \gamma xy^3 = 0$$

$$\frac{\partial f}{\partial x} = 2xz^2 + 2y^2z + 3x^2y + 2\beta xy^2 + \gamma y^3 = 0 \qquad \text{(i)}$$

$$\frac{\partial f}{\partial y} = 4xyz + 4y^3 + x^3 + 2\beta x^2y + 3\gamma xy^2 = 0 \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial z} = 2x^2z + 2xy^2 = 0 \qquad \text{(iii)}$$

By equation (iii), we know that singularities must lie on $x=0$

or $\quad xz + y^2 = 0 \qquad$ (iv)

If $x = 0$, by (ii) $y = 0$, satisfying (i) $\Longrightarrow (0,0,1)$ singular as expected.

If $xz + y^2 = 0$, assuming $x \neq 0$, $y \neq 0$, multiply $2z$ to (iv)

that is $\quad 2xz^2 + 2y^2z = 0 \qquad$ (v)

(i)−(v) $\quad 3x^2y + 2\beta xy^2 + \gamma y^3 = 0$

i.e. $\quad 3x^2 + 2\beta xy + \gamma y^2 = 0 \qquad$ (vi)

Multiply $4y$ to (iv), i.e. $\quad 4xyz + 4y^3 = 0 \qquad$ (vii)

(ii)−(vii) $\quad x^3 + 2\beta x^2y + 3\gamma xy^2 = 0$

i.e. $\quad x^2 + 2\beta xy + 3\gamma y^2 = 0 \qquad$ (viii)

(vi)−(viii) $\quad -4\beta xy - 8\gamma y^2 = 0$

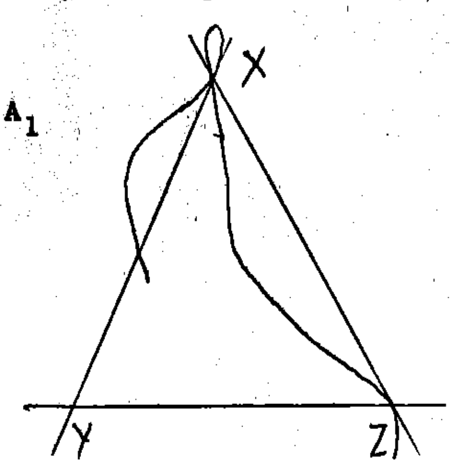
i.e. $\quad -\beta x - 2\gamma y = 0 \qquad \therefore x = -\frac{2\gamma}{\beta} y$

Let $y = 1$, then $x = -\frac{2\gamma}{\beta}$ and by (iv) $z = \frac{\beta}{2\gamma}$.

Therefore the singularities are at $(-4\gamma^2, 2\gamma\beta, \beta^2)$ where $\gamma \neq 0$, since assuming $x \neq 0$, $y \neq 0$.
Substituting into (vi) we have the condition $\gamma = \frac{\beta^2}{4}$ for another singularity. Hence for A_4 we should have $\beta^2 \neq 4\gamma$. And we can easily see that when $\beta^2 = 4\gamma$, the curve actually becomes A_1A_4. The extra singularity added is a node.

Note that we can replace +2 by −2 in the normal form since they are equivalent (take the transformation $z \to -z$).

A_1



Let the node be at X. Then x^4, x^3y, x^3z terms are absent. Suppose the curve Γ passes through Z. Then z^4 is absent. Consider the lowest non-homogeneous terms at Z. The equation of the tangent is $\mu X + \gamma Y = 0$. If we choose $y = 0$ to be the tangent to the curve at Z (possible because the cubic has class 4), then xz^3 term is absent too.

Similarly consider the lowest non-homogeneous terms at X. The equation of the two tangent is $\ell Z^2 + mZY + nY^2 = 0$. If we take $z = 0$ to be the tangent to the node at X, then x^2y^2 is absent. Now, we have the equation

$$ax^2z^2 + byz^3 + cx^2yz + dy^4 + exy^3 + fy^3z + gxy^2z + hy^2z^2 + kxyz^2 = 0$$

Take transformation $x \to x + \lambda y$

$y \to y$

$z \to z$

We have

$$a(x^2 + 2\lambda xy + \lambda^2 y^2)z^2 + byz^3 + c(x^2+2\lambda xy+\lambda^2 y^2)yz$$

$$+ dy^4+e(x+\lambda y)y^3+fy^3z+g(x+\lambda y)y^2z+hy^2z^2+k(x+\lambda y)yz^2 = 0.$$

i.e.
$$ax^2z^2 + (2\lambda a+k)xyz^2+(a\lambda^2+h+k\lambda)y^2z^2+byz^3$$

$$+ cx^2yz+(2\lambda c+g)xy^2z+(c\lambda^2+f+\lambda g)y^3z+(d+e\lambda)y^4+exy^3+hy^2z^2 = 0$$

Let $\lambda = -\frac{k}{2a}$, we can eliminate xyz^2.

We have the equation

$$ax^2z^2+byz^3+cx^2yz+d'y^4+exy^3+f'y^3z+g'xy^2z+h'y^2z^2 = 0$$

If $a = 0$, the equation has a factor y. Hence $a \neq 0$.

If $b = 0$, the curve Γ has a singular point at Z. Hence $b \neq 0$.

Consider the non-homogeneous leading terms at X,

$$aZ^2 + cYZ + eY^3, \quad \text{if } c \neq 0, \quad \mu = 1,$$

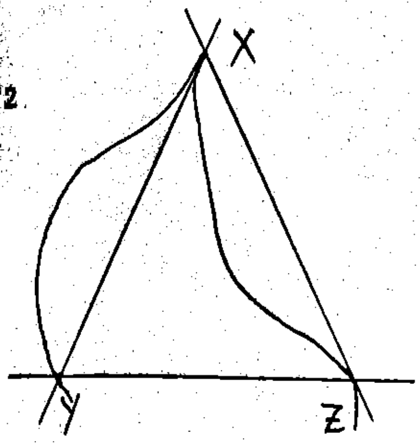
If $c = 0$, $\mu = (3-1)(2-1) = 2$. Therefore $c \neq 0$.

Then we can choose unit point to obtain normal form

$$A_1 \qquad x^2z^2 + yz^3 +x^2yz+\alpha y^4+\beta xy^2z+\gamma y^3z+\delta xy^2z+\varepsilon y^2z^2 = 0$$

N

A_2



Let the cusp be at X. Then x^4, x^3y, x^3y, x^2yz, x^2y^2 terms are absent. Suppose the curve Γ passes through both Y and Z. Then y^4 and g^4 terms are absent.

Choose $y = 0$ to be the tangent to the curve at Z. (This is possible since the class of the cubic is 3 and only two tangents are absorbed at the cusp). Then we must have x^3z term absent too.

Then we have the equation

$$ax^2z^2+byz^3+exy^3+fy^3z+gxy^2z+hy^2z^2+kxyz^2 = 0$$

If $a = 0$, the equation has a factor y. Hence $a \neq 0$.
If $b = 0$, the curves has a singular point at Z. Hence $b \neq 0$.
Consider the non-homogeneous leading terms at X,

$$aZ^2 + eY^3 + gY^2Z$$

If $e \neq 0$ $\mu = 2$

$e = 0$ $\mu = 3$ Hence $e \neq 0$ for A_2

Therefore, choosing unit point, we have normal form

$$A_2 \qquad x^2z^2+yz^3+xy^3+\alpha y^3z+\beta xy^2z+\gamma y^2z^2+\delta xyz^2 = 0$$

N

Note that there are always various ways of writing down the normal form for a singularity type. The above list of

ormal forms are specially chosen to have a certain property.
e shall discuss this in chapter 3 .

normal forms are specially chosen to have a certain property. We shall discuss this in chapter 3 .

2.3 Enumeration for Reducible Quartics

Although we also have a genus formula for reducible curves, it is not easy to make use of the formula as we have done in the irreducible case.

The easiest way to list the reducible cases is probably from the view point of geometry. We shall then proceed in this direction.

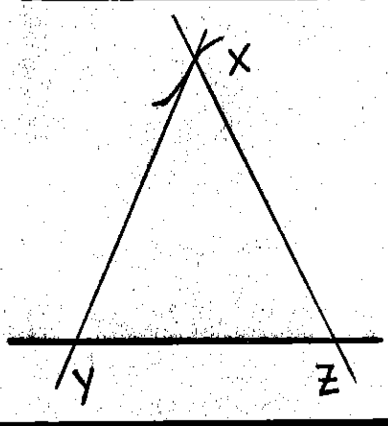
Geometrically, a quartic can only reduce into the following four kinds of combinations: (I) a cubic and a line, (II) two conics, (III) a conic and two lines, (IV) four lines. We shall discuss each of these four cases individually.

(I) A cubic and a line

If the cubic is reducible, the quartic will be included in the cases (II), (III) or (IV). Therefore we shall restrict ourselves to irreducible cubics in this case. From the classification of cubics we know that there are only three types of irreducible cubics - non-singular, nodal and cuspidal.

(i) We shall begin with non-singular cubic and a line. There are three possibilities.

A_1^3 Non-singular cubic and a general chord



Choose the chord to be the line $x = 0$. Choose X as an inflexion of the cubic and $z = 0$ as the inflexional tangent. Now consider the ten basis monomials for cubics. The term x^3 is absent since curve passes through X. Local equation at X is $Y+Z+\ldots\ldots = 0$. But we have already chosen $z = 0$ as the tangent to curve at X. Hence the term x^2y must be absent.

Also since $z = 0$ is an inflexional tangent, the curve should not cut the line $z = 0$ again. This implies xy^2 term goes too. The cubic has the form

$$ax^2z+bxz^2+cxyz+dy^3+ey^2z+fyz^2+gz^3 = 0.$$

We still have the freedom of choosing Z on $x = 0$. Therefore take transformation $x \to x$, $y \to y+\lambda z$, $z \to z$, we can get rid of the term z^3 and obtain

(*)
$$a'x^2z+b'xz^2+c'xyz+d'y^3+e'y^2z+f'yz^2 = 0$$

Note that $a' \neq 0$, since if $a' = 0$, X will be singular. Also $d' \neq 0$ in order to keep the cubic irreducible. If $f' = 0$, then $x = 0$ cuts the cubic at $y^2(d'y+e'z) = 0$. Hence to keep the three points of intersection distinct we must have $f \neq 0$ and also $e'^2 \neq 4d'f'$. Now choose unit point by taking $x \to mx$, $y \to ny$, $z \to kz$. We can make $a' = 1$, $d' = 1$ and $f' = 1$. Then the equation of the quartic has the form

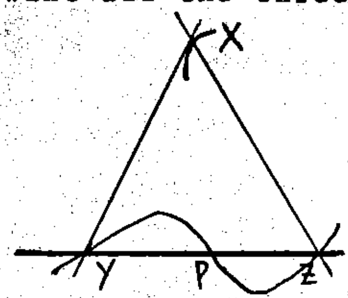
$$A_1^3 \qquad x(x^2z + \alpha xz^2 + \beta xyz + y^3 + \gamma y^2z + yz^2) = 0 \qquad \gamma^2 \neq 4$$

N

<u>Notes</u>

(i) There are still other conditions on α, β and γ in order to keep the cubic non-singular. But as we have mentioned before in the irreducible classification, these conditions may not be easy to obtain and also they will be too complicated to be of any use to us. Therefore, from here on whenever this situation arises, we shall omit the calculation of these further conditions.

(ii) Note that in this normal form the three nodes on $x = 0$ are not all fixed. We shall now describe another normal form for A_1^3. It has a certain special property which we shall mention in later chpaters. In this new normal form we want all the three nodes on $x = 0$ fixed, say, at Y, Z



and P(0,1,-1). That is, the cubic passes through the points (0,1,0), (0,1,-1) and (0,0,1) on $x = 0$. Let us consider the cubic basis monomials. Since (0,0,1) and (0,1,0) are solutions, y^3 and z^3 terms must be absent. Also because of the point (0,1,-1), the coefficients of y^2z and yz^2 must be equal. Now if we let the cubic pass through X, the x^3 term goes too. But now we cannot in general choose X to be an inflexion, so we choose $z = 0$ to be tangent to the cubic at X. This is possible because it is well known that for any non-singular cubic curve, four tangents can be drawn to the cubic from a point Q of the curve. Hence at least three lines through Y are tangent to the cubic at a point other than Y (even if Y is an inflexion). Then we have the term x^2y absent too. We result in the cubic equation

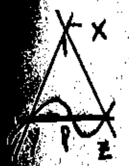
$$axy^2+bx^2z+cxz^2+dy^2z+dyz^2+exyz = 0$$

If $d = 0$, then x is a factor. Hence $d \neq 0$

If $b = 0$, X is a singular point. Hence $b \neq 0$.

If $a = 0$, then z is a factor. Hence $a \neq 0$.

Then by choosing unit point, we can make $d = a = 1$. We have the normal form for the quartic as



A_1^3

$$x(xy^2+y^2z+yz^2+\alpha x^2z+\beta xz^2+\gamma xyz) = 0 \qquad \alpha \neq 0$$

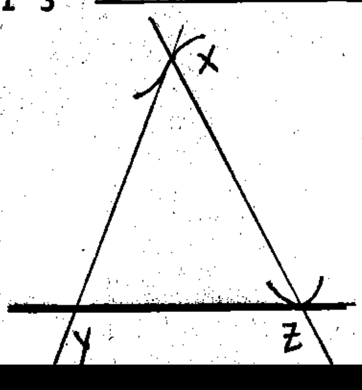
There are presumably other conditions for the cubic to be non-singular. These are omitted because of reasons mentioned before.

Also for a reason which will be mentioned in Chapter 5, we need neither of the tangents at P and Z to pass through X.
The tangent at Z passes through X iff $\beta=0$.
The tangent at P passes through X iff the line $y+z = 0$ is tangent to cubic at P. Now the line $z = -y$ meets the cubic where

$$xy^2 \cdot (1+\beta-\gamma) - \alpha x^2y = 0.$$

This has double root $x = 0$ iff $1+\beta-\gamma = 0$.
Therefore for the special purpose, we also want $\beta \neq 0$ and $1+\beta-\gamma \neq 0$.

A_1A_3 Non-singular cubic and a tangent line



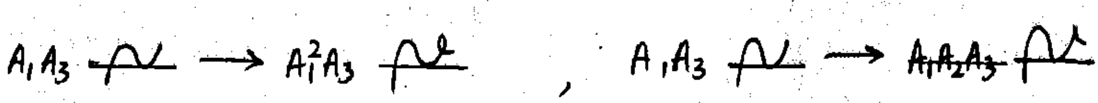
This is just the case when $x = 0$ cuts the cubic at two points, one repeated that is the case when $f'=0$ This time e' should not be equal to zero or else $x = 0$ would cut cubic three times at $(0,0,1)$. We could make $e' = 1$ by choosing unit point and have the normal form

 A_1A_3 $\quad x(x^2z + \alpha xz^2 + \beta xyz + y^3 + y^2z) = 0.$

$\alpha \neq 0$

N b S.

Note $\alpha = 0$ implies Z is a singular point, indeed a node provided $\beta^2 \neq 4$. We obtain specializations $A_1A_3 \to A_1^2A_3$ and $A_1A_3 \to A_1A_2A_3$



by letting $\alpha \to 0$.

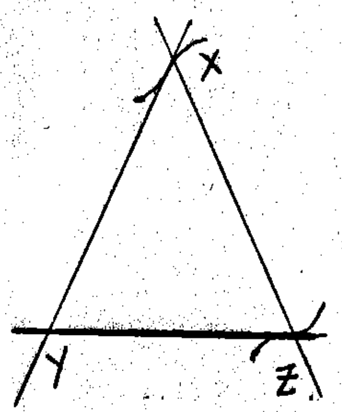
In the case A_1A_3 above it is a simple matter to determine the precise conditions for the cubic to be non-singular, and hence for the quartic to be of the type A_1A_3. The conditions are

$$\alpha \neq 0$$

$$\beta^4-3\alpha\beta^3+2\alpha\beta^2-8\beta^2+36\alpha\beta-27\alpha^2 + 16 \neq 0.$$

This illustrates the remark made above that the conditions are usually toocomplicated to be of use!

A_5 Non-singular cubic and inflexional tangent



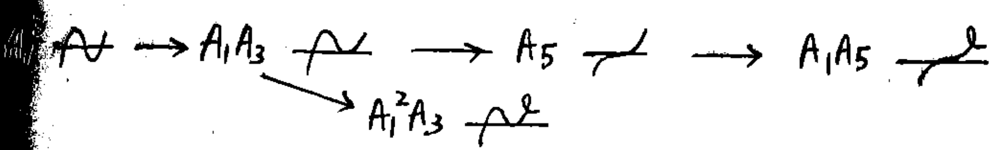
If e' does equal zero too in the above discussin, we then have the case A_5. That is, in the equation (*) in A_1^3, $e' = f' = 0$. Also $b' \neq 0$, so that Z will remain non-singular for the cubic. Choosing unit point, we have the normal form

A_5 $\quad x(x^2z + xz^2 + \beta xyz + y^3) = 0 \quad \beta^3+27\neq 0$

N & S

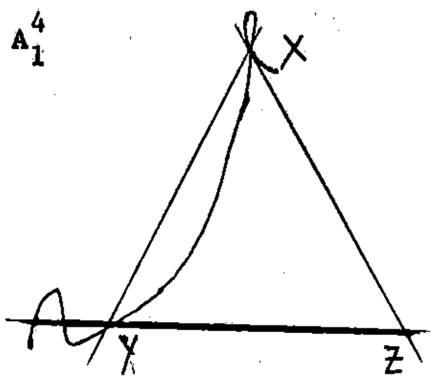
Let $x = 1$. Then, if $\beta^3+27=0$, the cubic has a singularity at $(1,1,1)$. We can easily check that it is a node. Therefore, for A_5, $\beta^3 \neq -27$. And when $\beta \to -3$ $A_5 \to A_1A_5$

Actually, up till now, we have already achieved the following specializations



(I) (ii) <u>Nodal cubic and a line</u>

A_1^4



There are five possibilities. Let $x = 0$ to be the line. This is the case when the node of the cubic is not on $x = 0$ and the cubic cuts the line in three distinct points. Position the node of the cubic at X so that x^3, x^2y, x^2z absent. Choose $y = 0$ to be the tangent to the node there, xz^2 absent. Also allow the cubic to go through the point Y, y^3 term gone. Then consider the ten basis monomials of the cubic curves. The equation remained

$$az^3 + bxy^2 + cy^2z + dyz^2 + exyz = 0.$$

If $a = 0$, y is a factor, contradiction to irreducibility.

Hence $a \neq 0$.

If $b = 0$, z is a factor. Hence $b \neq 0$.

If $c = 0$, $x = 0$ will cut the curve at $z^2(az+dy) = 0$

two points, one repeated at (0,1,0). This contradict our proposition to have three distinct intersections. Hence $c \neq 0$.

If $e = 0$, then there will be a cusp at X instead of a node. Hence $e \neq 0$.

Choosing unit point, we can make $a = b = e = 1$. Then we have normal form for the quartic as

A_1^4 $$x(z^3 + xy^2 + \alpha y^2 z + \beta y z^2 + xyz) = 0$$

where $\alpha \neq 0$, and $\beta^2 \neq 4\alpha, 1+\alpha-\beta \neq 0$

N & S

Note the line $x = 0$ also cuts the cubic at two points, one repeated when $\beta^2 - 4\alpha = 0$. This is because $x = 0$ meets the cubic where $z^3 + \beta y z^2 + \alpha y^2 z = 0$.

Now let us check whether there is any other conditions on α and β in the normal form for A_1^4 such that the curve will remain to be A_1^4. It is enough to check the condition for the cubic to remain nodal.

Let $g(x,y,z) = z^3 + xy^2 + \alpha y^2 z + \beta y z^2 + xyz = 0$

$$\frac{\partial g}{\partial x} = y^2 + yz = 0 \qquad \text{(i)}$$

$$\frac{\partial g}{\partial y} = 2xy + 2\alpha yz + \beta z^2 + xz = 0 \qquad \text{(ii)}$$

$$\frac{\partial g}{\partial z} = 3z^2 + 2y^2 + 2\beta yz + xy = 0 \qquad \text{(iii)}$$

By (i), singularities are either on $y = 0$ or $y+z = 0$.

If on $y = 0$, (iii) implies $z = 0$. And $y=z=0$ satisfies (ii). This means (1,0,0) is a singularity as expected. If $y+z = 0$, i.e. $y = -z$, substitute into (ii), we have

$$-2xz - 2\alpha z^2 + \beta z^2 + xz = 0$$

$$- xz + (\beta-2\alpha) z^2 = 0$$

$$z\ [(\beta-2\alpha)z-x] = 0 \qquad \text{(iv)}$$

Now $z = 0$ implies by (ii) $x = 0$ or $y = 0$.

If $z=x=0$, by (iii), since $\alpha \neq 0$, implies $y = 0$. Contradiction.

If $z=y=0$, we have singularity we expect. Hence we can assume $z \neq 0$ to find other singularities.

There by (iv) we have

$$x = (\beta-2\alpha) z \qquad \text{(v)}$$

Substitute $y = -z$ into (iii) we have

$$3z^2 + \alpha z^2 - 2\beta z^2 - xz = 0$$

$$(3+\alpha - 2\beta)\ z^2 - xz = 0$$

and since $z \neq 0$, we have

$$x = (3+\alpha - 2\beta)\ z \qquad \text{(vi)}$$

Compare (v) and (vi), we have

$$(\beta-2\alpha) = (3+\alpha-2\beta)$$

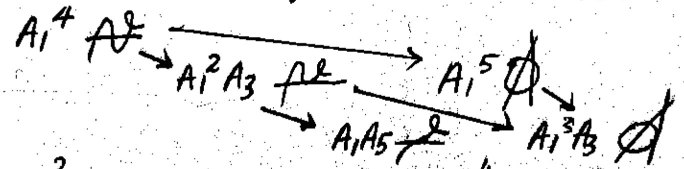
$$\text{i.e.} \quad 1+\alpha-\beta = 0.$$

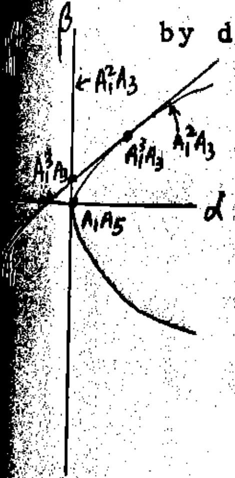
This is the condition for the cubic to have an extra singularity. Hence the cubic become reducible. Actually a conic and a chord. And if we let $z = 1$, the singularity is at

$$((\beta-2\alpha), -1, 1)$$

We can easily check that this is a node, by taking it to X through a transformation. Therefore for A_1^4, we must have $1+\alpha-\beta \neq 0$.

For the specialization in this case, it is best shown by drawing a picture





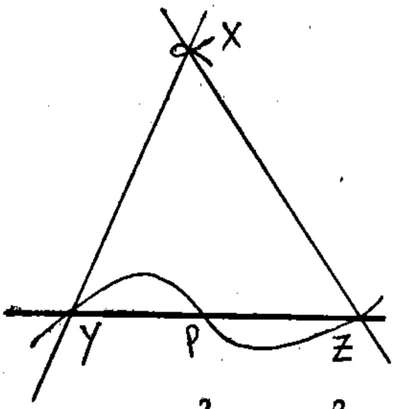
If $\alpha = 0$, $\beta \neq 0$, $A_1^4 \to A_1^2A_3$, $\alpha = 0$, $\beta = 0$, $A_1^4 \to A_1A_5$

If $\beta^2 = 4\alpha$, $A_1^4 \to A_1^2A_3$

If $1+\alpha-\beta = 0$, $A_1^4 \to A_1^5$. If $1+\alpha-\beta = 0$ and $\alpha = 0$, $\beta \neq 0$, i.e. $\alpha = 0$, $\beta = 1$, $A_1^4 \to A_1^3A_3$

If $1+\alpha-\beta=0$, and $\beta^2=4\alpha$ i.e. $\alpha=1$, $\beta=2$, then $A_1^4 \to A_1^3A_3$

Note: In the normal form for A_1^4 on p.54 the three nodes on $x = 0$ are again not fixed. We shall now give another normal form for A_1^4 which has a certain property that we shall need in later proofs. We want the nodes on $x = 0$ fixed in the new normal form. Let them be at Y, Z and P(0,1,-1).



And the node of the cubic is taken to X, but the tangent direction is not fixed. Then by considering the basis monomials, we have (see A_1^3 case) the cubic equation

$$axy^2 + bxz^2 + cy^2z + cyz^2 + dxyz = 0$$

If c = 0, x is a factor. Hence $c \neq 0$

If a = 0, z is a factor. Hence $a \neq 0$

If b = 0, y is a factor. Hence $b \neq 0$

By choosing the unit point, we have the quartic equation

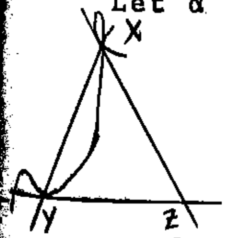
A_1^4 $\quad x(xy^2 + y^2z + yz^2 + \alpha xz^2 + \beta xyz) = 0$

$\alpha \neq 0$

$\alpha-\beta+1 \neq 0$ NbS

Note, $\beta^2 \neq 4$ for the nodal tangents to be distinct. Also, it is easily checked that $\alpha-\beta+1 \neq 0$ is also needed so that the cubic will remain irreducible.

Let $\alpha = 0$ in the normal form for A_1^4 on p. 54.



We can have the normal form for $A_1^2A_3$ as (using α as the only parameter)

$A_1^2A_3$ $\quad x(z^3 + xy^2 + \alpha yz^2 + xyz) = 0$

$\alpha \neq 0$

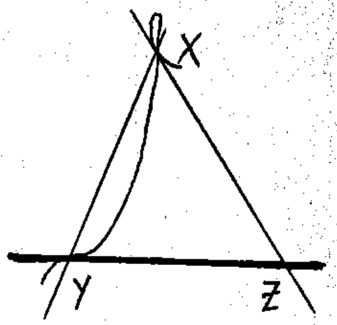
$\alpha \neq 1$ NbS

Here we use α instead of β as the only parameter.

If $\alpha = 0$ this becomes A_1A_5, if $\alpha = 1$ it becomes $A_1^3A_3$

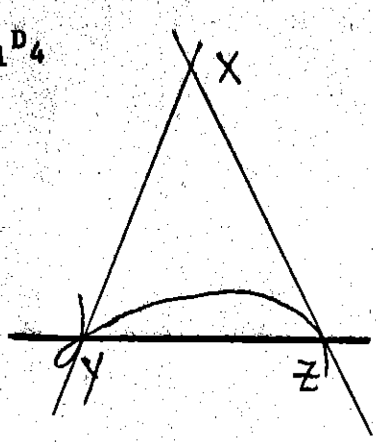
Also if we let $\alpha=\beta=0$ in the normal form for A_1^4, we have the normal form for a nodal cubic and inflexional tangent:

A_1A_5 $\qquad x(z^3 + xy^2 + xyz) = 0$



The above are the three cases when the node of the cubic is not on the line. Now we shall consider the other two cases when the node is actually on the line component of the quartic.

A_1D_4



Again choose $x = 0$ to be the line component of the quartic. Suppose the node of the cubic is at Y and it is positioned such that $z = 0$ is tangent to the node there. Also let the cubic pass through Z and $y = 0$ tangent to the cubic there (the line $x = 0$ will meet the cubic once away from Y). Then by considering the basis monomials, we have the following equation for the cubic

$$ax^3 + bx^2z + cyz^2 + dxyz = 0$$

We can see that if $a = 0$, z is a factor. This contradict our irreducibility. Hence $a \neq 0$.

If $c = 0$, x is a factor. Hence $c \neq 0$

If $d = 0$, then there will be a cusp at Y instead of a node. Hence $d \neq 0$.

Now we choose the unit point. The equation for the quartic of the type A_1D_4 is

$$A_1D_4 \qquad x(x^3 + \alpha x^2 z + yz^2 + xyz) \qquad \alpha \neq 1$$

N & S

We shall now check the condition on α for the curve to remain in A_1D_4. It is enough to check the condition on α for the cubic to remain nodal.

Let $g(y,z) = x^3 + \alpha x^2 z + yz^2 + xyz$

$$\frac{\partial g}{\partial x} = 3x^2 + 2\alpha xz + yz = 0 \qquad \text{(i)}$$

$$\frac{\partial g}{\partial y} = z^2 + xz = 0 \qquad \text{(ii)}$$

$$\frac{\partial g}{\partial z} = \alpha x^2 + 2yz + xy = 0 \qquad \text{(iii)}$$

By (ii), singularities must lie on $z = 0$ or $z+x = 0$.
If $z = 0$, by (i) implies $x = 0$. Then $x = z = 0$ also satisfies (iii). Hence $(0,1,0)$ is a singularity as we have expected.
Now we can assume $z \neq 0$ in order to find other singularities.
If $z+x = 0$ i.e. $x = -z$, substitute into (i), we have

$$3z^2 - 2\alpha z^2 + yz = 0$$

$$z\left[(3-2\alpha)z + y\right] = 0$$

Since $z \neq 0$ we have $y = (2\alpha - 3)z$ (iv)
Substitute $x = -z$ into (iii), we have

$$z(\alpha z + y) = 0$$

Hence $y = (-\alpha)z$ **(v)**

Compare (iv) and (v), we have

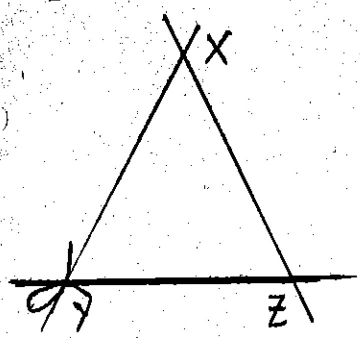
$$2\alpha - 3 = -\alpha$$

$$3\alpha = 3$$

$$\alpha = 1$$

That is, when $\alpha = 1$, we have an extra singularity - cubic becomes reducible, a conic and a chord. If we let $z = 1$, the singularity is at $(-1,-1,1)$. Moving this to one of the origins, we can easily check that it is a node. Therefore for A_1D_4, $\alpha \neq 1$. And when $\alpha \to 1$, $A_1D_4 \to A_1^2D_4$ $\emptyset$

D_6



This is the case when the node of the cubic is not only on the line component but also has it on one of the tangents.

Still chose $x = 0$ to be the line component. Let the node of the cubic be at Y and $x = 0$ be one of its tangents. Also choose $z = 0$ to be the other tangent to the node. Since the $x = 0$ cuts the cubic three times at Y, it can not cut the cubic again. Considering the basis monomials, we can have the equation

$$ax^3 + bz^3 + cx^2z + dxz^2 + exyz = 0$$

Taking the transformation

$$x \to x$$

$$y \to y + \lambda x + \mu z$$

$$z \to z$$

we can get rid of the terms x^2z and xz^2.
Hence we have the equation

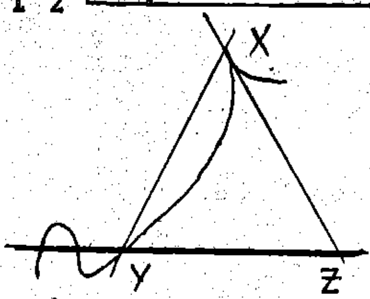
$$ax^3 + bz^3 + exyz = 0$$

Neither a nor b can equal zero because of irreducibility. Also $e \neq 0$, because if $e = 0$, there will be more degenerate singularity point at Y instead of a node. Therefore, choosing the unit point, we have the normal form for D_6

$$D_6 \qquad x(x^3 + z^3 + xyz) = 0$$

(I) (iii) Cuspidal cubic and a line. Similar to nodal cubic and a line, there are also five possibilities. The first three deal with the cases when the cusp is not on the line.

$A_1^3A_2$ Cuspidal cubic and a chord



This is the case when the cubic cuts the line in three distinct points. Choose $x = 0$ be the line component of the quartic. Let the cusp of the cubic be at X and $y = 0$ be the cuspidal tangent. Also let the cubic cut the line at Y and two other distinct points. Then by considering the basis monomials we have the equation

(*)

$$az^3 + bxy^2 + cy^2z + dyz^2 = 0$$

If $a = 0$, y is a factor. Hence $a \neq 0$ for irreducibility of the cubic.

If $b = 0$, z is a factor. Hence $b \neq 0$ for irreducibility of the cubic.

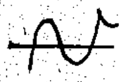
If $c = 0$, $x = 0$ is a tangent to the cubic at Y. This contradicts our assumption that the line $x = 0$ cuts the cubic at three distinct points. Hence $c \neq 0$.

Now by choosing the unit point, we can make $a = b = c = 1$. We then have the normal form for the quartic as

$A_1^3A_2$ $\quad x(z^3 + xy^2 + y^2z + \alpha yz^2) = 0$

$\alpha \neq \pm 2$

N & S

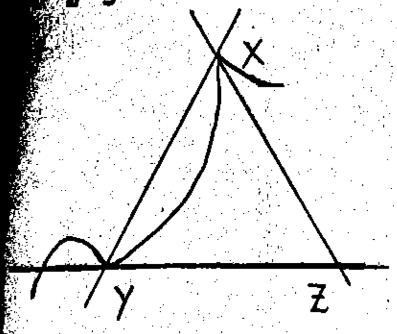


Note that when $\alpha^2 - 4 = 0$, the line $x = 0$ will cut the cubic at $z(z^2 + \alpha yz + y^2) = 0$,

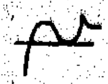
i.e. $z(z \pm y)^2 = 0$

two points, one repeated. Hence for $A_1^3A_2$ $\alpha \neq \pm 2$. In order for the quartic to remain to be $A_1^3A_2$, it is enough to check the condition for the cubic to remain cuspidal. We can easily check that (1,0,0) is the only singularity near the origin. Hence there is no other condition on α for the cubic to remain cuspidal.

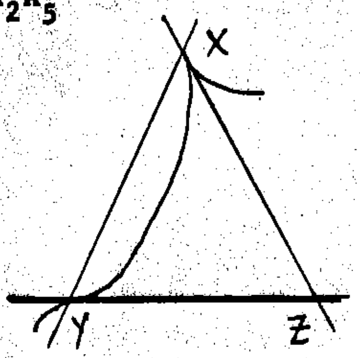
$A_1A_2A_3$ Cuspidal cubic and a tangent



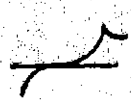
This case is when the cuspidal cubic cuts the line $x = 0$ at only two points, one repeated. That is, the case when $c = 0$ in the equation (*) in $A_1^3A_2$. Now consider the coefficient d. If $d = 0$, then $x = 0$ will make a three-point contact with the cubic — an inflexion at Y. Hence for $A_1A_2A_3$, $d \neq 0$. We can make it equal to 1 by choosing unit point. Then we have the normal form

$A_1A_2A_3$  $x(z^3 + xy^2 + yz^2) = 0$

A_2A_5

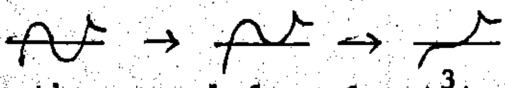


This is just the case when the line $x = 0$ cuts the cuspidal cubic at a three-point contact, i.e. the case when $c = d = 0$ in (*) of $A_1^3A_2$. We have the normal form

A_2A_5  $x(z^3 + xy^2) = 0$

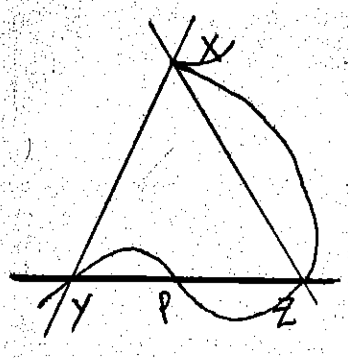
For specialization, we can see that

$$A_1^3A_2 \rightarrow A_1A_2A_3 \rightarrow A_2A_5$$



Note: Again in the normal form for $A_1^3A_2$, the three nodes on

$x = 0$ are not fixed. We would like to have another normal form for $A_1^3A_2$ in which the three collinear nodes are fixed. This normal form will also have certain properties which will be needed later on.



Let us now fix the three collinear nodes at Y, Z and P(0,1,-1) and take the cusp of the cubic to X, but without fixing the cuspidal tangent direction. Then by considering the basis monomials, we are left with the cubic equation (see A_1^3 Note P.50 and A_1^4 note P.56).

$$axy^2 + bxz^2 + cy^2z + cyz^2 + dxyz = 0$$

The two tangents at X to the cubic must be of the same direction because of a cusp. Hence tangents at X,

$$ay^2 + dyz + bz^2 = 0 \text{ implies } d^2 = 4ab$$

If $c = 0$ in the cubic equation, x is a factor. Hence $c \neq 0$.
If $b = 0$ in the cubic equation, y is a factor. Hence $b \neq 0$.
If $a = 0$ in the cubic equation, z is a factor. Hence $a \neq 0$.

Hence by $d^2 = 4ab$, $a \neq 0$, $b \neq 0$ implies $d \neq 0$.
Choosing unit point by the transformation

$$x \to \lambda x$$
$$y \to \mu y$$
$$z \to \zeta z$$

We have, $a\lambda\mu^2xy^2 + b\lambda\zeta^2xz^2 + c\mu^2\zeta y^2z + c\mu\zeta^2yz^2 + d\lambda\mu\zeta xyz = 0.$

We can choose λ, μ, ζ such that

$$c\mu^2\zeta = 1, \; c\mu\zeta^2 = 1 \text{ and } b\lambda\zeta^2 = 1.$$

These give $\mu = \zeta = 1/c^{1/3}$ and $\lambda = c^{2/3}/b.$

Hence the equation becomes

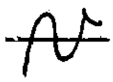
$$\frac{a}{b}\, xy^2 + xz^2 + y^2z + yz^2 + \frac{d}{b}\, xyz = 0.$$

But we have the relationship $d^2 = 4ab$ or $a = \dfrac{d^2}{4b}$.

Let $\alpha = \dfrac{d}{2b}$, we have the normal form of the quartic as

$A_1^3A_2$ $\quad x(\alpha^2xy^2 + xz^2 + y^2z + yz^2 + 2\alpha xyz) = 0$

$\alpha \neq 0 \quad \alpha \neq 1$



NbS.

Note that this is the first <u>non-linear</u> normal form we have encountered.

Now let us investigate when there is definitely a cusp at X. Let $x = 1$, leading terms are

$$z^2 + 2\,yz + \alpha^2y^2$$

$$= (z + \alpha y)^2 \qquad \text{perfect square}$$

Hence take the transformation $z \to z - \alpha y$. That is, replace z by $z - \alpha y$ in the normal form. It becomes

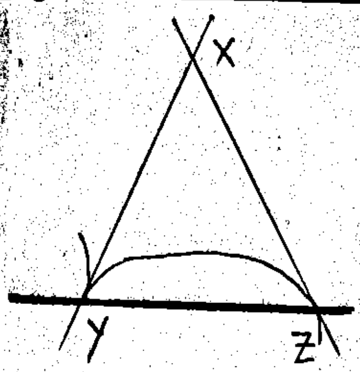
$$z^2 + y^2(z-\alpha y) + y(z-\alpha y)^2 = 0$$

$$\text{i.e. } z^2 + (\alpha^2-\alpha)y^3 + (1-2\alpha)y^2z + yz^2 = 0$$

Now if $(\alpha^2-\alpha) \neq 0$, then there is certainly a cusp at X and if $\alpha^2-\alpha = 0$ there is an A_3. Hence $\alpha \neq 0$, $\alpha \neq 1$ for $A_1^3A_2$. Further if $\alpha \neq 0$, $\alpha \neq 1$ then the cubic, having a cusp, must be cuspidal so this is the necessary and sufficient condition such that the normal form gives $A_1^3A_2$.

Now the other two possibilities deal with the cases when the cusps of the cubic is on the line component of the quartic.

A_1D_5 Cuspidal cubic and a cuspidal chord



Again we choose $x = 0$ to be line component of the quartic. Let the cusp of the cubic be at Y and $z = 0$ be the cuspidal tangent. Also we can allow the cubic to cut the line $x = 0$ again at Z and $y = 0$ be the tangent to the cubic there. Considering the basis monomials of the cubic, we result in the equation

$$ax^3 + bx^2z + cyz^2 = 0$$

Note that a and c will not be equal to zero for the irreducibility. Now by choosing unit point we can make $a = c = 1$ and if $b \neq 0$, $b = 1$ as well. Hence we have the equation for the quartic as

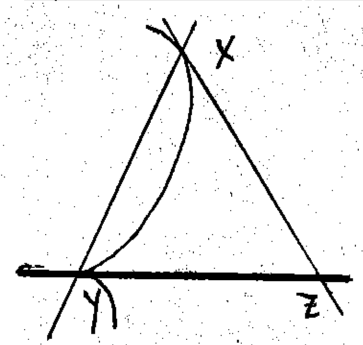
A_1D_5 $$x(x^3 + \alpha x^2 z + yz^2) = 0$$

where $\alpha = 0$ or 1

N & S

This result seems to imply that A_1D_5 has two orbits. But now the question is whether these two orbits are projectively distinct or not. This case is quite similar to E_6, and the answer here is also "Yes". Let us consider the situation carefully. If $\alpha = 1$, we can see that the line $y = 0$ cuts the cubic at $x^2(x+\alpha z) = 0$ — two points, one repeated. This implies that $y = 0$ is an ordinary tangent to the cubic at Z or Z is an ordinary point of the cubic. But if $\alpha = 0$, we can see $y = 0$ cuts the cubic at $x^3 = 0$ — a three-point contact. Thus $y = 0$ is actually an inflexional tangent to the cubic at Z. That is in case $\alpha = 0$, Z is an inflexion of the cubic. (Note that cuspidal cubic has an unique inflexion). Then in considering the quartic, i.e. the cubic together with the line component $x = 0$, we can see that the two cases are actually geometrically distinct.

E_7 Cuspidal cubic and cuspidal tangent



This is the case when the line component of the quartic is actually the cuspidal tangent. Let $x = 0$ be the line component, the cusp of the cubic be at Y and $x = 0$ be the cuspidal tangent. We can allow the cubic to pass through X and choose $y = 0$ be the tangent to the cubic there. Considering basis monomials of the cubic, we can reduce the equation to

$$az^3 + bx^2y + cxz^2 = 0$$

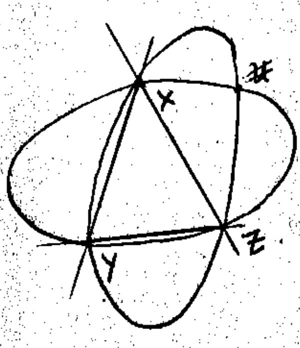
Note that neither a nor b is zero for irreducibility. c is not zero if we assume the inflexion of the cubic is not at X (or we can also take X to be the unique inflexion). Then choosing unit point we can make a = b = c = 1. Hence the quartic equation is

E_7  $x(z^3 + x^2y + xz^2) = 0$

Now, we have finished the enumeration of the combination of a cubic and a line. We shall go on to the second type - two conics.

(II) Two conics can have at most four intersections. Therefore we shall discuss the possibilities according to the number of intersections.

A_1^4 Two conics with 4 intersections



Let both of the conics go through the vertices X, Y, Z of the triangle of reference. Then the conics can be written as

$$ayz + bxz + cxy = 0 \quad \text{where } abc \neq 0$$

Divide through by c, we can write the conics as

$$a'yz + b'xz + xy = 0$$

choosing the unit point, we can have the equation of the quartic

A_1^4 $\qquad (\alpha yz + \beta xz + xy)(yz + xz + xy) = 0$

 $\alpha\beta \neq 0$ (irreducible conics), $\alpha \neq 1$, $\beta \neq 1$ and $\alpha \neq \beta$

N & S

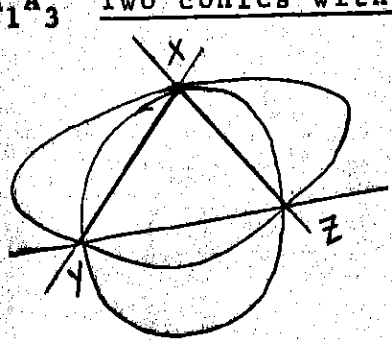
If $\alpha = 1$, at Y the line $x+z = 0$ would be tangent to both of the conics. Hence the two conics have only 3 intersections. This contradicts our assumptions. Therefore $\alpha \neq 1$.

Similar reasons apply to $\beta = 1$, because then at X, the line $y+z = 0$ would be tangent to both conics. Hence $\beta \neq 1$.

If $\alpha = \beta$, then at Z, the line $x+y = 0$ would be tangent to both conics. Hence $\alpha \neq \beta$.

It is obvious that for all non-excluded values of α, β the curve will indeed be A_1^4  .

$A_1^2A_3$ Two conics with three intersections



As we have discussed in A_1^4  , when $\alpha = 1$, $\beta = 1$ or $\alpha = \beta$, the two conics will have only three intersections. We shall choose the case when $\beta = 1$ to be our normal form. That is,

$A_1^2A_3$ $\qquad (\alpha yz + xz + xy)(yz + xz + xy) = 0$

$\alpha \neq 0$, $\alpha \neq 1$

N & S.

Note: If $\alpha = 1$, the two conics are exactly the same, hence repeated.

Now, before we go on to the next case, we want to go back and discuss the case A_1^4  again. Notice that in the given normal form the four nodes are not all fixed. This is a disadvantageous property for us in our future settings. So we shall now develop another normal form which will have all the four nodes fixed. Let them be fixed at X, Y, Z and P(1.-1,+1), say. This is that we want both of the conics to go through not only X, Y, Z but also P(1,-1,+1). Then the conics have the form $ayz + (a+1)\cdot xz + xy = 0$ (This conic is proper if and only if $a \neq 0$, $a \neq 1$). The quartic equation can therefore be written as

A_1^4  $(\alpha yz + (\alpha+1)\cdot xz+xy)(\beta yz + (\beta+1)\cdot xz+ xy) = 0$

$\alpha \neq 0,\ \beta \neq 0$

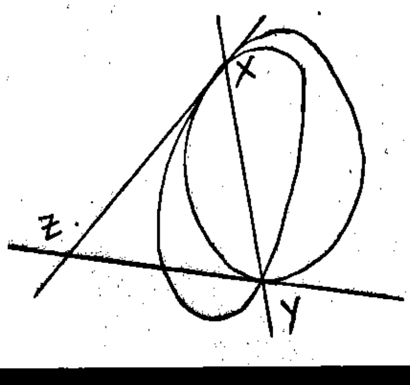
$\alpha \neq -1,\ \beta \neq -1,\ \alpha \neq \beta$

N & S

Note: If $\alpha = \beta$, the conics are repeated. This is the second non-linear normal form we have encountered.

There are two cases when the two conics have 2 intersections. One is when the multiplicity is three and one, the other is when the multiplicity is two and two.

A_1A_5 <u>Two conics with 2 intersections, multiplication three and one</u>



Let the three point contact be at X, and choose y = 0 to be the tangent to both of the conics at X. Let the other intersection be at Y and choose

0 to be the tangent to one conic at Y. This conic has the equation

$$z^2 + axy = 0 \qquad a \neq 0$$

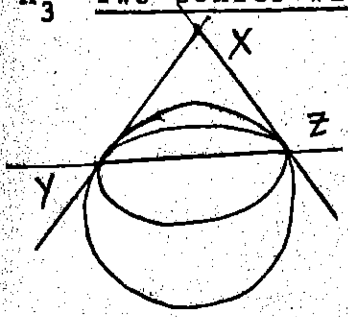
The other conic has the equation

$$z^2 + byz + cxy = 0$$

where $b \neq 0$ since the conics do not touch at Y. Also we want $a = c$ to avoid further points of intersection. Thus choosing unit point, we can make $b = c = a = 1$. And we have the normal form

$$A_1A_5 \qquad (z^2 + xy)(z^2 + yz + xy) = 0$$

A_3^2 Two conics with two intersections, multiplicity two and two



Let the two intersections be at Y and A. This means that both of the conics pass through Y and Z, and they touch each other there. Also choose $y = 0$ and $z = 0$ to be the tangents to both of the conics at Z and Y respectively. Then the conics are of the form

$$x^2 + ayz = 0 \quad \text{where} \quad a \neq 0.$$

And the quartic is $(x^2 + ayz)(x^2 + byz) = 0 \quad a \neq 0, b \neq 0.$ If $a = b$, we then have repeated conics. Hence $a \neq b$.

ter choosing unit point, we can make a = 1. The normal form is

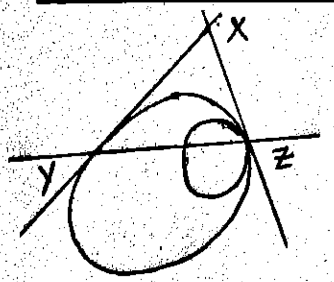
A_3^2 $\quad (x^2 + yx)(x^2 + \alpha yz) = 0 \qquad \alpha \neq 1$

$\alpha \neq 0$

N.S.

Next, we come to the case

A_7 Two conics with one intersection



Let the four-point-contact be at Z and take y = 0 to be the tangent to both of the conics at Z. Let one of the conics pass through Y and take z = 0 to be the tangent to that conic at Y. The conic is of the form

$$x^2 + ayz = 0, \quad a \neq 0 \qquad \text{(i)}$$

The other conic is of the form

$$bx^2 + cy^2 + exy + fyz = 0 \quad b \neq 0 \qquad \text{(ii)}$$

We can see, by substituting (i) into (ii), that for four-point-contact we must have

$$f - ab = 0 \text{ and } e = 0.$$

Hence the quartic is of the form

$$(x^2 + ayz)(bx^2 + cy^2 + abyz) = 0$$

or $$(x^2 + ayz)(x^2 + dy^2 + ayz) = 0, \text{ where } d = \frac{c}{b}.$$

By choosing λ in $z \to \lambda z$, we have

$$(x^2 + yz)(x^2 + yz + dy^2) = 0.$$

The general collineation taking $x^2 + yz = 0$ to itself and X to itself is

$$x \to +nkx - mky$$
$$y \to k^2 y$$
$$z \to n^2 z - m^2 y + 2nmx.$$

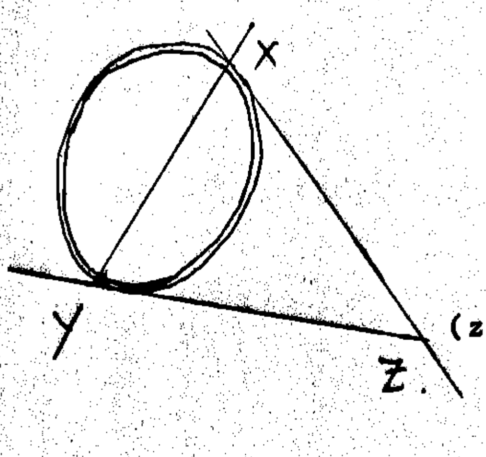
This is just multiplying $x^2 + yz$ by n^2k^2. So the other conic becomes

$$x^2 + yz + \frac{d}{n^2k^2} k^4 y^2 = 0$$

Hence we can choose k and n such that $\frac{dk^2}{n^2} = 1$. And the equation of the quartic becomes

A_7  $(x^2 + yz)(x^2 + yz + y^2) = 0$

Lastly, we have the case of two exactly the same conics— two repeated conics. Let the conics pass through X and Y, and $y = 0$ and $x = 0$ be the tangents to the conics at X and Y respectively. We have the normal form



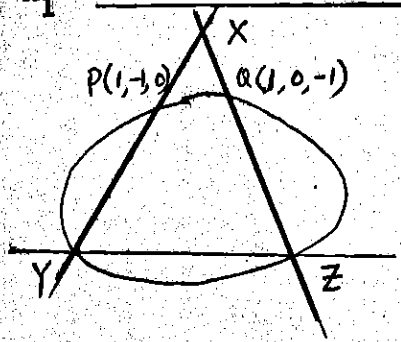
$(z^2 + xy)^2 = 0$ 

Notice that the singularities in this case are no longer isolated.

(III) A Conic and two lines

No doubt that two distinct lines will cut each other at exactly one point, and each of them can have at most two intersections with the conic. Hence again we will discuss the possibilities of singularity type according to the number of intersection these two lines make with the conic.

A_1^5 Conic and two chords



This is the case when the two lines cut the conic at four distinct points and their own intersection is off the conic. Choose $y = 0$ and $z = 0$ to be the two lines. Let the conic pass through Y and Z. Then it is represented by the equation

$$(*) \qquad ax^2 + byz + cxz + dxy = 0$$

Observe that $a \neq 0$, since X is not on the conic,

and $b \neq 0$, since the conic should be irreducible,

and $c \neq 0$, since $y = 0$ is not tangent to conic at Z,

and $d \neq 0$, since $z = 0$ is not tangent to conic at Y.

Hence choosing unit point, we can make $a = c = d = 1$. We have the normal form of

A_1^5  $yz(x^2 + \alpha yz + xz + xy) = 0 \qquad \alpha \neq 0, \alpha \neq 1$

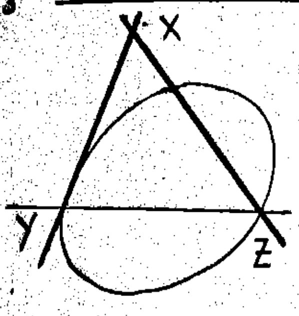
N & S

Notice that if $\alpha = 1$, the conic becomes two lines

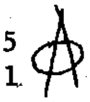
$$(x + z)(x + y) = 0.$$

The five nodes here are all fixed at X, Y, Z, P(1,-1,0) and Q(1,0,-1).

$A_1^3A_3$ Conic, a chord and a tangent

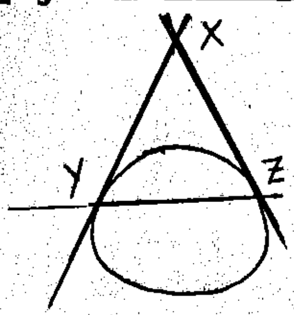


The two lines cut the conic one at two distinct points and the other at two coincident points. The intersection of the lines is still off the conic.

As discussed in equation (*) in A_1^5  case, but this time we have $z = 0$ tangent to the conic at Y. That is $d = 0$ in (*). Then the normal form of the quartic is

$A_1^3A_3$  $yz(x^2 + yz + xz) = 0$

$A_1A_3^2$ Conic and two tangents

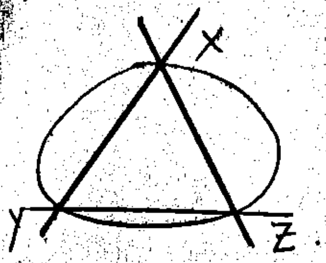


Still keeping the intersection of the lines off the conic, this time we have both of the lines cutting the conic at two coincident points. This is the case when we let $c = d = 0$ in equation (*) in A_1^5  . We have the normal form

$A_1A_3^2$  $yz(x^2 + yz) = 0$

Now, if the intersection of the two lines is actually on the conic, then we have two more cases.

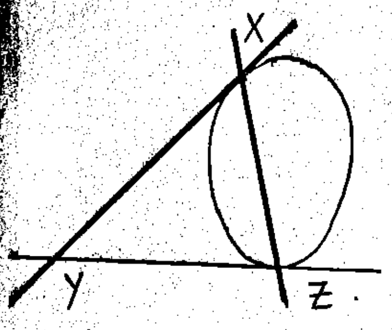
$A_1^2D_4$ Conic and two chords meeting on the conic



This is the case when we let $a = 0$ in equation (*) in A_1^5 . The normal form can be written as

$A_1^2D_4$ $yz(yz + xz + xy) = 0$

A_1D_6 Conic, chord and tangent meeting on conic

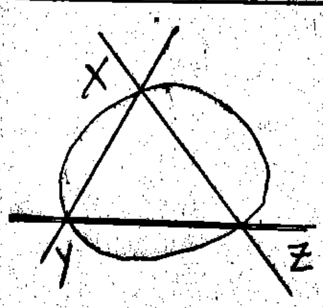


Let the conic pass through X and Z and $z = 0$ be the tangent to the conic at X. Also let $y = 0$ be the chord of the conic cutting it at X and Z. Choose $x = 0$ to be the tangent to the conic at Z. This situation is quite similar to $A_1A_3^2$ except that this time the line components are the chord and the tangent, and the intersection of the lines is on the conic. The quartic is of the form

A_1D_6  $yz(y^2 + xz) = 0$

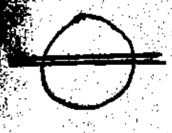
The following two cases are when the two lines are coincident. Note that the singularities are no longer isolated.

Conic and repeated chord



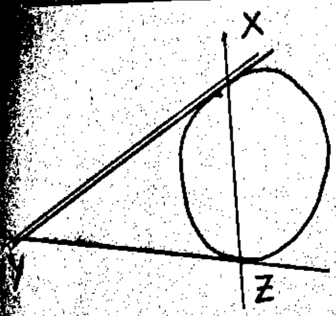
Let the conic pass through X, Y and Z, and $x = 0$ be the repeated chord cutting the conic at Y and Z.

have the equation



$$x^2(yz + xz + xy) = 0$$

Conic and repeated tangent



Let the conic pass through Y and Z, and x = 0, z = 0 be tangent to conic Z and X. Also let z = 0 be the repeated tangent components of the quartic.

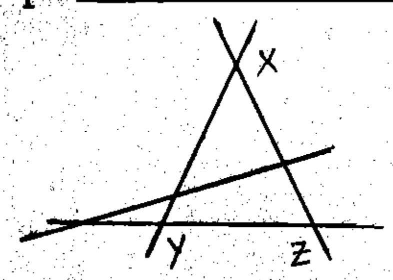
The equation for the quartic is

$$z^2(y^2 + xz) = 0$$

(IV) Four lines

We know that every n distinct lines will have at most $\binom{n}{2}$ intersections. If the lines are all distinct, then four lines can have at most six intersections.

A_1^6 Four general lines



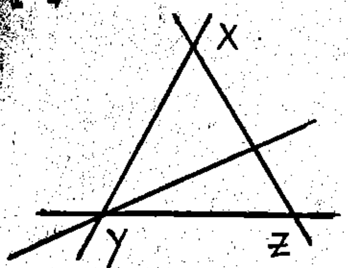
This is the case when the four lines are distinct and no three of them are concurrent. Choose x = 0, y = 0, z = 0 to be three of the lines. And a general line not passing through X, Y, Z can be written as

$$ax + by + cz = 0 \qquad a \neq 0 \quad b \neq 0 \quad c \neq 0$$

choosing unit point, we can write the quartic as

A_1^6  $$xyz(x + y + z) = 0$$

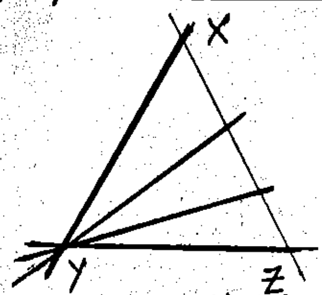
$A_1^3D_4$ <u>Three concurrent lines and one other</u>



Again choose $x = 0$, $y = 0$ and $z = 0$ to be three of the lines which are not concurrent. A general line passing through Y, but not X or Z can be written as $ax + cz = 0$, $a \neq 0$, $c \neq 0$. After choosing u, the equation of the quartic is

$A_1^3D_4$  $$xyz(x + z) = 0$$

$X_9(\tilde{E}_7)$ <u>Four concurrent lines</u>



Choose $x = 0$ and $z = 0$ be two of the concurrent lines. A general line passing through Y, but not X or Z is of the form $ax + cz = 0$, $a \neq 0$, $c \neq 0$. We want two <u>distinct</u> general lines of this form. Then after choosing unit point, the quartic is of the form

$X_9(\tilde{E}_7)$  $$xz(x + z)(x + \alpha z) = 0 \qquad \alpha \neq 1 \qquad \alpha \neq 0$$

NbS.

Note $\alpha \neq 0$, $\alpha \neq 1$, since we want the lines to be distinct. We can also have another normal form without choosing $x = 0$, $z = 0$ to be two of the lines.

distinct general lines all passing through Y, but not Z can be written as

$$ax^4 + bx^2z^2 + cz^4 = 0 \text{ where } b^2 \neq 4ac,\ a \neq 0,\ c \neq 0.$$

Choosing unit point, we can make a = c = 1. Hence the normal form is

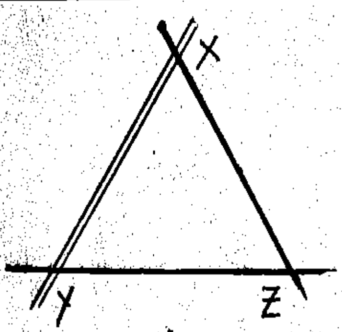
$$X_9(\tilde{E}_7) \quad x^4 + \alpha x^2 z^2 + z^4 = 0 \qquad \alpha \neq 2, \quad \alpha \neq -2$$

N b S.

An important feature of this quartic is that for different α, the singularity takes in general analytically distinct forms.

Now, we shall discuss the cases when some of the lines are repeated.

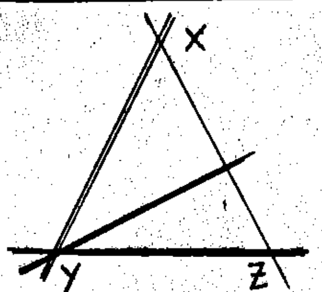
Three general non-concurrent lines, one repeated



Choose the three general lines to be x = 0, y = 0 and z = 0, and let z = 0 be the repeated one. We have the equation

$$xyz^2 = 0$$

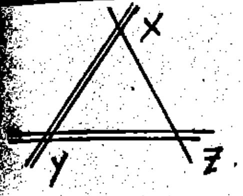
Three concurrent lines, one repeated



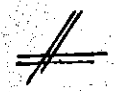
Choose x = 0 and z = 0 be two of the concurrent lines, and ax + cz = 0 be the general line passing through Y, but not X or Z. Let z = 0 be repeated. Choosing unit point, we have the equation

$$xz^2(x + z) = 0$$

Two pairs of repeated lines

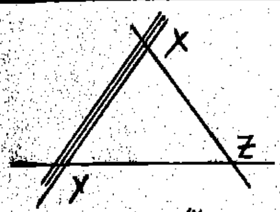


Let the two repeated lines be $x = 0$ and $z = 0$. The quartic equation is



$$x^2z^2 = 0$$

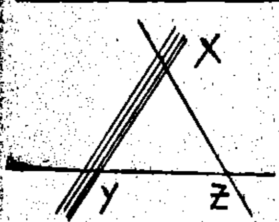
Three-folded line and one other



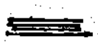
Choose $z = 0$ be the three folded line and $x = 0$ be the other. The equation is

$$xz^3 = 0$$

Four folded line



Choose $z = 0$ be the fourfold line. The equation is



$$z^4 = 0$$

This finishes our classification of quartics.

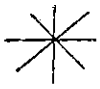
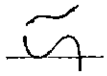
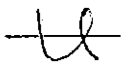
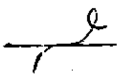
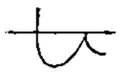
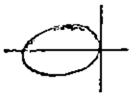
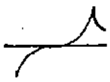
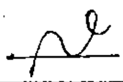
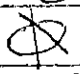
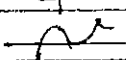
§ 2.4 The List of Classified Singularity Types

IRREDUCIBLE QUARTIC CURVES

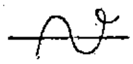
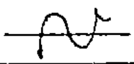
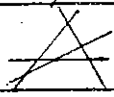
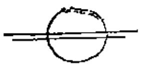
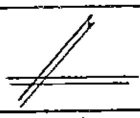
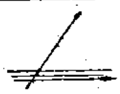
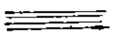
ities	Singularity types	No. of Moduli	dimension of the Stratum
	Non-Singular	6	15
	A_1	5	14
	A_2	4	13
	A_3	3	12
	A_4	3	11
	D_4	2	11
	A_5	1	10
	A_6	0	9
	E_6 (two orbits)	0	9
		0	8
	D_5	1	10
2	A_1 A_1	4	13
	A_1 A_2	3	12
	A_2 A_2	2	11
	A_1 A_3	2	11
	A_2 A_3	1	10
	A_1 A_4	1	10
	A_2 A_4	0	9
3	A_1^3	3	12
	A_1^2 A_2	2	11
	A_1 A_2^2	1	10
	A_2^3	0	9

The Codimension of the orbits of these strata are all 6.
There are a total of 21 types in the irreducible quartic curves.

REDUCIBLE QUARTIC CURVES

Singularity Type	Picture	Codimension of Orbits	Number of Moduli
A_5		6	1
D_6		6	0
A_7		7	0
E_7		7	0
X_9 ($\tilde{E}_7$)		9	1
A_1 A_3		6	2
A_1 D_4		6	1
A_1 A_5		6	0
A_1 A_5		6	0
A_1 D_5 (2 orbits)		6	0
A_1 D_6		7	0
A_2 A_5		7	0
A_3 A_3		7	1
A_1^3		6	3
A_1^2 A_3		6	1
A_1^2 A_3		6	0
A_1^2 D_4		6	0
A_1 A_2 A_3		6	0
A_1 A_3^2		7	0

REDUCIBLE QUARTIC CURVES (CONT)

Singularity Type	Picture	Codimension of Orbits	Number of Moduli
A_1^4		6	2
A_1^4		6	2
$A_1^3\ A_2$		6	1
$A_1^3\ A_3$		6	0
$A_1^3\ D_4$		7	0
A_1^5		6	1
A_1^6		6	0
Not isolated		7	0
"		8	0
"		8	0
"		9	0
"		9	0
"		10	0
"		10	0
"		12	0

A total number of 34 types in the Reducible quartic.

The number of singularity types altogether in the quartic classification is therefore 55.

Corrections

p. 2 The vertex O removed.
p. 28 Bottom third line " $x^2z^2 + 2xy^2z + \dots$"
p.107 Seventh line "$x^2z^2 + 2xy^2z + \dots$" ; Gens. for A_1 are $x^4, y^4, xy^3, y^3z, xy^2z, y^2z^2$
p.111 Twelfth line " $\mathcal{E}(n)$ "
p.115 Fourth line "quotient"
Fifth line "represented"
Bottom second line "$g \in \mathcal{E}(n)$"
Bottom third line "$f \in \mathcal{E}(n)$"
p.116 First line "(versal= k-transversal)"
p.118 Bottom fifth line ".... in $M \times V_f$ the singularity $f' = f +$ terms..."
p.119 Bottom line ".... ζxy^2z "
p.120 Third line "....+ ζxy^2 " $(f_{Z,x}, f_{Z,y})'$
Sixth line "....+ ζxy^2 "
Eleventh Line "......, xy^2 form aabase ....."
p.122 Tenth line " y, y^2, y^3, xy^2 form a base...."
Nineth line "$2x + 2y^2 - 2xy^2 \in J_{f_z}$ by $f_{z,x}$ "
p.123 Bottom line "... $3\varepsilon yz^2 + 2\zeta y^2z = 0$ "
p.124 Third line " $yz \in J_{f_x} + m^3$......"
p.132 Bottom third line "..... $k_i(y-1)(z+1)^3$ "
Bottom line "...$+r_i(z+1)^2 + s_i(y-1)^2(z+1)^2$ "
p.133 Third line " $\sum_i [a_i + \dots$ "
Fourth line " $+(-4b_i + \dots$"
Fifth line " ..... $+2s_i)z]$ "
Eleventh line " $-4b_i + \dots$"
p.134 Eleventh line "......$+ (\beta(\alpha_0+1) + \alpha(\beta_0+1) + \alpha\beta)x^2z^2$ "
p.146 Fourth line "Proof : We can write (M^m, 0)....."
p.150 Foruth line " ......$j_1^k(x_0, u, v) \pitchfork \mathcal{R}$-orbit of h "
p.156 Fifteenth line " ....iff $\mathcal{J}$ "
p.159 Third line " $\gamma = \delta = \varepsilon = \zeta = 0$ "
p.160 Bottom third line "..... $+ \beta_0'' x^3z$ where "
163 Fourteenth line "p.118, when $\mathcal{J}$ is ...."
p.163 Bottom second line, omit " Let $N \times U \times V \subset \mathcal{G}$, and "
put in " Working within neighborhood $\mathcal{G}$, let
$\eta : N \times U \to \mathbb{C}$ "
p.164 First line "$F: (N \times U \times V, 0) \to \mathbb{C}$ "
p.167 Bottom fifth line "$\mathcal{D}$: (UxV...... "
Bottom third line " (u,v,w) $S_\chi(G)$ iff $\mathcal{D}$ (UxV............ $\in S_\chi(F)$ "
p.170 Fourth line " As a versal...."
Eleventh line " $+ \gamma_0 xy^3 + \varepsilon y + \zeta y^2$ "

Corrections (cont.)

p.173 Nineth line " $\left.\begin{array}{l} y_i \to x \\ x_i \to x \end{array}\right\}$ where $x \in X$ "

p.174 Bottom fourth line, omit "(Cross Cap) "

p.178 Bottom Fourth line " ....: $u_2 = w_2 = 0$ "

p.179 Eighth line "argument on p. 166, we have "

p.180 First line "c_1 :($U_1 \times U_2 \times M$....."

Second line " c_2 : ($U_1 \times U_2 \times M$ ......"

p.183 Bottom third line " On p. 167 in Section 6.3 "

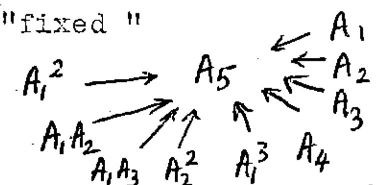
p.188 Sixth line ".....+ ζy^2 "

p.195 Seventh line " (see p.166) "

p.198& p.199 Wrong order. (Binding mistake)

p.201 Fourth line "fixed "

p.205 Diagram "



"

p.207 Thirteenth line "(**) $\frac{1}{3}x^3 - \frac{1}{3}\left(\frac{x^2+\beta}{\delta}\right)^3 + \ldots\ldots$"

Eleventh line " $\frac{(x^2+\beta)^2}{\delta^2} + \gamma + \delta x = 0$ "

p.213 Third line " This implies δ big, a contradiction (δ small)"

p.223 Bottom second line " Consider χ_1^0 case first "

p.225 Third line " a_2: $U_1 \times U_2 \times M \times W_1 \times W_2 \to$........."

Fourth line " a_2: $\pi_2' \circ \vartheta_2^{-1} \circ \pi_2 \circ \vartheta_2$ "

Eleventh line " $a_1 \times a_2$: $U_1 \times U_2 \times M \times W_1 \times W_2 \to$........"

Bottom third line "..... hence a submersion at "

p.238 Thirteenth line " (iv).....$-3\delta y_3^2 \ -4\varepsilon y_3^3$"

p.251 Eighth line " So the condition for A_4 is $\left[\delta\left(-\frac{\alpha}{2\beta}\right) - \frac{\alpha^2}{4\beta^2}\right] \neq 0$"

p.259 Fourth line " Similarly $n_3 - 3\alpha \geq 0$ and ...."

p.260 Bottom second line "...... $A_1^3A_3$, $A_1^2D_4$, A_1^6 "

CHAPTER 3

3.1 Orbits

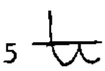
Let us start by recalling the definition of projective equivalence. Consider the group action

$$GL(3,\mathbb{C}) \times \mathbb{C}^{15}-\{o\} \to \mathbb{C}^{15}-\{o\}$$

$$(\theta,f) \to \theta.f = f o \theta^{-1}$$

The action is on the left by substitution. We can regard $f \in \mathbb{C}^{15}-\{o\}$ as a map $\mathbb{C}^3 \to \mathbb{C}$ and $\gamma \in G$ as a linear map $\mathbb{C}^3 \to \mathbb{C}^3$. Then $\theta.f$ is the map $f o \theta^{-1}$. Orbits are cones.

Note also that we can regard the space of quartics as $\mathbb{C}P^{14}$ and the group acting as $PGL(3,\mathbb{C})$. Then the orbits of the former action can be regarded as cones on orbits of this latter action.

In our classification, in Chapter 2, we can divide the strata into three kinds, according to the number of orbits they have. First, if the normal form of the stratum has no modulus, then it consists of only one orbit. For example, A_7, A_1^6, etc. (full list see §2.4). Second, if the normal form has moduli, then the stratum is composed of infinite number of orbits. For example A_1^4, A_1A_3, etc (see §2.4). Thirdly, there are cases when the modulus is only allowed to have a finite (but more than one) number of values. In fact, we found only two cases in which this situation happens. They are E_6 and A_1D_5  and they have precisely two orbits corresponding to the two allowed modulus value of 0 and 1.

The main interest in this section is to set up some foundation for proving in general that the strata we have chosen are manifolds. The exact proof will be given in Chapter 5.

Now, let us consider the case of single-orbit strata. The following well-known theorem (Martin Golubitsky and Victor Guillemin 1973) and the algebraic nature of our action enable us to show directly that strata of this type are manifolds.

Theorem 3.1.1.

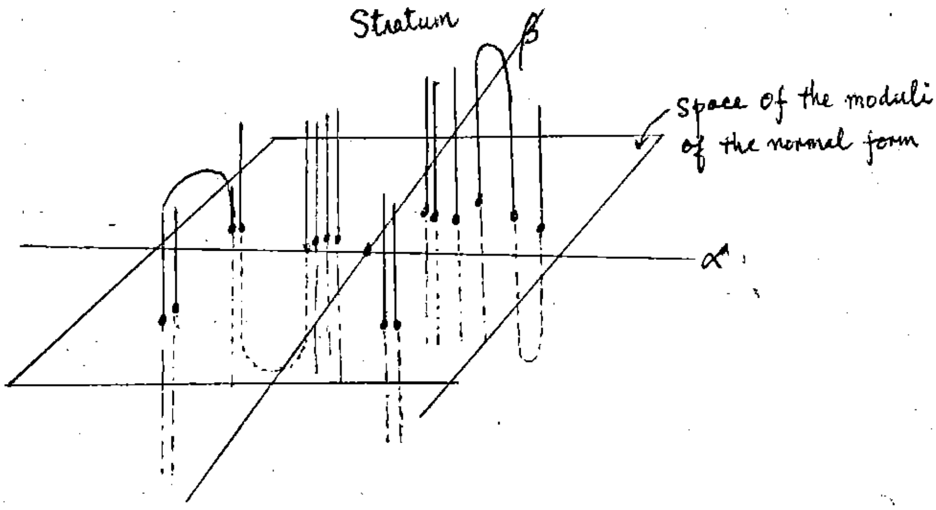
Let G be a Lie group acting on smooth manifold M such that $G \times M \to M$ is smooth. Then the orbits are immersed submanifolds

And since both the group $GL(3,\mathbb{C})$ and M are semi-algebraic sets, and $GL(3,\mathbb{C})$ a Lie group and M a smooth manifold, we can be sure that the induced topology on the orbit from M under the smooth action is the same as the induced manifold topology on the orbit. Then the immersed submanifold would actually be a submanifold of M.

Hence in our cases all single-orbit strata are manifolds and no more is needed to say about these cases. The real problem comes up with the second and third cases when there are more than one orbit. The strata now are unions of orbits. Although each orbit is a manifold, we still don't know whether the whole stratum is also a manifold or not. It is obvious in order to show the stratum is a manifold, further properties are needed (see Section 3.4). From now on when we refer to stratum, we refer to those having moduli in their normal forms, i.e. they have more than one orbits.

The following picture shows how the orbits of a stratum might cut the normal form:

For illustrative purpose, we assume a 2-dimensional space, say (α, β)- space, of moduli of the normal form.



The conditions on the moduli would be some excluded curves on this space of moduli.

Note that the orbits might cut the normal form in more than one value of the moduli. But this number of intersections is finite. This is based on the fact that in our process of finding the normal forms we had already made use of all the degrees of freedom of the group $GL(3,\mathbb{C})$ in each case. So, if two values on the space of moduli of the normal form are on the same orbit, then any transformation taking one to another must be of the form $(x,y,z) \to (\lambda x,\mu y,\nu z)$ where $\lambda,\mu,\nu \in \mathbb{C}$ and since we demand the same form again, on equating the coefficients we can see that λ,μ,ν can only be allowed finite number of values. For example, look at A_1A_3(irr.).

Suppose $y^4 + x^2y^2 + x^2yz + \alpha xy^2z + \beta xy^3 = 0$ (1)

$\alpha \neq 2$

$\alpha \neq -2$

$\beta^2-\alpha\beta+1 \neq 0$

and $y^4 + x^2z^2 + x^2yz + \alpha' xy^2z + \beta' xy^3 = 0$ (2)

are projective equivalent. The transformation taking one to the other must be of the form $(x,y,z)\to(\lambda x,\mu y,\nu z)$ making (1) into

$$\lambda^4y^4 + \lambda^2\nu^2x^2z^2 + \lambda^2\mu\nu x^2yz + \alpha\lambda\mu^2\nu xy^2z + \beta\lambda\mu^3xy^3 = 0 \quad (3)$$

Equating coefficient of (3) with (2), we have

$$\lambda^4 = 1, \quad \lambda^2\nu^2 = 1, \quad \lambda^2\mu\nu = 1$$
$$\to \lambda = \pm 1, \quad \nu = \pm 1, \quad \mu = \pm 1$$

So $\alpha' = \pm\alpha$, $\beta' = \pm\beta$.

And the transformations are $(x,y,z)\to(\pm x,\pm y,\pm z)$. Therefore there are only finite number of transformations which can take (α,β) to some values (α',β') on the normal form again. If the normal form is in some way symmetric, the transformation may be accompanied by a permutation as well. The number of permutation is of course finite.

For example, let us look at the trinodal irreducible case.

Suppose $y^2z^2+x^2z^2+x^2y^2 - xyz(\alpha x+\beta y+\gamma z) = 0$, $\alpha^2,\beta^2,\gamma^2 \neq 4$ (1)

and $y^2z^2+x^2z^2+x^2y^2 - xyz(\alpha' x+\beta' y+\gamma' z) = 0$, $\alpha'^2,\beta'^2,\gamma'^2 \neq 4$ (2)

are projectively equivalent.

Since both of them are trinodal and with nodes at X, Y, Z, any transformation taking one to the other must take

{X,Y,Z} to {X,Y,Z}. Hence the transformation is a composite of $(x,y,z)\to(\lambda x,\mu y,\nu z)$ with a permutation. The transformation $(x,y,z)\to(\lambda x,\mu y,\nu z)$ takes (1) into

$$\mu^2\nu^2y^2z^2 + \lambda^2\nu^2x^2z^2 + \lambda^2\mu^2x^2y^2 - \lambda\mu\nu xyz(\alpha\lambda x+\beta\mu y+\gamma\nu z) = 0 \tag{3}$$

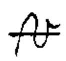
If we equate the coefficients of (3) with (2), we have

$$\mu = \pm\nu, \qquad \lambda = \pm\mu .$$

Taking $\lambda = 1$, we have $\mu = \pm 1$, $\nu = \pm 1$.
Hence the transformation is $(x,y,z)\to(x,\pm y,\pm z)$.
Therefore $\alpha' = \pm\alpha$, $\beta' = \pm\beta$, $\gamma' = \pm\gamma$.

Since the number of permutation also finite, the number of intersection of the orbits with the normal form must be finite. Therefore in all cases the number of transformation that can serve our purpose is finite. This implies the number of intersection of the orbit with the normal form is finite.

3.2 Normal Forms are Manifolds

From the classification in Chapter 2, we notice that all except two cases the spaces formed by the moduli of the normal forms are linear. The two non-linear cases are A_1^4  and $A_1^3A_2$ . Now we want to show that all these spaces are submanifolds of $\mathbb{C}^{15}-\{o\}$. For the linear ones, since an open subset of an affine space is a submanifold, this is obvious. But for the non-linear cases, this takes up more work. The following is a commonly-known theorem and we shall omit the proof. (Golubitsky and Guillemin 1973)

Theorem 3.2.1.

Let X, Y be manifolds and $\phi: X \to Y$ be a 1:1 proper immersion. Then $\phi(X)$ is a submanifold of Y.

Note (i) ϕ is proper if for every compact subset K in Y, $\phi^{-1}(K)$ is a compact subset of X;

(ii) it is easy to see that if ϕ^{-1} is continuous, then ϕ is proper.

We shall start with A_1^4 . It is fixed-nodes case we are discussing. It has the normal form

$$(\alpha yz-(\alpha+1)xz+xy)(\beta yz-(\beta+1)xz+xy) = 0.$$

That is

$$x^2y^2 + (\alpha+\beta)xy^2z + (\alpha\beta)y^2z^2 + (\alpha+1)(\beta+1)x^2z^2$$

$$- (\alpha+\beta+2\alpha\beta)xyz^2 - (\alpha+\beta+2)x^2yz = 0$$

with the restriction $\alpha \neq 0$, $\beta \neq 0$, $\alpha \neq -1$, $\beta \neq -1$ and $\alpha \neq \beta$. The space formed by the normal form can be identified with the subset $(\alpha+\beta, \alpha\beta, (1+\alpha)(1+\beta), \alpha+\beta+2\alpha\beta, 2+\alpha+\beta)$ of $\mathbb{C}^5$. Now let us consider the function

$$F : \frac{\mathbb{C}^2 \setminus E'}{\mathbb{Z}_2} \to \mathbb{C}^5$$

(Because of symmetry of the normal form $\mathbb{Z}_2$ is used to identify (α,β) with (β,α) in the set $\mathbb{C}^2 \setminus E'$)

$$(\alpha,\beta) \rightarrow (\alpha+\beta,\ \alpha\beta,\ (1+\alpha)(1+\beta),\ \alpha+\beta+2\alpha\beta,\ 2+\alpha+\beta)$$

where $E' = \{(\alpha,\beta) : \alpha = 0,\ \beta = 0,\ \alpha = 1,\ \beta = -1 \text{ or } \alpha = \beta\}$. We can easily see that $\mathbb{C}^2\backslash E'/\mathbb{Z}_2$ is a manifold and F is injective. Let us check whether F is an immersion. The Jacobian matrix is

$$\begin{pmatrix} 1 & 1 \\ \beta & \alpha \\ (\beta+1) & (\alpha+1) \\ (1+2\beta) & (1+2\alpha) \\ 1 & 1 \end{pmatrix}$$

Since we have $\alpha \neq \beta$, $(1, 1)$ and (β,α) is linearly independent. Hence F is an immersion.

$$F^{-1} : (A,B,C,D,E) \mapsto \left(\frac{A+\sqrt{A^2-4B}}{2},\ \frac{A-\sqrt{A^2-4B}}{2}\right).$$

It is also easy to check that F^{-1} is continuous. Hence F is proper. This implies that the space formed by the moduli of the normal form is a submanifold of $\mathbb{C}^5$, and hence of $\mathbb{C}^{15}-\{o\}$.

For $A_1^3A_2$, the normal form is

$$xy^2z + xyz^2 + x^2z^2 + 2\alpha x^2yz + \alpha^2x^2y^2 = 0$$

with restriction $\alpha \neq 0$, $\alpha \neq 1$.

The spaces formed can be identified with the points $(2\alpha,\alpha^2)$

in $\mathbb{C}^2$ with the exclusion $\alpha \neq 0, 1$.

Consider function $G : \mathbb{C} \setminus E'' \to \mathbb{C}^2$

$$\alpha \mapsto (2\alpha, \alpha^2)$$

where $E'' = \{\alpha = 0,1\}$.

This is obviously an injective immersion and also G^{-1} is continuous. Hence the space $(2\alpha,\alpha^2)$ forms a submanifold in $\mathbb{C}^2$ as well.

$$G^{-1} : (A,B) \mapsto \frac{A}{2} \quad \text{where} \quad \frac{A^2}{4} = B.$$

3.3 Tangent Space of orbits of strata

Since orbits are manifolds (see P.82) we could also consider their tangent spaces. The theorem in this section enables us to find the generators for these tangent spaces explicitly. This will give us more convenience in considering local properties in our later context.

Let $M = \mathbb{C}^{15}-\{o\}$ and $G = GL(3,\mathbb{C})$. Also let $f \in M$ and Gf be the orbit of f under the given group action. We know that Gf is a manifold. Now we want a set of vectors in M which span the tangent space to Gf at f. First consider the differentiable map $\phi_f : G \to Gf$

$$m \mapsto m.f$$

We can easily see that the tangent map

$$T\phi_f(e) \quad TeG \to T_fG.f, \qquad e = I_3$$

is a surjection.

Therefore to obtain the generators for this tangent space $T_f Gf$ of the orbit Gf at f we can take the images of the generators of TeG under $T\phi_f(e)$.

Now let us consider maps $C_1 : \mathbb{C} \to G$

$$t \to \begin{pmatrix} 1+t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$C_2 : \mathbb{C} \to G$ ..... $C_5 : \mathbb{C} \to G$ ..... $C_9 : \mathbb{C} \to G$

$$t \to \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad t \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+t & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad t \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+t \end{pmatrix}$$

defined near 0.

These define tangent vectors to G at I_3.

Consider the composition

$$\mathbb{C} \xrightarrow{C_i} G \xrightarrow{\phi_f} Gf$$

$$t \overset{C_1}{\longmapsto} \begin{pmatrix} 1+t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\phi_{f1}}{\longmapsto} f((1+t)x,y,z)$$

$$t \overset{C_2}{\longmapsto} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\phi_{f2}}{\longmapsto} f(x+ty,y,z)$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$t \overset{C_9}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+t \end{pmatrix} \overset{\phi_{f9}}{\longmapsto} f(x,y,(1+t)z)$$

where $\phi_{fi} = \phi_f \big|_{\text{image } C_i}$

The above curves, give tangent vectors to Gf at f which are exactly the images of the tangents to G under $T\phi_f(e)$.

Let $\psi_i = \phi_{fi} \circ C_i(t) : \mathbb{C} \to Gf$

$$\psi_i(o) = f$$

The image of unit tangent vector to $\mathbb{C}$ at 0 under $d\psi_i$ is

$$\left.\frac{d\psi_i}{dt}\right|_{t = 0}$$

Using the chain rule, we get, for example when i = 1.

$$\frac{\partial f}{\partial x} \cdot \frac{\partial(1+t)x}{\partial t} = \frac{\partial f}{\partial x} \cdot x.$$

Correspondingly, we can also get

$$\frac{\partial f}{\partial x}\, y \text{ when } i = 2$$

$$\frac{\partial f}{\partial z}\, z \text{ when } i = 9, \text{ as tangent vectors to } Gf \text{ at } f.$$

Now since $t \mapsto \begin{pmatrix} 1+t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ etc. are basis vectors for T_eG,

$$x\frac{\partial f}{\partial x} \ldots z\frac{\partial f}{\partial z} \text{ span } T_f(Gf)$$

So we have the theorem

Theorem 3.3.1 The tangent space of the orbit of f at f is spanned by vectors

$x\frac{\partial f}{\partial x}$, $y\frac{\partial f}{\partial x}$, $z\frac{\partial f}{\partial x}$, $x\frac{\partial f}{\partial y}$, $y\frac{\partial f}{\partial y}$, $z\frac{\partial f}{\partial z}$, $x\frac{\partial f}{\partial z}$, $y\frac{\partial f}{\partial z}$, $z\frac{\partial f}{\partial z}$,

§3.4 "Good" Normal Forms

For the purpose of proving the strata are manifolds, we should also realize another property of the normal forms.

Def. Any normal form $\mathcal{N}$ of a stratum which is a submanifold of $\mathbb{C}^{15}-\{o\}$ and has the following properties is called a "good" normal form of the stratum.

(i) $\mathcal{N}$ intersects every orbit of the stratum finite number of times.

(ii) Let $f \in \mathcal{N}$. Then $T_f(G.f) \cap T_f(\mathcal{N}) = \{o\}$.

(Tangent space to the orbit at any point of intersection) $\cap$ (tangent space to $\mathcal{N}$ at the same point of intersection) $= \{o\}$ —— (zero vector).

We claim that all the normal forms we get from our classification of quartics in Chapter 2 are actually "good normal forms" of each of their strata. This result is checked to be true by going through every case using the following method.

Since we have already shown that the moduli spaces of all the normal forms are actually manifolds in Section 3.2, the only thing left behind for us to check is the condition (ii). The method used is best shown by example. Let us take

$$A_1^3(\text{irr})\ f(x,y,z) = x^2y^2 + x^3z^2 + y^2z^2 - \alpha x^2yz - \beta xy^2z - \gamma xyz^2$$

$$\alpha^2, \beta^2, \gamma^2 \neq 4$$

Let us choose the intersection point to be at $(\alpha_o, \beta_o, \gamma_o) = v$ then

$$(f_v)_x = 2xy^2 + 2xz^2 - 2\alpha_o xyz - \beta_o y^2z - \gamma_o yz^2$$

$$\left(f_v\right)_y = 2x^2y + 2yz^2 - \alpha_o x^2z - 2\beta_o xyz - \gamma_o xz^2$$

$$\left(f_v\right)_z = 2x^2z + 2y^2z - \alpha_o x^2y - \beta_o xy^2 - 2\gamma_o xyz$$

The tangent space of the orbit $G.f_v$ at f_v is as in the table

		x^4	y^4	z^4	x^3y	xy^3	x^3z	xz^3	y^3z	yz^3	x^2yz	xy^2z	xyz^2	x^2y^2	x^2z^2	y^2z^2
λ_1	xf_x										$-2\alpha_o$	$-\beta_o$	$-\gamma_o$	2	2	
λ_2	yf_x					2			$-\beta_o$			$-2\alpha_o$	2			$-\gamma_o$
λ_3	zf_x							2		$-\gamma_o$		2	$-2\alpha_o$			$-\beta_o$
λ_4	xf_y				2		$-\alpha_o$				$-2\beta_o$		2		$-\gamma_o$	
λ_5	yf_y										$-\alpha_o$	$-2\beta_o$	$-\gamma_o$	2		2
λ_6	zf_y							$-\gamma_o$		2	2		$-2\beta_o$		$-\alpha_o$	
λ_7	xf_z				$-\alpha_o$		2				$-2\gamma_o$	2		$-\beta_o$		
λ_8	yf_z					$-\beta_o$			2		2	$-2\gamma_o$		$-\alpha_o$		
λ_9	zf_z										$-\alpha_o$	$-\beta_o$	$-2\gamma_o$		2	2

The tangent space to the normal form at $(\alpha_o, \beta_o, \gamma_o) = v$ is spanned by x^2yz, xy^2z and xyz^2.

Therefore if we suppose

$$\lambda_1 xf_x + \ldots\ldots \lambda_9 zf_z = \lambda x^2yz + \mu xy^2z + \xi xyz^2$$

We must show that $\lambda = \mu = \xi = 0$.

Now refer back to the table

$$x^3y : 2\lambda_4 - \alpha_o\lambda_7 = 0$$

$$x^3z : -\alpha_o\lambda_4 + 2\lambda_7 = 0 \qquad \lambda_4 = \lambda_7 = 0 \text{ since } \alpha_o^2 \neq 4$$

Similarly

$$xz^3 : 2\lambda_3 - \gamma_o\lambda_6 = 0$$

$$yz^3 : -\gamma_o\lambda_3 + 2\lambda_6 = 0 \qquad \lambda_3 = \lambda_6 = 0, \ \ \gamma_o^2 \neq 4$$

$$xy^3 : 2\lambda_2 - \beta_o\lambda_8 = 0$$

$$y^3z : -\beta_o\lambda_8 + 2\lambda_2 = 0 \qquad \lambda_2 = \lambda_8 = 0, \ \beta_o^2 \neq 4$$

Hence we have

$$x^2y^2 : 2\lambda_1 + 2\lambda_5 = 0$$

$$x^2z^2 : 2\lambda_1 + 2\lambda_9 = 0$$

$$y^2z^2 : 2\lambda_5 + 2\lambda_9 = 0 \Rightarrow \lambda_1 = \lambda_5 = \lambda_9 = 0$$

Thus we have $\lambda = \mu = \xi = 0$.

Therefore only the zero vector is in the intersection. Hence the normal form is a "Good" one . All the other normal forms has this property, though some might be more difficult to show than the other, for example the case of non-linear normal forms, A_1^4 . We shall exclude the rest of the calculations from this context.

CHAPTER 4

TRANSVERSALS

4.1. Transversal (Slice)

In order to prove all the Strata Σ's are manifolds, we also need mechanisms called "transversals" (slice). They have properties which are useful to use even in proving regularities.

Definition. A transversal (slice) $\mathcal{J}$ at a point $f \in M = \mathbb{C}^{15}-\{0\}$ is a submanifold of M with $f \in \mathcal{J}$ where the tangent space to $\mathcal{J}$ at f is complementary to the tangent space of the orbit of f at f. That is,

$$T_f\mathcal{J} \oplus \underline{T_f(G.f)} = T_f M \qquad (*)$$

It follows from (*) that the map

$$\phi_f : G \to M = \mathbb{C}^{15}-\{0\}$$

$$m \to m.f$$

is transverse at e = I to $\mathcal{J}$. Also, if we choose $f' \in \mathcal{J}$ close to f, then

$$\phi_{f'} : G \to M$$

is still transverse at e = I to $\mathcal{J}$. (Since transversality is an open condition). Hence, we have the orbit through f' also meets $\mathcal{J}$ transversally at f'. Furthermore, consider the mapping

$$G \times \mathcal{J} \to M$$

$$(m,g) \to m \cdot g$$

The tangent map of this takes

$$T_eG \times \{0\} \rightarrow T_f(G.f)$$

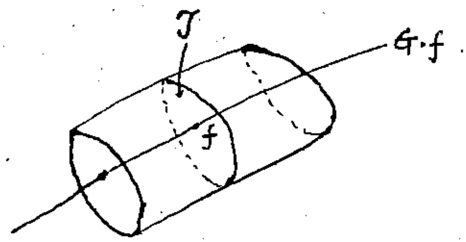
and $$\{0\} \times T_f\mathcal{J} \rightarrow T_f\mathcal{J}$$

And since $T_f(G.f) \oplus T_f\mathcal{J} = T_fM$, the mapping is a submersion at (e,f). This shows that given $f'' \in M$ near f, $\exists$ $m \in G$ close to e and $f' \in \mathcal{J}$ such that

$$m.f'' = f'$$

or in words, $G.\mathcal{J}$ "fills up a neighbourhood of f in M"

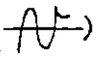
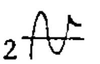
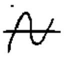
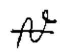
(Local picture)



So $\mathcal{J}$ intercepts all nearby orbits transversally.

Finally, note that in our examples, $\mathcal{J}$ will be semi-algebraic, and it follows that $\mathcal{J} \cap G.f$ is a finite collection of points. Thus if $\mathcal{J}$ is sufficiently small, $\mathcal{J} \cap G.f = \{f\}$.

Next, we are going to show how such a transversal in general can be found explicitly. The homogeneity property for orbits shows us that it is enough to look at only one point of the orbit $G.f_\lambda$ namely the point f_λ on the normal form f where $\lambda \in$ space of moduli $\bar{V}$ of f. Now we want to find a linear complementary space to the tangent space of the orbit $G.f_\lambda$ at f_λ in M. Because of the linearity this space is of course a submanifold and the tangent space is itself at any point. Hence we can choose this linear complementary space to be the transversal at f_λ. Since from Section 3·3 of Chapter 3 we know how to find the tangent space of the orbit at f_λ, it is a basic linear algebra calculation to find such a complementary space. In fact, we have found such transversals for all the normal forms and the generators of each of them are given in Table 4.2.1 in Section 2.

Note that in case when the normal form f is linear (in fact, all the normal forms are linear except two, A_1^4  and $A_1^3A_2$ ), from the fact that all the normal forms are "good", we can see that the space spanned by the moduli terms is "perpendicular" to the orbit $G.f_\lambda$. Therefore a complementary space can be found by "extending" this "perpendicular" space. More precisely, we can regard the moduli terms of the normal form f as part of the set of generators for the complementary space but when the moduli terms are not single monomials, we will have to use the general method e.g. A_3^2 For a reason which we will mention later, the cases A_1^4 , $A_1^3A_2$ , A_1^3 , $A_1^3A_3$ , A_1^5 , A_1^4  are treated separately in Section 5.3.

We can easily check that the result given in Table 4.2.1. are truly transversals. This amounts to checking transversality, that is to show transversals are transverse to the orbits in the strata. We do this by showing that the tangent vectors of the orbits together with the generating vectors of the tangent space of the transversal will span the whole of M. Actually what we have checked here is a stronger result,

$$T_f(G.f) \cap T_f(\mathcal{T}) = \{0\}$$

This is just a further step after checking the property of "Good" normal forms. The following are three examples:

Examples 4.1.1.

A_3 $\quad f(x,y,z) = x^2z^2 + y^4 + x^3y + \alpha_0 xy^2z + \beta_0 x^2y^2 + \gamma_0 xy^3 \qquad \alpha_0^2 \neq 4.$

$f_x = 2xz^2 + 3x^2y + \alpha_0 y^2z + 2\beta_0 xy^2 + \gamma_0 y^3$

$f_y = 4y^3 + x^3 + 2\alpha_0 xyz + 2\beta_0 x^2y + 3\gamma_0 xy^2$

$f_z = 2x^2z + \alpha_0 xy^2$

The tangent space of the orbit G.f at f is as in the table

		x^4	y^4	z^4	x^3y	xy^3	x^3z	xz^3	y^3z	yz^3	x^2yz	xy^2z	xyz^2	x^2y^2	x^2z^2	y^2z^2
λ_1	xf_x				3	γ_0						α_0		$2\beta_0$	2	
λ_2	yf_x		γ_0			$2\beta_0$			α_0				2	3		
λ_3	zf_x							2	γ_0		3	$2\beta_0$				α_0
λ_4	xf_y	1			$2\beta_0$	4					$2\alpha_0$			$3\gamma_0$		
λ_5	yf_y		4		1	$3\gamma_0$						$2\alpha_0$		$2\beta_0$		
λ_6	zf_y						1		4		$2\beta_0$	$3\gamma_0$	$2\alpha_0$			
λ_7	xf_z						2							α_0		
λ_8	yf_z					α_0					2					
λ_9	zf_z											α_0			2	

The tangent space to the normal form at $(\alpha_0, \beta_0, \gamma_0)$ is spanned by xy^2z, x^2y^2 and xy^3.

Therefore if we suppose

$$\lambda_1 x f_x + \ldots\ldots + \lambda_9 z f_z = \alpha xy^2z + \beta x^2y^2 + \gamma xy^3$$

We must show that $\alpha = \beta = \gamma = 0$.

Now let us refer back to the table

$x^4 : \quad \lambda_4 = 0$

$xz^3 : \quad 2\lambda_3 = 0 \implies \lambda_3 = 0$

$y^2z^2 : \quad \alpha_0 \lambda_3 = 0$

$xyz^2 : \quad 2\lambda_2 + 2\alpha_0 \lambda_6 = 0$

$y^3z : \quad \alpha_0 \lambda_2 + 4\lambda_6 = 0 \implies \lambda_2 = \lambda_6 = 0$ since $\alpha_0^2 \neq 4$

$y^4 : \quad \gamma_0 \lambda_2 + 4\lambda_5 = 0 \implies \lambda_5 = 0$

$x^3y : \quad 3\lambda_1 + \lambda_5 = 0 \implies \lambda_1 = 0$

$x^2z^2 : \quad 2\lambda_1 + 2\lambda_9 = 0 \implies \lambda_9 = 0$

$x^3z : \quad \lambda_6 + 2\lambda_7 = 0 \implies \lambda_7 = 0$

$x^2yz : \quad 2\beta_0 \lambda_6 + 2\lambda_8 = 0 \implies \lambda_8 = 0$

Hence we have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = 0$

(Note that this means the codimension of the orbit is 6.)

Thus $\alpha = \beta = \gamma = 0$. This implies that this is a "good" normal form.

By finding the complementary space, we can put the transversal $\mathcal{J}_f$ in the following form

$$\mathcal{J}_f = x^2z^2 + y^4 + x^3y + \alpha_0 xy^2z + \beta_0 x^2y^2 + \gamma_0 xy^3$$
$$+ \alpha xy^2z + \beta x^2y^2 + \gamma xy^3 + \lambda z^4 + \mu yz^3 + \varepsilon y^2z^2$$

The property

$$T_f(G.f) \cap T_f(\mathcal{J}_f) = \{0\}$$

can be checked. Now suppose

$$\lambda_1 x f_x + \ldots\ldots + \lambda_9 z f_z = \alpha xy^2z + \beta x^2y^2 + \gamma xy^3 + \lambda z^4$$
$$+ \mu yz^3 + \varepsilon y^2z^2$$

We can use the same equations again, only that this time we have $\alpha_0 \lambda_3 = \varepsilon$, but from xz^3 : $2\lambda_3 = 0$, we still have $\lambda_3 = 0$, hence $\varepsilon = 0$.

So we can easily see that

$$\alpha = \beta = \gamma = \lambda = \mu = \varepsilon = 0$$

Examples 4.1.2.

A_3^2 ⊖ $f(x,y,z) = (x^2 + yz)(x^2 + \alpha_0 yz)$ $\qquad \alpha_0 \neq 1 \quad \alpha_0 \neq 0.$

$$= x^4 + (\alpha_0 + 1)x^2yz + \alpha_0 y^2z^2$$

$$f_x = 4x^3 + 2(\alpha_0 + 1)xyz$$

$$f_y = (\alpha_0 + 1)x^2z + 2\alpha_0 yz^2$$

$$f_z = (\alpha_0 + 1)x\,y + 2\alpha_0 y^2z$$

The tangent space to the normal form at (α_0) is spanned by $x^2yz + y^2z^2$. Therefore if we suppose

$$\lambda_1 x f_x + \ldots\ldots + \lambda_9 z f_z = \alpha(x^2yz + y^2z^2)$$

We must show that $\alpha = 0$.

Now let us refer to the table,

The tangent space of the orbit G.f is as in the table

		x^4	y^4	z^4	x^3y	xy^3	x^3z	xz^3	y^3z	yz^3	x^2yz	xy^2z	xyz^2	x^2y^2	x^2z^2	y^2z^2
λ_1	xf_x	4									$2(\alpha_0+1)$					
λ_2	yf_x				4							$2(\alpha_0+1)$				
λ_3	zf_x						4						$2(\alpha_0+1)$			
λ_4	xf_y						(α_0+1)						$2\alpha_0$			
λ_5	yf_y										(α_0+1)					$2\alpha_0$
λ_6	zf_y									$2\alpha_0$					(α_0+1)	
λ_7	xf_z				(α_0+1)							$2\alpha_0$				
λ_8	yf_z								$2\alpha_0$					(α_0+1)		
λ_9	zf_z										(α_0+1)					$2\alpha_0$

x^4 : $4\lambda_1 = 0 \Longrightarrow \lambda_1 = 0$.

y^3z : $2\alpha_0\lambda_8 = 0 \Longrightarrow \lambda_8 = 0$

x^2y^2: $(\alpha_0+1)\lambda_8 = 0$

yz^3 : $2\alpha_0\lambda_6 = 0 \Longrightarrow \lambda_6 = 0$

x^2z^2: $(\alpha_0+1)\lambda_6 = 0$

x^3y : $4\lambda_2 + (\alpha_0+1)\lambda_7 = 0$

xy^2z: $2(\alpha_0+1)\lambda_2 + 2\alpha_0\lambda_7 = 0 \Longrightarrow \lambda_2 = \lambda_7 = 0$ since $\alpha_0 \neq 1$

Similarly,

x^3z : $4\lambda_3 + (\alpha_0+1)\lambda_4 = 0$

xyz^2: $2(\alpha_0+1)\lambda_3 + 2\alpha_0\lambda_4 = 0 \Longrightarrow \lambda_3 = \lambda_4 = 0$

x^2yz: $(\alpha_0+1)\lambda_5 + (\alpha_0+1)\lambda_9 = \alpha$ (i)

y^2z^2: $2\alpha_0\lambda_5 + 2\alpha_0\lambda_9 = \alpha$ (ii)

Eq.(ii)-(i), we have $(\alpha_0-1)\lambda_5 + (\alpha_0-1)\lambda_9 = 0$.

Therefore $\lambda_5 + \lambda_9 = 0$, since $\alpha_0 \neq 1$.

(We are left with one relation. This implies the codimension of the orbit is 7.)

Hence $\alpha = 0$.

Thus this is a "good" normal form.

By finding the complementary space we can put the transversal in the form

$$\mathcal{J}_f = x^4 + x^2yz + \alpha_0(x^2yz + y^2z^2) + \alpha x^2yz + \beta y^4 + \gamma z^4 + \lambda xy^3 + \mu xz^3 + \varepsilon x^2y^2 + \zeta x^2z^2$$

For checking the property

$$T_f(G.f) \cap T_f(\mathcal{J}_f) = \{0\} \quad ,$$

we suppose

$$\lambda_1 x f_x + \dots\dots + \lambda_9 z f_z = \alpha x^2yz + \beta y^4 + \gamma z^4 + \lambda xy^3 + \mu xz^3 + \varepsilon x^2y^2 + \zeta x^2z^2$$

Again we can use the same equations as above, only that this time we have $x^2y^2 : (\alpha_0 + 1)\lambda_8 = \varepsilon$, $x^2z^2 : (\alpha_0 + 1)\lambda_6 = \zeta$, butt from $y^3z : 2\alpha_0\lambda_8 = 0$ and $yz^3 : 2\alpha_0\lambda_6 = 0$, we still have $\lambda_8, \lambda_6 = 0$. Hence $\varepsilon = \zeta = 0$.

So we can easily check that $\alpha = \beta = \gamma = \lambda = \mu = \varepsilon = \zeta = 0$.

Examples 4.1.3.

A_1^4  $f(x,y,z) = (yz + \alpha_0 xz + (1+\alpha_0)xy)(yz + \beta_0 xz + (1+\beta_0)xy)$

$$\alpha_0 \neq -1, \quad \beta_0 \neq -1, \quad \alpha_0 \neq \beta_0, \quad \alpha_0 \neq 0, \quad \beta_0 \neq 0$$

$$= y^2z^2 + (\alpha_0 + \beta_0)xyz^2 + \alpha_0\beta_0 x^2z^2 + (1+\alpha_0)(1+\beta_0)x^2y^2 + (\alpha_0 + \beta_0 + 2\alpha_0\beta_0)x^2yz + (2 + \alpha_0 + \beta_0)xy^2z$$

$$f_x = (\alpha_0+\beta_0)yz^2 + 2\alpha_0\beta_0 xz^2 + 2(1+\alpha_0)(1+\beta_0)xy^2 + 2(\alpha_0+\beta_0+2\alpha_0\beta_0)xyz + (2 + \alpha_0+\beta_0)y^2z$$

$$f_y = 2yz^2 + (\alpha_0+\beta_0)xz^2 + 2(1+\alpha_0)(1+\beta_0)x^2y + (\alpha_0+\beta_0+2\alpha_0\beta_0)x^2z + 2(2+\alpha_0+\beta_0)xyz$$

$$f_z = 2y^2z + 2(\alpha_0+\beta_0)xyz + 2\alpha_0\beta_0 x^2z + (\alpha_0+\beta_0+2\alpha_0\beta_0)x^2y + (2+\alpha_0+\beta_0)xy^2$$

The tangent space of the orbit G.f at f is as in the table

	x^4	y^4	z^4	x^3y	xy^3	x^3z	xz^3	y^3z	yz^3	x^2yz	xy^2z	xyz^2	x^2y^2	x^2z^2	y^2z^2
xf_x										$2(\alpha_0+\beta_0+2\alpha_0\beta_0)$	$(2+\alpha_0+\beta_0)$	$(\alpha_0+\beta_0)$	$2(1+\alpha_0)(1+\beta_0)$	$2\alpha_0\beta_0$	
yf_x					$2(1+\alpha_0)(1+\beta_0)$			$(2+\alpha_0+\beta_0)$			$2(\alpha_0+\beta_0+2\alpha_0\beta_0)$	$2\alpha_0\beta_0$			$(\alpha_0+\beta_0)$
zf_x							$2\alpha_0\beta_0$		$(\alpha_0+\beta_0)$		$2(1+\alpha_0)(1+\beta_0)$	$2(\alpha_0+\beta_0+2\alpha_0\beta_0)$			$(2+\alpha_0+\beta_0)$
xf_y				$2(1+\alpha)(1+\beta)$		$(\alpha_0+\beta_0+2\alpha_0\beta_0)$				$2(2+\alpha_0+\beta_0)$		2		$(\alpha_0+\beta_0)$	
λ_5 yf_y										$(\alpha_0+\beta_0+2\alpha_0\beta_0)$	$2(2+\alpha_0+\beta_0)$	$(\alpha_0+\beta_0)$	$2(1+\alpha_0)(1+\beta_0)$		2
λ_6 zf_y							$(\alpha_0+\beta_0)$		2	$2(1+\alpha_0)(1+\beta_0)$		$2(2+\alpha_0+\beta_0)$		$(\alpha_0+\beta_0+2\alpha_0\beta_0)$	
λ_7 xf_z				$(\alpha_0+\beta_0+2\alpha_0\beta_0)$		$2\alpha_0\beta_0$				$2(\alpha_0+\beta_0)$	2		$(2+\alpha_0+\beta_0)$		
λ_8 yf_z					$(2+\alpha_0+\beta_0)$			2		$2\alpha_0\beta_0$	$2(\alpha_0+\beta_0)$		$(\alpha_0+\beta_0+2\alpha_0\beta_0)$		
λ_9 zf_z										$(\alpha_0+\beta_0+2\alpha_0\beta_0)$	$(2+\alpha_0+\beta_0)$	$2(\alpha_0+\beta_0)$		$2\alpha_0\beta_0$	2

The tangent space to the normal form at (α_0, β_0) spanned by

$$xyz^2 + \beta_0 x^2z^2 + (1+\beta_0)x^2y^2 + (1+2\beta_0)x^2yz + xy^2z$$

$$xyz^2 + \alpha_0 x^2z^2 + (1+\alpha_0)x^2y^2 + (1+2\alpha_0)x^2yz + xy^2z$$

Therefore if we suppose

$$\lambda_1 xf_x + \ldots\ldots + \lambda_9 zf_z = \alpha(xyz^2 + \beta_0 x^2z^2 + (1+\beta_0)x^2y^2 + (1+2\beta_0)x^2yz + xy^2z)$$
$$+ \beta(xyz^2 + \alpha_0 x^2z^2 + (1+\alpha_0)x^2y^2 + (1+2\alpha_0)x^2yz + xy^2z)$$

We must show that $\alpha = \beta = 0$.

Now let us refer back to the table

$$x^3y \;: 2(1+\alpha_0)(1+\beta_0)\lambda_4 + (\alpha_0+\beta_0+2\alpha_0\beta_0)\lambda_7 = 0$$

$$x^3z \;: (\alpha_0+\beta_0+2\alpha_0\beta_0)\lambda_4 + 2\alpha_0\beta_0\lambda_7 = 0 \implies \lambda_4 = \lambda_7 = 0$$

since $\alpha_0 \neq \beta_0$ and $\alpha_0 \neq 0, \beta_0 \neq 0$.

Similarly,

$$xy^3 \;: 2(1+\alpha_0)(1+\beta_0)\lambda_2 + (2+\alpha_0+\beta_0)\lambda_8 = 0$$

$$y^3z \;: (2+\alpha_0+\beta_0)\lambda_2 + 2\lambda_8 = 0 \implies \lambda_2 = \lambda_8 = 0$$

$$xz^3 \;: 2\alpha_0\beta_0\lambda_3 + (\alpha_0+\beta_0)\lambda_6 = 0$$

$$yz^3 \;: (\alpha_0+\beta_0)\lambda_3 + 2\lambda_6 = 0 \implies \lambda_3 = \lambda_6 = 0$$

Hence we have $\lambda_4 = \lambda_7 = \lambda_2 = \lambda_8 = \lambda_3 = \lambda_6 = 0$

Let us now consider the following equations

$$y^2z^2 : 2\lambda_5 + 2\lambda_9 = 0 \qquad \text{(i)}$$

$$x^2z^2 : 2\alpha_0\beta_0\lambda_1 + 2\alpha_0\beta_0\lambda_9 = \alpha\beta_0 + \beta\alpha_0 \qquad \text{(ii)}$$

x^2y^2 : $2(1+\alpha_0)(1+\beta_0)\lambda_1 + 2(1+\alpha_0)(1+\beta_0)\lambda_5 = \alpha(1+\beta_0) + \beta(1+\alpha_0)$ (iii)

x^2yz : $2(\alpha_0+\beta_0+2\alpha_0\beta_0)\lambda_1 + (\alpha_0+\beta_0+2\alpha_0\beta_0)\lambda_5 + (\alpha_0+\beta_0+2\alpha_0\beta_0)\lambda_9$
$= \alpha(1+2\beta_0) + \beta(1+2\alpha_0)$ (iv)

xy^2z : $(2+\alpha_0+\beta_0)\lambda_1 + 2(2+\alpha_0+\beta_0)\lambda_5 + (2+\alpha_0+\beta_0)\lambda_9 = \alpha + \beta$ (v)

xyz^2 : $(\alpha_0+\beta_0)\lambda_1 + (\alpha_0+\beta_0)\lambda_5 + 2(\alpha_0+\beta_0)\lambda_9 = \alpha + \beta$ (vi)

Eq.(v)-Eq.(vi), we have $2\lambda_1 + (4+\alpha_0+\beta_0)\lambda_5 + (2-\alpha_0-\beta_0)\lambda_9 = 0$ (vii)

Eq. (iv)- Eq.(iii), we have

$$(2\alpha_0\beta_0-2)\lambda_1 + (-\alpha_0-\beta_0-2)\lambda_5 + (\alpha_0+\beta_0+2\alpha_0\beta_0)\lambda_9 = \alpha\beta_0+\beta\alpha_0 \quad \text{(viii)}$$

Eq.(viii)-Eq.(ii), we have $-2\lambda_1 - (2+\alpha_0+\beta_0)\lambda_5 + (\alpha_0+\beta_0)\lambda_9 = 0$ (ix)

Eq.(ix)+Eq.(vii), $\lambda_5 + \lambda_9 = 0$ (x)

Eq.(iii)-Eq.(v), we have $(\alpha_0+\beta_0+2\alpha_0\beta_0)\lambda_1 + (\alpha_0+\beta_0+2\alpha_0\beta_0)\lambda_5 = \alpha\beta_0+\beta\alpha_0$ (xi)

Eq.(xi)-Eq.(ii), using Eq.(x), we have

$$(\alpha_0+\beta_0)\lambda_1 - (\alpha_0+\beta_0+4\alpha_0\beta_0)\lambda_9 = 0 \quad \text{(xii)}$$

Eq.(iii)-Eq.(vi), we get $(2+\alpha_0+\beta_0+2\alpha_0\beta_0)\lambda_1 + (2+3\alpha_0+3\beta_0+2\alpha_0\beta_0)\lambda_5$
$= \alpha\beta_0 + \beta\alpha_0$ (xiii)

Eq.(xiii)-Eq.(ii), using Eq.(x),

$$(2+\alpha_0+\beta_0)\lambda_1 - (2+3\alpha_0+3\beta_0+4\alpha_0\beta_0)\lambda_9 = 0 \quad \text{(xiv)}$$

Eq.(xiv)-Eq.(xii), $\lambda_1 - (1+\alpha_0+\beta_0)\lambda_9 = 0$ (xv)

Eq.(xv) into Eq.(xii), we have $\lambda_1 = \lambda_9 = 0$, since $\alpha_0 \neq \beta_0$.

By Eq.(x) again we get $\lambda_1 = \lambda_5 = \lambda_9 = 0$.

(The codimension of the orbit is 6)

This means that $\alpha = \beta = 0$, since $\alpha\beta_0+\beta\alpha_0 = 0$ and $\alpha+\beta = 0$ and $\beta_0 \neq \alpha_0$.

Hence this is a "good" normal form.

The Transversal $\mathcal{T}_f$ for this stratum at f is found to be (see Example 5.3.1.)

$$\mathcal{T}_f = y^2z^2 + (\alpha_0+\beta_0)xyz^2 + \alpha_0\beta_0x^2z^2 + (1+\alpha_0)(1+\beta_0)x^2y^2 + (\alpha_0+\beta_0+2\alpha_0\beta_0)x^2yz$$
$$+ (2+\alpha_0+\beta_0)xy^2z$$
$$+ \alpha(xyz^2 + \beta_0x^2z^2 + (1+\beta_0)x^2y^2 + (1+2\beta_0)x^2yz + xy^2z)$$
$$+ \beta(xyz^2 + \alpha_0x^2z^2 + (1+\alpha_0)x^2y^2 + (1+2\alpha_0)x^2yz + xy^2z)$$

$$+ \gamma(3y^4 + 2y^3z - 2xy^3 + x^2z^2)$$
$$+ \lambda(z^4 + 2xz^3 + x^2z^2)$$
$$+ \mu(x^4 + 2x^3z + x^2z^2)$$
$$+ \varepsilon(x^2z^2)$$

For checking the property

$$T_f(G.f) \cap T_f(\mathcal{J}_f) = \{0\} ,$$

we suppose

$$\begin{aligned} \lambda_1 x f_x + \ldots\ldots + \lambda_9 z f_z &= \alpha(xyz^2 + \beta_0 x^2z^2 + (1+\beta_0)x^2y^2 + (1+2\beta_0)x^2yz + xy^2z) \\ &+ \beta(xyz^2 + \alpha_0 x^2z^2 + (1+\alpha_0)x^2y^2 + (1+2\alpha_0)x^2yz + xy^2z) \\ &+ \gamma(3y^4 + 2y^3z - 2xy^3 + x^2z^2) \\ &+ \lambda(z^4 + 2xz^3 + x^2z^2) \\ &+ \mu(x^4 + 2x^3z + x^2z^2) \\ &+ \varepsilon(x^2z^2) \end{aligned}$$

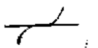
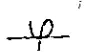
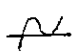
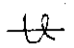
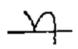
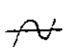
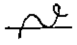
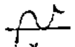
Again we can use the same equations as above, only that this time we have x^2z^2: $2\alpha_0\beta_0\lambda_1 + 2\alpha_0\beta_0\lambda_9 = \alpha\beta_0 + \beta\alpha_0 + \varepsilon$, but from y^2z^2: $2\lambda_5 + 2\lambda_9 = 0$, we still have $\lambda_5 + \lambda_9 = 0$. By going through the calculation again adding the following equations: y^4 : $3\gamma = 0$, z^4 : $\lambda = 0$, and x^4 : $\mu = 0$, we can easily see that

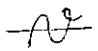
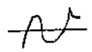
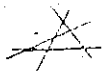
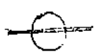
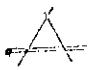
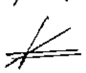
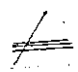
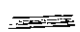
$$\alpha = \beta = \gamma = \lambda = \mu = \varepsilon = 0.$$

§42. Generators of the

Types	Normal Forms	Generators of the Transversals :
A_1	$x^2z^2+yz^3+x^2yz+\alpha y^4+\beta xy^3+\gamma y^3z+\delta xy^2z+\varepsilon y^2z^2=0$	x^4 y^4 ~~z^4~~ ~~x^2yz~~ ~~x^3z~~ ~~x^3y~~ or xy^2z or ~~xyz^2~~ $xy^3, y^3z, xy^2z, y^2z^2$
A_2	$x^2z^2+yz^3+xy^3+\alpha y^3z+\beta xy^2z+\gamma y^2z^2+\delta xyz^2=0$	x^4 y^2z^2 y^3z xy^2z xyz^2 x^3y
A_3	$x^2z^2+y^4+x^3y+\alpha xy^2z+\beta x^2y^2+\gamma xy^3=0$	z^4 yz^3 xy^2z x^2y^2 xy^3 y^2z^2
A_4	$x^2z^2+2xy^2z+y^4+x^3y+\beta x^2y^2+\gamma xy^3=0$	z^4 yz^3 y^2z^2 y^3z x^2y^2 xy^3
D_4	$xyz(x+y)=x^4+\alpha x^2y^2+\beta y^4$	y^4 z^4 xz^3 yz^3 x^2y^2 xyz^2
A_5	$x^2z^2+xy^2z+y^4=x^2y^2+\alpha x^3y$	z^4 x^3y yz^3 xy^2z xz^3 or y^2z^2 y^3z or xyz^2
A_6	$(xz+y^2)^2=x^3y$	z^4 yz^3 x^2z^2 or xy^2z xy^3 or x^2yz y^3z or xyz^2 xz^3 or y^2z^2
E_6	$x^3z=x^2y^2+y^4$	z^4 xz^3 yz^3 xy^2z xyz^2 y^2z^2
	$x^3z=y^4$ (cod. 7)	z^4 xz^3 yz^3 xy^2z xyz^2 y^2z^2 x^2y^2
D_5	$x^2yz=x^4+\alpha xy^3+y^4$	z^4 xy^3 y^2z^2 xz^3 yz^3 x^2z^2 or y^3z
A_1^2	$x^2y^2+x^2z^2+y^2z^2+z^4+\alpha xyz^2+\beta xz^3+\gamma yz^3+\delta z^4=0$	x^4 y^4 xyz^2 xz^3 yz^3 z^4
A_1A_2	$x^2y^2+y^2z^2+\alpha xyz^2+xz^3+\gamma yz^3+\delta z^4=0$	x^4 y^4 x^3z xyz^2 yz^3 z^4
A_2^2	$x^2y^2+\alpha xyz^2+xz^3+yz^3+\delta z^4=0$	x^4 y^4 x^3z y^3z xyz^2 z^4
A_1A_3	$y^4+x^2z^2+x^2yz+\alpha xy^2z+\beta xy^3=0$	x^4 z^4 yz^3 xy^2z xy^3 y^2z^2
A_2A_3	$y^4+x^2z^2+\alpha xy^2z+xy^3=0$	x^4 z^4 xy^2z x^3y yz^3 y^2z^2
A_1A_4	$(xz+y^2)^2=\alpha x^2y^2+xy^3$	x^4 z^4 x^2y^2 yz^3 y^3z or xyz^2 y^2z^2 or xz^3
A_2A_4	$(xz+y^2)^2=xy^3$	x^4 z^4 x^3y yz^3 y^3z or xyz^2 y^2z^2 or xz^3
A_1^3	$x^2y^2+x^2z^2+y^2z^2=xyz(\alpha x+\beta y+\gamma z)$	x^4 y^4 z^4 x^2yz xy^2z xyz^2
$A_1^2A_2$	$x^2y^2+x^2z^2+y^2z^2=xyz(2x+\beta y+\gamma z)$	x^4 y^4 z^4 xyz^2 xy^2z x^3y or x^3z
$A_1A_2^2$	$x^2y^2+x^2z^2+y^2z^2=xyz(2x+2y+\gamma z)$	x^4 y^4 z^4 xyz^2 x^3y or x^3z xy^3 or y^3z
A_2^3	$x^2y^2+x^2z^2+y^2z^2=xyz(2x+2y+2z)$	x^4 y^4 z^4 x^3y or x^3z xy^3 or y^3z xz^3 or yz^3

Generators of [illegible]

Types	Pictures	Normal Forms	Generators of the Transversals
A_5		$x(x^2z + xz^2 + \beta xyz + y^3) = 0$	$y^4 \quad z^4 \quad x^2yz \quad yz^3 \quad y^2z^2 \quad y^3z$
D_6		$x(x^3 + z^3 + xyz) = 0$	$y^4 \quad y^2z^2 \quad y^3z \quad xy^3 \quad yz^3 \quad z^4$
A_7		$(x^2+yz)(x^2+yz+y^2) = 0$	$z^4 \quad xz^3 \quad x^2z^2 \quad x^3z \quad y^2z^2$ or $x^2yz \quad xyz^2$ or $xy^2z \quad xy^3$ or x^3y
E_7		$x(z^3 + x^2y + xz^2) = 0$	$y^4 \quad y^3z \quad y^2z^2 \quad yz^3 \quad xy^2z \quad xy^3 \quad z^4$
$X_9(\tilde{E}_7)$		$x^4 + \alpha x^2z^2 + z^4 = 0$	$y^4 \quad x^2y^2 \quad y^2z^2 \quad xy^2z \quad xy^3 \quad y^3z \quad xyz^2$ or $x^3y \quad x^2yz$ or $yz^3 \quad x^2z^2$
A_1A_3		$x(x^2z + \alpha xz^2 + \beta xyz + xy^2 + y^2z) = 0$	$y^4 \quad z^4 \quad yz^3 \quad y^2z^2 \quad x^2yz \quad x^2z^2$
A_1D_4		$x(x^3 + \alpha x^2z + yz^2 + xyz) = 0$	$y^4 \quad z^4 \quad y^3z \quad x^3z \quad xy^3 \quad xy^2z$
A_1A_5		$(z^2 + xy)(z^2 + yz + xy) = 0$	$x^4 \quad y^4 \quad x^3z \quad x^2z^2$ or $x^3y \quad xz^3$ or $x^2yz \quad xy^2z$
A_1A_5		$x(z^3 + xy^2 + xyz) = 0$	$x^4 \quad y^4 \quad y^3z \quad y^2z^2 \quad yz^3 \quad xy^2z$
A_1D_5		$x(x^3 + \alpha x^2z + yz^2) = 0$	$y^4 \quad z^4 \quad x^2y^2 \quad xy^2z$ or $x^3y \quad y^3z \quad xy^3$
A_1D_6		$yz(y^2 + xz) = 0$	$x^4 \quad y^4 \quad z^4 \quad x^3z \quad x^3y \quad xy^3 \quad x^2y^2$
A_2A_5		$x(z^3 + xy^2) = 0$	$x^4 \quad y^4 \quad z^4$ or $xy^2z \quad y^2z^2 \quad x^3z \quad y^3z \quad xy^3$ or yz^3
A_3^2		$(x^2+yz)(x^2+\alpha yz) = 0$	$y^4 \quad z^4 \quad xz^3 \quad xy^3 \quad x^2yz \quad x^2y^2 \quad x^2z^2$
A_1^3		$x(xy^2 + y^2z + yz^2 + \alpha x^2z + \beta xz^2 + \gamma xyz) = 0$	See chapter 5
$A_1^2A_3$		$x(xz^2 + xy^2 + y^2z + \alpha xyz) = 0$	$x^4 \quad y^4 \quad z^4 \quad yz^3 \quad y^2z^2 \quad x^2yz$
$A_1^2A_3$		$(\alpha yz + xz + xy)(yz + xz + xy) = 0$	$x^4 \quad y^4 \quad z^4 \quad xyz^2 \quad x^3y$ or $x^3z \quad xy^2z$
$A_1^2D_4$		$yz(yz + xz + xy) = 0$	$x^4 \quad y^4 \quad z^4 \quad x^3y \quad x^3z \quad x^2yz$
$A_1A_2A_3$		$x(xz^2 + 2xyz + xy^2 + y^2z) = 0$	$x^4 \quad y^4 \quad z^4 \quad yz^3 \quad y^2z^2 \quad x^3y$
$A_1A_3^2$		$yz(x^2 + yz) = 0$	$x^4 \quad y^4 \quad z^4 \quad xy^3 \quad xz^3 \quad y^3z \quad yz^3$

Types	Pictures	Normals Forms	Generators of the Transversals
A_1^4		$x(xy^2+y^2z+yz^2+\alpha xz^2+\beta xyz)=0$	See Chapter 5
A_1^4		$(\alpha yz+(\alpha+1)xz+xy)(\beta yz+(\beta+1)xz+xy)=0$	" "
$A_1^3A_2$		$x(\alpha^2xy^2+xz^2+y^2z+yz^2+2\alpha xyz)=0$	" "
$A_1^3A_3$		$yz(x^2+yz+xz)=0$	x^4 y^4 z^4 xy^3 y^3z x^3z or xz^3 or x^2z^2 x^2y^2 or y^2z^2 or xy^2z
$A_1^3D_4$		$xyz(x+z)=0$	See Chapter 5
A_1^5		$yz(x^2+\alpha yz+xz+xy)=0$	" "
A_1^6		$xyz(x+y+z)=0$	y^4 z^4 xy^3 xz^3 y^3z yz^3 y^2z^2
		$x^2(yz+xz+xy)=0$	x^4 y^4 x^3z x^3y xy^3 x^2z^2 x^2y^2 x^2yz
		$z^2(y^2+xz)=0$	x^4 y^4 z^4 y^3z x^3z xy^3 x^3y x^2y^2
		$xyz^2=0$	x^4 y^4 x^3y xy^3 x^2y^2 y^2z^2 xy^2z y^3z yz^3 or x^2yz or xyz^2
		$xz^2(x+z)=0$	x^4 y^4 z^4 x^3z y^3z xy^3 x^3y xz^3 yz^3
		$(z^2+xy)^2=0$	x^4 y^4 z^4 x^3y xy^3 y^3z yz^3 x^2y^2 y^2z^2 xy^2z
		$x^2z^2=0$	x^4 y^4 x^3y xy^3 x^3z y^3z x^2y^2 y^2z^2 x^2yz xy^2z
		$xz^3=0$	x^4 y^4 x^3y xy^3 x^3z y^3z x^2y^2 x^2z^2 y^2z^2 x^2yz xy^2z xyz^2
		$z^4=0$	

CHAPTER 5

STRATA ARE MANIFOLDS

5.1 Unfolding Theory

In order to study a singularity, we need to know something about the unfolding theory of singularities. The following is just a general statement of the theory in the complex analytic case. Most of the definitions and theorems we adopt here are formulated in the real case by BRÖCKER in his book "Differentiable germs and Catastrophes" (translated by Lander in 1975) Chapter 14. We will not include any proofs in this context, since they will be quite similar to the ones in the real case as presented in the book. If reference is needed for the proofs in the complex analytic case, we refer to Bruce (Liverpool Ph.D. Thesis, 1978).

Let f be a singularity in E(n) = the ring of complex analytic germs: $(\mathbb{C}^n, 0) \to \mathbb{C}$ (our case is n = 2). Hence $f \in m(n)^2$, where m(n) is the maximal ideal $\{f \in \mathcal{E}(n) \mid f(0) = 0\}$. Also let $\mathbb{C}^n \subset \mathbb{C}^{n+r}$ be the subspace where the last r coordinates vanish. A point of $\mathbb{C}^{n+r}$ is then denoted by $(x,u) = (x_1,\dots x_n, u_1,\dots,u_r)$, $x \in \mathbb{C}^n$, $u \in \mathbb{C}^r$.

5.1.1. Def. An (r-parameter) <u>unfolding</u> or <u>deformation</u> of f is a germ $F \in \mathcal{E}(n+r)$ such that $F/\mathbb{C}^n = f$. This unfolding will then be denoted by (r,F).

Let $\pi_r: \mathbb{C}^{n+r} \to \mathbb{C}^r$ be the projection. We can construct a category of unfoldings of a singularity, the objects being unfoldings and the morphisms being defined as follows.

5.1.2. Def. If (r,F) and (s,F') are unfoldings of f, a <u>morphism</u> is a mapping

$$(\phi,c): (r,F) \to (s,F')$$

as follows:

(I) The germ $\phi: (\mathbb{C}^{n+r},0) \to (\mathbb{C}^{n+s},0)$ with $\phi / \mathbb{C}^n \times \{0\} = \mathrm{id}$.

(II) The germ $c: (\mathbb{C}^r,0) \to (\mathbb{C},0)$ such that

$$F = F' \circ \phi + c \circ \pi_r$$

$$\text{i.e.} \quad F(x,u) = F'(\phi(x,u)) + c(u)$$

(III) There exists a germ $\Phi: (\mathbb{C}^r,0) \to (\mathbb{C}^s,0)$ such that

$$\pi_s \phi = \Phi \, \pi_r$$

$$\text{i.e.} \quad \pi_s \phi(x,u) = \Phi(u)$$

That is, the following diagram commutes

$$\begin{array}{ccc} (\mathbb{C}^{n+r},0) & \xrightarrow{\phi} & (\mathbb{C}^{n+s},0) \\ \pi_r \downarrow & & \downarrow \pi_s \\ (\mathbb{C}^r,0) & \xrightarrow{\Phi} & (\mathbb{C}^s,0) \end{array}$$

Now, for a representative of c, we may assign a translation a_u to each $u \in \mathbb{C}^r$ near the origin. Suppose $a_u: \mathbb{C} \to \mathbb{C}$ is given by

$$a_u(t) = t + c(u)$$

then by (II), F and F' are related by

$$\begin{aligned} F(x,u) &= F' \circ \phi(x,u) + c \circ \pi_r(x,u) \\ &= F'(\phi(x,u)) + c(u) \\ &= a_u(F'(\phi(x,u))) \\ &= a_u \circ F' \circ \phi(x,u) \end{aligned}$$

Suppose (ϕ,c) is a morphism between (r,F) and (s,F') and (ψ,c') is a morphism between (s,F') and $(t,F")$, then the composite

$$(\psi,c')(\phi,c) = (\psi \circ \phi,\ c+c' \circ \Phi)$$

where $\Phi: \mathbb{C}^r \to \mathbb{C}^t$ comes from ψ as in III, is a morphism between (r,F) and $(t,F")$. If b is a translation associated with c' for each $v \in \mathbb{C}^s$ as described above for c, then $c(u) + c' \circ \Phi(u)$, $u \in \mathbb{C}^r$ describes the composition of translations, namely $b_{\Phi(u)} \circ a_u$. Also, we call a morphism (ϕ,c) invertible if and only if there exists a morphism (ψ,c') such that

$$\psi \circ \phi = id. \text{ and } c+c' \circ \Phi = 0$$

Observe that $\psi \circ \phi = id.$ implies Φ is also invertible and hence we can always solve $c+c' \circ \Phi = 0$ for c'. Therefore a morphism (ϕ,c) is invertible (an isomorphism) exactly when ϕ is invertible.

Now, suppose we have an unfolding (s,F') of f and there are germs ϕ, c and Φ satisfying Def. 5.1.2., then by equation II, the unfolding (r,F) is determined.

5.1.3. Def. An unfolding (r,F) determined in this way as described above is called the unfolding of f *induced* by (ϕ,c) from (s,F').

Let us look at a simple example. Suppose we have two unfoldings (r,F), (s,G) of f having the following relationship: $s = r-1$ and

$$F(x,u_1 \ldots u_r) = G(x,u_1 \ldots u_{r-1}) + u_r$$

We claim that F can be induced from G by a suitable morphism. According to the definitions, all we need to do is to choose ϕ and c. Let us take

$$\phi = \pi^r_s : \mathbb{C}^{n+r} \to \mathbb{C}^{n+r-1} \text{ the projection and}$$

$$c = \pi^r_1 : \mathbb{C}^r \to \mathbb{C}$$

$$(u_1 \ldots u_r) \mapsto u_r$$

Then, we have

$$F(x,u_1,\ldots,u_r) = G(\pi^r_s(x,u_1,\ldots,u_r)) + \pi^r_1(u_1 \ldots u_r)$$

and this is exactly equation II. Hence F is induced from G by (π^r_s, π^r_1).

5.1.4. Def. An unfolding (r,F) of f is called *versal* if every unfolding of f is induced from (r,F) by a suitable morphism.

Note that a versal unfolding of f may be characterized by a transversality condition as follows:

Let (r,F) be an unfolding of f. Suppose $\mathbb{F}$ is a representative of F. Consider the following diagram

$$\mathcal{E}(n+r) \xrightarrow{\;j^k\;} \mathcal{E}(n+r)/m(n+r)^{k+1}$$

$$\mathcal{E}(n+r) \xrightarrow{\;j^l\;} \mathcal{E}(n+r)/m(n+r)^{l+1}$$

$$\vdots$$

$$\mathcal{E}(n+r)/m(n+r) = \mathbb{C}$$

The image $j^k(\mathbb{F})$ of $F \in \mathcal{E}(n+r)$ is called the k-jet of F, denoted by $\hat{F}$. The guotient $\frac{(n+r)}{m(n+r)^{k+1}}$ is the space of k-jets. Any k-jet is represtned by a polynomial of degree $\leq k$. Let $J_o^k(n,1) = \frac{m(n)}{m(n)^{k+1}}$ be the space of k-jets whose zero-th (or constant) term vanishes. Consider a germ $j_1^k F: (\mathbb{C}^{n+r},0) \to J_o^k(n,1)$ whose representative $j_1^k \mathbb{F}$ is the map defined as

$$(x,u) \mapsto \text{k-jet of } (y \longmapsto \mathbb{F}(x+y,u) - \mathbb{F}(x,u))$$

Thus, $j_1^k \mathbb{F}$ is the partial Taylor expansion of $\mathbb{F}$ at the point (x,u) with respect to the first variables.

5.1.5 Def. F is called k-transversal if $j_1^k\ \mathbb{F}$ is transversal at the origin to the orbit $\hat{f}.\hat{\mathcal{B}}_k(n)$ of $\hat{f}$ ($\hat{f}$ is the k-jet of f, $\hat{\mathcal{B}}_k(n)$ = group of invertible k-jets) under right equivalence.

Obviously, $j_1^k\ \mathbb{F}(0) = j^k f(0) = \hat{f} \in \hat{f}.\hat{\mathcal{B}}_k(n)$.

5.1.6. Def. A germ f (n) is called k-determined if every germ g (n) with the same k-jet as f, is right equivalent to f.

Then, we have the theorem

5.1.7. Theorem. (versal = K-transversal). If f is k-determind, then an unfolding F of f is versal if and only if it is k-transversal.

5.1.8. Def. A versal unfolding (r,F) with <u>minimal</u> r is called <u>universal</u>.

5.1.9. Def. The codimension of a singularity f is defined as

$$\text{codim } f = \dim_{\mathbb{C}} \left(\frac{m(n)}{\langle \frac{\partial f}{\partial x_j} \rangle} \right)$$

where $\langle \frac{\partial f}{\partial x_j} \rangle$ is the ideal generated by the partial derivatives of f with respect to variables x. This is equal to μ-1, where μ is the Milnor number of the singularity.

The main theorem on unfolding theory is the following.

5.1.10. Theorem (Mather)

(I) A singularity $f \in m(n)$ has a versal unfolding iff codimension f is finite.

(II) Any two r-parameter versal unfoldings are isomorphic. (Uniqueness of unfoldings).

(III) Every versal unfolding is isomorphic to (r,F) + constant where (r,F) is universal.

(IV) If $\{b_1,\ldots b_r\}$ m(n) is a basis of $\frac{m(n)}{\langle \frac{\partial f}{\partial x_j} \rangle}$, then the unfolding F of f defined by

$$F(x,u) = f(x) + b_1(x)u_1 + \ldots b_r(x)u_r$$

is universal.

[For proofs of 5.1.7 and 5.1.10 in the real case, refer to BRÖCKER, Chapter 16].

[For complex analytic proofs, refer to Bruce (Liverpool Ph.D. Thesis, 1978)]

5.2 Special properties of Transversals

The transversals we got in Chapter 4 are all global equations describing a submanifold of $M = \mathbb{C}^{15} - \{0\}$. They are formed by adding the "complementary space" (to the tangent space of the orbit) to the normal forms, that is, adding the set of generators of the "complementary spaces" to the normal forms in the following way. Let the normal form be $\Gamma(x)$ and $b_1(x), \ldots, b_n(x)$ be the set of generators, then the transversal $\mathcal{T}$ is the set of quartic polynomials

$$\Gamma_o(x) + u_1 b_1(x) + \ldots u_n b_n(x) \qquad (*)$$

where $u_1, \ldots u_n \in \mathbb{C}$ are parameters and Γ_o is the normal form with fixed values of moduli (see examples).

It is a general practice that if we want to discuss a particular singularity of the normal form, we would have to move it (by transformation) to one of the vertices of the triangle of reference and look at the local equation there. In our classification, we have tried our best to choose normal forms for the strata such that they will have the singularities at the vertices. In fact, most of our normal forms have this property. There are only a few exceptions. We will deal with them in Section 5.3. Also in what follows we exclude the singularity $\tilde{E}_7$.

Now if we take the local equation of a transversal $\mathcal{T}$ at a singularity f of the normal form which is at one of the vertices, we can easily see that the Transversal $\mathcal{T}$ becomes an unfolding of the singularity f (by definition of unfolding).

We claim that the Transversal we got in Chapter 4 have the following properties:

(I) Let us denote the space with coordinates $u_1, \ldots, u_n$ as in (*) by $\mathcal{U}$. If we consider one of the singularities of Γ_0, which is assumed to be at X, Y or Z, putting x, y or z equal to 1 in Γ_0, we obtain a singularity f in 2 variables and putting x, y or z = 1 in $\mathcal{T}$, we obtain an unfolding f' of f and the space $\mathcal{U}$ is the direct sum of four subspace. That is

$$\mathcal{U} = M \oplus U \oplus V \oplus W \text{ (disjoint)}$$

where M corresponds to the parameters of the moduli terms of Γ.

U_f corresponds to the parameters of the universal unfolding terms of the singularity f

V_f corresponds to the parameters of the universal unfolding terms of other singularities.

W_f corresponds to the constant term (for f).

To explain this more precisely, in fact, we have for any point (m,v) close to $(\underline{0},\underline{0})$ in $M \times V_f$ the function f' = f + terms given by (m,v) is R-equivalent to f. Also f + terms given by U_f is a universal unfolding of f. The term given by W_f appears simply as an additive constant, f + terms given by U_f+w. where $w \in W_f$.

(II) For different singularities of the normal form, the U_f's and W_f's are disjoint. For example, for singularities f and g on the normal form

$$(U_f \oplus W_f) \cap (U_g \oplus W_g) = \{0\}.$$

In other words, the universal unfolding parameters corresponding to one singularity do not correspond to another. Notice that this property is the crucial point which we need for the proof of strata being manifolds. It enables us to show

Transversal $\cap$ stratum = Normal form

The formal proof of this will be shown in Section 5.4.

5.2.1. Theorem All the transversals listed in Tables have the properties I and II mentioned above.

We have gone through the checking of the properties I and II with each of the transversals in Table 4.2.1. Since it would be too lengthy to include all the calculations here, we choose three examples to show the technique.

Example 5.2.2 A_5(irr)

$$\Gamma = x^2z^2 + 2xy^2z + y^4 - x^2y^2 - \alpha x^3y$$

The singularity is at Z.

The transversal $\mathcal{T} = x^2z^2 + 2xy^2z + y^4 - x^2y^2 - \alpha_0 x^3y$

$$+ \alpha x^3y + \beta z^4 + \gamma yz^3 + \delta y^2z^2 + \varepsilon y^3z + \zeta y^4$$

Let us look at Z locally. Putting $z = 1$

$$\mathcal{J}_Z = x^2 + 2xy^2 + y^4 - x^2y^2 - \alpha_0 x^3 y$$

$$+ \alpha x^3 y + \beta + \gamma y + \delta y^2 + \varepsilon y^3 + \zeta y^4$$

$$M = (\alpha),\ U_Z = (\gamma, \delta, \varepsilon, \zeta),\ V_Z = (0),\ W_Z = (\beta)$$

Let $\mathcal{J}'_Z = x^2 + 2xy^2 + y^4 - x^2y^2 - \alpha_0 x^3 y$

$$+ \alpha x^3 y + \gamma y + \delta y^2 + \varepsilon y^3 + \zeta y^4$$

Then consider $f_Z = x^2 + 2xy^2 + y^4 - x^2y^2 - \alpha_0 x^3 y + \alpha x^3 y = 0$,

$f_Z \in m(2)$ and has singularity A_5 at the origin.

Now we want to show $\mathcal{J}'_Z$ is a universal unfolding of f_Z. Using Theorem 5.1.10, part IV, we can do this by showing $\{y, y^2, y^3, y^4\}$ form a base for $\dfrac{m(2)}{\langle \frac{\partial f}{\partial x_j} \rangle} = J_{f_Z}$

$$J_{f_Z} = \langle f_{z.x}, f_{z.y} \rangle \text{ where}$$

$$(f_Z)_x = 2x + 2y^2 - 2xy^2 - 3\alpha_0 x^2 y + 3\alpha x^2 y = 0$$

$$(f_Z)_y = 4xy + 4y^3 - 2x^2 y - \alpha_0 x^3 + \alpha x^3 = 0$$

Let $A = \langle x^4, x^3y, xy^3, x^2y^2 \rangle$. We start by showing

$A \subset J_{f_Z} + mA$

$x^4 \in J_{f_Z} + mA$ since $x^3(f_Z)_x \in J_{f_Z}$

$x^3y \in J_{f_Z} + mA$ since $x^2y(f_Z)_x \in J_{f_Z}$

$x^2y^2 \in J_{f_Z} + mA$ since $xy^2(f_Z)_x \in J_{f_Z}$

also $x^3 \in J_{f_Z} + mA$ since $x^2(f_Z)_x \in J_{f_Z}$

Now consider

$$yf_{Z\cdot x} = 2xy + 2y^3 - 2xy^3 - 3\alpha_o x^2y^2 + 3\alpha x^2y^2 \in J_{f_Z}$$

$$f_{Z\cdot y} = 4xy + 4y^3 - 2x^2y - \alpha_o x^3 + \alpha x^3 \in J_{f_Z}$$

$$2yf_{Z\cdot x} - f_{Z\cdot y} = -4xy^3 + 2x^2y - 6\alpha_o x^2y^2 + 6\alpha x^2y^2 + \alpha_o x^3 - \alpha x^3 \in J_{f_Z}$$

Hence $-4xy^3 + 2x^2y \in J_{f_Z} + mA$ (i)

By $xyf_{Z\cdot x} = 2x^2y + 2xy^3 - 2x^2y^3 - 3\alpha_o x^3y^2 + 3\alpha x^3y^2 \in J_{f_Z}$

we have $2x^2y + 2xy^3 \in J_{f_Z} + mA$ (ii)

Let (i)-(ii), we have $xy^3 \in J_{f_Z} + mA$

Therefore $A \subset J_{f_Z} + mA$

By Nakayama's Lemma $A \subset J_{f_Z}$ and $x^3 \in J_{f_Z}$.

Also, since $y^2 f_{Z \cdot y} = 4xy^3 + 4y^5 - 2x^2y^3 - \alpha_0 x^3 y^2 + \alpha x^3 y^2 \in J_{f_Z}$

$$\therefore \; y^5 \in J_{f_Z}$$

Then we have $m(2)^5 \subset J_{f_Z}$

Similarly, we can show

$$x^2 y \in J_{f_Z} \text{ by } xyf_{Z \cdot x} \in J_{f_Z}$$

Also, $\quad 2xy^2 + 2y^4 \in J_{f_Z}$ by $y^2 f_{Z \cdot x}$

$$4xy + 4y^3 \in J_{f_Z} \text{ by } f_{Z \cdot x}$$

$$2x^2 + 2xy^2 \in J_{f_Z} \text{ by } xf_{Z \cdot x}$$

$$2x + 2y^2 \in J_{f_Z} \text{ by } f_{Z \cdot x}$$

Therefore, y, y^2, y^3, y^4 form a base for $\frac{m(2)}{J_{f_Z}}$. That is, $\mathcal{J}'_Z$ is a universal unfolding of f_Z. Hence, we have the property that there exist a neighbourhood N of $\underline{0}$ of the $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-space such that if $\beta = \gamma = \delta = \varepsilon = \zeta = 0$, $\mathcal{J}$ has A_5 at Z.

Example 5.2.3. A_2A_3 (irr)

$$\Gamma = y^4 + x^2z^2 + \alpha xy^2 z + xy^3 = 0 \qquad \alpha \neq \pm 2$$

The singularities A_2 and A_3 are at X and Z respectively. The transversal

$$\mathcal{J} = y^4 + x^2z^2 + xy^3 + \alpha_0 xy^2 z + \alpha xy^2 z + \beta x^4 + \gamma x^3 y + \delta z^4 + \varepsilon yz^3 + \zeta y^2 z^2$$

Let us look at X locally. Putting $x = 1$

$$\mathcal{J}_X = y^4 + z^2 + y^3 + \alpha_0 y^2 z + \alpha y^2 z + \beta + \gamma y + \delta z^4 + \varepsilon yz^3 + \zeta y^2 z^2$$

$M = (\alpha)$, $U_X = (\gamma)$, $V_X = (\delta, \varepsilon, \zeta)$ ($\mathbb{C}^3$ with coordinates $\delta, \varepsilon, \zeta$), $W_X = (\beta)$.

Let $$\mathcal{J}'_X = y^4 + z^2 + y^3 + \alpha_0 y^2 z + \alpha y^2 z + \gamma y + \delta z^4 + \varepsilon yz^3 + \zeta y^2 z^2$$

Then consider $f_X = y^4 + z^2 + y^3 + \alpha_0 y^2 z + \alpha y^2 z + \delta z^4 + \varepsilon yz^3 + \zeta y^2 z^2 = 0, \qquad ((\alpha_0 + \alpha) \neq \pm 2)$

we have $f_X \in m(2)$ and has singularity A_2 at the origin. Now we want to show $\mathcal{J}'_X$ is a universal unfolding of f_X. That is, to show $\{y\}$ form a base for $\dfrac{m(2)}{\langle \frac{\partial f_X}{\partial x_j} \rangle = J_{f_X}}$.

$$J_{f_X} = \langle f_{X \cdot y}, f_{X \cdot z} \rangle$$

$$f_{X \cdot y} = 4y^3 + 3y^2 + 2\alpha_0 yz + 2\alpha yz + \varepsilon z^3 + 2\zeta yz^2 = 0$$

$$f_{X \cdot z} = 2z + \alpha_0 y^2 + \alpha y^2 + 4\delta z^3 + 3\ yz^2 + 2\zeta y^2 z = 0$$

We want to show $m(2)^2 \subset J_{f_X} + m(2)^3$

$$z^2 \in J_{f_X} + m^3 \text{ since } zf_{X \cdot z} \in J_{f_X}$$

$$yz \quad J_{f_X} + m^3 \text{ since } yf_{X \cdot z} \in J_{f_X}$$

Hence, $\quad y^2 \in J_{f_X} + m^3$ by $f_{X \cdot y} \in J_{f_X}$

Therefore, $m(2)^2 \subset J_{f_X} + m(2)^3$.

By Nakayama's Lemma $m(2)^2 \subset J_{f_X}$

Also by $f_{X \cdot z} \in J_{f_X}$, we have $z \in J_{f_X}$.

Therefore $\{y\}$ form a base for $\frac{m(2)}{J_{f_X}}$

That is, $\mathcal{J}'_X$ is a universal unfolding of f_X.
Hence we have a neighbourhood N of $\underline{0}$ of the space $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-space such that when $\beta = \gamma = 0$, then $\mathcal{J}$ has A_2 at X.
Similarly look at Z locally. Putting z = 1

$$\mathcal{J}_Z = y^4 + x^2 + xy^3 + \alpha_0 xy^2$$

$$+ \alpha xy^2 + \beta x^4 + \gamma x^3 y + \delta + \varepsilon y + \zeta y^2$$

$$M = (\alpha),\ U_Z = (\varepsilon,\zeta),\ V_Z = (\beta,\gamma),\ W_Z = (\delta)$$

Let $\mathcal{J}'_Z = y^4 + x^2 + xy^3 + \alpha_0 xy^2$

$$+ \alpha xy^2 + \beta x^4 + \gamma x^3 y + \varepsilon y + \zeta y^2$$

Then consider $f_Z = y^4 + x^2 + xy^3 + \alpha_o xy^2 + \alpha xy^2 + \beta x^4 + \gamma x^3 y = 0$,

we have $f_Z \in m(2)$ and has singularity A_3 at the origin.

We now want to show $\{y, y^2\}$ form a basis for $\frac{m(2)}{J_{f_Z}}$.

$$f_{Z \cdot x} = 2x + y^3 + \alpha_o y^2 + \alpha y^2 + 4\beta x^3 + 3\gamma x^2 y = 0$$

$$f_{Z \cdot y} = 4y^3 + xy^2 + 2\alpha_o xy + 2\alpha xy + \gamma x^3 = 0$$

Let us show $m(2)^3 \subset J_{f_Z} + m(2)^4$

$$x^3 \in J_{f_Z} + m(2)^4 \text{ since } x^2 f_{Z \cdot x} \in J_{f_Z}$$

$$x^2 y \in J_{f_Z} + m(2)^4 \text{ since } xy f_{Z \cdot x} \in J_{f_Z}$$

$$xy^2 \in J_{f_Z} + m(2)^4 \text{ since } y^2 f_{Z \cdot x} \in J_{f_Z}$$

Hence, $x^2 \in J_{f_Z} + m(2)^4$ since $x f_{Z \cdot x} \in J_{f_Z}$

Now consider $y f_{Z \cdot x} = 2xy + y^4 + (\alpha_o + \alpha) y^3 + 4\beta x^3 y + 3\gamma x^2 y^2 \in J_{f_Z}$

$$\text{Hence} \quad 2xy + (\alpha_o + \alpha) y^3 \in J_{f_Z} + m(2)^4 \qquad \text{(i)}$$

Also $f_{Z \cdot y} = 4y^3 + 3xy^2 + 2(\alpha_o + \alpha) xy + \gamma x^3 \in J_{f_Z}$

$$\text{Hence} \quad 4y^3 + 2(\alpha_o + \alpha) xy \in J_{f_Z} + m(2)^4 \qquad \text{(ii)}$$

By $4x(i)-(\alpha_o+\alpha)x(ii)$, $\quad 8xy - 2(\alpha_o+\alpha)^2xy \in J_{f_Z} + m(2)^4$

That is, $$2\left[4-(\alpha_o+\alpha)^2\right]xy \in J_{f_Z} + m(2)^4$$

Since $\alpha_o+\alpha \neq \pm 2$, $\quad xy \in J_{f_Z} + m(2)^4$

Then by $f_{Z\cdot y} \in J_{f_Z}$, $y^3 \in J_{f_Z} + m(2)^4$

Therefore $m(2)^3 \subset J_{f_Z} + m(2)^4$

By Nakayama's Lemma $m(2)^3 \subset J_{f_Z}$, also $x^2 \in J_{f_Z}$ and $xy \in J_{f_Z}$

By $f_{Z\cdot x} \in J_{f_Z}$, we can also see that

$$2x + (\alpha_o+\alpha)y^2 \in J_{f_Z}$$

Therefore $\{y,y^2\}$ form a basis for $\dfrac{m(2)}{J_{f_Z}}$

That is, $\mathcal{J}'_Z$ is a universal unfolding of f_Z. Hence we have a neighbourhood N of $\underline{0}$ of the $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-space such that when $\delta = \varepsilon = \zeta = 0$, has A_3 at Z.

Note that $(U_X \oplus W_X) \cap (U_Z \oplus W_Z) = \{0\}$.

Example 5.2.4 A_3^2

$$\Gamma = (x^2 + yz)(x^2 + \alpha yz) = 0 \qquad \alpha\neq 1,\ \alpha\neq 0$$

$$= x^4 + x^2yz + \alpha x^2yz + \alpha y^2z^2$$

The singularities are at Y and Z.

The transversal is

$$\mathcal{J} = x^4 + x^2yz + \alpha_0 x^2yz + \alpha_0 y^2z^2$$

$$+ \alpha(x^2yz+y^2z^2) + \beta y^4 + \gamma xy^3 + \delta x^2y^2 + \varepsilon z^4 + \zeta xz^3 + \eta x^2z^2$$

(codimension 7)

Locally at Y, the transversal becomes

$$\mathcal{J}_Y = x^4 + x^2z + \alpha_0 x^2z + \alpha_0 z^2 + \alpha x^2z + \alpha z^2$$

$$+ \beta + \gamma x + \delta x^2 + \varepsilon z^4 + \zeta xz^3 + \eta x^2z^2 = 0$$

$$M = (\alpha),\ U_Y = (\gamma,\delta),\ V_Y = (\varepsilon,\zeta,\eta),\ W_Y = (\beta)$$

Let $$\mathcal{J}_Y' = x^4 + x^2z + \alpha_0 x^2z + \alpha_0 z^2 + \alpha x^2z + \alpha z^2$$

$$+ \gamma x + \delta x^2 + \varepsilon z^4 + \zeta xz^3 + \eta x^2z^2 = 0$$

Then consider $$f_Y = x^4 + x^2z + \alpha_0 x^2z + \alpha_0 z^2 + \alpha x^2z + \alpha z^2$$

$$+ \varepsilon z^4 + \zeta xz^3 + \eta x^2z^2 = 0.$$

we have $f_Y \in m(2)$ and has singularity A_3 at the origin ($\alpha_0 \neq 0$). We want to show $\{x,x^2\}$ form a basis for $\frac{m(2)}{J_{f_Y}}$

$$f_{Y\cdot x} = 4x^3 + 2xz + 2(\alpha_0+\alpha)xz + \zeta z^3 + 2\eta xz^2 = 0$$

$$f_{Y\cdot z} = x^2 + (\alpha_0+\alpha)x^2 + 2(\alpha_0+\alpha)z + 4\varepsilon z^3 + 3\zeta xz^2 + 2\eta x^2z = 0$$

First we want $m(2)^3 \in J_{f_Y} + m(2)^4$

$$z^3 \in J_{f_Y} + m(2)^4 \text{ since } z^2 f_{Y\cdot z} \in J_{f_Y}$$

$$x^2 z \in J_{f_Y} + m(2)^4 \text{ since } x^2 f_{Y\cdot z} \in J_{f_Y}$$

$$xz^2 \in J_{f_Y} + m(2)^4 \text{ since } xz f_{Y\cdot z} \in J_{f_Y}$$

Now consider $xf_{Y\cdot z} = x^3 + (\alpha_0+\alpha)x^3 + 2(\alpha_0+\alpha)xz + 4\varepsilon xz^3 + 3\zeta x^2 z^2 + 2\eta x^3 z \in J_{f_Y}$

Hence, $(\alpha_0+\alpha+1)x^3 + 2(\alpha_0+\alpha)xz \in J_{f_Y} + m(2)^4$ (i)

And $f_{Y\cdot x} = 4x^3 + 2xz + 2(\alpha_0+\alpha)xz + \zeta z^3 + 2\eta xz^2 \in J_{f_Y}$

Hence, $4x^3 + 2(\alpha_0+\alpha+1)\, xz \in J_{f_Y} + m(2)^4$ (ii)

$4\times$(i) $-$ $(\alpha_0+\alpha+1)\times$(ii) gives $8(\alpha_0+\alpha)xz - 2(\alpha_0+\alpha+1)^2 xz \in J_{f_Y} + m(2)^4$

That is $(\alpha_0+\alpha-1)^2 xz \in J_{f_Y} + m(2)^4$

Since $(\alpha_0+\alpha) \neq 1$, we have $xz \in J_{f_Y} + m(2)^4$ and also $x^3 \in J_{f_Y} + m(2)^4$.

Hence, $m(2)^3 \subset J_{f_Y} + m(2)^4$.

This implies $m(2)^3 \subset J_{f_Y}$ by Nakayama's Lemma. Hence also $xz \in J_{f_Y}$ and by $zf_{Y\cdot z} \in J_{f_Y}$, we have $z^2 \in J_{f_Y}$ and $f_{Y\cdot z} \in J_{f_Y}$ implies $(\alpha_0+\alpha+1)x^2 + 2(\alpha_0+\alpha)z \in J_{f_Y}$. Therefore, $\{x, x^2\}$ form a basis for $\frac{m(2)}{J_{f_Y}}$. That is, by Theorem 5.1.10. $\mathcal{J}_Y'$ is a universal unfolding for f_Y.

Then, we have the property that there exist a neighbourhood N of $\underline{0}$ of the $(\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-space such that when

$$\beta = \gamma = \delta = 0, \ \mathcal{J} \text{ has } A_3 \text{ at } Y.$$

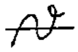
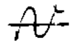
We can obtain similar result locally at Z.

$$M = (\alpha),\ U_Z = (\zeta,\eta),\ V_Z = (\beta,\gamma,\delta),\ W_Z = (\varepsilon)$$

$\mathcal{J}'_Z$ is an universal unfolding of f_Z.
And there exist neighbourhood N of $\underline{0}$ of $(\beta,\gamma,\ ,\varepsilon,\zeta,\eta)$-space such that when $\varepsilon = \zeta = \eta = 0$, $\mathcal{J}$ has A_3 at Z. Also, that

$$(U_Y \oplus W_Y) \cap (U_Z \oplus W_Z) = \{0\}.$$

3. Transversal of Normal Forms with Singularities at linearly dependent points

All the transversal we dealt with in Section 2 were those of the normal forms which have all their singularities fixed at the origins. The list of exceptions are actually those which have collinear singularities (A_1^3 , A_1^4 , $A_1^3A_2$ ,  A_1^5) or have more than three singularities (A_1^4 ) (three is the number of origins). Therefore we would not be able to look at each of the singularities locally without making at least one linear transformation. And, as from our experience, if we choose the set of generators $b_1(x) \ldots b_n(x)$ for the complementary space as before, any linear transformation would easily disturb the properties (I), (II), we claimed on the transversals in Section 2. Hence, we have to develop another method to choose the generators for the complementary spaces for this special list of normal forms, taking particular attention in satisfying the claimed properties I and II of transversals. Also for the purpose of easier to attain such properties, we have choosen the normal forms for this list to have their singularities fixed, the best possible at the vertices.

Again, the method is best shown by examples.

Example 5.3.1. A_1^4 

$$\Gamma = (xy + \alpha yz + (\alpha+1)xz)(xy + \beta yz + (\beta+1)xz)$$

$$\alpha \neq 0, -1$$

$$\beta \neq 0, -1 \qquad \alpha \neq \beta \qquad (\text{cod } 6)$$

$$= x^2y^2 + (\alpha+\beta)xy^2z + \alpha\beta y^2z^2 + (\alpha+1)(\beta+1)x^2z^2$$

$$+ (\alpha+\beta+2\alpha\beta)xyz^2 + (2+\alpha+\beta)x^2yz$$

The singularities are at X, Y, Z, P (1,-1,+1). Let us number the singularities say, 1,2,3,4 respectively. We start off by finding "an unfolding F for Γ" in a sense that when taken locally (e.g. when discussing singularity at X let x = 1 in F) at each singularity, F will be the unfolding of the singularity and also having the following properties: Suppose $b_1(x)$, $b_2(x)$, $b_3(x)$, $b_4(x)$ are the unfoldings for each of the singularities, with parameter $\gamma, \delta, \varepsilon, \zeta$ respectively. (Note that the sum of Milnor number + number of moduli in the normal form = codimension of the orbits)

We want the unfoldings such that there exist a neighbourhood N of 0 in $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-space such that

$\gamma = 0$, F_X has a node at X

$\delta = 0$, F_Y has a node at Y

$\varepsilon = 0$, F_Z has a node at Z

$\zeta = 0$, F_P has a node at P.

Let us take the $b_i(x)$ to be arbitrary, that is,

$$b_i = a_i x^4 + b_i y^4 + c_i z^4 + d_i x^3 y + e_i x y^3 + f_i x^3 z + g_i x z^3$$

$$+ h_i y^3 z + k_i y z^3 + m_i x^2 y z + n_i x y^2 z + p_i x y z^2 + q_i x^2 y^2$$

$$+ r_i x^2 z^2 + s_i y^2 z^2$$

$$= (a_i x^4 + \ldots\ldots + s_i y^2 z^2) \qquad i = 1,2,3,4$$

(1) At X. For a node to exist at X, we need when putting $x = 1$, the linear and constant terms be missing from b_2, b_3, and b_4. Allowing $a_1 = 1$, we then can decide

$$a_2 = a_3 = a_4 = 0$$
$$d_2 = d_3 = d_4 = 0$$
$$f_2 = f_3 = f_4 = 0$$

(2) At Y. For a node to exist at Y, we need when putting $y = 1$, the linear and constant terms be missing from b_1, b_3 and b_4. Allowing $b_2 = 1$, we then can decide

$$b_1 = b_3 = b_4 = 0$$
$$e_1 = e_3 = e_4 = 0$$

$$h_1 = h_3 = h_4$$

(3) At Z. Similarly, allowing $c_3 = 1$, we can decide

$$c_1 = c_2 = c_4 = 0$$
$$g_1 = g_2 = g_4 = 0$$
$$k_1 = k_2 = k_4 = 0$$

(4) At P. In order to observe the singularity, we have to take it to X by the following transformation

$$(**) \qquad \begin{aligned} x &\to x \\ y &\to y-x \\ z &\to z+x \end{aligned}$$

Then putting $x = 1$ there. This amounts to replacing y by $y-1$ and z by $z+1$ in F. That is,

$$(y-1)^2 + (\alpha+\beta)(y-1)^2(z+1) + \alpha\beta(y-1)^2(z+1)^2$$
$$+ (\alpha+1)(\beta+1)(z+1)^2 + (\alpha+\beta+2\alpha\beta)(y-1)(z+1)^2$$
$$+ (2+\alpha+\beta)(y-1)(z+1)$$
$$+ \sum_i \Big[a_i + b_i(y-1)^4 + c_i(z+1)^4 + d_i(y-1) + e_i(y-1)^3$$
$$+ f_i(z+1) + g_i(z+1)^3 + h_i(y-1)^3(z+1) + k_i(y-1)(z+1)^2$$
$$+ m_i(y-1)(z+1) + n_i(y-1)^2(z+1) + p_i(y-1)(z+1)^2$$
$$+ q_i(y-1)^2 + r_i(z+1) + s_i(y-1)^2(z+1)\Big].$$

The constant and linear terms are (those in normal form cancelled)

$$\sum_i \Big| (a_i+b_i+c_i-d_i-e_i+f_i+g_i-h_i-k_i-m_i+n_i-p_i+q_i+r_i+s_i)$$

$$+ (4b_i+d_i+3e_i+3h_i+k_i+m_i-2n_i+p_i-2q_i-2s_i)y$$

$$+ (+4c_i+f_i+3g_i-h_i-3k_i-m_i+n_i-2p_i+2r_i+2s_i)z \Big|$$

We require the constant and linear terms be missing from b_1, b_2 and b_3.

Allowing $a_4+b_4+c_4-d_4-e_4+f_4+g_4-h_4-k_4-m_4+n_4-p_4+q_4+r_4+s_4 = 1$

we then can decide that for i = 1,2,3

$$a_i+b_i+c_i-d_i-e_i+f_i+g_i-h_i-k_i-m_i+n_i-p_i+q_i+r_i+s_i = 0$$

$$4b_i+d_i+3e_i+3h_i+k_i+m_i-2n_i+p_i-2q_i-2s_i = 0$$

$$+ 4c_i+f_i+3g_i-h_i-3k_i-m_i+n_i-2p_i+2r_i+2s_i = 0$$

Then by the four sets of equations in (1), (2), (3) and (4), we can find a general solution as in the following diagram

	a	b	c	d	e	f	g	h	k	m	n	p	q	r	s
	x^4	y^4	z^4	x^3y	xy^3	x^3z	xz^3	y^3z	yz^3	x^2yz	xy^2z	xyz^2	x^2y^2	x^2z^2	y^2z^2
i=1	1	0	0	$+2/3$	0	$-2/3$	0	0	0						$1/3$
i=2	0	1	0	0	0	0	0	2	0						1
i=3	0	0	1	0	0	0	0	0	2						1
i=4	0	0	0	0	0	0	0	0	0						1

where m_i, n_i, p_i, g_i, r_i all zero, $i = 1,2,3,4$. Hence, we have the unfoldings

$$b_1 = 3x^4 + 2x^3y - 2x^3z + y^2z^2$$

$$b_2 = y^4 + 2y^3z + y^2z^2$$

$$b_3 = z^4 + 2yz^3 + y^2z^2$$

$$b_4 = y^2z^2$$

The next thing, we want to check whether they are actually universal unfoldings for each of the singularities locally.

Now

$$F = x^2y^2 + (\alpha_0+\beta_0)xy^2z + \alpha_0\beta_0y^2z^2 + (\alpha_0+1)(\beta_0+1)x^2z^2$$

$$+ (\alpha_0+\beta_0+2\alpha_0\beta_0)xyz^2 + (2+\alpha_0+\beta_0)x^2yz$$

$$+ \Big[(\alpha+\beta)xy^2z + (\alpha_0\beta+\alpha\beta_0+\alpha\beta)y^2z^2 + (\beta(\alpha_0+1)+\alpha(\beta_0+1)+\alpha\beta)\,{}^2z^2$$

$$+ (\alpha+\beta+2(\alpha_0\beta+\alpha\beta_0+\alpha\beta))xyz^2 + (\alpha+\beta)x^2yz\Big]$$

$$+ \gamma(3x^4 - 2x^3y + 2x^3z + y^2z^2)$$

$$+ \delta(y^4 + 2y^3z + y^2z^2)$$

$$+ \varepsilon(z^4 + 2yz^3 + y^2z^2)$$

$$+ \zeta(y^2z^2)$$

Putting $x = 1$, we have

$$F_X = \Gamma_{oX} + \big[(\alpha+\beta)y^2z + \cdots + (\alpha+\beta)yz\big]$$

$$+ \gamma(3 - 3y + 2z + y^2z^2)$$

$$+ \delta(y^4 + 2y^3z + y^2z^2)$$

$$+ \varepsilon(z^4 + 2yz^3 + y^2z^2)$$

$$+ \zeta(y^2z^2)$$

Let $f_X = \Gamma_{oX} + \left[(\alpha+\beta)y^2z + \cdots + (\alpha+\beta)yz\right]$

$$+ \delta(y^4 + 2y^3z + y^2z^2)$$

$$+ \varepsilon(z^4 + 2yz^3 + y^2z^2)$$

$$+ \zeta(y^2z^2).$$

$f_X \in m(2)$ and has singularity A_1 at the origin. We now want to show $\frac{m(2)}{J_{f_X}}$ is spanned by the vector $\{(3-3y+2z + y^2z^2)\}$

$$f_{X \cdot y} = (2+\alpha_o+\beta_o)z + (\alpha+\beta)z$$

$$+ 2y \cdots\cdots + \text{higher terms} \in J_{f_X}$$

$$f_{X \cdot z} = 2(\alpha_o+1)(\beta_o+1)z + 2(\beta(\alpha_o+1)+\alpha(\beta_o+1)+\alpha\beta)z$$

$$+ (2+\alpha_o+\beta_o)y+(\alpha+\beta)y + \text{higher terms} \in J_{f_X}$$

This implies $\left[2 + (\alpha_o+\alpha) + (\beta_o+\beta)\right] z + 2y \in J_{f_X} + m(2)^2$

$$2\left[(\alpha_o+1+\alpha)(\beta_o+1+\beta)\right]z + \left[2+(\alpha_o+\alpha) + (\beta_o+\beta)\right]y \in J_{f_X} + m(2)^2$$

Hence y and z $\in J_{f_X} + m(2)^2$

provided $4\left[(\alpha_o+\alpha+1)(\beta_o+\beta+1)\right] \neq \left[2+(\alpha_o+\alpha)+ (\beta_o+\beta)\right]^2$

That is $\left[(\alpha_o+\alpha) - (\beta_o+\beta)\right]^2 \neq 0$, which is our condition on the normal form.

Therefore, by Nakayama's Lemma $m \subset J_{f_X}$. The element $3-3y+2z+y^2z^2$ obviously spans $\frac{m(2)}{J_{f_X}}$. That is, F_X is a universal unfolding of f_X and we have the property that there exist a neighbourhood N of $\underline{0}$ in $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-space such that $\gamma = 0$, F has an A_1 at X. Similarly, we can do the same thing at Y and Z. Putting y = 1, we have

$$F_Y = \Gamma_{oY} + \left[(\alpha+\beta)xz + \dots\dots + (\alpha+\beta)x^2z\right]$$
$$+ \gamma(3x^4 - 2x^3 + 2x^3z + z^2)$$
$$+ \delta(1 + 2z + z^2).$$
$$+ \varepsilon(z^4 + 2z^3 + z^2)$$
$$+ \zeta(z^2)$$

Let

$$f_Y = \Gamma_{oY} + \left[(\alpha+\beta)xz + \dots\dots\right]$$
$$+ \gamma(3x^4 - 2x^3 + 3x^3z + z^2)$$
$$+ \varepsilon(z^4 + 2z^3 + z^2)$$
$$+ \zeta(z^2).$$

$f_Y \in m(2)$ and has singularity A_1 at the origin. We now want to show $\frac{m(2)}{J_{f_Y}}$ is spanned by the vector $\{(1+2z+z^2)\}$

$$f_{Y \cdot x} = 2x + (\alpha_0+\alpha+\beta_0+\beta)z + \text{higher terms} \in J_{f_Y}$$

$$f_{Y \cdot z} = 2(\alpha_0+\alpha)(\beta_0+\beta)z + (\alpha_0+\alpha+\beta_0+\beta)x$$
$$+ 2\gamma z + 2\varepsilon z + 2\zeta z \in J_{f_Y}$$

Hence, we have

$$2x + (\alpha_0+\alpha+\beta_0+\beta)z \in J_{f_Y} + m^2$$

$$(\alpha_0+\alpha+\beta_0+\beta)x + 2\left[(\alpha_0+\alpha)(\beta_0+\beta)+\gamma+\varepsilon+\zeta\right]z \in J_{f_Y} + m^2$$

This implies x and $z \in J_{f_Y} + m^2$ provided that

$$4\left[(\alpha_0+\alpha)(\beta_0+\beta)+\gamma+\varepsilon+\zeta\right] \neq (\alpha_0+\alpha+\beta_0+\beta)^2$$

when $\gamma = \varepsilon = \zeta = 0$, this condition becomes $\left[(\alpha_0+\alpha) - (\beta_0+\beta)\right]^2 \neq 0$, which is our condition for normal form. Since it is true when $\gamma = \varepsilon = \zeta = 0$, there exist a neighbourhood of $\underline{0}$ in which it remains true. Therefore we have $m \subset J_{f_Y} + m^2$, which implies $m \subset J_{f_Y}$. Then the vector $(1+2z+z^2)$ spans $\frac{m(2)}{J_{f_Y}}$. That is, F_Y is a universal unfolding of f_Y and we have the property that there exist a neighbourhood N of $\underline{0}$ in $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-space such that when $\delta = 0$, F has an A_1 at Y. The result for Z is similar: There exist neighbourhood N of $\underline{0}$ in $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-space such that when $\varepsilon = 0$, F has an A_1 at Z. But as for P, we have to take it to one of the vertices, say X. That is, by taking the transformation (**). Then putting x = 1, and let $\alpha' = \alpha_0+\alpha$, $\beta' = \beta_0+\beta$ we have

$$F_p = (y-1)^2 + (\alpha'+\beta')(y-1)^2(z+1) + \alpha'\beta'(y-1)^2(z+1)^2$$

$$+ (\alpha'+1)(\beta'+1)(z+1)^2 + (\alpha'+\beta'+2\alpha'\beta')(y-1)(z+1)^2$$

$$+ (2+\alpha'+\beta')(y-1)(z+1)$$

$$+ \gamma\left[3 + 2(y-1) - 2(z+1) + (y-1)^2(z+1)^2\right]$$

$$+ \delta\left[(y-1)^4 + 2(y-1)^3(z+1) + (y-1)^2(z+1)^2\right]$$

$$+ \varepsilon\left[(z+1)^4 + 2(y-1)(z+1)^3 + (y-1)^2(z+1)^2\right]$$

$$+ \zeta\left[(y-1)^2(z+1)^2\right]$$

Let $f_P = \Gamma_P$

$$+ \gamma\left[3 + 2(y-1) - 2(z+1) + (y-1)^2(z+1)^2\right]$$

$$+ \delta\left[(y-1)^4 + 2(y-1)^3(z+1) + (y-1)^2(z+1)^2\right]$$

$$+ \varepsilon\left[(z+1)^4 + 2(y-1)(z+1)^3 + (y-1)^2(z+1)^2\right].$$

We want to show the vector $(y-1)^2(z+1)^2$ spans $\frac{m(2)}{J_{f_P}}$.

Let us now write f_P in a simple form

$$f_P = y^2 + (\alpha'+\beta')(y^2-2yz) + \alpha'\beta'(y^2-4yz+z^2)$$

$$+ (\alpha'+1)(\beta'+1)z^2 + (\alpha'+\beta'+2\alpha'\beta')(2yz-z^2)$$

$$+ (2+\alpha'+\beta')(yz) + \text{higher terms}$$

$$+ \gamma\left[y^2-4yz+z^2 + \text{higher terms}\right]$$

$$+ \delta\left[y^2+2yz+z^2 + \text{higher terms}\right]$$

$$+ \varepsilon\left[y^2+2yz+z^2 + \text{higher terms}\right]$$

Now

$$f_{P\cdot y} = \left[2+2(\alpha'+\beta') + 2\alpha'\beta'\right]y$$

$$+ \left[2(\alpha'+\beta')-4\alpha'\beta' + 2(\alpha'+\beta'+2\alpha'\beta')\right.$$

$$\left. + (2+\alpha'+\beta')\right] z + \text{higher terms}$$

$$+ \left[2\gamma+2\delta+2\varepsilon\right]y + \left[-4\gamma+2\delta+ 2\varepsilon\right]z + \text{higher terms.}$$

$$f_{P\cdot z} = \left[-2(\alpha'+\beta')-4\alpha'\beta' + 2(\alpha'+\beta'+2\alpha'\beta') + 2(\alpha'+\beta')\right]y$$

$$+ \left[2\alpha'\beta'+2(\alpha'+1)(\beta'+1)-2(\alpha'+\beta'+2\alpha'\beta')\right]z + \text{higher terms}$$

$$+ \left[-4\gamma+2\delta+2\varepsilon\right]y$$

$$+ \left[+2\gamma+2\delta+2\ \ \right]z + \text{higher terms}$$

Therefore

$$f_{P\cdot y} = 2\left[(1+\alpha'+\beta'+\alpha'\beta') + \gamma+\delta+\xi\right]y$$

$$+ \left[(2+\alpha'+\beta') - 4\gamma+2\delta+2\xi\right]z + \text{higher terms} \in J_{f_P}$$

$$f_{P\cdot z} = \left[(2+\alpha'+\beta') - 4\gamma+2\delta+2\varepsilon\right]y$$

$$+2\left[1 + \gamma + \delta + \varepsilon\right]z \qquad + \text{higher terms} \in J_{f_P}$$

That is, $2\left[(1+\alpha'+\beta'+\alpha'\beta')+\gamma+\delta+\xi\right]y$

$$+\left[(2+\alpha'+\beta') - 4\gamma+2\delta+2\xi\right]z \in J_{f_P} + m(2)^2$$

and $\left[(2+\alpha'+\beta') - 4\gamma+2\delta+2\xi\right]y$

$$+ 2\left[1+\gamma+\delta+\varepsilon\right]z \qquad \in J_{f_P} + m(2)^2$$

This implies y and z $\in J_{f_P} + m(2)^2$ provided that

$$\left[(2+\alpha'+\beta') - 4\gamma+2\delta+2\xi\right]^2 \neq 4\left[(1+\alpha'+\beta'+\alpha'\beta') + \gamma+\delta+\xi\right]\cdot\left[1+\gamma+\delta+\varepsilon\right]$$

When $\alpha = \beta = \gamma = \delta = \xi = 0$, this condition becomes

$$(2+\alpha_o+\beta_o)^2 \neq 4(1+\alpha_o+\beta_o+\alpha_o\beta_o)$$

That is, $(\alpha_o-\beta_o)^2 \neq 0$, which is our condition on the normal form.

Hence $m(2) \subset J_{f_P} + m(2)^2$ and by Nakayama's Lemma $m(2) \subset J_{f_P}$.

Then, the vector $(y-1)^2(z+1)^2$ does span $\dfrac{m(2)}{J_{f_P}}$.

Then F_p is a universal unfolding of f_p. We have the property that there exists a neighbourhood N of $\underline{0}$ in $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ - space such that when $\zeta = 0$, F has an A_1 at P. This shows that F is the unfolding we want for Γ.

Then we are left to check that F is actually a transversal for Γ, i.e. F is transverse to every orbit of the stratum Σ. Actually, we can prove a slightly stronger result, if $f \in F \cap \Sigma$,

$$T_f(G.f) \cap T_f(F) = \{0\} \text{ - zero vector.}$$

Let us look at the tangent space to the normal form at (α_0, β_0). It is spanned by

$$xy^2z + \beta_0 y^2z^2 + (\beta_0+1)x^2z^2 + (1+2\beta_0)xyz^2 + x^2yz \text{ —— tangent ①}$$

$$xy^2z + \alpha_0 y^2z^2 + (\alpha_0+1)x^2z^2 + (1+2\alpha_0)xyz^2 + x^2yz \text{ —— tangent ②}$$

And tangent space of the transversal at (α, β) is spanned by these two vectors together with the four unfolding vectors.

We can easily find the tangent space to the orbits by the following

$$f_x = 2xy^2 + (\alpha_0+\beta_0)y^2z + 2(\alpha_0+1)(\beta_0+1)xz^2 + (\alpha_0+\beta_0+2\alpha_0\beta_0)yz^2 + 2(2+\alpha_0+\beta_0)xyz$$

$$f_y = 2x^2y + 2(\alpha_0+\beta_0)xyz + 2\alpha_0\beta_0 yz^2 + (\alpha_0+\beta_0+2\alpha_0\beta_0)xz^2 + (2+\alpha_0+\beta_0)x^2z$$

$$f_z = (\alpha_0+\beta_0)xy^2 + 2\alpha_0\beta_0 y^2z + 2(\alpha_0+1)(\beta_0+1)x^2z + 2(\alpha_0+\beta_0+2\alpha_0\beta_0)xyz + (2+\alpha_0+\beta_0)x^2y$$

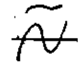
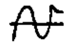
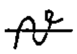
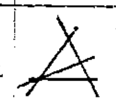
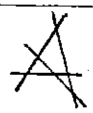
So the above claimed result, is equivalently shown by the nonsingularity of the following matrix,

		x^4	y^4	z^4	x^3y	x^3z	x^3z	xz^3	y^3z	yz^3	x^2yz	xy^2z	xyz^2	x^2y^2	x^2z^2	y^2z^2
λ_1	xf_x										$2(2+\alpha_0+\beta_0)$	$(\alpha_0+\beta_0)$	$(\alpha_0+\beta_0+2\alpha_0\beta_0)$	2	$2(\alpha_0+1)(\beta_0+1)$	
λ_2	yf_x					2			$\alpha_0+\beta_0$			$2(2+\alpha_0+\beta_0)$	$2(\alpha_0+1)(\beta_0+1)$			$(\alpha_0+\beta_0+2\alpha_0\beta_0)$
λ_3	zf_x							$2(\alpha_0+1)(\beta_0+1)$		$(\alpha_0+\beta_0+2\alpha_0\beta_0)$		2	$2(2+\alpha_0+\beta_0)$			$(\alpha_0+\beta_0)$
λ_4	xf_y				2		$(2+\alpha_0+\beta_0)$				$2(\alpha_0+\beta_0)$		$2\alpha_0\beta_0$		$(\alpha_0+\beta_0+2\alpha_0\beta_0)$	
λ_5	yf_y										$(2+\alpha_0+\beta_0)$	$2(\alpha_0+\beta_0)$	$(\alpha_0+\beta_0+2\alpha_0\beta_0)$	2		$2\alpha_0\beta_0$
λ_6	zf_y							$(\alpha_0+\beta_0+2\alpha_0\beta_0)$		$2\alpha_0\beta_0$	2		$2(\alpha_0+\beta_0)$		$(2+\alpha_0+\beta_0)$	
λ_7	xf_z				$(2+\alpha_0+\beta_0)$		$2(\alpha_0+1)(\beta_0+1)$				$2(\alpha_0+\beta_0+2\alpha_0\beta_0)$	$2\alpha_0\beta_0$		$(\alpha_0+\beta_0)$		
λ_8	yf_z					$(\alpha_0+\beta_0)$			$2\alpha_0\beta_0$		$2(\alpha_0+1)(\beta_0+1)$	$2(\alpha_0+\beta_0+2\alpha_0\beta_0)$		$(2+\alpha_0+\beta_0)$		
λ_9	zf_z										$(2+\alpha_0+\beta_0)$	$(\alpha_0+\beta_0)$	$2(\alpha_0+\beta_0+2\alpha_0\beta_0)$		$2(\alpha_0+1)(\beta_0+1)$	$2\alpha_0\beta_0$
	b_1	3			2		-2									1
	b_2		1						2							1
	b_3			1						2						1
	b_4															1
Tangent(1)											1	1	$1+2\beta_0$		β_0+1	β_0
Tangent(2)											1	1	$1+2\alpha_0$		α_0+1	α_0

We can find the determinant of this matrix is equal to $(\alpha_0-\beta_0)^2$ which is definitely not zero since we have the condition $\alpha_0\neq\beta_0$. Hence the matrix is nonsingular and the transversality is established. Therefore F is a transversal for Γ.

The Transversals of all the normal form with singularities at linearly dependent points had been found and checked. The following is the list.

List 5.3.2.

Types	Picture	Normal Forms.	Generators of the Transversals.
A_1^3		$x(xy^2+y^2z+yz^2+\gamma x^2z+\alpha xz^2+\beta xyz)=0$ $\gamma\neq 0$	$x^2z^2,\ x^2yz,\ x^3z,\ y^2z^2,$ $y^4+2y^3z+y^2z^2,\ z^4+2yz^3+y^2z^2$
$A_1^3A_2$		$x(y^2z+yz^2+xz^2+2\alpha xyz+\alpha^2xy^2)=0$ $\alpha\neq 0 \quad \alpha\neq 1$	$x^4,\ x^3z+x^2y^2,\ y^2z^2,\ x^2yz+\alpha x^2y^2,$ $y^4+2y^3z+y^2z^2,\ z^4+2yz^3+y^2z^2$
A_1^4		$x(xy^2+y^2z+yz^2+\alpha xz^2+\beta xyz)=0$ $\beta^2\neq 4\alpha,\ \alpha\neq 0,\ \alpha-\beta+1\neq 0$	$x^2z^2,\ x^2yz,\ x^4,\ y^2z^2$ $y^4+2y^3z+y^2z^2,\ z^4+2yz^3+y^2z^2.$
A_1^4		$[xy+\alpha yz+(\alpha+1)xz][xy+\beta yz+(\beta+1)xz]=0$ $\alpha\neq 0,-1$ $\beta\neq 0,-1. \quad \alpha\neq\beta$	$xy^2z+\beta y^2z^2+(\beta+1)x^2z^2+(1+2\beta)xyz^2+x^2yz,$ $xy^2z+\alpha y^2z^2+(\alpha+1)x^2z^2+(1+2\alpha)xyz^2+x^2yz,$ $3x^4\pm 2x^3y\mp 2x^3z+y^2z^2,$ $y^4+2y^3z+y^2z^2,\quad y^2z^2.$ $z^4+2yz^3+y^2z^2$
A_1^5		$yz(x^2+\alpha yz+xz+xy)=0$ $\alpha\neq 0,1$	$y^2z^2,\ x^4+2x^3y+2x^3z+2x^2yz+x^2y^2+x^2z^2,$ $y^4+2xy^3+x^2y^2,\ z^4+2xz^3+x^2z^2$ $x^2y^2,\ x^2z^2.$
$A_1^3A_3$		$yz(x^2+yz+xz)=0$	$x^4+2x^3z+x^2z^2,\ y^4,\ xy^3,\ x^2y^2,$ $z^4+2xz^3+x^2z^2,\ x^2z^2$
$A_1^3D_4$		$xyz(x+y)=0$ (cod 7)	$x^4+2x^3y+x^2y^2,\ y^4+2xy^3+x^2y^2,$ $z^4,\ xz^3,\ yz^3,\ xyz^2,\ x^2y^2$
A_1^6		$xyz(x+y+z)=0$	$x^4+2x^3y+2x^3z+2x^2yz+x^2y^2+x^2z^2$ $y^4+2xy^3+2y^3z+2xy^2z+x^2y^2+y^2z^2$ $z^4+2xz^3+2yz^3+2xyz^2+x^2z^2+y^2z^2$ $x^2y^2,\ x^2z^2,\ y^2z^2.$

5.4 Strata are Manifolds

The main object of this section is to prove that the strata we have chosen for the stratification are all manifolds. Since we know orbits are submanifolds, we need only to show this for cases where there are moduli, and note that in all these cases the singularities are isolated. To achieve our object, first we need a lemma.

Lemma 5.4.1. Let M be a smooth manifold under the action of a group G and $\Sigma \subseteq M$ be an invariant set under the action. Then Σ is a submanifold of M iff for every point in Σ there is a transversal $\mathcal{J}$ such that $\Sigma \cap \mathcal{J}$ is a submanifold of $\mathcal{J}$, [Gibson, C.G. 1976].

Let Γ be the normal form of the stratum Σ. By Lemma 5.4.1. and the homogeneity of orbits, to prove the stratum is a manifold, it is enough to show that for every point Γ_λ on the normal form Γ, $\lambda \in$ space of moduli of Γ, there is a transversal $\mathcal{J}_\lambda$ such that $\Sigma \cap \mathcal{J}_\lambda =$ a submanifold of $\mathcal{J}$.

Let $\mathcal{J}$ be the transversal we had chosen for the normal form Γ. Note that $\mathcal{J}$ can be taken as transversal for every point of the normal form. Now if Γ_λ is a point on the normal form Γ and we take $\mathcal{J}_\lambda$ to be a neighbourhood of Γ_λ in $\mathcal{J}$, we claim that we have the following property: If the neighbourhood $\mathcal{J}_\lambda$ is chosen small enough, then

$$\Sigma \cap \mathcal{J}_\lambda = \Gamma \cap \mathcal{J}_\lambda \qquad (*)$$

This means that, a point in the neighbourhood $\mathcal{J}_\lambda$ of Γ_λ is in the stratum Σ, that is having the same singularity type as the curves in the normal form, iff it is in the normal form Γ.

Notice that the size of the neighbourhood $\mathcal{J}_\lambda$ may be different for each λ, and the neighbourhood is tranverse to the orbit through Γ_λ at Γ_λ and hence to all nearby orbits. If (*) is true, then since we know (from Chapter 3) that all the normal forms are manifolds (actually submanifolds of their transversals) we have $\Gamma \cap \mathcal{J}_\lambda$ is a submanifold of $\mathcal{J}_\lambda$. So this will prove the main result.

Let us now start to prove our claim (*). It is easy to see that for the points in the neighbourhood $\mathcal{J}_\lambda$, if they are in the normal form Γ, they are in the stratum, that is

$$\Sigma \cap \mathcal{J}_\lambda \supset \Gamma \cap \mathcal{J}_\lambda$$

But the reverse is not as easy. This involves working in the jet space and we need a further lemma.

Lemma 5.4.2. Let $p: (X \times Y, 0) \to (M, 0)$ be a smooth map germ, where X^d, Y^r are Euclidean spaces and M^m smooth manifold, and $(Z,0) \subset (M^m,0)$ is a submanifold of M with $\operatorname{codim} Z = \dim Y = r$. Suppose.

$p(X \times \{0\}) \subset Z$ and p is transverse to Z at $\underline{0}$,

then there exists a neighbourhood A of $\underline{0}$ in $X \times Y$ such that for all $(a,b) \in A$, $p(a,b) \in Z$ iff $b = 0$.

Proof. We can write $(M^n, 0)$ locally as $(\mathcal{U} \times Z, 0)$ where $\dim \mathcal{U} = r$. Let $\pi: (\mathcal{U} \times Z, 0) \to (\mathcal{U}, 0)$ be the projection onto $\mathcal{U}$. Consider the composite $\pi \circ p: (X \times Y, 0) \to (\mathcal{U}, 0)$, we have $\pi \circ p(X \times \{0\}) = \{0\}$ and by the transversality the tangent map $T_0(\pi \circ p)$ is surjective.

Then by Hadamard lemma, we can write

$$\pi \circ p(x_1, \cdots x_d, y_1, \cdots y_r)$$

$$= \begin{pmatrix} p_{11}(x,y) & \cdots & p_{1r}(x,y) \\ \vdots & & \vdots \\ p_{r1}(x,y) & \cdots & p_{rr}(x,y) \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}$$

where the p_{ij} are smooth and

$$\det (p_{ij}(0,0)) \neq 0 \qquad \text{(max. rank)}$$

Let A be a neighbourhood inside which $\det(p_{ij}(x.y)) \neq 0$. Then for every $(a,b) \in A$,

$$p(a,b) \in Z \text{ iff } \pi \circ p(a,b) = 0 \text{ iff } (p_{ij}(a,b)) \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{iff } b_1 = \cdots = b_r = 0 \qquad \text{Q.E.D.}$$

Before we start to prove the reverse, we also need to know several facts. Suppose the normal form Γ has singularities $\sigma_1 \ldots \sigma_\ell$ at points $P_1 \ldots P_\ell$ where $\ell \leq 6$. To discuss the conditions on the points in the neighbourhood $\mathcal{T}_\lambda$ such that they have the same singularity type as the curves in the normal form, we have to look at each of the singularities locally. As expalined before, to look at a singularity σ_i of the normal form Γ locally, we take Γ through a transformation which will move the singularity σ_i to a vertex X_i of the triangle of reference and then let $x_i = 1$ in the resulting equation. Note that actually in most cases, the singularities are already at the vertices and hence we do not have to take any transformations. Then the normal form becomes a representative of a germ

$$h = \Gamma(\sigma_i) : (N, 0) \to \mathbb{C}$$

where $N \subset \mathbb{C}^2$ in our case.

Fact (i). h is k-determined (some k), since all the singularities under discussion are isolated.

If we do the same thing to the transversal $\mathcal{J}$ of the normal form Γ, taking it through the above mentioned transformation if necessary and letting $x_i = 1$, we have a representative of the germ

$$H = \mathcal{J}(\sigma_i) : (N \times M \times U \times V \times W, 0) \to \mathbb{C}$$

By the properties of transversals (Theorem 5.2.1), we can always have the transversal written locally in this form, where M corresponds to the parameters of the moduli terms of Γ;

U corresponds to the parameters of the universal unfolding term of σ_i;

V corresponds to the parameters of the universal unfolding terms of the rest of singularities;

W corresponds to the constant term.

For the matter of later convenience, we shall now combine V and M into just V. We can do so because both V and M would not affect the unfolding and we can regard them as one. Therefore we write

$$H = \mathcal{J}(\sigma_i) : (N \times U \times V \times W, 0) \to \mathbb{C}$$

Fact (ii). By Property I of $\mathcal{J}$, we know for $(x,u,v,w) \in (N\times U\times V\times W)$ $H(x,u,v,0)$ is a universal unfolding for $H(x,0,v,0)$.

Fact (iii). By fact (ii) and Part III of Theorem 5.1.10, we have $H(x,u,v,w)$ a versal unfolding of h.

Fact (iv). For each v close to $\underline{0}$, $H_{0,v,0}(x)$ is R-equivalent to h, that is, $H_{0,v,0}(x)$ has the same singularity as h. In all our cases, it is easy to check that $H_{0,v,0}(x)$ has the same Milnor number μ as h. And also by looking at the lowest terms of the equation we can always tell the number of distinct tangents and the multiplicity of the singularity. Hence it enables us to check whether the singularity is of the same type as h or not. For example (see Table 2.4), there are only two singularity types with $\mu = 4$, A_4 and D_4. But A_4 has multiplicity 2 and D_4 has 3. Similarly, we can differentiate between A_5 and D_5. For $\mu = 6$, we have A_6, E_6 and D_6. But A_6 has 1 tangent-direction and multiplicity 2

D_6 has 3 tangents, multiplicity 3

E_6 has 1 tangent, multiplicity 3.

Hence we can always tell the difference.

Now consider the following map which was introduced in Section 5.1.

$$j_1^k(H) : N \times U \times V \to J_o^k(n,1)$$

$(x,u,v) \mapsto$ the k-jet of function $H_{u.v}$ expanded about x,

where n = dim N (n = 2 in our case).

Notice that the V now includes the parameter of the universal unfolding terms of the other singularities and also the moduli parameters M. Since $J_o^k(n,1)$ is the k-jet space with the constant term zero, we shall deal with the case with the constant term W = 0 in all the mappings first. The case with the $W \neq 0$ will be shown later.

Let us put $j_1^k(H) = P$, $V = X^d$, $N^2 \times U = Y^r$, $J_o^k(n,1) = M^m$, R-orbit of h = Z in the Lemma 5.4.2.

Using Fact (i) and Fact (iii), h is k-determined, H is versal. Then by Theorem 5.1.7, H is k-transversal, that is, $j_1^k(H)$ is transverse to the $\mathbb{R}^{(k)}$-orbit of h at $\underline{0}$.

Using Fact (ii) and Fact (iv), when W = 0, $N \times U \to \mathbb{C}$ is the universal unfolding of h. Hence N x U is the minimum dimensional vector space which can be transverse to the tangent space to the orbit. Therefore the dimension of this vector space is exactly the codimension of the orbit, that is, dim N x U = codim Z.

Then we can conclude from Lemma 5.4.2. that there exists a neighbourhood A of $\underline{0}$ in N x U x V such that for every (x,u,v) $\in$ A,

if $j_1^k(H)$ (x,u,v) $\in$ $R^{(k)}$-orbit of h

then $x = \underline{0}$, $u = \underline{0}$.

That is, the only point which can be the same singularity as he is (0,0,v).

Let us now deal with the case when $W \neq 0$. Let $H_{u,v,w}(x)$ =

$H_{u,v}(x)+w$, that is $H_{u,v}(x)$ is the function without the constant term. Obviously $H_{u,v}(\underline{0}) = 0$. Suppose $H_{u,v}(x)+w$ has a singularity at x_o $\mathcal{R}$-equivalent to h. Thus $H_{u,v}(x_o)+w=0$. By definition of $j_1^k(H)$, $j_1^k(x_o, u, v) \in \mathcal{R}$-orbit of h. Hence again by lemma, we have $x_o = \underline{0}$ and $u = \underline{0}$, so $w = -H_{u,v}(\underline{0}) = 0$. Therefore, provided x_o, u, v, w are sufficiently close to $\underline{0}$, we have $H_{u,v}(x)+w$ has a singularity at x_o $\mathcal{R}$-equivalent to $h \Rightarrow x_o = \underline{0}$, $u = \underline{0}$, $w = 0$.

Finally if we want to ensure that x_o, u, v, are within a given neighbourhood of $\underline{0}$, we can choose u,v,w sufficiently close to $\underline{0}$, since the singularities are isolated.

The above argument can be applied to every singularity σ_i of the normal form.

And we have the property

$$(U_{\sigma_i} \oplus W_{\sigma_i}) \cap (U_{\sigma_j} \oplus W_{\sigma_j}) = \{0\} \text{ for } i \neq j$$

That is, the unfolding parameters corresponding to one singularity will not correspond to another. If we write

$$\mathcal{U} = (U_{\sigma_1} \oplus W_{\sigma_1}) \oplus (U_{\sigma_2} \oplus W_{\sigma_2}) \ldots \oplus (U_{\sigma_\ell} \oplus W_{\sigma_\ell})$$

then for a small enough neighborhood $\mathcal{J}_\lambda$ of Γ_λ a point $(x, v, u_\sigma, w_\sigma, \ldots, u_{\sigma_\ell}, w_{\sigma_\ell})$ is in the Stratum Σ, (that is has singularities $\sigma_1 \ldots \sigma_\ell$ at the points $P_1 \ldots P_\ell$), if and only if $u_{\sigma_1} = w_{\sigma_1} = 0, \ldots, u_{\sigma_\ell} = w_{\sigma_\ell} = 0$.

That is, just a point on the normal form.

Hence $$\Sigma \cap \mathcal{J}_\lambda \subset \Gamma \cap \mathcal{J}_\lambda$$

Therefore we have $$\Sigma \cap \mathcal{J}_\lambda = \Gamma \cap \mathcal{J}_\lambda$$

Since we have proved in Chapter 3 that all the normal forms are manifolds (actually submanifolds of their transversals) we know that $\Gamma \cap \mathcal{J}_\lambda$ is a submanifold of $\mathcal{J}_\lambda$. Also because $\mathcal{J}$ can be taken transversal for every point of the normal form Γ, for every point Γ_λ of the normal form Γ, there exist transversal (neighbourhood $\mathcal{J}_\lambda$ of $\Gamma_\lambda) \subset \mathcal{J}$ such that $\Sigma \cap \mathcal{J}_\lambda = \Gamma \cap \mathcal{J}_\lambda$ = submanifold of $\mathcal{J}_\lambda$. Therefore, by Lemma 5.4.1, we have

Theorem 5.4.3. All the strata are manifolds (for $X_9(\tilde{E}_7)$ see Remark (ii)).

Remark (i). Furthermore the strata are connected. This is because the group $PGL(3,\mathbb{C})$ is connected and by the group action, any two point on the manifold can be taken along the orbits to the normal form. Since the normal form itself is the complement of a subvariety in an affine space, it is connected. Therefore we can find a path for any two points on the manifold.

Remark (ii).

We have checked that the Transversal $\mathcal{J}$ chosen for the normal form $x^4 + \alpha x^2z^2 + z^4 = 0$ for $X_9(\tilde{E}_7)$ in the list 4.2 also has the property that for a point Γ_λ on the normal form, these exist a transversal (neighbourhood $\mathcal{J}_\lambda$ of $\Gamma_\lambda) \subset \mathcal{J}$ such that $\Sigma \cap \mathcal{J}_\lambda = \Gamma \cap \mathcal{J}_\lambda$ = submanifold of $\mathcal{J}_\lambda$. Therefore, Lemma 5.4.1. can also be applied and gives us the stratum $X_9(\tilde{E}_7)$ also a manifold.

Chapter 6

Regularity (Whitney)

A Whitney stratification of $M = \mathbb{C}^{15}-\{0\}$ is a partition of the manifold into a finite, disjoint union of manifolds Σ_i such that $M = \Sigma_1 \cup \ldots \cup \Sigma_s$ so that they have desirable properties. We call Σ_i strata. The nice properties are

(i) Whitney regularity condition: Any stratum Σ_i is "regular over" any other stratum Σ_j. The definition of regularity will be given in Section 6.1.

(ii) Frontier condition: If $\Sigma_i \cap \bar{\Sigma}_j \neq \phi$, it implies $\Sigma_i \subset \bar{\Sigma}_j$. This follows from (i), but we can usually verify it directly (see Chapter 7).

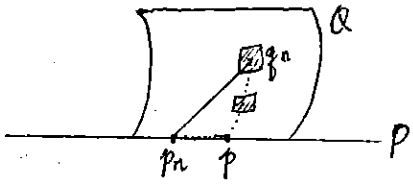
So far in this context, we had already had a candidate for the stratification of M in which all the strata had been proved to be manifolds. In our case now, the strata Σ_i are the different classes of singularity types as shown in Chapter 2, e.g. $\Sigma(A_1A_3)$, $\Sigma(A_5)$.

Now we would like to see whether our stratification has the Whitney Regularity condition.

6.1. Definitions of Regularity

Let P and Q be disjoint submanifolds of a smooth manifold M, and $p \in P \cap \bar{Q}$ where $\bar{Q}$ is the closure of Q. By choosing suitable local coordinates at p, we can make P a linear subspace. It can be verified that our definitions are independent of this choice of coordinate.

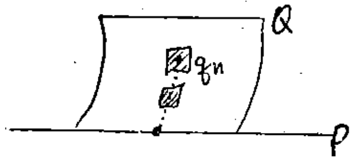
Definition 6.1.1. (C)-regularity (normally called just Regularity)
Q is said to be (C)-regular over P at p if given sequences p_n, q_n of points in P,Q, each tending to p, such that the unit vector from p_n to q_n has limit v and the tangent space to Q at q_n, $T_{q_n}Q$ has limit $\mathcal{T}$, then $v \in \mathcal{T}$. Q is said to be (C)-regular over P if Q is (C)-regular over every point of P.



(C)-regularity.

(A)-regularity:

Q is said to be (A)-regular over P at p if given a sequence q_n of points of Q, tending to p, such that Tq_nQ has limit $\mathcal{T}$, then $T_pP \subset \mathcal{T}$. Q is said to be (A)-regular over P if Q is (A)-regular over every point of P.

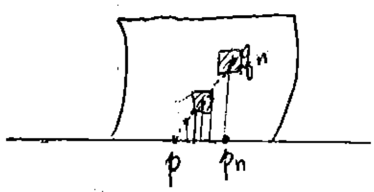


(A)-regularity.

Let π_p : M → P be the orthogonal projection of M on P and for any q ∈ M - P, write η(q) for the unit vector in the direction from $\pi_p(q)$ to q.

(B)-regularity:

Q is say to be (B)-regular over P at p if given a sequence q_n of points of Q, tending to p, such that $\eta(q_n)$ has limit v_o, and Tq_nQ has limit $\mathcal{T}$, then $v_o \in \mathcal{T}$. Q is said to be (B)-regular over P if Q is (B)-regular over every point of P.



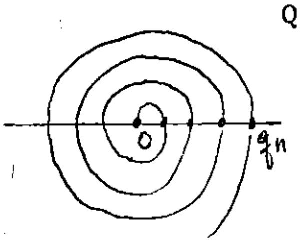
(B)-regularity.

Note

(i) (B)-regularity is just a special case of (C)-regularity with p_n being the orthogonal projection of q_n on P.

(ii) (A)-regularity does not imply (B)-regularity. For example, $\{\text{a spiral}\} \times \mathbb{R}$. Let $P = \{0\} \times \mathbb{R}$

$$Q = \{\text{spiral} - \{0\}\} \times \mathbb{R}$$



Infinite Spiral.

Q is (A)-regular over P, but is not (B)-regular over P.

(iii) Wall [WALL 1974]

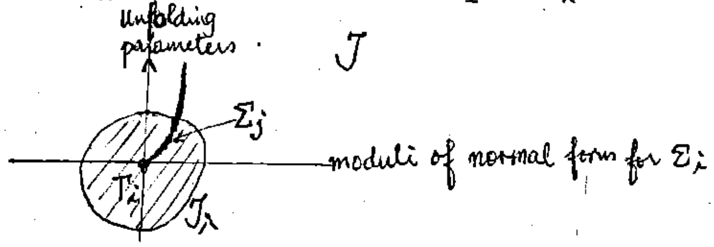
(C) $\Leftrightarrow$ (A) and (B).

Normally it is not easy to prove regularity between strata. We have a lemma which will reduce the proof of regularity over the whole of the strata to only those points in the Transversal.

Lemma 6.1.2 Let $P, Q \subseteq M$ be submanifolds invariant under the action of the group G. Q is (C)-regular over P iff $\forall p \in P$ there is a transversal $\mathcal{T}$ of P at p such that $Q \cap \mathcal{T}$ is (C)-regular over $P \cap \mathcal{T}$ at p. [Gibson 1976].

6.2. Examples of Regularity

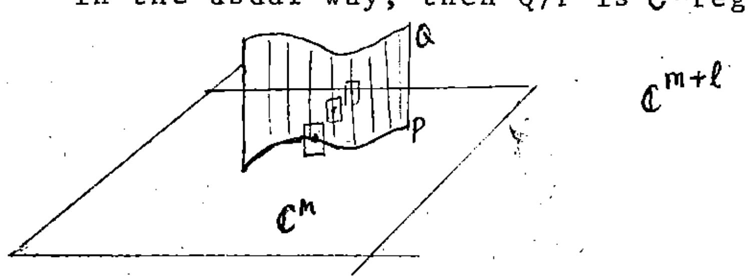
Let Σ_i, Σ_j be strata with normal forms Γ_i, Γ_j. Suppose $\mathcal{J}$ is the transversal for Σ_i, then if we want to prove that Σ_j is regular over Σ_i, by Lemma 6.1.2. it is enough to show for every point Γ_λ of Γ_i, there exists a neighbourhood $\mathcal{J}_\lambda$ of Γ_λ in $\mathcal{J}$ such that $\Sigma_j \cap \mathcal{J}_\lambda$ is regular over $\Sigma_i \cap \mathcal{J}_\lambda$ which is equivalent to $\Sigma_j \cap \mathcal{J}_\lambda$ is regular over $\Gamma_i \cap \mathcal{J}_\lambda$ (see p.151).



In general, it is very hard to find out the condition on points of $\mathcal{J}_\lambda$ that they should lie in Σ_j. But there are a few special cases in which we can not only realize the conditions but also prove the regularity directly. We shall deal with the general argument for proving regularity in the next section. The following lemma serves as a tool for proving simple cases of regularity.

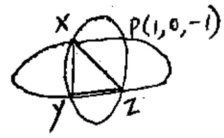
Lemma 6.2.1. If P is a submanifold of $\mathbb{C}^m$ and $Q = P \times \mathbb{C}^\ell \subset \mathbb{C}^m \times \mathbb{C}^\ell$ where $\mathbb{C}^m \subset \mathbb{C}^{m+\ell}$ in the usual way, then Q/P is C-regular over P.

Proof:



Both (A) and (B)-regularity is obvious. Hence (C)-regularity follows.

Example 6.2.2. Consider $\Sigma(A_1^4 \oplus)$, all the singularities are nodes. Since nodes can only despecialize into non-singular points, we can be sure that the only strata which can specialize to $\Sigma(A_1^4 \oplus)$ are $\Sigma(A_1^3)$ (irr), $\Sigma(A_1^2)$, $\Sigma(A_1)$ (the other $\Sigma(A_1^3 \widetilde{\nVdash})$ can not specialize to $A_1^4 \oplus$ because the line component of $A_1^3 \widetilde{\nVdash}$ cannot degenerate into a conic). The transversal of $\Sigma(A_1^4 \oplus)$ at α_0, β_0 is



$$\mathcal{J} = (xy + (\alpha_0+\alpha)yz + (\alpha_0+\alpha+1)xz)(xy+(\beta_0+\beta)yz + (\beta_0+\beta+1)xz)$$

$$+ \gamma(3x^4 + 2x^3y - 2x^3z + y^2z^2)$$

$$+ \delta(y^4 + 2y^3z + y^2z^2)$$

$$+ \varepsilon(z^4 + 2yz^3 + y^2z^2)$$

$$+ \zeta(y^2z^2)$$

We have the property (P. 118) that there exists a sufficiently small neighbourhood $\mathcal{J}_{\alpha_0\beta_0}(X)$ of $\Gamma_{\alpha_0\beta_0}$ in $\mathcal{J}$, that is a neighbourhood N(X) of $\underline{O}$ in $(\underline{x},\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-space such that $\gamma = 0$ iff $\mathcal{J}$ has A_1 at X.

Similarly, we have $\mathcal{J}_{\alpha_0\beta_0}(Y)$, N (Y) $\delta = 0$ iff A_1 at Y

$\mathcal{J}_{\alpha_0\beta_0}(Z)$, N (Z) $\varepsilon = 0$ iff A_1 at Z

$\mathcal{J}_{\alpha_0\beta_0}(P)$, N (P) $\zeta = 0$ iff A_1 at P

Hence there exists a neighbourhood $\mathcal{J}_{\alpha_0\beta_0}$ of $\Gamma_{\alpha_0\beta_0}$ in $\mathcal{J}$ such that $\mathcal{J}_{\alpha_0\beta_0} \cap \Sigma(A_1^3$ irr) is the union of the disjoint linear subspaces

$\gamma = \delta = \varepsilon = 0, \quad \zeta \neq 0$

$\delta = \varepsilon = \zeta = 0, \quad \gamma \neq 0$

$\varepsilon = \zeta = \gamma = 0, \quad \delta \neq 0$

$\zeta = \gamma = \delta = 0, \quad \varepsilon \neq 0$

By Lemma 6.2.1., $\mathcal{T}\alpha_o\beta_o \cap \Sigma(A_1^3 \text{ irr})$ is regular over $\mathcal{T}_\lambda \cap \Gamma$, then by Lemma 6.1.2 regularity of $\Sigma(A_1^3(\text{irr}))$ over $\Sigma(A_1^4\, \Phi)$. Similarly, there exists a neighbourhood $\mathcal{T}'\alpha_o\beta_o$ of $\Gamma\alpha_o\beta_o$ in $\mathcal{T}$ such that $\mathcal{T}'\alpha_o\beta_o \cap \Sigma(A_1^2)$ is the union of the disjoint linear subspace

$\gamma = \delta = 0$	$\varepsilon \neq 0$	$\zeta \neq 0$
$\gamma = \varepsilon = 0$	$\delta \neq 0$	$\zeta \neq 0$
$\gamma = \zeta = 0$	$\delta \neq 0$	$\varepsilon \neq 0$
$\delta = \varepsilon = 0$	$\gamma \neq 0$	$\zeta \neq 0$
$\delta = \zeta = 0$	$\gamma \neq 0$	$\varepsilon \neq 0$
$\varepsilon = \zeta = 0$	$\gamma \neq 0$	$\delta \neq 0$

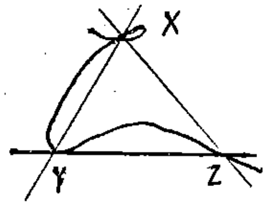
Again by Lemma 6.2.1. and Lemma 6.1.2, we have $\Sigma(A_1^2)$ regular over $\Sigma(A_1^4\, \Phi)$.

Also, there exist neighbourhood $\mathcal{T}''\alpha_o\beta_o$ of $\Gamma\alpha_o\beta_o$ in $\mathcal{T}$ such that $\mathcal{T}''_{\alpha_o\beta_o} \cap \Sigma(A_1)$ is the union of the disjoint linear subspaces

$\gamma = 0$	$\delta, \varepsilon, \zeta \neq 0$
$\delta = 0$	$\gamma, \varepsilon, \zeta \neq 0$
$\varepsilon = 0$	$\gamma, \delta, \zeta \neq 0$
$\zeta = 0$	$\gamma, \delta, \varepsilon \neq 0$

As before, regularity of $\Sigma(A_1)$ over $\Sigma(A_1^4\, \Phi)$ is established.

Example 6.2.3. Let X be a certain singularity type. In this example we intend to discuss the regularity of X over $A_1^n X$. Again we use the fact that nodes can only despecialize to non-singular points and nothing else. Let us look at a specific case $\Sigma(A_1A_3\,\widetilde{\mathcal{A}\ell})$ over $\Sigma(A_1^2A_3\,\mathcal{A}^{\ell})$. The transversal of $\Sigma(A_1^2A_3\,\mathcal{A}^{\ell})$ is



$$\mathcal{J} = x^2y^2 + x^2z^2 + xyz^2 + \alpha_0 x^2yz + \alpha x^2yz$$

$$+ \beta x^4 + \gamma y^4 + \delta y^3z + \varepsilon y^2z^2 + \zeta z^4$$

We have the property (P.118) that there exists a small enough neighbourhood $\mathcal{J}_{\alpha_0\beta_0}$ of Γ_{α_0} in $\mathcal{J}$ such that

$$\gamma = \delta = \varepsilon = 0 \text{ iff } \mathcal{J} \text{ has } A_3 \text{ at } Y$$

Also $\exists$ a small enough neighbourhood $\mathcal{J}_{\alpha_0}(X)$ of Γ_{α_0} in such that

$$\beta = 0 \text{ iff } \mathcal{J} \text{ has } A_1 \text{ at } X.$$

Similarly $\exists$ a small enough neighbourhood $\mathcal{J}_{\alpha_0}(Z)$ of Γ_{α_0} at such that

$$\zeta = 0 \text{ iff } \mathcal{J} \text{ has } A_1 \text{ at } Z.$$

Thus, there exist a neighbourhood $\mathcal{J}_{\alpha_0}$ of Γ_{α_0} in $\mathcal{J}$ such that $\mathcal{J}_{\alpha_0} \cap \Sigma(A_1A_3\,\widetilde{\mathcal{A}\ell})$ is the linear subspace

$$\gamma = \delta = \varepsilon = \beta = 0 \quad , \quad \zeta \neq 0$$

(This amounts to desingularizing the node at X).

Also, there exist a neighbourhood $\mathcal{J}'_{\alpha_0}$ of Γ_{α_0} in $\mathcal{J}$ such that $\mathcal{J}'_{\alpha_0} \cap \Sigma(A_1A_3$ irr) is the linear subspace

$$\gamma = \delta = \varepsilon = \zeta\ 0 \qquad , \qquad \beta \neq 0$$

(This amounts to desingularizing the node at Z).
So, by Lemma 6.2.1 and Lemma 6.1.2, both $A_1A_3\ \widetilde{\mathcal{A}}$ and A_1A_3(irr) are Whitney regular over $A_1^2A_3$.

Furthermore, we can also see $\Sigma(A_3)$ is regular over $\Sigma(A_1^2A_3)$, since there would also exist a neighbourhood $\mathcal{J}''_{\alpha_0}$ of Γ_{α_0} in $\mathcal{J}$ such that $\mathcal{J}''_{\alpha_0} \cap \Sigma(A_3$ irr) is the linear subspace

$$\gamma = \delta = \varepsilon = 0 \qquad , \qquad \beta, \zeta \neq 0$$

(Here the nodes at X and Z are both desingularized).

Example 6.2.4. In this example, we shall show another method of proving regularity. Notice that this only applies to very special cases.

Let us consider regularity of $\Sigma(A_1A_3$ irr) over $\Sigma(A_1D_4$ [illegible]). The normal form Γ of $\Sigma(A_1D_4$ [illegible]) is

$$\Gamma = x^4 + xyz^2 + x^2yz + \alpha_0x^3z \qquad \alpha \neq 1$$

(from p.59 Chapter 2)

The normal form Γ' of $\Sigma(A_1A_3$ irr) is

$$\Gamma' = y^4 + x^2z^2 + x^2yz + \alpha xy^2z + \beta xy^3 \qquad \alpha \neq 2, \quad \alpha \neq -2$$

$$\beta^2 - \alpha\beta + 1 \neq 0.$$

(from p.16 Chapter 2)

Taking Γ' through the transformation

$$x \to z$$
$$y \to x$$
$$z \to y$$

we have $x^4 + y^2z^2 + xyz^2 + \alpha x^2yz + \beta x^3z = 0$.

Again, taking the transformation $y \to \lambda y$, $z \to \mu z$, we have

$$\Gamma'' = x^4 + \lambda^2\mu^2y^2z^2 + \lambda\mu^2xyz^2 + \alpha\lambda\mu x^2yz + \beta\mu x^3z$$

We can make $\lambda\mu^2 = 1$, $\alpha\lambda\mu = 1$,

by taking $\mu = \alpha$, $\lambda = \frac{1}{\alpha^2}$.

Hence $\Gamma'' = x^4 + \frac{1}{\alpha^2}y^2z^2 + xyz^2 + x^2yz + \alpha\beta x^3z$

This is a form of $\Sigma(A_1A_3 \text{ irr})$, but not necessary the whole of $\Sigma(A_1A_3 \text{ irr})$.

If $\alpha \to \infty$, and $\alpha\beta \to \alpha'$ (avoiding $\beta^2-\alpha\beta+1 = 0$) Γ'' becomes

$$x^4 + xyz^2 + x^2yz + \alpha' x^3z$$

which is the complete normal form for $\Sigma(A_1D_4$ [illegible])(see Γ).
Now let us check the transversality of the orbits of $\Sigma(A_1A_3 \text{ irr})$ to the form Γ'', writing $\alpha'' = \frac{1}{\alpha^2}$, $\alpha \neq \pm 2$, $\beta'' = \alpha\beta$

$$\Gamma'' = x^4 + xyz^2 + x^2yz + \alpha_0''y^2z^2 + \beta_0''x^3y \text{ where}$$

$$\Gamma''_x = 4x^3 + yz^2 + 2xyz + 3\beta_0''x^2z \qquad \Big| \quad \alpha_0'' \neq \frac{1}{4} \text{ since } \alpha_0 \neq \pm 2$$

$$\Gamma''_y = xz^2 + x^2z + 2\alpha_0''yz^2$$

$\Gamma''_z = 2xyz + x^2y + 2\alpha''_o y^2z + \beta''_o x^3$

		x^4	y^4	z^4	x^3y	xy^3	x^3z	xz^3	y^3z	yz^3	x^2yz	xy^2z	xyz^2	x^2y^2	x^2z^2	y^2z^2
λ_1	$x\Gamma''_x$	4					$3\beta''_o$				2		1			
λ_2	$y\Gamma''_x$				4						$3\beta''_o$	2				1
λ_3	$z\Gamma''_x$						4			1			2		$3\beta''_o$	
λ_4	$x\Gamma''_y$						1						$2\alpha''_o$		1	
λ_5	$y\Gamma''_y$										1		1			$2\alpha''_o$
λ_6	$z\Gamma''_y$							1		$2\alpha''_o$					1	
λ_7	$x\Gamma''_z$	β''_o			1						2	$2\alpha''_o$				
λ_8	$y\Gamma''_z$				β''_o				$2\alpha''_o$			2		1		
λ_9	$z\Gamma''_z$						β''_o				1		2			$2\alpha''_o$

Suppose $\lambda_1 x\Gamma''_x + \dots \lambda_9 z\Gamma''_z = \mu_1 y^2z^2 + \mu_2 x^3 z$, from above table we have $\lambda_8 = 0$, $\lambda_6 = 0$, $\lambda_3 = 0$, $\lambda_4 = 0$ and

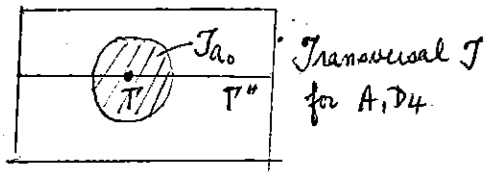
$$\left.\begin{array}{l} 4\lambda_2 + \lambda_7 = 0 \\ 2\lambda_2 + 2\alpha''_o\lambda_7 = 0 \end{array}\right\} \Rightarrow \quad \lambda_2 = \lambda_7 = 0 \text{ since } \alpha''_o \neq \frac{1}{4}$$

By $4\lambda_1 + \beta''_o\lambda_7 = 0$, we have $\lambda_1 = 0$

And
$$\left.\begin{array}{l} \lambda_5 + \lambda_9 = 0 \\ \lambda_5 + 2\lambda_9 = 0 \end{array}\right\} \Rightarrow \quad \lambda_5 = \lambda_9 = 0$$

This proves that there is no common tangent between the orbits and the form Γ", except the vector {0}. Hence Γ" transverse to orbits.

Next, put α" = 0. The same calculation shows that the $\Sigma(A_1D_4)$ orbits are transverse not only to their normal form, that is when α" = 0, but actually to the $\Sigma(A_1A_3$ irr) form Γ" too.



This shows that we can choose the same transversal for both strata.

Suppose we choose

$$\mathcal{J}a_o = x^4 + xyz^2 + x^2yz + a_ox^3z + ax^3z$$

$$+ by^4 + cz^4 + dy^3z + exy^3 + fy^2z^2$$

to be the transversal for $\Sigma(A_1D_4)$ at a_o, where a,b,c, ... f are small. Since this is also a transversal for Γ", $\mathcal{J}a_o \cap \Sigma(A_1A_3$ irr) = $\mathcal{J}a_o \cap$ Γ" (see p. 118). Hence the points of $\mathcal{J}a_o$ which lie in the $\Sigma(A_1A_3)$ are precisely the points of the normal form Γ". Using Theorem 6.1.2., $\Sigma(A_1A_3$ irr) is regular over $\Sigma(A_1D_4)$ iff $\forall a_o \in \Sigma(A_1D_4)$, $\mathcal{J}a_o \cap \Sigma(A_1A_3$ irr) is regular over $\mathcal{J}a_o \cap \Sigma(A_1D_4)$. But $\mathcal{J}a_o \cap \Sigma(A_1A_3$ irr) = $\mathcal{J}a \cap$ Γ" appears as a plane (a,f) and $\mathcal{J}a_o \cap \Sigma(A_1D_4)$ as just a line (a) in the plane. Hence regularity follows immediately.

Notice that this method only applies to cases when we can choose a common transversal for both strata concerned.

We have checked that in two other cases $A_1^4 \to A_1^5$

$A_1^4 \to A_1^5$

we can also use this technique.

6.3. The Uniqueness of Unfolding Technique: General Theory for Proving Regularity

The general argument for proving regularity depends on the facts that "any two r-parameter versal unfoldings are isomorphic" (Theorem 5.1.10 III) and "Regularity holds for products" (Lemma 6.2.1).

Let Γ be the normal form for the stratum Σ. As explained before (p. 118) when the normal form Γ is taken locally at a singularity σ_i, and fixed values are taken for the moduli it becomes a germ for the singularity of the form

$$g : (N,0) \to (\mathbb{C},0)$$

(Thus, if the singularity is at X, then $g(y,z) = \Gamma(1,y.z)$).

Let $\mathcal{T}$ be the transversal of Γ. Again as explained in p. , when $\mathcal{T}$ is taken locally at a singularity σ_i of Γ, it becomes a versal unfolding of g. Omitting the constant term (with the singularity at X, this comes from the term in x^4) $\mathcal{T}$ is of the form

$$G : (N \times U \times V, \underline{0}) \to (\mathbb{C},0)$$

where U is the universal unfolding parameter space and

V is the parameter space of the moduli and the

unfolding terms relating to other singularities.

By properties of transversals, (p. 118) there exists a neighbourhood $\mathcal{G}$ of $\underline{0}$ in N x U x V such that if $(x,u,v) \in \mathcal{G}$ $G(x,u,v)$ has σ_i near $\underline{0}$ iff x = 0 and u = 0.

Secondly, let us introduce another versal unfolding of g. Let $N \times U \times V \subset \mathcal{G}$, and $\eta = N \times U \to \mathbb{C}$ be a standard universal unfolding of g. Now define

$$F = (N \times U \times V, 0) \to \mathbb{C}$$

$$(x,u,v) \mapsto \eta(x,u)$$

Then F is a versal unfolding of g with the same number of parameters as G, only that it is independent of the V-parameters. That is

$$F(x,u,v) = F(x,u,v')$$

for any points $\in \mathcal{G}$

For a singularity type $\mathcal{X}$ which can specialize to that of g (for local specialization see Chapter 7; note that $\mathcal{X}$ may be a cluster of several singularities such as $A_1A_1A_2$) we write

$$S_{\mathcal{X}}(G) = \{(u,v,w) \in U \times V \times \mathbb{C} : G_{u,v}(x)+w \text{ has a singularity type } \mathcal{X} \text{ near } \underline{0}\}$$

$$S_{\mathcal{X}}(F) = \{(u,v,w) \in U \times V \times \mathbb{C} : F_{u,v}(x)+w \text{ has a singularity type } \mathcal{X} \text{ near } \underline{0}\}$$

If σ_o is the singularity type (up to right equivalence) of g then by the argument in §5.4 (x_o simple), we have

$$S_{\sigma_o}(G) = \{(u,v,w) \in U \times V \times \mathbb{C} : G_{u.v}(x)+w \text{ has a singularity type } \sigma_o \text{ near } \underline{0}\}$$

$$= \{(u,v,w) : u = w = 0\}$$

$$S_{\sigma_o}(F) = \{(u,v,w) \in U \times V \times \mathbb{C} : F_{u,v}(x)+w \text{ has a singularity type } \sigma_o \text{ near } \underline{0}\}$$

$$= \{(u,v,w) : u = w = 0\}.$$

Now by Theorem 5.1.10 III, F and G are isomorphic, that is there exists

(i) diffeomorphic germ

$$\phi : (N \times U \times V, \underline{0}) \to (N \times U \times V, \underline{0})$$

written as $\phi(x,u,v) = (\psi(x,u,v), \rho_1(u,v), \rho_2(u,v))$ where $\psi_{o,o}$ = identity

(ii) a germ $c : (U \times V, 0) \to (\mathbb{C}, 0)$

such that

$$G(x,u,v) = F(\phi(x,u,v)) + c(u,v)$$

$$(*) \qquad = F(\psi(x,u,v), \rho_1(u,v), \rho_2(u,v)) + c(u,v)$$

Now since F is independent of the third set of coordinates, we are allowed to change ρ_2 as long as we keep ϕ a germ of diffeomorphism.

We claim that we can make $\rho_2(u,v) = v$.

First, since ϕ is a diffeomorphism, its Jacobian

$$\begin{pmatrix} \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \\ 0 & \frac{\partial\rho_1}{\partial u} & \frac{\partial\rho_1}{\partial v} \\ 0 & \frac{\partial\rho_2}{\partial u} & \frac{\partial\rho_2}{\partial v} \end{pmatrix}$$

must have non-zero determinant.

That is,

$$\begin{vmatrix} \frac{\partial\rho_1}{\partial u} & \frac{\partial\rho_1}{\partial v} \\ \frac{\partial\rho_2}{\partial u} & \frac{\partial\rho_2}{\partial v} \end{vmatrix}_{u=v=0} = 0 \text{ or } (\rho_1, \rho_2) \text{ is a diffeomorphic germ.}$$

After the replacement $\rho_2(u,v) = v$, $\begin{vmatrix} \frac{\partial\rho_1}{\partial u} & \frac{\partial\rho_1}{\partial u} \\ \frac{\partial\rho_2}{\partial u} & \frac{\partial\rho_2}{\partial v} \end{vmatrix}$ becomes

$$\begin{vmatrix} \frac{\partial \rho_1}{\partial u} & \frac{\partial \rho_1}{\partial v} \\ 0 & I \end{vmatrix}$$ and we want this to remain non zero,

that is, the germ ϕ will still be a diffeomorphism.

Notice that if $\left(\frac{\partial \rho_1}{\partial v}\right)_{\underline{0}} = 0$, $\begin{vmatrix} \frac{\partial \rho_1}{\partial u} & \frac{\partial \rho_1}{\partial v} \\ \frac{\partial \rho_2}{\partial u} & \frac{\partial \rho_2}{\partial u} \end{vmatrix}_{\underline{0}} = \left|\frac{\partial \rho_1}{\partial u}\right|_{\underline{0}} \cdot \left|\frac{\partial \rho_2}{\partial v}\right|_{\underline{0}}$

Since $\begin{vmatrix} \frac{\partial \rho_1}{\partial u} & \frac{\partial \rho_1}{\partial v} \\ \frac{\partial \rho_2}{\partial u} & \frac{\partial \rho_2}{\partial v} \end{vmatrix}_{\underline{0}}$ is non zero, this implies $\left|\frac{\partial \rho_1}{\partial u}\right|_{\underline{0}} \neq 0$

Therefore if we can show $\left(\frac{\partial \rho_1}{\partial v}\right)_{\underline{0}} = 0$, then

$$\begin{vmatrix} \frac{\partial \rho_1}{\partial u} & \frac{\partial \rho_1}{\partial v} \\ 0 & I \end{vmatrix} \neq 0$$ as required.

To prove $\left(\frac{\partial \rho_1}{\partial v}\right)_{\underline{0}} = 0$, first we have

$$S_{\sigma_o}(G) = \{(u,v,w) : u=w=0\}$$

By (*), this implies $F(\psi(x,u,v),\ \rho_1(u,v),\ \rho_2(u.v)) + c(u,v) + w$ has (as a function of x) a singularity of type σ_o near $\underline{0}$ iff $u=w=0$ (**)

But since $\psi_{o,o}$ = identity, for each fixed (u,v) near $\underline{0}$, $\psi_{u,v}$ is a diffeomorphism and F is independent of ρ_2. So the condition for $F(\psi(x,u,v),\ \rho_1(u,v),\ \rho_2(u,v)) + c(u,v) + w$ to have a σ_o-singularity near $\underline{0}$ can also be written as

$$\rho_1(u,v) = c(u,v) + w = 0$$

Combined with (**), we have for all small v,

$$\rho_1(0,v) = c(0,v) = 0$$

Now consider the composite

$$V \to U \times V \xrightarrow{\rho_1} U$$

$$v \mapsto (0,v)$$

It is identically zero. Hence the Jacobian of the composite at $\underline{0}$ is zero, that is

$$\left(\frac{\partial \rho_1}{\partial u} , \frac{\partial \rho_1}{\partial v} \right) \begin{pmatrix} 0 \\ I \end{pmatrix} = 0$$

Therefore $\left(\frac{\partial \rho_1}{\partial v} \right)_{\underline{0}} = 0$ as required.

(Similarly, we can also show $\left(\frac{\partial c}{\partial v} \right)_{\underline{0}} = 0$).

Thus, we can write

$$G(x,u,v) = F(\psi(x,u,v), \ \rho_1(u,v),v) + c(u,v)$$

Now we define a germ

$$\mathcal{D} = (U \times V \times \mathbb{C} , \underline{0}) \to (U \times V \times \mathbb{C}, \underline{0})$$

$$(u,v,w) \mapsto (\rho_1(u,v),v, \ w+c(u,v))$$

It follows that $(u,v,w) \in S_x(G)$

$$\text{iff } \mathcal{D}(u,v,w) = (\rho_1(u,v),v, \ w+c(u,v)) \in S_x(F).$$

The Jacobian matrix of $\mathcal{D}$ at $\underline{0}$ is

$$\begin{pmatrix} \frac{\partial \rho_1}{\partial u} & \frac{\partial \rho_1}{\partial v} & 0 \\ 0 & I & 0 \\ \frac{\partial c}{\partial u} & \frac{\partial c}{\partial v} & 1 \end{pmatrix}_{\underline{0}}$$

Since $\left(\frac{\partial \rho_1}{\partial v}\right)_{\underline{0}} = 0$ $\left(\frac{\partial c}{\partial v}\right)_{\underline{0}} = 0$ and $\left|\frac{\partial \rho_1}{\partial u}\right| \neq 0$, this matrix is invertible. This implies $\mathcal{D}$ is a germ of diffeomorphism. By the above, $\mathcal{D}$ preserves strata, so regularity is invariant under it. Therefore, we have the following lemma.

Lemma 6.3.1. For each simple singularity of a normal form, there exists a diffeomorphic germ $\mathcal{D}$ as defined in the previous paragraph, such that it preserves strata and therefore regularity is invariant under it.

Also, we can easily deduce that

Corollary 6.3.2. $S_{\chi}(G)$ is regular over the V-space u=w=0.
Proof: Since $\mathcal{D}$ takes $S_{\chi_o}(G)$ to $S_{\chi_o}(F)$, the space u=w=0, (that is V-space) is taken to itself by $\mathcal{D}$. Therefore, $S_x(G)$ is regular over V iff $S_{\chi}(F)$ is regular over $\mathcal{D}(\{0\} \times V \times \{0\}) = V$. Now since F is independent of V-parameters, we have

$(a,b,c) \in S_{\chi}(F) \Leftrightarrow (a,b',c) \in S_{\chi}(F)$, that is $S_{\chi}(F)$ is a product. Therefore, by Theorem 6.2.1, (products are regular) $S_{\chi}(F)$ is regular over the V-space. Hence by above $S_{\chi}(G)$ is regular over V.

The above lemma and corollary apply, of course, only to <u>one</u> singular point of the normal form of $\Sigma(\sigma_o)$. In the case when there is only one singular point on the normal form of $\Sigma(\sigma_o)$ the V-space will correspond only to the parameter space of the moduli, that is, the normal form itself. Therefore, the above Corollary 6.3.2 proves the regularity of $S_{\chi}(G)$ over the normal form in the transversal. Then by Lemma 6.1.2, we have the regularity of the stratum of singularity type χ $(\Sigma(\chi))$ over the stratum of singularity type $\sigma_o\left(\Sigma(\sigma_o)\right)$.

Let us now look at an example

Example 6.3.3.

$\chi \to A_3$, $\chi = A_1, A_1^2$, or A_2 (see table 7.4 for local despecialization).

Normal form for quartic curves with A_3 is

$$\Gamma_{\alpha,\beta,\gamma}(x,y,z) = x^2z^2 + x^3y + y^4 + \alpha xy^2z + \beta x^2y^2 + \gamma xy^3$$

Putting $z = 1$, we have the germ

$$g_o(x.y) = x^2 + x^3y + y^4 + \alpha_o xy^2 + \beta_o x^2y^2 + \gamma_o xy^3$$

with A_3 at $\underline{O}$.

The transversal at point $\Gamma_{\alpha_o\beta_o,\gamma_o}$ is

$$\mathcal{J}_o\ (x,y,z,\ \alpha,\beta,\gamma,\delta,\varepsilon,\zeta) = \Gamma_{\alpha_o\beta_o\gamma_o}(x,y,z)$$
$$+ \alpha xy^2z + \beta x^2y^2 + \gamma xy^3$$
$$+ \delta z^4 + \varepsilon yz^3 + \zeta y^2z^2$$

By putting $z = 1$, we have a representative of the germ of the versal unfolding G_o of g_o

$$G_o(x,y,\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) = x^2 + x^3y + y^4 + \alpha_o xy^2 + \beta_o x^2y^2 + \gamma_o xy^3 + \alpha xy^2 + \beta x^2y^2 + \gamma xy^3 + \varepsilon y + \zeta y^2$$

as a versal unfolding we have omitted the constant δ.
By properties of transversal (p. 118), there exists a neighbourhood $\mathcal{G}$ of $\underline{0}$ in $(x,y,z,\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-space such that $\mathcal{J}_o$ has an A_3 at (x,y,z) near Z iff $x=y=\delta=\varepsilon=\zeta=0$.

Following the general argument, we define another versal unfolding

$$F_o(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta) = x^2 + x^3y + y^4 + \alpha_o xy^2 + \beta_o x^2y^2 + \gamma_o xy^2 + \varepsilon y + \zeta y^2$$

of g_o where $(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta) \in \mathcal{G}$. Note that F_o is independent of the (α,β,γ)-parameters.
Here now the (x,y)-space corresponds to the N-space in the general argument, (α,β,γ)-space corresponds to the V-space and (ε,ζ)-space to the U-space. Then, as $F_o, G_o : N \times V \times U \to N \times V \times U$ in the general argument we have, by isomorphism of unfoldings

$$\begin{aligned} G_o(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta) = F_o\big(&\psi(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta), \\ &\rho_{11}(\alpha,\beta,\gamma,\varepsilon,\zeta),\ \rho_{12}(\alpha,\beta,\gamma,\varepsilon,\zeta), \\ &\rho_{13}(\alpha,\beta,\gamma,\varepsilon,\zeta),\ \rho_{21}(\alpha,\beta,\gamma,\varepsilon,\zeta), \\ &\rho_{22}(\alpha,\beta,\gamma,\varepsilon,\zeta)\big) + c(\alpha,\beta,\gamma,\varepsilon,\zeta) \end{aligned}$$

where ρ_{11}, ρ_{12} and ρ_{13} are components of ρ_1, which is now corresponding to the V-space and ρ_{21}, ρ_{22} are components of ρ_2, which is now corresponding to the U-space. (Note this is a different notation from the general argument).

By the same proof as in the general argument (p. 166) we can write

$$G_o(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta) = F_o(\mathcal{Y}(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta),\alpha,\beta,\gamma,$$

$$\rho_{21}(\alpha,\beta,\gamma,\varepsilon,\zeta),\ \rho_{22}(\alpha,\beta,\gamma,\varepsilon,\zeta)) + c(\alpha,\beta,\gamma,\varepsilon,\zeta) \qquad (*)$$

Write $S_{A_3}(G_o) = \{(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) : G_o(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta) = -\delta$ has an A_3 near $\underline{0}\}$

$= \{(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) : \varepsilon = \zeta = \delta = 0\}$

$S_{A_3}(F_o) = \{(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) : F_o(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta) = -\delta$ has an A_3 near $\underline{0}\}$

$= \{(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) : \varepsilon = \zeta = \delta = 0\}$

$S_{\chi}(G_o) = \{(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) : G_o(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta) = -\delta$ has singularity type χ near $\underline{0}\}$

$S_{\chi}(F_o) = \{(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) : F_o(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta) = -\delta$ has singularity type χ near $\underline{0}\}$

Define $\mathcal{D} : (V \times \mathbb{C} \times U, 0) \to (V \times \mathbb{C} \times U, 0)$

$$(\alpha,\beta,\gamma,\delta,\xi,\zeta) \mapsto (\alpha,\beta,\gamma,\delta+c(\alpha,\beta,\gamma,\xi,\zeta)$$

$$\rho_{21}(\alpha,\beta,\gamma,\xi,\zeta),\ \rho_{22}(\alpha,\beta,\gamma,\xi,\zeta))$$

(Note the different position of U, V, $\mathbb{C}$ from the general argument. This is for the purpose of keeping $\alpha,\beta,\gamma,\delta,\xi,\zeta$ in the same order, i.e. the same space all through). The Jacobian matrix of $\mathcal{D}$ is invertible. Hence $\mathcal{D}$ is a germ of diffeomorphism.

We can easily see that the (α,β,γ)-space goes to itself under $\mathcal{D}$ and by (*) $(\alpha,\beta,\gamma,\delta,\xi,\zeta) \in S_{\mathcal{X}}(G_o)$

$$\text{iff } \mathcal{D}(\alpha,\beta,\gamma,\delta,\xi,\zeta) = (\alpha,\beta,\gamma,\delta+c(\alpha,\beta,\gamma,\xi,\zeta),$$

$$\rho_{21}(\alpha,\beta,\gamma,\xi,\zeta),\ \rho_{22}(\alpha,\beta,\gamma,\xi,\zeta)) \in S_{\mathcal{X}}(F_o)$$

But since F_o is independent of (α,β,γ)-parameters variables, we have $S_{\mathcal{X}}(F_o)$ regular over α,β,γ-space. That is, by $\mathcal{D}^{-1}$ one have $S_{\mathcal{X}}(G_o)$ regular over (α,β,γ)-space. Hence the stratum $\Sigma(\mathcal{X})$ is regular over $\Sigma(A_3)$, where $\mathcal{X}$ can in fact be A_1, A_1^2 or A_2.

As for the case when there are several singularities on the normal form, we have to apply the theory to each of the singularities separately. The regularity follows from the following lemma.

Lemma 6.3.4. Let Y_1, Y_2, X_1 and X_2 be sub-manifolds of a euclidean space $\mathbb{R}^N$.

Suppose Y_1 is regular over X_1

Y_2 is regular over X_2 and $Y_1 \pitchfork Y_2$ and $X_1 \pitchfork X_2$

Then $Y_1 \cap Y_2$ is regular over $X_1 \cap X_2$.

Proof. Suppose Y_1 is regular over X_1 and Y_2 is regular over X_2. Also $Y_1 \pitchfork Y_2$ and $X_1 \pitchfork X_2$.

Let $Y_1 \cap Y_2 = Y$, $X_1 \cap X_2 = X$; then by transversality, X and Y are submanifolds of the euclidean space.

Let $y_i \in Y$, $x_i \in X$ be sequences such that

$$\left.\begin{matrix} y_i \to x \\ x_i \to x \end{matrix}\right\} \text{ where } x \to X$$

We prove regularity of Y over X at x.

Suppose $T_{y_i}Y \to \tau$ and $\frac{x_i - y_i}{|x_i - y_i|} \to \ell$.

We have to show $\ell \subset \tau$.

Surely, we have $y_i \in Y_1$ and $y_i \in Y_2$

$x_i \in X_1$ and $x_i \in X_2$

and $\frac{x_i - y_i}{|x_i - y_i|}$ has limit ℓ in both cases.

Now suppose $T_{y_i}Y_1$ has limit τ_1 and $T_{y_i}Y_2$ has limit τ_2 (Existence of limit is guaranteed by the fact that Grassmannian is compact, so at any rate a suitable subsequence of the sequence will have a limit). We claim that

$$T_{y_i}Y = T_{y_i}Y_1 \cap T_{y_i}Y_2$$

It is clear that $T_{y_i}Y_1 \cap T_{y_i}Y_2 \supset T_{y_i}Y$.

For $T_{y_i}Y_1 \cap T_{y_i}Y_2 \subset T_{y_i}Y$, we have to use the dimension formula for transversality in $\mathbb{R}^N$, dim Y = dim Y_1 + dim Y_2-N. Also N = dim ($T_{y_i}Y_1 + T_{y_i}Y_2$) = dim ($T_{y_i}Y_1$) + dim ($T_{y_i}Y_2$) - dim ($T_{y_i}Y_1 \cap T_{y_i}Y_2$).

Therefore we have dim (Ty_iY) = dim ($Ty_iY_1 \cap Ty_iY_2$).

Hence $Ty_iY = Ty_iY_1 \cap Ty_iY_2$.

Also, we claim $\tau = \tau_1 \cap \tau_2$.

The inclusion $\tau \subset \tau_1 \cap \tau_2$ is always true. But for the other inclusion, we know $\tau_1 \supset T_xX_1$, $\tau_2 \supset T_xX_2$ by A-regularity. Hence τ_1 is transverse to τ_2 since $X_1 \pitchfork X_2$. Hence

$$\begin{aligned}\dim(\tau_1 \cap \tau_2) &= \dim \tau_1 + \dim \tau_2 - N \\ &= \dim Y_1 + \dim Y_2 - N \\ &= \dim Y = \dim \tau\end{aligned}$$

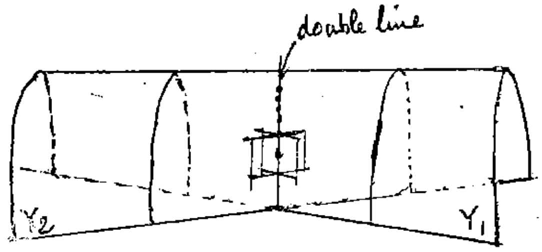
Hence $\tau = \tau_1 \cap \tau_2$.

By regularity given, we have

$$\ell \subset \tau_1 \quad \text{and} \quad \ell \subset \tau_2$$

So that $\ell \subset \tau$ Q.E.D.

Example 6.3.5. This example is to show a case where $\tau \neq \tau_1 \cap \tau_2$. The diagram show a Whitney Umbrella. (Cross Cap)



$\tau_1 = \tau_2 \quad \therefore \tau_1 \cap \tau_2 = \tau_1$

τ = { the double line }

$\therefore \tau \neq \tau_1 \cap \tau_2$

Remark 6.3.6. With this lemma, the regularity over singularity type with more than one singularity can be proved in the following way. The same technique as in the single singularity case can be applied to each of the singularities separately. Using the above lemma, all we need then is the condition that $Y_1 \pitchfork Y_2$ and $X_1 \pitchfork X_2$. We will give the general argument in Section 6.4.

6.4. The application of the Uniqueness of Unfolding Technique to the proof of regularity over strata with two singularities or more

For illustrative purposes, we shall show the general argument in the case of two singularities. The regularity of those with more singularities can be deduced from the same method. More details will be given in Section 6.6. Note that this method only applies to simple isolated singularities. Hence $\tilde{E}_7$ should be dealt with differently, see Chapter 8.

Consider a stratum Σ which consists of curves with two singularities of types σ_1 and σ_2. The normal form Γ has these singularities fixed say at the vertices X and Z of the triangle of reference. Suppose we have a stratum Σ' specializing to Σ, that is, for some point Γ_0 on Γ obtained by giving fixed values to the moduli; suppose we have a sequence of points in Σ' (that is, a sequence of curves in $\mathbb{C}P^2$ all giving points in Σ') with limit Γ_0. These curves will have singularities close to X and Z; let those close to X form a type X_1 (which may be several singularities e.g. $A_1A_1A_2$ if $\sigma_1 = A_6$) and those close to Z form a type X_2.

We shall verify regularity of Σ' over Σ at Γ_0. The proof will not require us to know what X_1 and X_2 are, nor which point Γ_0 of Γ has been chosen. So we are in fact verifying regularity of Σ' over the whole stratum Σ for any Σ' specializing to Σ.

We shall as usual work in a transversal $\mathcal{T}$ to Γ, and verify regularity of $\mathcal{T} \cap \Sigma'$ over $\mathcal{T} \cap \Sigma = \Gamma$ at Γ_0. An explicit example is given in Section 6.5 to help the reader follow the argument.

Letting $x = 1$ in Γ_o, it becomes a representative of the germ $g_1(N_1,0) \to (\mathbb{C},0)$ of singularity σ_1. Similarly, putting $z = 1$, Γ_o becomes a representative of the germ $g_2(N,0) \to (\mathbb{C},0)$ of singularity σ_2.

The transversal $\mathcal{T}$ of Γ at Γ_o, taken locally at X by letting $x = 1$, (see p. 118) becomes a versal unfolding G_1 of σ_1 and taken locally at Z by putting $z = 1$, becomes a versal unfolding G_2 of σ_2 (see p. 118). Also, follow ing Section 6.3.3, we can define other versal unfoldings F_1 and F_2 for σ_1 and σ_2 respectively. And furthermore, we can write the versal unfoldings as follows:

For σ_1, $G_1 : (N \times U_1 \times U_2 \times M \times W_2, 0) \to (\mathbb{C},0)$

$$F_1 : (N \times U_1 \times U_2 \times M \times W_2, 0) \to (\mathbb{C},0)$$

where N is a neighbourhood of (0,0) in $\mathbb{C}^2$ (coordinates y,z).

U_1 is the universal unfolding parameter space for σ_1.

U_2 is the universal unfolding parameter space for σ_2.

W_2 is the constant parameter corresponding to the unfolding of σ_2, i.e. corresponding to the term in y^4.

M is the parameter space of the moduli of the normal form.

Let $V_1 = U_2 \times M \times W_2$, this is the V-space for σ_1 as in Section 6.3. And F_1 is independent of the $U_2 \times M \times W_2$ parameters. It is chosen to be a standard universal unfolding of σ_1.

For σ_2, we have

$$G_2 : (N \times U_1 \times U_2 \times M \times W_1, 0) \to (\mathbb{C},0)$$

$$F_2 : (N \times U_1 \times U_2 \times M \times W_1, 0) \to (\mathbb{C}, 0)$$

where N, U_1, U_2 and M are defined as for σ_1, and W_1 is the constant parameter corresponding to the unfolding of σ_1, i.e., corresponding to the term in x^4. Let $V_2 = U_1 \times M \times W_1$, this is the V-space for σ_2 as in Section 6.3.3. F_2 is independent of the $U_1 \times M \times W_1$ parameters. It is the standard universal unfolding of σ_2.

By properties of transversals (p. 118), there exist small enough neighbourhoods $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\underline{0}$ in $(U_1 \times U_2 \times M \times W_1 \times W_2)$ respectively such that

$$S_{\sigma_1}(G_1) = \{(u_1, u_2, m, w_1, w_2) : u_1 = w_1 = 0\}$$

(recall that this means the function $G_1(-, u_1, u_2, m, w_2) + w_1$ has σ_1 near $(0,0) \Leftrightarrow u_1 = w_1 = 0$).

$$S_{\sigma_1}(F_1) = \{(u_1, u_2, m, w_1, w_2) : u_1 = w_1 = 0\}$$

and $$S_{\sigma_2}(G_2) = \{(u_1, u_2, m, w_1, w_2) : u_2 = w_2 = 0\}$$

$$S_{\sigma_2}(F_2) = \{(u_1, u_2, m, w_1, w_2) : u_2 = \;\; {}_2 = 0\}$$

Define $S_{\chi_1}(G_1) = \{(u_1, u_2, m, w_1, w_2) \in \mathcal{G}_1 : G_{1(u_1,u_2,m,w_2)}(x) + W_1$

has a singularity type χ_1 near $\underline{0}\}$.

(i.e. the function $G_1(-,u_1,u_2,m,w_2) + w_1$ has χ_1 near $\underline{0}$).

$S_{\mathcal{X}_1}(F_1) = \{(u_1, u_2, m, w_2, w_1) \in \mathcal{Y}_1 ; F_{1(u_1,u_2,m,w_2)}(x)+w_1$ has a singularity type $\mathcal{X}_1$ near $\underline{0}\}$.

and $S_{\mathcal{X}_2}(G_2) = \{(u_1, u_2, m, w_1, w_2) \in \mathcal{Y}_2 ; G_{2(u_1,u_2,m,w_1)}(x)+w_2$ has a singularity type $\mathcal{X}_2$ near $\underline{0}\}$.

$S_{\mathcal{X}_2}(F_2) = \{(u_1, u_2, m, w_1, w_2) \in \mathcal{Y}_2 ; F_{2(u_1,u_2,m,w_1)}(x)+w_2$ has a singularity type $\mathcal{X}_2$ near $\underline{0}\}$.

Then by the isomorphism between the versal unfoldings and argument on p.168, we have

$$G_1(x, u_1, u_2, m, w_2) = F_1(\mathcal{Y}_1(x, u_1, u_2, m, w_2), \rho(u_1, u_2, m, w_2), u_2, m, w_2) + c_1(u_1, u_2, m, w_2)$$

(recall that the variables (u_2, m, w_2) appear as just v on p.166)

and $G_2(x, u_2, u_1, m, w_1) = F_2(\mathcal{Y}_2(x, u_1, u_2, m, w_1)$

$$u_1, \rho'(u_1, u_2, m, w_1), u_1, m, w_1) + c_2(u_1, u_2, m, w_1)$$

(recall that the variables (u_1, m, w_1) appear as just v on p.166)

where (i) $\mathcal{Y}_{1(0,0,0,0)}$ = identity and $\mathcal{Y}_{2(0,0,0,0)}$ = identity

(ii) $c_1 = (U_1 \times U_2 \times M \times W_2, 0) \rightarrow (\mathbb{C}, 0)$

$c_2 = (U_1 \times U_2 \times M \times W_1, 0) \rightarrow (\mathbb{C}, 0)$

Then by Lemma 6.3.1, we have diffeomorphisms $\mathcal{D}_1$ and $\mathcal{D}_2$

$$\mathcal{D}_1 : (U_1 \times U_2 \times M \times W_1 \times W_2, 0) \longrightarrow (U_1 \times U_2 \times M \times W_1 \times W_2, 0)$$
$$(u_1; u_2, m, w_1, \omega_2) \longmapsto (\rho(u_1, u_2, m, \omega_2), u_2, m, \omega_1 + c_1(u_1, u_2, m, \omega_2), \omega_2)$$

and
$$\mathcal{D}_2 : (U_1 \times U_2 \times M \times W_1 \times W_2, 0) \longrightarrow (U_1 \times U_2 \times M \times W_1 \times W_2, 0)$$
$$(u_1, u_2, m, \omega_1, \omega_2) \longmapsto (u_1, \rho'(u_1, u_2, m, \omega_1), m, \omega_1, \omega_2 + c_2(u_1, u_2 m, \omega_2))$$

having the property that

$$(S_{\chi_1}(G_1)) = \mathcal{D}_1^{-1}(S_{\chi_1}(F_1))$$

and
$$(S_{\chi_2}(G_2)) = \mathcal{D}_2^{-1}(S_{\chi_2}(F_2))$$

(that is, $\mathcal{D}_1$ and $\mathcal{D}_2$ preserve strata).

Then by Corollary 6.3.2., we have the regularity of

$S_{\chi_1}(G_1)$ over $S_{\sigma_1}(G_1) = U_2 \times M \times W_2$ space

and $S_{\chi_2}(G_2)$ over $S_{\sigma_2}(G_2) = U_1 \times M \times W_1$ space.

Now to prove the regularity of $S_{\varkappa_1}(G_1) \cap S_{\varkappa_2}(G_2)$ over $S_{\sigma_1}(G_1) \cap S_{\sigma_2}(G_2)$, we need the transversality $S_{\varkappa_1}(G_1) \pitchfork S_{\varkappa_2}(G_2)$ and $S_{\sigma_1}(G_1) \pitchfork S_{\sigma_2}(G_2)$ in the space $U_1 \times U_2 \times M \times W_1 \times W_2$, by Lemma 6.3.3. But

$$S_{\sigma_1}(G_1) \cap S_{\sigma_2}(G_2) = U_2 \times M \times W_2 \cap W_1 \times M \times W_1 = M$$

since the variables in the 5 spaces U_1, U_2, W_1, W_2, M are all different.

Then, the transversality $S_{\sigma_1}(G_1) \pitchfork S_{\sigma_2}(G_2)$ is obvious.

Therefore, all we need is $S_{\varkappa_1}(G_1) \pitchfork S_{\varkappa_2}(G_2)$.

Let us now try to find the tangent vector to $S_{\varkappa_1}(F_1)$:

Consider the restriction

$$\mathcal{D}_1^{-1} ; \ S_{\varkappa_1}(F_1) \to S_{\varkappa_1}(G_1)$$

$S_{\varkappa_1}(F_1)$ is a product over $U_2 \times M \times W_2$, that is, $(u_1, u_2, m, w_1, w_2) \in S_{\varkappa_1}(F_1) \Rightarrow (u_1, u_2', m', w_1, w_2') \in S_{\varkappa_1}(F_1)$ provided all points are close to $\underline{0}$.

Consider the curves given parametrically by

$$(u_1,\ u_2 + te_i,\ m,\ w_1,\ w_2), \quad i = 1,2 \ldots, \dim U_2$$

$$(u_1,\ u_2,\ m + te_j,\ w_1,\ w_2), \quad j = 1,2 \ldots, \dim M$$

and $(u_1,\ u_2,\ m,\ w_1,\ w_2 + t)$

where e_i stands for a unit vector with 1 in the ith place and 0 elsewhere. These curves lie in $S_{\varkappa_1}(F_1)$ for small $|t|$ provided $(u_1,u_2,m,w_1,w_2) \in S_{\varkappa_1}(F_1)$ (by product structure of $S_{\varkappa_1}(F_1)$). The tangent vectors to these curves at (u_1,u_2,m,w_1,w_2) are

$$\left.\begin{array}{l}
(\underbrace{0 \cdots\cdots 0}_{\dim U_1 \text{ coordinates}},\ \underbrace{e_i}_{\dim U_2 \text{ coordinates}},\ \underbrace{0 \cdots\cdots 0}_{\dim M \text{ coordinates}},\ \underbrace{0}_{W_1},\ \underbrace{0}_{W_2}) \\
(0 \cdots\cdots 0,\ 0 \cdots\cdots 0,\ e_j,\ 0,\ 0) \\
(0 \cdots\cdots 0,\ 0 \cdots\cdots 0,\ 0 \cdots\cdots 0,\ 0,\ 1)
\end{array}\right\} \quad (**)$$

$$i = 1, 2, \cdots \dim U_2 \quad , \quad j = 1, 2, \cdots \dim M$$

Then the images of these tangent vectors of $S_{\varkappa_1}(F_1)$ under the Jacobian of ϑ_1^{-1} are just tangent vectors of $S_{\varkappa_1}(G_1)$ at $\vartheta_1^{-1}(u_1,u_2,m,w_1,w_2)$. We want to verify that these $\dim U_2 + \dim M+1$ tangent vectors are linearly independent. Now the Jacobian matrix of ϑ_1 at (u_1,u_2,m,w_1,w_2) is

$$\begin{pmatrix}
\frac{\partial \rho}{\partial u_1} & \frac{\partial \rho}{\partial u_2} & \frac{\partial \rho}{\partial m} & 0 & \frac{\partial \rho}{\partial w_2} \\
0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
\frac{\partial c_1}{\partial u_1} & \frac{\partial c_1}{\partial u_2} & \frac{\partial c_1}{\partial m} & 1 & \frac{\partial c_1}{\partial w_2} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

and the inverse of this is

$$\begin{pmatrix} \left(\frac{\partial \rho}{\partial u_1}\right)^{-1} & -\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial u_2}\right) & -\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial m}\right) & 0 & -\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial w_2}\right) \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ -\left(\frac{\partial c_1}{\partial u_1}\right)\left(\frac{\partial \rho}{\partial u_1}\right)^{-1} & \left(\frac{\partial c_1}{\partial u_1}\right)\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial u_2}\right)-\left(\frac{\partial c_1}{\partial u_2}\right) & \left(\frac{\partial c_1}{\partial u_1}\right)\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial m}\right)-\left(\frac{\partial c_1}{\partial m}\right) & 1 & \left(\frac{\partial c_1}{\partial u_1}\right)\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial w_2}\right)-\left(\frac{\partial c_1}{\partial w_2}\right) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence the tangent vectors (**) under the inverse become

$$\left(-\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial u_2}\right)e_i,\ e_i,\ 0,\ \left[\left(\frac{\partial c_1}{\partial u_1}\right)\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial u_2}\right)-\left(\frac{\partial c_1}{\partial u_2}\right)\right]e_i,\ 0\right)_{\mathcal{D}_1^{-1}(u_1,u_2,m,w_1,w_2)}$$

$$\left(-\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial m}\right)e_j,\ 0,\ e_j,\ \left[\left(\frac{\partial c_1}{\partial u_1}\right)\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial m}\right)-\left(\frac{\partial c_1}{\partial m}\right)\right]e_j,\ 0\right)_{\mathcal{D}_1^{-1}(u_1,u_2,m,w_1,w_2)}$$

and $$\left(-\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial w_2}\right),\ 0,\ 0,\ \left[\left(\frac{\partial c_1}{\partial u_1}\right)\left(\frac{\partial \rho}{\partial u_1}\right)^{-1}\left(\frac{\partial \rho}{\partial w_2}\right)-\left(\frac{\partial c_1}{\partial w_2}\right)\right],\ 1\right)_{\mathcal{D}_1^{-1}(u_1,u_2,m,w_1,w_2)}$$

respectively.

On p.168 in Section 6.3. , we have proved

$$\left(\frac{\partial \rho}{\partial v}\right)_{\underline{0}} = \left(\frac{\partial c}{\partial v}\right)_{\underline{0}} = 0$$

That is, in our case now, we have

$$\left(\frac{\partial \rho}{\partial u_2}\right)_{\underline{0}} = \left(\frac{\partial \rho}{\partial m}\right)_{\underline{0}} = \left(\frac{\partial \rho}{\partial w_2}\right)_{\underline{0}} = \left(\frac{\partial c_1}{\partial u_2}\right)_{\underline{0}} = \left(\frac{\partial c_1}{\partial m}\right)_{\underline{0}} = \left(\frac{\partial c_1}{\partial w_2}\right)_{\underline{0}} = 0$$

Therefore at $(u_1, u_2, m, w_1, w_2) = \underline{0}$, these tangent vectors of $S_{\chi_1}(G_1)$ becomes

$$(0, \cdots 0, e_i, 0 \cdots 0, 0, 0)$$
$$(0 \cdots 0, 0 \cdots 0, e_j, 0, 0)$$

and $(0 \cdots 0, 0 \cdots 0, 0 \cdots 0, 0, 1)$ respectively, hence are clearly linearly independent. Therefore for small (u_1, u_2, m, w_1, w_2) the images of the vectors under the Jacobian of $\mathcal{D}_1^{-1}$ will remain linearly independent.

Similarly, we have the same result for $\mathcal{D}_2$. That is, we can find tangent vectors for $S_{\chi_2}(G_2)$ in the same way. The resulting tangent vectors with $(u_1, u_2, m, w_1, w_2) = \underline{0}$ would be

$$(\underbrace{e_k}_{\dim U_1 \text{ coordinates}}, \underbrace{0 \cdots 0}_{\dim U_2 \text{ coordinates}}, \underbrace{0 \cdots 0}_{\dim M \text{ coordinates}}, \underbrace{0}_{w_1}, \underbrace{0}_{w_2}).$$

$$(0 \cdots 0, 0 \cdots 0, e_j, 0, 0)$$

and $(0, \cdots 0, 0 \cdots 0, 0 \cdots 0, 1, 0)$

$$k = 1, 2, \cdots \dim U_1 \quad , \quad j = 1, 2, \cdots \dim M.$$

which again will be clearly linearly independent.

Now let us look at the following table of tangent vectors at $\underline{0}$

$T_0 S_{x_1}(G_1)$	U_1	U_2	M	W_1	W_2
$i = 1, 2 \cdots \dim U_2$	(0	e_1	0	0	0)
		$\vdots$			
	(0	e_i	0	0	0)
		$\vdots$			
$j = 1, 2 \cdots \dim M$	(0	0	e_1	0	0)
			$\vdots$		
	(0	0	e_j	0	0)
			$\vdots$		
	(0	0	0	0	1)
$T_0 S_{x_2}(G_2)$					
$k = 1, 2, \cdots \dim U_1$	(e_1	0	0	0	0)
	$\vdots$				
	(e_k	0	0	0	0)
	$\vdots$				
$j = 1, 2 \cdots \dim M$	(0	0	e_1	0	0)
			$\vdots$		
	(0	0	e_j	0	0)
			$\vdots$		
	(0	0	0	1	0)

Clearly, they span the whole of the space $U_1 \times U_2 \times M \times W_1 \times W_2$, and hence the tangent space. Therefore for small (u_1, u_2, m, w_1, w_2) these vectors still span the tangent space to the amibient space $U_1 \times U_2 \times M \times W_1 \times W_2$, thereby establishing the transversality of the intersection. Hence provided we restrict ourselves to a suitably small neighbourhood of $\underline{0}$ the two strata $S_{\varkappa_1}(G_1)$ and $S_{\varkappa_2}(G_2)$ intersect transversally. Therefore by Lemma 6.3.1, $S_{\varkappa_1}(G_1) \cap S_{\varkappa_2}(G_2)$ is regular over the $U_1 \times M \times W_1 \cap U_2 \times M \times W_2 = M$ space, that is the normal form.

6.5 Example

In order to gain a more explicit version of the general argument in Section 6.4, here we give an example of proving regularity over a stratum with two singularities in full details. It is advisable to study this together with the general argument.

Let Σ be the A_2A_3 stratum. The normal form Γ is

$$y^4 + x^2z^2 + xy^3 + \alpha xy^2z \qquad \text{where } \alpha \neq \pm 2$$

in which A_2 is at X and A_3 at Z.
The transversal $\mathcal{J}$ of Γ at Γ_o is

$$\mathcal{J} = y^4 + x^2z^2 + xy^3 + \alpha_o xy^2z + \alpha xy^2z$$
$$+ \beta x^4 + \gamma x^3y + \delta z^4 + \varepsilon yz^3 + \zeta y^2z^2$$

At X (putting $x = 1$ in $\mathcal{J}$)

$$G_1(y,z,\alpha,\gamma,\delta,\varepsilon,\zeta)$$

$$= y^4 + z^2 + y^3 + \alpha_o y^2z + \alpha y^2z + \gamma y + \delta z^4 + \varepsilon yz^3 + \zeta y^2z^2$$

is a versal unfolding of A_2, where in the notation of Section 6.4

(y,z)-space is the N-space
(α)-space is the M-space
(γ)-space is the U_1-space

(δ)-space is the W_2-space

(ε,ζ)-space is the U_2-space

Note that we have omitted the constant β for versal unfolding. Similarly at Z (putting z = 1 in $\mathcal{J}$), we have

$$G_2(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta)$$

$$= y^4 + x^2 + x^3y + \alpha_0 xy^2 + \alpha xy^2 + \beta x^4 + \gamma x^3 y + \varepsilon y + \quad y^2$$

where now in the notation of Section 6.4 (x,y)-space is the N-space

(α)-space is the M-space

(β)-space is the W_1-space

(γ)-space is the U_1-space

(ε,ζ)-space is the U_2-space

The constant δ has been omitted.

Then by properties of transversal (p. 118), there exist neighbourhoods $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\underline{0}$ $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-space respectively such that if $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \in \mathcal{G}_1$

$G_1(-,\alpha,\gamma,\delta,\varepsilon,\zeta)$ has A_2 $\underline{0}$ iff $\gamma = 0$

(and the A_2 is necessarily at (0,0))

and if $(\alpha,\beta,\gamma,\varepsilon,\zeta) \in \mathcal{G}_2$, then

$G_2(-,\alpha,\beta,\gamma,\varepsilon,\zeta)$ has A_3 near $\underline{0}$ iff $\varepsilon = \zeta = 0$

(A_3 necessarily at (0,0)).

Then we define versal unfoldings F_1 and F_2 for A_2 and A_3 respectively.

$$F_1(y,z,\alpha,\gamma,\delta,\varepsilon,\zeta) = y^4 + z^2 + y^3 + \alpha_o y^2 z + \gamma y$$ - standard

universal unfolding of A_2 and independent of ε, ζ, α, δ.

$$F_2(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta) = y^4 + x^2 + xy^3 + \alpha_o xy^2 + \varepsilon y + \zeta y^2$$ - standard

universal unfolding of A_3 which is independent of γ,α,β. Then by isomorphism of versal unfoldings and argument in Section 6.3. , we have

$$G_1(y,z,\alpha,\gamma,\delta,\varepsilon,\zeta) = F_1(\varphi_1(y,z,\alpha,\gamma,\delta,\varepsilon,\zeta), \alpha, \rho(\alpha,\gamma,\delta,\varepsilon,\zeta), \delta, \varepsilon, \zeta) + \beta + c_1(\alpha,\gamma,\delta,\varepsilon\zeta)$$

and $$G_2(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta) = F_2(\varphi_2(x,y,\alpha,\beta,\gamma,\varepsilon,\zeta), \alpha, \beta, \gamma, \rho'(\alpha,\beta,\gamma,\varepsilon,\zeta), \rho''(\alpha,\beta,\gamma,\varepsilon,\zeta)) + \delta + c_2(\alpha,\beta,\gamma,\varepsilon,\zeta)$$

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be as defined in Section 6.4. Then, by Lemma 6.3.1., there exists diffeomorphism $\mathcal{D}_1$ and $\mathcal{D}_2$

$$(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \xmapsto{\mathcal{D}_1} (\alpha, \beta + c_1(\alpha, \gamma, \delta, \varepsilon, \zeta), \rho(\alpha, \gamma, \delta, \varepsilon, \zeta), \delta, \varepsilon, \zeta)$$

$$(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \xmapsto{\mathcal{D}_2} (\alpha, \beta, \gamma, \delta + c_2(\alpha, \beta, \gamma, \varepsilon, \zeta), \rho'(\alpha, \beta, \gamma, \varepsilon, \zeta), \rho''(\alpha, \beta, \gamma, \varepsilon, \zeta))$$

such that $\quad S_{\varkappa_1}(G_1) = \mathcal{D}_1^{-1}(S_{\varkappa_1}(F_1))$

and $\quad S_{\varkappa_2}(G_2) = \mathcal{D}_2^{-1}(S_{\varkappa_2}(F_2))$

Then by Corollary 6.3.2, we have the regularity of

$S_{\varkappa_1}(G_1)$ over the $(\alpha, \delta, \varepsilon, \zeta)$-space

and $\quad S_{\varkappa_2}(G_2)$ over the (α, β, γ)-space

provided $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ are small enough. It is obvious that $(\alpha, \delta, \varepsilon, \zeta)$-space intersects (α, β, γ)-space transversely in the α-space of moduli. Then, if we can show that $S_{\varkappa_1}(G_1)$ and $S_{\varkappa_2}(G_2)$ intersect transversally it will follow from Lemma 6.3.1, that $S_{\varkappa_1}(G_1) \cap S_{\varkappa_2}(G_2)$ is regular over α-space, which is the required result.

The following is how we find the tangent vectors of $S_{\varkappa_1}(G_1)$ and $S_{\varkappa_2}(G_2)$. It now follows from the general argument that this transversality holds however there follows an alternative version of the argument in which we work explicitly with diffeomorphisms $\mathcal{D}$ and $\mathcal{D}^{-1}$. Let us concentrate only on $\mathcal{D}_1$ at the moment. Suppose we write

$$\mathcal{D}_1 : (\alpha, \underset{M}{\beta}, \underset{W_1}{\gamma}, \underset{U_1}{\delta}, \underbrace{\underset{W_2}{\varepsilon}, \zeta}_{U_2}) \longmapsto (a, b, c, d, e, f)$$

where

$$a = \alpha$$
$$b = \beta + c_1(\alpha, \gamma, \delta, \varepsilon, \zeta)$$
$$c = c\ (\alpha, \gamma, \delta, \varepsilon, \zeta)\ \text{(write this instead of } \rho(\alpha, \gamma, \delta, \varepsilon, \zeta))$$
$$d = \delta$$
$$e = \varepsilon$$
$$f = \zeta$$

Consider the inverse map $\mathcal{D}_1^{-1} = (a,b,c,d,e,f) \to (\alpha, \beta, \gamma, \sigma, \varepsilon, \zeta)$

Now by taking the identity map $(\mathcal{D}_1 \cdot \mathcal{D}_1^{-1})$, we have

$$0 = \frac{\partial \gamma}{\partial \beta} = \frac{\partial \gamma}{\partial a} \cdot \frac{\partial a}{\partial \beta} + \frac{\partial \gamma}{\partial b} \cdot \frac{\partial b}{\partial \beta} + \cdots + \frac{\partial \gamma}{\partial f} \cdot \frac{\partial f}{\partial \beta} = \frac{\partial \gamma}{\partial b}$$

Hence γ is independent of b.

So we can write $\mathcal{D}_1^{-1}$ as

$$\alpha = a$$
$$\beta = b - c_1(a,\ \gamma(a,c,d,e,f)\ d,e,f)$$
$$= \beta(a,b,c,d,e,f)$$
$$\gamma = \gamma(a,c,d,e,f)$$
$$\delta = d$$
$$\varepsilon = e$$
$$\zeta = f$$

For any $(a,b,c,d,e,f) \in S_{\chi_1}(F_1)$, the curve $(a+t,b,c,d,e,f)$

($|t|$ small) lies in $S_{\mathcal{X}_1}(F_1)$. The image of this curve under $\mathcal{D}_1^{-1}$ is $(a+t,\ \beta(a+t,b,c,d,e,f),\ \gamma(a+t,c,d,e,f)\ d,e,f)$ and the tangent vector to this curve at $t = 0$ is

$$\left(1,\ \frac{\partial\beta}{\partial a},\ \frac{\partial\gamma}{\partial a},\ 0,\ 0,\ 0\right)$$

Writing $\frac{\partial\beta}{\partial a}$ as β_a this shows that

$$(1,\ \beta_a,\ \gamma_a,\ 0,\ 0,\ 0)$$

is a tangent vector to $S_{\mathcal{X}_1}(G_1)$. Similarly we find

$$(0,\ \beta_d,\ \gamma_d,\ 1,\ 0,\ 0)$$

$$(0,\ \beta_e,\ \gamma_e,\ 0,\ 1,\ 0)$$

$$(0,\ \beta_f,\ \gamma_f,\ 0,\ 0,\ 1)$$

are tangent vectors to $S_{\mathcal{X}_1}(G_1)$ at $\mathcal{D}_1^{-1}(a,b,c,d,e,f)$ where the partial derivatives are evaluated at $(a,b,c,d,e,f) \in S_{\mathcal{X}_1}(F_1)$.

In the G_2, F_2 situation we find similar results.

$$\mathcal{D}_2 : (\alpha,\beta,\gamma,\delta,\xi,\zeta) \mapsto (a,b,c,d,e,f)$$

where

$$a = \alpha$$

$$b = \beta$$

$$c = \gamma$$

$$d = \delta + c_2(\alpha,\beta,\gamma,\xi,\zeta)$$

$e = e(\alpha,\beta,\gamma,\varepsilon,\zeta)$ (writing this for $\rho'(\alpha,\beta,\gamma,\varepsilon,\zeta)$)

$f = f(\alpha,\beta,\gamma,\varepsilon,\zeta)$ (writing this for $\rho''(\alpha,\beta,\gamma,\varepsilon,\zeta)$)

By considering the inverse map ϑ_2^{-1} and the identity $(\vartheta_2 \cdot \vartheta_2^{-1})$ we have

$$0 = \frac{\partial \varepsilon}{\partial \delta} = \frac{\partial \varepsilon}{\partial a} \cdot \frac{\partial a}{\partial \delta} + \cdots + \frac{\partial \varepsilon}{\partial f} \cdot \frac{\partial f}{\partial \delta} = \frac{\partial \varepsilon}{\partial d}$$

and

$$0 = \frac{\partial \zeta}{\partial \delta} = \frac{\partial \zeta}{\partial a} \cdot \frac{\partial a}{\partial \delta} + \cdots + \frac{\partial \zeta}{\partial f} \cdot \frac{\partial f}{\partial \delta} = \frac{\partial \zeta}{\partial d}$$

Hence both ε and ζ are independent of d.
Then we can write ϑ_2^{-1} as

$$\begin{aligned}
\alpha &= a \\
\beta &= b \\
\gamma &= c \\
\delta &= d - c_2(a,b,c,\varepsilon(a,b,c,e,f),\ \zeta(a,b,c,e,f)) \\
&= \delta(a,b,c,d,e,f) \\
\varepsilon &= \varepsilon(a,b,c,e,f) \\
\zeta &= \zeta(a,b,c,e,f)
\end{aligned}$$

Hence we can obtain as before vectors

$$(1,\ 0,\ 0,\ \delta_a,\ \varepsilon_a,\ \zeta_a)$$

$$(0,\ 1,\ 0,\ \delta_b,\ \varepsilon_b,\ \zeta_b)$$

$$(0,\ 0,\ 1,\ \delta_c,\ \varepsilon_c,\ \zeta_c)$$

as tangent vector to $S_{\varkappa_2}(G_2)$ at $\mathcal{D}_2^{-1}(a,b,c,d,e,f)$ where the partial derivatives are evaluated at $(a,b,c,d,e,f) \in S_{\varkappa_2}(F_2)$.

We now claim that the seven vectors we have obtained span the ambient space $\mathbb{C}^6$, provided a,b,c,d,e,f are sufficiently small.

Consider now the Jacobian matrix of $\mathcal{D}_1$ and its inverse $\mathcal{D}_1^{-1}$

$$\mathcal{D}_1 \qquad\qquad \mathcal{D}_1^{-1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ b_\alpha & 1 & b_\gamma & b_\delta & b_\varepsilon & b_\zeta \\ c_\alpha & 0 & c_\gamma & c_\delta & c_\varepsilon & c_\zeta \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_a & 1 & \beta_c & \beta_d & \beta_e & \beta_f \\ \gamma_a & 0 & \gamma_c & \gamma_d & \gamma_e & \gamma_f \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{D}_1 \cdot \mathcal{D}_1^{-1} = I_6$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ b_\alpha + \beta_a + b_\gamma \gamma_a & 1 & \beta_c + b_\gamma \gamma_c & \beta_d + b_\delta + b_\gamma \gamma_d & \beta_e + b_\varepsilon + b_\gamma \gamma_e & \beta_f + b_\zeta + b_\gamma \gamma_f \\ c_\alpha + c_\gamma \gamma_a & 0 & c_\gamma \gamma_c & c_\delta + c_\gamma \gamma_d & c_\varepsilon + c_\gamma \gamma_e & c_\zeta + c_\gamma \gamma_f \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence

$$\begin{aligned} & b_\alpha + \beta_a + b_\gamma \gamma_a = 0 & \qquad & c_\alpha + c_\gamma \gamma_a = 0 \\ & \beta_c + b_\gamma \gamma_c = 0 & & c_\gamma \gamma_c = 1 \\ & \beta_d + b_\delta + b_\gamma \gamma_d = 0 & \text{and} \qquad & c_\delta + c_\gamma \gamma_d = 0 \\ & \beta_e + b_\varepsilon + b_\gamma \gamma_e = 0 & & c_\varepsilon + c_\gamma \gamma_e = 0 \\ & \beta_f + b_\zeta + b_\gamma \gamma_f = 0 & & c_\zeta + c_\gamma \gamma_f = 0 \end{aligned}$$

From these equations and the fact that $c_\gamma \neq 0$ when at $\underline{O}$ (see p.167), we can show that

$$b_\alpha = b_\gamma = b_\delta = b_\varepsilon = b_\zeta = 0 \qquad \text{(all evaluated at } \underline{O}\text{)}$$

and $$c_\alpha = c_\delta = c_\varepsilon = c_\zeta = 0 \qquad \text{"}$$

Hence $$\beta_a = \beta_d = \beta_e = \beta_f = 0 \qquad \text{"}$$

$$\gamma_a = \gamma_d = \gamma_e = \gamma_f = 0 \qquad \text{"}$$

Similarly for G_2, F_2 case, we have

$$\delta_a = \delta_b = \delta_c = 0$$

$$\varepsilon_a = \varepsilon_b = \varepsilon_c = 0$$

$$\zeta_a = \zeta_b = \zeta_c = 0$$

Hence at $\underline{O}$, all the partial derivatives in the vectors are zero. Then it is obviously that these seven vectors span $\mathbb{C}^6$ at $\underline{O}$. It follows that, provided (a,b,c,d,e,f) is sufficiently close to $\underline{O}$, the partial derivatives will also be so small that the seven vectors still span $\mathbb{C}^6$. Hence for small enough a,b,c,d,e,f, $S_{\varkappa_1}(G_1)$ and $S_{\varkappa_2}(G_2)$ intersect transversely.

§6.6.

Note that in the case when there are more than two singularities, we have more to check than just transversality, between the $S_{\varkappa}(G_i)$'s where $i = 1, \ldots, n$, $n > 2$. Let us now introduce the notion of general position.

Def. 6.6.1. Smooth sub-manifolds $X_1, \ldots, X_n$ of a smooth manifold Z are in general position when the natural map $T_x Z \to \bigoplus_i T_x Z / T_x X_i$ is surjective for each point x in the intersection $\bigcap_i X_i$.

This is the generalized transversality in the case when $n > 2$. In the case when $n = 2$, this says simply that X_1, X_2 intersect transversally in Z. We can show this easily. Let us have the surjectivity of the map

$$T_x Z \to \frac{T_x Z}{T_x X_1} \oplus \frac{T_x Z}{T_x X_2}$$

$$\xi \mapsto (\xi + T_x X_1, \ \xi + T_x X_2)$$

$\xi \in T_x Z$ That is, given $(\eta + T_x X_1, \ \zeta + T_x X_2) \in \frac{T_x Z}{T_x X_1} \oplus \frac{T_x Z}{T_x X_2}$ there exists ξ such that

$$\xi - \eta \in T_x X_1, \quad \xi - \zeta \in T_x X_2.$$

For transversality, we have to show

$$T_x X_1 + T_x X_2 = T_x Z \quad \text{for each } x \in X_1 \cap X_2.$$

Now for every $\theta \in T_x Z$, let $\eta = \theta$, $\zeta = -\theta$, then by surjectivity there exists $\xi \in T_x Z$ such that

$$\xi - \eta = \alpha_1 \in T_x X_1$$

$$\xi - \zeta = \alpha_2 \in T_x X_2.$$

Therefore
$$\begin{aligned} \alpha_1 - \alpha_2 &= (\xi - \eta) - (\xi - \zeta) \\ &= \zeta - \eta \\ &= -2\theta \end{aligned}$$

Hence $\theta = -\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$ as required.

The converse of this is also true. The transversality also implies the surjectivity of the map $T_x Z \to \frac{T_x Z}{T_x X_1} \oplus \frac{T_x Z}{T_x X_2}$

Assume $T_x X_1 + T_x X_2 = T_x Z$, then given η and $\zeta \in T_x Z$ we can write

$$\eta = \eta_1 + \eta_2 \text{ where } \eta_1 \in T_x X_1,\ \eta_2 \in T_x X_2$$

$$\zeta = \zeta_1 + \zeta_2 \text{ where } \zeta_1 \in T_x X_1,\ \zeta_2 \in T_x X_2$$

Then choosing $\xi = \eta_2 + \zeta_1$, we have $\xi - \eta \in T_x X_1$ and $\xi - \zeta \in T_x X_2$ as required.

With this generalized notion of transversality we have a similar lemma to Lemma 6.3.4.

Lemma 6.6.2. Let $Y_1, \ldots, Y_n$ and $X_1, \ldots, X_n$ be submanifolds of a smooth manifold Z, where $n \geq 2$. Suppose

$$Y_i \text{ is regular over } X_i$$

and $Y_1,\dots,Y_n$ and $X_1,\dots,X_n$ are both in general positions. Then $Y_1 \cap \dots \cap Y_n$ is regular over $X_1 \cap \dots \cap X_n$.

The proof is similar to that of Lemma 6.3.4.

Hence, the extra condition we need to check for regularity in the case when there are more than two singularities is the surjectivity of the map $T_x Z \to \oplus_i \frac{T_x Z}{T_x X_i}$ for every point $x \in \bigcap_i X_i$ and the map $T_y Z \to \oplus_i \frac{T_y Z}{T_y Y_i}$ for each $y \in \bigcap_i Y_i$.

The way we check this surjectivity is by considering the tangent vectors we found for each $S_x(G_i)$ in the general argument (see p.185). As for $S_{x_0}(G_i)$'s, the <u>general position</u> is obvious. We shall now look at the case when n=3. Then similar as in p.186, we have a table of tangent vectors at $\underline{0}$. (If this claim of general position is true at $\underline{0}$, it will also be true within small neighborhood of $\underline{0}$.)

	U_1	U_2	U_3	M	W_1	W_2	W_3
$T_y S_x(G_1)$	0	e_1 ⋮	0	0	0	0	0
	0	e_i ⋮	0	0	0	0	0
	0	0	e_1 ⋮	0	0	0	0
	0	0	e_j ⋮	0	0	0	0
	0	0	0	e_1 ⋮	0	0	0
	0	0	0	e_k ⋮	0	0	0
	0	0	0	0	0	1	0
	0	0	0	0	0	0	1
$T_y S_x(G_2)$	e_1 ⋮	0	0	0	0	0	0
	e_l ⋮	0	0	0	0	0	0
	0	0	e_1 ⋮	0	0	0	0
	0	0	e_j ⋮	0	0	0	0

	U_1	U_2	U_3	M	W_1	W_2	W_3
	0	0	0	e_1	0	0	0
				⋮			
	0	0	0	e_k	0	0	0
				⋮			
	0	0	0	0	1	0	0
	0	0	0	0	0	0	1
$T_y S_x(G_3)$	e_1	0	0	0	0	0	0
	⋮						
	e_l	0	0	0	0	0	0
	⋮						
	0	e_1	0	0	0	0	0
		⋮					
	0	e_i	0	0	0	0	0
	0	0	0	e_1	0	0	0
				⋮			
	0	0	0	e_k	0	0	0
				⋮			
	0	0	0	0	1	0	0
	0	0	0	0	0	1	0

Now given $\left(\xi_1 + T_y S_x(G_1),\ \xi_2 + T_y S_x(G_2),\ \xi_3 + T_y S_x(G_3) \right)$

$$\in \frac{T_y Z}{T_y S_x(G_1)} \oplus \frac{T_y Z}{T_y S_x(G_2)} \oplus \frac{T_y Z}{T_y S_x(G_3)}$$

where $\xi_1 ; \xi_2 , \xi_3 \in T_y Z = U_1 \times U_2 \times U_3 \times M \times W_1 \times W_2 \times W_3$

We want to find $\xi \in U_1 \times U_2 \times U_3 \times M \times W_1 \times W_2 \times W_3$

such that
$$\xi - \xi_1 \in T_y S_x(G_1)$$
$$\xi - \xi_2 \in T_y S_x(G_2)$$
$$\xi - \xi_3 \in T_y S_x(G_3)$$

Now let

$$\xi_1 = (u_1, u_2, u_3, m, \omega_1, \omega_2, \omega_3)$$
$$= (0, u_2, u_3, m, 0, \omega_2, \omega_3) + (u_1, 0, 0, 0, \omega_1, 0, 0)$$

But $(0, u_2, u_3, m, 0, \omega_2, \omega_3) \in T_y S_x(G_1)$

and $(u_1, 0, 0, 0, \omega_1, 0, 0) \in T_y S_x(G_2) \cap T_y S_x(G_3)$

Similarly let $\xi_2 = (u_1', u_2', u_3', m', \omega_1', \omega_2', \omega_3')$

$$= \underbrace{(u_1', 0, u_3', m', \omega_1', 0, \omega_3')}_{\in T_y S_x(G_2)} + \underbrace{(0, u_2', 0, 0, 0, \omega_2', 0)}_{\in T_y S_x(G_1) \cap T_y S_x(G_3)}$$

and $\xi_3 = (u_1'', u_2'', u_3'', m'', \omega_1'', \omega_2'', \omega_3'')$

$$= \underbrace{(u_1'', u_2'', 0, m'', \omega_1'', \omega_2'', 0)}_{\in T_y S_x(G_3)} + \underbrace{(0, 0, u_3'', 0, 0, 0, \omega_3'')}_{\in T_y S_x(G_1) \cap T_y S_x(G_2)}$$

Let us choose $\xi = (u_1, u_2', u_3'', m, \omega_1, \omega_2', \omega_3'') \in T_y Z$

Then, we have
$$\xi - \xi_1 \in T_y S_x (G_1)$$
$$\xi - \xi_2 \in T_y S_x (G_2)$$
$$\xi - \xi_3 \in T_y S_x (G_3) \quad \text{as required.}$$

Hence $S_x(G_1)$, $S_x(G_2)$ and $S_x(G_3)$ are in general position at 0. Hence it is true also for small neighbourhood of $\underline{0}$ in $U_1 \times U_2 \times U_3 \times M \times W_1 \times W_2 \times W_3$), and by Lemma 6.6.2, regularity of

$$S_x(G_1) \cap S_x(G_2) \cap S_x(G_3) \text{ over}$$

$$S_{x_0}(G_1) \cap S_{x_0}(G_2) \cap S_{x_0}(G_3) \text{ is established.}$$

CHAPTER 7

DESPECIALIZATION

7.1 Local Despecialization

In Section 6.4, p.176, we encountered the following situation: Consider a stratum Σ which consists of curves with two singularities of types σ_1 and σ_2. The normal form Γ has these singularities fiexed say at vertices X and Z of the triangle of reference. Suppose we have a stratum Σ' specializing to Σ, that is, for some point Γ_0 on Γ obtained by giving fixed values to the moduli, and suppose we have a sequence of points in Σ' (that is, a sequence of curves in $\mathbb{C}P^2$ all giving points in Σ') with limit Γ_0. These curves will have singularities close to X and Z; let those close to X form a type χ_1 (which may be composed of several singularities e.g. $A_1A_1A_2$ if $\sigma_1 = A_6$) and those close to Z form a type χ_2 (similarly χ_2 may be of the type which has more than one singularities).

It is in this Chapter that we discuss the problem of exactly what stratum Σ' will specialize to Σ, that is, what stratum Σ' can have a sequence of curves with limit Γ_0.

We start off by looking at what type of singularity χ_1 can occur close to $X(\sigma_1)$ and similarly what type χ_2 can occur close to $Z(\sigma_2)$. That is, we want to look at how a singularity σ_i can despecialize (break up) locally. We claim, for this, it is enough to look at the universal unfolding of the singularity σ_i with the unfolding parameters being sufficiently small.

To prove the claim, we look at singularity f where

$$f : \mathbb{C}^2, 0 \to \mathbb{C}, 0$$

and a family of functions

$$F : \mathbb{C}^2 \times \mathbb{C}^k \to \mathbb{C}$$

with $F(x,0) = f(x)$.

We can regard F as an unfolding of f and induce it from the universal unfolding G of f, where

$$G : \mathbb{C}^2 \times \mathbb{C}^r \to \mathbb{C}$$

with $\phi(x,u) = (\phi_1(x,u), \Phi(u))$ where $\phi_1(x,0) = x$ and $\Phi : \mathbb{C}^k \to \mathbb{C}^r$, and also a germ

$$c : \mathbb{C}^k, 0 \to \mathbb{C}, 0$$

such that $F(x,u) = G(\phi(x,u)) + c(u)$.

Hence $F_u(x) - c(u) = G_{\Phi(u)}(\phi_1(x,u)) = G_{\Phi(u)}(\phi_{1,u}(x))$

Since $\phi_{1,0}$ = identity, $\phi_{1,u}$ will be, for small u, a nonsingular change of coordinates, depending on u. Now $S_\chi(F) =$ $\{(u,v) \in \mathbb{C}^k \times \mathbb{C} : F(-,u)+v$ has singularities of type χ near $\underline{0}\}$.

$S_\chi(G) = \{(w,v) \in \mathbb{C}^r \times \mathbb{C} : G(-,w)+v$ has singularities of type χ near $\underline{0}\}$.

(The dashes indicates that this is the variables).

Then $(u,v) \in S_\chi(F) \Leftrightarrow G(\phi_1(x,u), \Phi(u)) + c(u)+v$ has singularity of type χ near $\underline{0} \Leftrightarrow (\Phi(u), c(u)+v) \in S_\chi(G)$

Hence $(u,v) \mapsto (\Phi(u),\ c(u)+v)$ is a smooth map taking $S_\chi(F)$ to $S_\chi(G)$ and taking $\underline{0}$ to $\underline{0}$. Hence every type of singularity occurring arbitrarily close to $\underline{0}$ in any analytic family F also occurs arbitrarily close to $\underline{0}$ in the particular family G. So it is enough to look at only the universal unfoldings.

Let us now look at two examples using elementary methods (the general result is most conveniently obtained by using Dynkin diagram, see later p.214).

Example 7.1.1 Local despecialization of A_3

Let us use the standard formula and the universal unfolding for A_3. That is

$$F = x^2 + \frac{1}{4}z^4 + \alpha + \beta z + \frac{1}{2}\gamma z^2$$

The conditions for singular points are

$$F = 0,\ F_x = 2x = 0 \text{ and } F_z = z^3 + \beta + \gamma z = 0$$

That is, $\frac{1}{4}z^4 + \alpha + \beta z + \frac{1}{2}\gamma z^2 = 0$

$$x = 0$$

and $\quad z^3 + \gamma z + \beta = 0$

Let $(0,z_1)$ be a singularity of F. Expand F about $(0,z_1)$:

$$F(x,z+z_1) = \frac{1}{2}\left(\left.\frac{\partial^2 F}{\partial x^2}\right|_{(0,z_1)} x^2 + \left.\frac{2\partial^2 F}{\partial x \partial z}\right|_{(0,z_1)} xz + \left.\frac{\partial^2 F}{\partial z^2}\right|_{(0,z_1)} z^2\right) + \ldots$$

$$= \frac{1}{2}(2x^2 + (3z_1^2+\gamma)z^2) + \frac{1}{6}(6z_1z^3) + \frac{1}{24} \cdot 6z^4 + \cdots$$

$$= x^2 + \frac{1}{2}(3z_1{}^2+\gamma)z^2 + z_1z^3 + \frac{1}{4}z^4 + \cdots\cdots$$

Therefore, we have a node iff $3z_1^2 + \gamma \neq 0$

a cusp iff $3z_1^2 + \gamma = 0$, $z_1 \neq 0$

Now let us consider the roots of the quartic

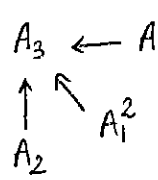
$$\frac{1}{4}z^4 + \alpha + \beta z + \frac{1}{2}\gamma z^2 = 0$$

The absence of the term z^3 ensures that there is no quadruple root. For a double root, it must have a root z_1 common with its derivative, which is the cubic

$$z^3 + \beta + \gamma z = 0$$

But then this is just the condition for a singularity of F. Hence, for general β, γ as long as $3z_1^2 + \gamma \neq 0$, the singularity is a node. That is, in short, double root of $F \Rightarrow A_1$. For a triple root, the quartic must have a root z_1 common not only with its first, but also the second derivative which $3z^2 + \gamma = 0$.
But, then this is exactly the condition for the singularity being a cusp. Hence, triple root of $F \Rightarrow A_2$.

Since a quartic can have at most two double roots or one triple root, the possibilities of local despecialization are as in the following diagram



Note that the above technique can be applied to all A_k singularities. For example with A_5, we will have a sextic equation and again

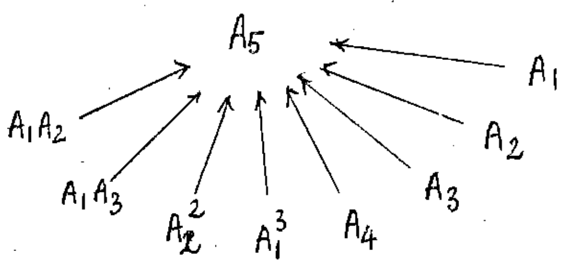
double root of the equation $\Rightarrow A_1$

triple root of the equation $\Rightarrow A_2$

4 ple root of the equation $\Rightarrow A_3$

5 ple root of the equation $\Rightarrow A_4$

The absence of the term z^5 would ensure that there is no 6 ple root. Then, by considering the possible roots for a sextic, we can deduce the following local despecializations easily.



Example 7.1.2 Local Despecialization of D_4

Taking the standard formula for D_4 and its universal unfolding, we have

$$F = \frac{1}{3}x^3 + \frac{1}{3}z^3 + \alpha + \beta x + \gamma z + \delta xz$$

The conditions for singular points are

$$F = 0, \quad F_x = x^2 + \beta + \delta z = 0$$

$$F_z = z^2 + \gamma + \delta x = 0$$

Of course, by choosing $\beta = \gamma = \delta = 0$, $\alpha \neq 0$, we can ensure that F has no singularities at all. Let us next consider the case when $\delta = 0$, β, $\gamma \neq 0$. By F_x and F_z' we have

$$x^2 = -\beta \quad \text{and} \quad z^2 = -\gamma$$

We can see there are actually up to four possible singularities. If (b,c) is one singularity, the other possibilities are (-b,c), (b,-c), (-b,-c) where

$$b^2 = -\beta, \qquad c^2 = -\gamma$$

Let us consider

(i) (b,c) is a singularity iff $F(b,c) = 0$ iff $\alpha = \frac{2}{3}(b^3+c^3)$.

(ii) (-b,-c) is also a singularity together with (b,c) iff we can replace b by -b and c by -c in the equation $\alpha = \frac{2}{3}(b^3+c^3)$ and get the same α iff $\alpha = 0$.

(iii) (b,-c) is also a singularity together with (b,c) we can replace c by -c in this equation and get the same α iff $c = 0$ i.e. $\gamma = 0$ and $\alpha = \frac{2}{3}b^3$.

(iv) (-b,c) is also a singularity together with (b,c) iff we can replace b by -b in the equation and get same α iff $b = 0$ i.e. $\beta = 0$ and $\alpha = \frac{2}{3}c^2$.

Hence, in case (i) $\beta \neq 0$, $\gamma \neq 0$, $\alpha \neq 0$, then (b,c) is the only singularity. (ii) If $\beta \neq 0$, $\gamma \neq 0$, $\alpha = 0$, then (b,c) and (-b,-c) are the only singularities. (iii) If $\beta \neq 0$, $\gamma = 0$, $\alpha \neq 0$, then (b,0) is the only singularity. (iv) If $\beta = 0$, $\gamma \neq 0$, $\alpha \neq 0$, then (0,c) is the only singularity.

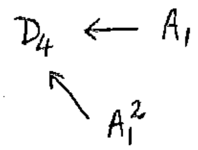
In case (ii) we have two singularities and by symmetry they must be of the same kind, so it is enough to expand F about one of them to see what type of singularity it is.

Now let us expand F about (b,c).

$$F(x+b,\ z+c) = \frac{1}{2}(2bx^2 + 2cz^2) + \ldots$$

Hence it is always a node when β, $\gamma \neq 0$. Therefore we have two nodes in case (ii) and one node in case (i). In case (iii) and (iv), we have to expand F about (b,0) and (0,c), we can easily see that it is always a node as well.

So taking $\delta = 0$, we can deduce



Now for the case when $\delta \neq 0$, we have by F_x

$$z = \frac{-x^2-\beta}{\delta}$$

Substituting this into F_z, and F = 0, we have

$$\frac{(x^2+\beta)^2}{\delta} + \gamma + \delta x = 0$$

(*) i.e. $x^4 + 2\beta x^2 + \delta^3 x + \gamma\delta^2 + \beta^2 = 0$

and

(**) $\frac{1}{3}x^2 + \frac{1}{3}\left(\frac{x^2+\beta}{\delta}\right)^3 + \alpha + \beta x - \gamma\left(\frac{x^2+\beta}{\delta}\right) - \delta x\left(\frac{x^2+\beta}{\delta}\right) = 0$

The Equation (**) determines α once x is given.

Now suppose (u,v) is a critical point, on the curve. Expand F about (u,v)

$$F(u+x, v+z) = \frac{1}{2}(2ux^2+2\delta xz+2vz^2) + \frac{1}{6}(2x^3+2z^3)$$

$$= ux^2 + \delta xz + vz^2 + \frac{1}{3}(x^3+z^3)$$

If $\delta^2 \neq 4uv$ then the quadratic part is not a perfect square and this is a node.

If $\delta^2 = 4\,uv$, then u and v must both be nonzero since $\delta \neq 0$, and the equation becomes

$$u\left(x + \frac{\delta}{2u}\,z\right)^2 + \frac{1}{3}(x^3+z^3)$$

Let $x' = x + \frac{\delta}{2u}\,z$

$z' = z$

then $x = x' - \frac{\delta}{2u}\,z$, $z = z'$

Hence by substitution, and taking away the primes, we have

$$ux^2 + \frac{1}{3}\left(x-\frac{\delta}{2u}\,z\right)^3 + \frac{1}{3}z^3$$

This is a cusp provided $\frac{1}{3}\left(-\frac{\delta}{2u}\right)^3 + \frac{1}{3} \neq 0$ i.e. $\delta^3 \neq 8u^3$

If $\delta^3 = 8u^3$, then the leading terms becomes

$$ux^2 + \frac{\delta}{4u^2}\,xz^2 + \ldots$$

which is a tacnode since $\delta \neq 0$.

Thus the conditions for A_1, A_2, A_3 are:

A_1 iff $\delta^2 \neq 4uv$

A_2 iff $\delta^2 = 4uv$ and $\delta^3 \neq 8u^3$

A_3 iff $\delta^2 = 4uv$ and $\delta^3 = 8u^3$.

Now given β, γ, δ, if u is a root of (*) and $v = \frac{-u^2-\beta}{\delta}$, then for the unique α determined by (**) we have a function with singularity at (u,v). The next task is to determine what kind of singularity it will be and which kinds can occur simultaneously.

Now if u is a repeated root of (*), then

$$4u^3 + 4\beta u + \delta^3 = 0 \text{ (first derivative)}$$

and $u^4 + 2\beta u^2 + \delta^3 u + \gamma\delta^2 + \beta^2 = 0$

From above, the condition for a cusp is just $\delta^2 = 4uv$, which is by $v = \frac{-u^2-\beta}{\delta}$, the same as

$$4u^3 + 4\beta u + \delta^3 = 0.$$

Hence we have repeated root of (*) implies a cusp. For triple root we also have the condition $3u^2 + \beta = 0$ (second derivative).

Since $v = \frac{-u^2-\beta}{\delta}$, we have $v = -\frac{2\beta}{3\delta} = \frac{2u^2}{\delta}$

And satisfying first derivative, we have $\delta^2 = 4uv$ so that eliminating v we get $\delta^3 = 8u^3$, which is precisely the condition for a tacnode.

Hence, when $\delta \neq 0$, considering the roots of (*)

$$\left.\begin{array}{l} \text{simple root } u \Rightarrow \text{node} \\ \text{double root } u \Rightarrow \text{cusp} \\ \text{triple root } u \Rightarrow \text{tacnode} \end{array}\right\} \text{ at } (u.v) \text{ where } v = \frac{-u^2-\beta}{\delta}$$

For $D_4 \leftarrow A_2$, we require F vanishes at a double root of (*). We can always choose α (when $\delta \neq 0$) to ensure F vanishes. So we just need (*) to have a double root when $\delta \neq 0$. No doubt the discriminant of (*) can vanish when $\delta \neq 0$.

For $D_4 \leftarrow A_3$, we need (*) to have a triple root. Again this is possible. Of course (*) can not have a quadrupole root since there is no x^3 term, unless $\beta = \gamma = \delta = 0$.

Note also that, since $\delta \neq 0$, we can make (*) into any quartic equation with the x^3 term absent (and the x term present), by choosing β, γ, δ. Hence all the above possibilities certainly occur.

We now go on to determine which combinations of singularities can occur. Now

$$F \equiv \frac{1}{3}x^3 + \frac{1}{3}z^3 + \alpha + \beta x + \gamma z + \delta xz = 0$$

when $z = \frac{-x^2-\beta}{\delta}$ at a singular point.

We also have

$$x^2 + \beta + \delta z = 0$$

$$z^2 + \gamma + \delta x = 0$$

By substitution,

$$F \equiv \frac{1}{3}\,x(-\beta-\delta z) + \frac{1}{3}z(-\gamma-\delta x) + \beta x + \gamma z + \delta xz + \alpha$$

Using $z = \frac{-x^2-\beta}{\delta}$, we have

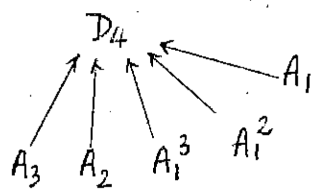
$$F\delta = \frac{2}{3}\,\beta\delta x + \frac{2}{3}\gamma(-x^2-\beta) + \frac{1}{3}\delta x(-x^2-\beta) + \alpha\delta$$

This gives a cubic equation which must be satisfied by u for any singular point (u,v). We are interested in how many of the distinct roots of (*) can also be roots of F = 0,

(***) i.e. $\delta x^3 + 2\gamma x^2 - \beta\delta x + (2\beta\gamma - 3\alpha\delta) = 0$

It is certainly clear that not all four can be. Hence we can rule out A_1^4. By examining the condition for (*) to be of the form (linear factor) × (***) we find that it is possible to have 3 nodes. In fact the simplest example of this is $\beta = \gamma = 0$, in which case the 3 (distinct) roots shared by (*) and (***) are $x = \xi\delta$ where $\xi^3 = -1$. The value of α is $-\frac{1}{3}\delta^3$ and $z = -\xi^2\delta$. Similarly it is possible to have 2 nodes. Hence we have $D_4 \leftarrow A_1^3$ and $D_4 \leftarrow A_1^2$.

Hence, finally we can now write



Also, we need to show that, provided there is an A_2 or worse in the despecialization, there cannot be any other singularities.

Let (u,v) be the singular point on the curve. Let us expand F about (u,v), we have

$$F(x+u,\ z+v) = ux^2 + \delta xz + vz^2 + \frac{1}{3}(x^3+z^3)$$

Suppose that this is a cusp or worse on the curve. Imposing the condition $\delta^2 = 4uv$, the equation becomes (as before)

$$ux^2 + \frac{1}{3}\left(x - \frac{\delta}{2u}z\right)^3 + \frac{1}{3}z^3$$

Denote this by

$$f(x,y) = ux^2 + \frac{1}{3}x^3 - \frac{\delta}{2u}x^2z + \frac{\delta^2}{4u^2}xz^2 - \frac{1}{3}\left(\frac{\delta^3}{8u^3} - 1\right)z^3$$

We want to see whether we can have another singularity near $(0,0)$ for small u, v, δ.

Consider

$$\frac{\partial f}{\partial x} = 2ux + x^2 - \frac{\delta}{u}xz + \frac{\delta^2}{4u^2} z^2 = 0 \qquad \text{(i)}$$

$$\frac{\partial f}{\partial z} = - \frac{\delta}{2u} x^2 + \frac{\delta^2}{2u^2}xz - (\frac{\delta^3}{8u^3} -1) z^2 = 0 \qquad \text{(ii)}$$

Multiplying (i) by $\frac{\delta}{2u}$ and adding to (ii), we have

$$2ux\frac{\delta}{2u} + (\frac{\delta^2}{4u^2}.\frac{\delta}{2u} - \frac{\delta^3}{8u^3} + 1) z^2 = 0$$

i.e. $\delta x + z^2 = 0$

Hence $x = \frac{-z^2}{\delta}$, since $\delta \neq 0$.

Hence any singularity besides the cusp must have $x \neq 0$, $z \neq 0$. Substitute into (ii), we have

$$- \frac{1}{2u\delta} z^2 - \frac{\delta}{2u^2} z - (\frac{\delta^3}{8u^3} - 1) = 0 \qquad \text{(iii)}$$

Substitute $x = \frac{-z^2}{\delta}$ into f(x.y), we have

$$- \frac{1}{3\delta^3} z^3 - \frac{1}{2u\delta} z^2 + (\frac{u}{\delta^2} - \frac{\delta}{4u^2}) z - \frac{1}{3}(\frac{\delta^3}{8u^3} - 1) = 0$$

(iv)

(-3)x(iv) + (iii)

$$\frac{z^3}{\delta^3} + (\frac{3}{2u\delta} - \frac{1}{2u\delta}) z^2 + (\frac{-3u}{\delta^2} + \frac{3\delta}{4u^2} - \frac{\delta}{2u^2}) z = 0$$

i.e. $\frac{z^2}{\delta^3} + \frac{1}{u\delta} z + (\frac{-3u}{\delta^2} + \frac{\delta}{4u^2}) = 0$ (v)

$\frac{1}{\delta^2} \times$ (iii) $+ \frac{1}{2u} \times$ (v)

$$-\frac{1}{\delta^2}\left(\frac{\delta^3}{8u^3} - 1\right) + \frac{1}{2u}\left(\frac{-3u}{\delta^2} + \frac{\delta}{4u^2}\right) = 0$$

This implies $\delta = 0$, a contradiction.

Hence, there can be no other singularities together with a cusp or anything worse, hence rule out the cases A_1A_2, $A_1^2A_2$, A_2^2 and A_1A_3.

The above are two examples in which we use elementary calculation to find the despecializations and also for showing cases which are not possible. Note that the general results of despecialization are shown in §7.2 and here we shall give several lemmas which can help us to rule out easily some cases which were so troublesome in the examples.

Lemma 7.1.3. The sum of the milnor numbers of the singularities of the "upper stratum" is always <u>less than</u> the milnor number of the "lower stratum". That is

$$\Sigma\mu_i < \mu$$

GABRIELOV (1974), Th.3 [~~A'Campo 1975~~]

For example, $D_4 \not\leftarrow A_2^2, A_1^4$

Lemmas 7.1.4. The number of singularities t in the "upper stratum" is less than or equal to the number

$$\tfrac{1}{2}(\mu + r - 1)$$

where μ is the milnor number of the singularity of the "lower stratum" and r is the number of branches of the singularity.

That is,

$$t \leqslant \tfrac{1}{2}(\mu + r - 1) \qquad \text{[Teissier 1975]}$$

For example, for E_6, $\frac{1}{2}(\mu + r - 1) = 3$,

hence $\qquad E_6 \not\leftrightarrow A_1^4, A_1^5$, etc.

Note 7.1.5. One other fact, in the global specialization, it is obvious that reducible singularity type can never specialize into irreducible ones.

§7.2 General Results

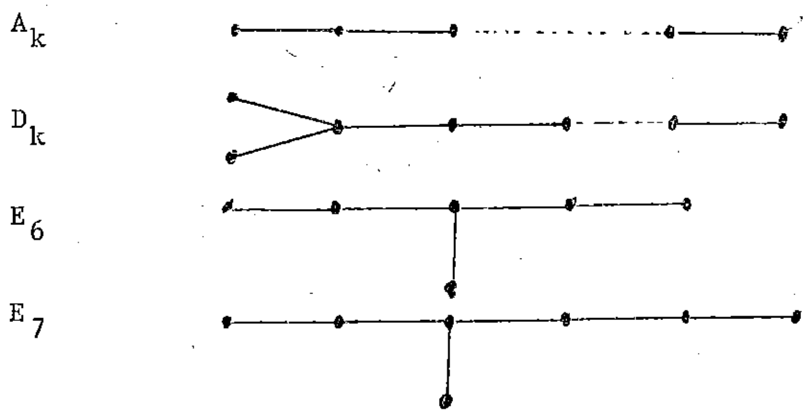
Now we proceed to the general result of Despecialization of simple singularities. This is shown in the paper by O.V. Lyashko [Lyashko 1976]. The arguments make use of the Dykin Diagrams. A Dykin diagram of a singularity of milnor number μ is a connnected graph with μ vertices numbered 1, 2, ..., μ that corresponds to the vanishing cycles of a distinguished basis (see [Grabrielov 1973]). Two vertices are connected by k simple edges, if the intersection index of the corresponding vanishing circles is equal to k (or minus k). The Dykin diagrams of the simple singularities are shown as follows in 7.2.2.

7.2.1. First we give a list of simple singularities which actually occur on quartic curves:

A_1	A_2	A_3	A_4	A_5	A_6	A_7
			D_4	D_5	D_6	
					E_6	E_7

To the right of the dotted lines are singularities occurring only on reducible quartics.

7.2.2. With each singularity A_k, D_k, E_k we associate a Dynkin Diagram



We quote a theorem from Lyashko [Lyashko 1976].

Theorem 7.2.3. A collection of singularities σ_1, σ_2 ... σ_n specialise into a simple singularity σ iff there exist vertices of the Dynkin Diagram for σ such that removing them and the edges coming from them, we are left with Dynkin Diagrams for the singularities σ_1, σ_2 ..., σ_n.

By this theorem, we can work out a list of despecialization for simple singularities.

7.2.4. Realizable local Despecializations of Simple singularities on quartic curves.

ϕ (no singularity)

$A_1 \leftarrow \phi$

$A_2 \leftarrow \phi$

$\quad\quad A_1$

$A_3 \leftarrow \phi$

A_1

$A_2,\ A_1^2$

$A_4 \leftarrow \phi$

A_1

$A_2,\ A_1^2$

$A_3,\ A_1A_2$

$A_5 \leftarrow \phi$

A_1

$A_2,\ A_1^2$

$A_3,\ A_1A_2,\ A_1^3$

$A_4,\ A_1A_3,\ A_2^2$

$A_6 \leftarrow \phi$

A_1

$A_2,\ A_1^2$

$A_3,\ A_1A_2,\ A_1^3$

$A_4,\ A_1A_3,\ A_2^2,\ A_1^2A_2,$

$A_5,\ A_2A_3,\ A_1A_4$

$A_7 \leftarrow \phi$

A_1

$A_2,\ A_1^2$

$A_3,\ A_1A_2,\ A_1^3$

$A_4,\ A_1A_3,\ A_2^2,\ A_1^2A_2,\ A_1^4$

$A_5,\ A_2A_3,\ A_1A_4,\ A_1^2A_3,\ A_1A_2^2$

$A_6,\ A_2A_4,\ A_1A_5,\ A_3^2$

$D_4 \leftarrow \phi$

A_1

$A_2,\ A_1^2$

$A_3,\ A_1^3$

$D_5 \leftarrow \phi$

A_1

$A_2,\ A_1^2$

$A_3,\ A_1A_2,\ A_1^3$

$A_4,\ A_1A_3,\ A_1^2A_2,\ D_4$

$D_6 \leftarrow \phi$

A_1

$A_2,\ A_1^2$

$A_3,\ A_1A_2,\ A_1^3$

$A_4,\ A_1A_3,\ A_2^2,\ A_1^2A_2,\ D_4$

$A_5,\ A_2A_3,\ A_1^2A_3,\ A_1D_4,\ D_5$

$E_6 \leftarrow \phi$

A_1

$A_2,\ A_1^2$

$A_3,\ A_1A_2,\ A_1^3$

$A_4,\ A_1A_3,\ A_2^2,\ A_1^2A_2,\ D_4$

$A_5,\ A_1A_4,\ A_1A_2^2,\ D_5$

$E_7 \leftarrow \phi$

A_1

$A_2,\ A_1^2$

$A_3,\ A_1A_2,\ A_1^3$

$A_4,\ A_1A_3,\ A_2^2,\ A_1^2A_2,\ A_1^4,\ D_4$

$A_5,\ A_2A_3,\ A_1A_4,\ A_1^2A_3,\ A_1A_2^2,\ A_1D_4,\ D_5,\ A_1^3A_2$

$A_6,\ A_2A_4,\ A_1A_5,\ D_6,\ E_6,\ A_1A_2A_3,\ A_1D_5$

7.3 Global Despecialization: Isolated Singularities Case

Before we go into Global Despecialization, we like to give a clear meaning of specialization. If C_n is a sequence of curves all in one stratum Σ' and if $C_n \to C$ where C is in a different stratum Σ, then we say that C is a specialization of curves in Σ'. This simply says that $\Sigma \cap \mathrm{cl}(\Sigma') \neq \phi$.

If we restrict our attention to the strata with isolated singularities we can rule out many specializations by local considerations (using general result Th. 7.2.3.) For example if C has type A_5, then it cannot be a limit of curves of type A_2^3 since the singularity A_5 cannot break up locally into three cusps (see Section 7.2). Likewise if C has type $A_1A_2A_3$ then C cannot be a limit of curves of type A_1^5, since A_2 can only produce locally one node and A_3 can only produce locally 2 nodes, making a maximum of 4. What is not so clear now is whether say A_1^4 can specialize to $A_1A_2A_3$: even though in theory 4 nodes can come from $A_1A_2A_3$, can they all occur on the same quartic curve? That is, can all the local despecializations be put together to form a global one?

We shall show in what follows that all specialization not ruled out by local considerations do actually occur.

Theorem 7.3.1. Any number of separate local despecializations of simple singularities can be realized simultaneously on quartic curves.

To illustrate the meaning of the theorem, let us look at the case of Despecialization of A_2A_3. We know that ϕ, $A_1 \to A_2$ and ϕ, A_1, A_2, $A_1^2 \to A_3$ locally at each singularity. Then we can deduce from the theorem that

$$\phi,\ A_1,\ A_2,\ A_1^2,\ A_1A_2,\ A_1^3 \to A_2A_3$$

We must note that the arguments we use to prove Theorem 7.3.1 actually prove the stronger result that whenever $\Sigma \cap cl(\Sigma') \neq \phi$, then $\Sigma \subset cl(\Sigma')$, i.e. if one curve of type Σ is a limit of curves of type Σ' then all are. Hence in the above example the theorem actually implies that the A_2A_3 stratum is wholly contained in the closure of each of the 6 strata on the left.

In order to prove, for a particular pair of strata, that $\Sigma \subset cl(\Sigma')$ it is clearly enough to show that the normal form Γ for Σ is contained in $cl(\Sigma')$, that is for any curve of Σ in the normal form we can find a Σ' curve arbitrarily close to it in the $\mathbb{C}^{15}$ of all quartic curves. This is because every curve of Σ is projectively equivalent to one in the normal form (Given any Σ curve, C say, take a projective transformation P so that PC is in the normal form. Given a sequence C_n in Σ' tending to PC, we simply take $P^{-1}C_n$ as a sequence tending to C). In fact, we show that $\Gamma \subset cl(\Sigma' \cap \mathcal{J})$ where $\mathcal{J}$ is a transversal for Γ. That is, we work entirely in the transversal.

By properties of transversal (p. 118), each of the singularities of Γ is allocated with the appropriate versal unfolding terms and the terms corresponding to one singularity will not correspond to another.

In the case of a "lower" stratum with only one singularity the specialization follows without difficulty from the local despecialization technique in Section 7.1. Or we can use the general result in Section 7.2 to prove the specialization. Note that in both cases the argument is independent of which curve of Σ we start from so it really does prove $\Sigma \subset cl(\Sigma')$. But is this true in the case when the "lower" stratum has two or more singularities? Here we meet with some difficulties. Let us look at an example in the case of two singularities.

Example 7.3.2. Suppose the "lower" stratum is $\Sigma(A_2A_3)$

Normal form $\Gamma = x^2z^2 + xy^3 + y^4 + \alpha xy^2z$ (A_2 at X, A_3 at Z)

The transversal $\mathcal{T} = x^2z^2 + xy^3 + y^4 + \alpha_0 xy^2z$

$$+ \alpha xy^2z + \beta x^4 + \gamma x^3y + \delta z^4 + \varepsilon yz^3 + \zeta y^2z^2$$

Now we want to see whether $\Sigma'(A_1^3) \to \Sigma(A_2A_3)$, that is, whether for any neighbourhood U of $\underline{0}$ in $(\alpha, \ldots\zeta)$-space, there exists a point of U which is an A_1^3?

We know that given α,β,γ $\exists$ δ,ε,ζ arbitrarily close to $\underline{0}$ such that there is A_1^2 close to Z and given $\alpha,\delta,\varepsilon,\zeta$ $\exists$ β,γ arbitrarily close to $\underline{0}$ such that there is A_1 close to X. But the δ,ε,ζ depend on β,γ and β,γ depend on δ,ε,ζ. So it is not clear that we can always find points of U giving A_1 near X <u>and</u> A_1^2 near Z.

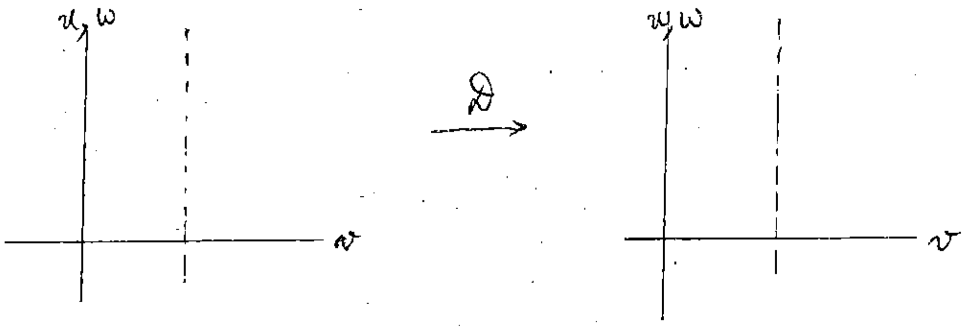
This is the difficulty we meet with in all the cases with the lower stratum having two or more singularities. To avoid this difficulty we introduce a map $\mathcal{A}$ which serves the purpose of fixing the "redundant" unfolding parameters (v in previous notation). We make use of the diffeomorphism $\mathcal{D}$ which was defined in Section 6.3.

$$\mathcal{D}: (U \times V \times W, 0) \to (U \times V \times W, 0)$$

$$(u, v, w) \mapsto (\rho(u,v)\ v,\ w+c(u,v))$$

It has the property of preserving strata, i.e.,

$$S_\chi(G) = \mathcal{D}^{-1}(S_\chi(F))$$



The "lines" v = constant go to themselves under $\mathcal{D}$.
Let us have that projection

$$\pi(u', v', w') = (u', 0, w')$$

Since we know that F (p.164) is independent of v, we can change v and keep u,w constant in $S_\chi(F)$.
Hence

$$\pi(\rho(u,v),\ v, w + c(u.v)) \in S_\chi(F).$$

that is, $(\rho(u,v),\ 0,\ w + c(u,v)) \in S_\chi(F)$.

So $\mathcal{D}^{-1}(\rho(u,v),\ 0,\ w + c(u,v)) \in S_\chi(G)$.
Now we introduce $\mathcal{Q}$.

$$\mathcal{Q} : U \times V \times W \to U \times W.$$

$$\mathcal{Q} = \pi' \circ \mathcal{D}^{-1} \circ \pi \circ \mathcal{D}$$

where $\pi'(u,v,w) = (u,w)$.

Then $\mathcal{A}$ has the property that

$$(u,v,w) \in S_{\chi}(G) \Leftrightarrow (u',0\ w') \in S_{\chi}(G)$$

where $(u',w') = \mathcal{A}(u,v,w)$

This enables us to change the "redundant" unfolding parameters v to 0 provided we make a certain change in the "universal" unfolding prameters u and the "constant" parameter w.

The Jacobian matrix of $\mathcal{D}$ at $\underline{0}$ is $\begin{pmatrix} \frac{\partial \rho}{\partial u} & 0 & 0 \\ 0 & I & 0 \\ \frac{\partial c}{\partial u} & 0 & 1 \end{pmatrix}_{\underline{0}}$

Since $\left.\frac{\partial \rho}{\partial v}\right|_{\underline{0}} = 0$ and $\left.\frac{\partial c}{\partial v}\right|_{\underline{0}} = 0$

The inverse of this is $\begin{pmatrix} \left(\frac{\partial \rho}{\partial u}\right)^{-1} & 0 & 0 \\ 0 & I & 0 \\ -\left(\frac{\partial c}{\partial u}\right)\left(\frac{\partial c}{\partial u}\right)^{-1} & 0 & 1 \end{pmatrix}_{\underline{0}}$

So that the Jacobian of $\mathcal{A}$ at $\underline{0}$ is $\begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

which has maximal rank, hence $\mathcal{A}$ is a <u>submersion</u>, and <u>locally</u> at $\underline{0}$ a <u>surjection</u>.

Now we are ready to prove Theorem 7.3.1. For illustrative purpose we shall perform the proof in the case where there are only two singularities in the singularity type of the lower stratum. For cases with more than two singularities, similar results are true and will be stated afterwards.

Proof 7.3.1.

Let $\chi = \chi_1^o \chi_2^o$. Suppose $\chi_1 \to \chi_1^o$ and $\chi_2 \to \chi_2^o$, we want to show $\chi_1\chi_2 \to \chi_1^o\chi_2^o$, where χ_1^o, χ_2^o are single simple singularity types and χ_1, χ_2 are singularity types that may consist of several singularities.

According to Section 6.4, for χ_1^o, we will have G_1, F_1 and then by Lemma 6.3.1 $\mathcal{D}_1$ can be defined (see p.167). And similarly $\mathcal{D}_2$ for χ_2^o. That is

$$\mathcal{D}_1 : (U_1 \times U_2 \times M \times W_1 \times W_2, 0) \to (U_1 \times U_2 \times M \times W_1 \times W_2, 0)$$

$$(u_1, u_2, m, w_1, w_2) \to (\rho(u_1, u_2, m, w_2), u_2, m, w_1 + c_1(u_1, u_2, m, w_2), w_2).$$

$$\mathcal{D}_2 : (U_1, U_2 \times M \times W_1 \times W_2, 0) \to (U_1 \times U_2 \times M \times W_1 \times W_2, 0)$$

$$(u_1, u_2, m, w_1, w_2) \to (u_1, \rho'(u_1, u_2, m, w_1), m, w_1, w_2 + c_2(u_1, u_2, m, w_2))$$

have the property that

$$(S_{\chi_1}(G_1) = \mathcal{D}_1^{-1}(S_{\chi_1}(F_1))$$

$$(S_{\chi_2}(G_2) = \mathcal{D}_2^{-1}(S_{\chi_2}(F_2))$$

i.e. $\mathcal{D}_1$, $\mathcal{D}_2$ preserve strata.

Consider χ^o, case first.

Let $\pi_1 : (u_1, u_2, m, w_1, w_2) \mapsto (u_1, 0, 0, w_1, 0)$

$$\pi_1' : (u_1, u_2, m, w_1, w_2) \to (u_1, w_1).$$

Then we can define a function $\mathcal{A}_1$

$$U_1 \times U_2 \times M \times W_1 \times W_2 \xrightarrow{\mathcal{A}_1} U_1 \times W_1 .$$

where $\mathcal{A}_1 = \pi_1' \circ \mathcal{D}_1^{-1} \circ \pi_1 \circ \mathcal{D}_1$

With the property that

$$(u_1, u_2, m, w_1, w_2) \in S_{\chi_1}(G_1) \iff$$

$$(\underbrace{\mathcal{A}_{11}(u_1, u_2, m, w_1, w_2)}_{u_1'}, 0, 0, \underbrace{\mathcal{A}_{14}(u_1, u_2, m, w_1, w_2)}_{w_1'}, 0)$$

$$\in S_{\chi_1}(G_1)$$

Let us use the result on P.182 to compute the Jacobian of $\mathcal{A}_1$ at 0.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{\partial \rho}{\partial u_1}\right)_0^{-1} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ -\left(\frac{\partial \rho}{\partial u_1}\right)_0^{-1}\left(\frac{\partial C_1}{\partial u_1}\right)_0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{\partial \rho_1}{\partial u_1}\right)_0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ \left(\frac{\partial C_1}{\partial u_1}\right)_0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and this has rank equal to $(\dim U_1 + 1)$.

We now do a similar thing in the χ_2^0 case. Let

$$\pi_2 : (u_1, u_2, m, w_1, w_2) \mapsto (0, u_2, 0, 0, w_2)$$

$$\pi_2' : (u_1, u_2, m, w_1, w_2) \mapsto (u_2, w_2)$$

We shall define $\mathcal{A}_2 = U_1 \times U_2 \times M \times W_1 \times W_2 \to U_2 \times W_2$

where $$\mathcal{A}_2 = \pi_2' \circ \mathcal{D}_2^{-1} \circ \pi_2 \circ \mathcal{D}_2$$

which has the property

$$(u_1, u_2, m, w_1, w_2) \in S_{\chi_2}(G_2)$$

$$(0, \underbrace{\mathcal{A}_{21}(u_1, u_2, m, w_1, w_2)}_{u_2'}, 0, 0, \underbrace{\mathcal{A}_{22}(u_1, u_2, m, w_1, w_2)}_{\omega_2'}) \in S_{\chi_2}(G_2).$$

The Jacobian matrix of $\mathcal{A}_2$ at $\underline{0}$ is

$\begin{pmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ having rank $(\dim U_2 + 1)$

Consequently the map

$$\mathcal{A}_1 \times \mathcal{A}_2 = U_1 \times U_2 \times M \times W_1 \times W_2 \to (U_1 \times W_1 \times U_2 \times W_2)$$

$$(u_1, u_2\ m, w_1, w_2) \mapsto (\mathcal{A}_1(u_1, u_2, m, w_1, w_2),$$

$$\mathcal{A}_2(u_1, u_2, m, w_1, w_2))$$

has rank $(\dim U_1 + \dim U_2 + 2)$ at $\underline{0}$, hence a surjection at $\underline{0}$. Certainly then $\mathcal{A}_1 \times \mathcal{A}_2$ is locally a surjection, so that if we can find points $(u_1, w_1, u_2, w_2) \in U_1 \times W_1 \times U_2 \times W_2$

arbitrarily close to $\underline{0}$ for which

$$(u_1, 0, 0, w_1, 0) \in S_{\chi_1}(G_1)$$

and $(0, u_2, 0, 0, w_2) \in S_{\chi_2}(G_2)$

then we can find points (u_1, u_2, m, w_1, w_2) arbitrarily close to $\underline{0}$ for which

$$(u_1, u_2, m, w_1, w_2) \in S_{\chi_1}(G_1) \cap S_{\chi_2}(G_2)$$

[Then with the property of transversal (p. 118), we know that definite unfolding parameters are attached to definite singularities and they do not overlap]. Also the unfolding for each of the singularities are actually versal unfoldings. Therefore we can find such $(u_1, 0, 0, w_1, 0)$ so that the unfolding formula has a type χ_1 singularity near $\underline{0}$ and similarly we can find such $(0, u_2, 0, 0, w_2)$ so that the unfolding formula has a type χ_2 singularity near $\underline{0}$. Hence by the surjectivity of $a_1 \times a_2$, we can realize χ_1 and χ_2 simultaneously. Q.E.D.

Remark (i). There are ambiguous cases like $A_5 \to D_6$ when there are two candidates for the singularity type in the upper stratum (in this case A_5(irr) and A_5 [illegible]). The Theorem only implies that at least one of these two must specialize to D_6. Therefore we have to distinguish the possibilities that both of them specialize or only one does. Checking is required in these ambiguous cases. Further detail will be shown later in Section 7.4.

Remark (ii). In the cases when there are more than two singularities in the "lower" stratum, we shall have the map $\mathcal{a}_1 \times \mathcal{a}_2 \times \mathcal{a}_3 \times \ldots \times \mathcal{a}_n$ being still locally a surjection at $\underline{0}$. The same arguement proves the result.

Finally, we note that

<u>Every quartic curve without repeated component specializes to</u> X_9. To see this, take any line meeting the curve in 4 distinct points and change coordinates so that this line is $z = 0$. The curve has the form

$$\alpha z^4 + Az^3 + Bz^2 + Cz + D = 0$$

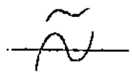
where $\alpha \in \mathbb{C}$, and A, B, C, D are linear, quadratic, cubic, quartic in x, y respectively. Furthermore D has 4 distinct factors since $z = 0$ meets the curve in 4 distinct points. Now replace z by tz (a projective change of coordinates for $t \neq 0$) and let $t \to 0$. The resulting limiting curve $D = 0$ is an X_9.

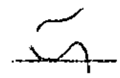
With more care it can be shown that the whole of the X_9 stratum is in the closure of every stratum of curves without repeated component, i.e. that the cross-ratio of the 4 points $D = 0$ above can take any value. An argument is sketched in Bruce & Giblin (to appear).

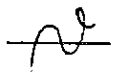
§7.4 Ambiguous Cases

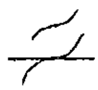
Theorem 7.3.1 has shown that all the global despecializations can be realized, but as explained in Remark 7.3.3, there are cases when one singularity type can have two candidates and this is where ambiguous cases arise. If there is only one candidate for the singularity type in the "upper" stratum, by Theorem 7.3.1, we are sure that if a despecialization occurs, the "upper" stratum must be the only candidate we can have. But if there are two candidates , we are doubtful, because the Theorem only implies at least one of them is a despecialization. There is the possibility that both of them can specialize or only one can. This needs further checking.

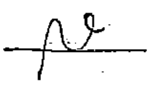
The following is a list of singularity types having two candidates.

7.4.1. A_1^3 irreducible — A_1^3 

A_1A_3 irreducible — A_1A_3 

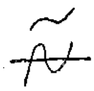
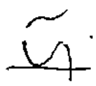
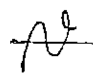
A_1^4  — A_1^4 

A_5 irreducible — A_5 

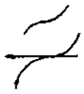
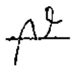
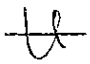
$A_1^2A_3$  — $A_1^2A_3$ 

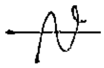
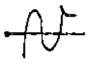
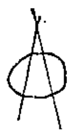
A_1A_5  — A_1A_5 

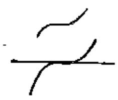
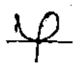
If we can check that one does not specialize, then by Theorem, the other must. That is what we are going to do with the ambiguous cases. A lot of the possibilities are ruled out by the Lemmas in §7.3 and the fact that a reducible curve can not degenerate into an irreducible one. The following are the only outstanding ambiguous cases. Note that we only look at cases where $\Sigma\mu$ goes up by one. This is because of the fact that despecializations are transitive, i.e. if $\Sigma_1 \to \Sigma_2$ and $\Sigma_2 \to \Sigma_3$, then $\Sigma_1 \to \Sigma_3$. And we have checked through all the cases with $\Sigma\mu$ goes up by more than one in the big table of specialization, again all of them can either be proved by transitivity or disproved obviously by Lemmas or facts mentioned above.

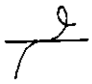
	A_1^3(irr)	A_1^3 
$\downarrow$		
A_1A_3(irr)	✓ (Th.7.3.1)	X (red. $\not\to$ irr)
A_1A_3 	X (Check (9))	✓ (Obvious by geometry)
$A_1^2A_2$(irr)	✓ (Th.7.3.1)	X (red. $\not\to$ irr)
D_4(irr)	✓ (Th.7.3.1)	X (red. $\not\to$ irr)
A_1^4 	✓ (Check (1))	✓ (Obvious by geometry)
A_1^4 	✓ (Th.7.3.1)	X (line $\not\to$ conic)

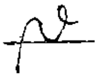
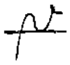
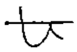
	A_1A_3(irr)	A_1A_3 
$\downarrow$		
A_5(irr)	✓ (Th.7.3.1)	X (red. $\not\to$ irr)
A_1A_4(irr)	✓ (Th.7.3.1)	X (red. $\not\to$ irr)

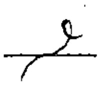
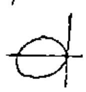
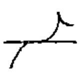
A_2A_3(irr)	✓ (Th.7.3.1)	✗ (red. $\not\to$ irr)
D_5(irr)	✓ (Th.7.3.1)	✗ (red. $\not\to$ irr)
A_5 	✗ (Check (8))	✓ (Obvious by geometry)
$A_1^2A_3$ 	✓ (Check (2))	✓ (Obvious by geometry)
$A_1^2A_3$ 	✓ (Th.7.3.1)	✗ (line $\not\to$ conic)
A_1D_4 	✓ (Check (3))	✓ (Obvious by geometry)

	A_1^4 	A_1^4 
$A_1^2A_3$ 	✓ (Obvious by geometry)	✗ (conic $\not\to$ cubic)
$A_1^2A_3$ 	✗ (line $\not\to$ conic)	✓ (Obvious by geometry)
$A_1^3A_2$ 	✓ (Obvious by geometry	✗ (conic $\not\to$ cubic)
A_1^5 	✓ (Check (4))	✓ (Obvious by geometry)

	A_5(irr)	A_5 
A_6(irr)	✓ (Th.7.3.1)	✗ (red. $\not\to$ irr)
E_6(irr)	✓ (Th.7.3.1)	✗ (red. $\not\to$ irr)
D_6 	✓ (Check (7))	✓ (Obvious by geometry)

A_1A_5 	X (Check(10))	✓ (Obvious by geometry)
A_1A_5 	✓ (Th.7.3.1)	X (line $\not\to$ conic)

	$A_1^2A_3$ 	$A_1^2A_3$ 
D_6 	✓ (By geom. or Th.7.3.1)	X (conic $\not\to$ cubic)
A_1A_5 	✓ (Th.7.3.1)	X (conic $\not\to$ cubic)
A_1A_5 	X (line $\not\to$ conic)	✓ (Th.7.3.1)
$A_1A_2A_3$ 	✓ (Th.7.3.1)	X (conic $\not\to$ cubic)
A_1D_5 	✓ (Th.7.3.1)	X (conic $\not\to$ cubic)
$A_1^3A_3$ 	✓ (Check (5))	✓ (Obvious by geom.)
$A_1^2D_4$ 	X (tangent $\not\to$ non-tangent)	✓ (Obvious by geom. or Th.7.3.1)
A_3^2 	X (line $\not\to$ conic)	✓ (Th.7.3.1)

	A_1A_5 	A_1A_5 
A_7 	X (line $\not\to$ conic)	✓ (Th.7.3.1)
E_7 	✓ (Th.7.3.1)	X (conic $\not\to$ cubic)
A_1D_6 	✓ (Check (6))	✓ (Obvious by geom.)
A_2A_5 	✓ (Th.7.3.1)	X (conic $\not\to$ cubic)

There are a total of ten cases that we need to check. Some of them, can be proved to be true by showing an explicit family of the "upper stratum" Σ' degenerating into the "lower stratum" Σ. In the general argument about specializations no mention is made of the point in the lower stratum. Once we have found a family in Σ' with limit in Σ it will follow that the whole of Σ is in the closure of Σ' (this "frontier condition" also follows from regularity). The cases which are not covered by the above list (where there are non-isolated singularities) are all orbits, so with Σ one of those strata, $\bar{\Sigma}' \cap \Sigma \neq \phi$ automatically implies $\Sigma \subset \bar{\Sigma}'$. Also for some cases in the above list, we have to disprove the property by some special technique, for example by showing that there is only one connected family of Σ' in the universal unfolding space of Σ and since we know one of them is true, the other must not.

We shall proceed with the checking.

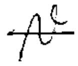
Check (1) $A_1^3(\mathrm{irr}) \to A_1^4$ is true.
Consider the general form for $A_1^3(\mathrm{irr})$.

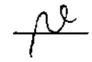
$$ax^2y^2+bx^2z^2+cy^2z^2 = xyz\ (\alpha x+\beta y+\gamma z)$$

$$a \neq 0,\ b \neq 0,\ c \neq 0,\ \alpha \neq \pm 2,\ \beta \neq \pm 2,\ \gamma \neq \pm 2$$

Now let $c \to 0$, we have

$$x(axy^2 + bxz^2 - yz\ (\alpha x + \beta y + \gamma z)) = 0$$

Clearly $x = 0$ cut the cubic at three distinct points for $\beta \neq 0$, $\gamma \neq 0$, since $-yz(\beta y + \gamma z) = 0$. And we can easily check that the cubic is a nodal cubic with the node at X. Hence when $c \to 0$, $A_1^3(\text{irr}) \to A_1^4$  .

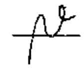
Check (2) $A_1A_3(\text{irr}) \to A_1^2A_3$  is true.
Consider the general form for $A_1A_3(\text{irr})$

$$ay^4 + bx^2z^2 + cx^2yz + dxy^2z + exy^3 = 0$$

where $a \neq 0$, $b \neq 0$, $c \neq 0$

Let $a \to 0$, then we have

$$x(bxz^2 + cxyz + dy^2z + ey^3) = 0$$

The line $x = 0$ cuts the cubic at two points, one repeated, since $y^2(dz + ey) = 0$
i.e. at $(0,0,1)$ twice and $(0,-d,e)$ once.
It is easy to check that the cubic is a nodal cubic if $b \neq d$ or $c \neq e$, with the node at X.
Hence when $a \to 0$, $A_1A_3(\text{irr}) \to A_1^2A_3$  .

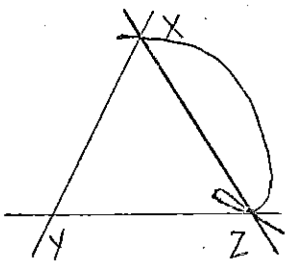
Check (3) $A_1A_3(\text{irr}) \to A_1D_4$  is true.

If we let $b \to 0$ in the general form for $A_1A_3(\text{irr})$ as in Check (2), we have

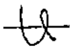
$$y(ay^3 + cx^2z + dxyz + exy^2) = 0$$

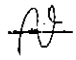
The line $y = 0$ cuts the cubic at $(0,0,1)$ twice and $(1,0,0)$

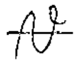
once. And the cubic has the node at (0,0,1) and the tangent directions at the node are



$x(cx + dy) = 0$

Hence when $b \to 0$, $A_1A_3(\text{irr}) \longrightarrow A_1D_4$  .

Check (4) A_1^4  $\to A_1^5$  is true

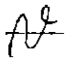
Consider the normal form for A_1^4 

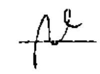
$$x(xy^2 + y^2z + yz^2 + \alpha xz^2 + \beta xyz) = 0$$

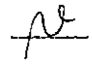
$$\alpha \neq 0, \ \alpha - \beta + 1 \neq 0, \ \beta^2 \neq 4\alpha$$

Let $\alpha \to 0$, we have

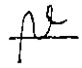
$$xy(xy + yz + z^2 + \beta xz) = 0$$

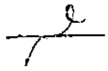
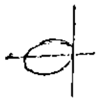
If $\beta \neq 1$, the conic is genuine. The line $x = 0$ cuts the conic at (0,1,0) and (0,1,-1). The line $y = 0$, cuts conic at (1,0,0) and $(1,0,-\beta)$. Hence when $\alpha \to 0$, A_1^4  $\to A_1^5$  .

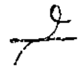
Check (5) $A_1^2A_3$  $\to A_1^3A_3$  is true.

Consider the normal form for $A_1^2A_3$ 

$$x(z^3 + xy^2 + \alpha yz^2 + xyz) = 0 \qquad \alpha \neq 0, \ \alpha \neq 1$$

Let $\alpha \to 1$, we have $x(y+z)(z^2+xy) = 0$ where $x = 0$ cuts conic twice at (0,1,0), $y+z = 0$ cuts conic at (1,0,0) and (1,-1,1). Hence when $\alpha \to 1$, we have $A_1^2A_3$  $\to A_1^3A_3$ 

Check (6) A_1A_5  $\to A_1D_6$  is true.

Consider the general form for A_1A_5 

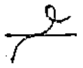
$$x(az^3 + bxy^2 + cxyz) = 0$$

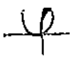
where $a \neq 0$, $b \neq 0$, $c \neq 0$

Let $b \to 0$, we have $xz(az^2 + cxy) = 0$

The line $x = 0$, cuts the conic twice at $(0,1,0)$

The line $z = 0$ cuts the conic at $(0,1,0)$ and $(1,0,0)$

Hence when $b \to 0$, A_1A_5  $\to A_1D_6$ 

Check (7) $A_5(\text{irr}) \to D_6$  is true.

Consider the normal form for $A_5(\text{irr})$

$$x^2z^2 - 2xy^2z + y^4 - \alpha x^4 - x^2y^2 = 0$$

Take the transformation
$$x \to x$$
$$y \to y$$
$$z \to z + ax + by$$

we have

$$x^2(z + ax + by)^2 - 2xy^2(z + ax + by) + y^4 - \alpha x^4 - x^2y^2 = 0$$

i.e.
$$x^2z^2 + 2ax^3z + 2bx^2yz + 2ax^3y + (a^2-\alpha)x^4$$
$$+ (b^2-2a-1)x^2y^2 - 2xy^2z - 2\gamma xy^3 + y^4 = 0$$

Take the transformation
$$x \to \lambda x$$
$$y \to \mu y$$
$$z \to \nu z$$

Then we have

$$\lambda^2\nu^2x^2z^2+2a\lambda^3\nu x^3z+2b\lambda^2\mu\nu x^2yz+2ab\lambda^3\mu x^3y$$

$$+ (a^2-\alpha)\lambda^4x^4+(b^2-2a-1)\lambda^2\mu^2x^2y^2-2\lambda\mu^2\nu xy^2z-2b\lambda\mu^3xy^3+\mu^4y^4 = 0$$

Let us choose $a = \dfrac{\mu^2}{\lambda^2}$, $b = \dfrac{1}{2\lambda\mu^3}$, $\nu = \dfrac{\mu^2}{\lambda}$, $\alpha = a^2 - \dfrac{1}{\lambda^4}$

and $\lambda^2 = \dfrac{1-4\mu^4-8\mu^8}{4\mu^6}$

The equation becomes

$$\mu^4x^2z^2 + 2\mu^4x^3z + x^2yz + x^3y + x^4 + x^2y^2 - 2\mu^4xy^2z$$

$$- xy^3 + \mu^4y^4 = 0$$

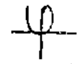
This is still A_5(irr)

Let $\mu \to 0$, we have

$$x^2yz + x^3y + x^4 + x^2y^2 - xy^3 = 0$$

i.e. $x(xyz + x^2y + x^3 + xy^2 - y^3) = 0$

The line $x = 0$, cuts the cubic three times at $(0,0,1)$. And we can easily check that the cubic is nodal with the node at $(0,0,1)$ and tangent direction $x = 0$, $y = 0$.

Hence we find a family of A_5(irr) $\to D_6$ 

Check (8) A_1A_3(irr) $\not\to A_5$ 

Let us look at the universal unfolding of A_5 

$$f(x.y) = x^2+y^6 + \alpha + \beta y + \gamma y^2 + \delta y^3 + \varepsilon y^4$$

We want to look for A_1A_3 in this unfolding space and see whether it is connected as a piece.

Let us look at the singularities of f

$$f = 0 \Longrightarrow x^2 + y^6 + \alpha + \beta y + \gamma y^2 + \delta y^3 + \varepsilon y^4 = 0$$

$$f_x = 0 \Rightarrow 2x = 0 \qquad (1)$$

$$f_y = 0 \Rightarrow 6y^5 + \beta + 2\gamma y + 3\delta y^2 + 4\varepsilon y^3 = 0 \qquad (2)$$

Substituting (1) into $f = 0 \Rightarrow y^6 + \beta + \beta y + \gamma y^2 + \delta y^3 + \varepsilon y^4 = 0$

(3)

For A_1A_3, there must be a triple root y_1 of (2) and a simple root y_3 of (2) where $y_3 \neq y_1$. And also y_1 and y_3 both satisfy (3). (There is of course another root y_2 in (2)).

Condition for triple root y_1 of (2)

(i) y_1 satisfies (2), hence β is determined

$$\beta = -6y_1^5 - 2\gamma y_1 - 3\delta y_1^2 - 4\varepsilon y_1^3$$

(ii) $f_{yy}(y_1) = 0$, γ is determined

$$\gamma = -15y_1^4 - 3\delta y_1 - 6\varepsilon y_1^2$$

(iii) $f_{yyy}(y_1) = 0$, $\delta = -20y_1^3 - 4\varepsilon y_1$ (triple root)

Condition for simple root y_3 of (2)

(iv) $f_y(y_3) = 0$, $\beta = -6y_3^5 - 2\gamma y_3 - 2\delta y_3^2 - 4\varepsilon y_3^3$

Thus β, γ, δ all determined by ε, y_1 and y_3. So it is enough to concentrate on these variables.

The four conditions are

$$\beta + 2y_3\gamma + 3y_3^2\delta + 4y_3^3\varepsilon = -6y_3^5$$

$$\beta + 2y_1\gamma + 3y_1^2\delta + 4y_1^3\varepsilon = -6y_1^5$$

$$\gamma + 3y_1\delta + 6y_1^2\varepsilon = -15y_1^4$$

$$\delta + 4y_1\xi = -20y_1^3$$

Let us look at the determinant

$$\begin{vmatrix} 1 & 2y_3 & 3y_3^2 & 4y_3^3 \\ 1 & 2y_1 & 3y_1^2 & 4y_1^3 \\ 0 & 1 & 3y_1 & 6y_1 \\ 0 & 0 & 1 & 4y_1 \end{vmatrix} = 4(y_1-y_3)^3$$

Thus the determinant is never zero and β, γ, δ, ξ are uniquely determined by y_1 and y_3. So we can analyse the set of solution by looking at the remaining equation. That is the condition for y_1 and y_3 both satisfying (3)

$$y_1^6 + \beta y_1 + \gamma y_1^2 + \delta y_1^3 + \xi y_1^4 = y_3^6 + \beta y_3 + \gamma y_3^2 + \delta y_3^3 + \xi y_3^4$$

i.e. $y_1^6-y_3^6+\beta(y_1-y_3)+\gamma(y_1^2-y_3^2)+\delta(y_1^3-y_3^3)+\xi(y_1^4-y_3^4) = 0$ (4)

Now from (iii) $\delta = -20y_1^3 - 4\xi y_1$

from (ii) $\gamma = -15y_1^4 - 3(-20y_1^3-4\xi y_1)y_1 - 6\xi y_1^2$

$= 45y_1^4 + 6\xi y_1^2$

from (i) $\beta = -6y_1^5 - 2(45y_1^4 + 6\xi y_1^2)y_1 - 3(-20y_1^3-4\xi y_1)y_1^2$
$-4\xi y_1^3$

$= -36y_1^5 - 4\xi y_1^3$

from (iv) $\beta = -6y_3^5 - 2(45y_1^4-6\xi y_1^2)y_3 - 3(-20y_1^3-4\xi y_1)y_3^2-4\xi y_3^3$

$= -6y_3^5 - 90y_1^4y_3 + 60y_1^3y_3^2 - 12\xi y_1^2y_3$

$$+ 12\xi y_1 y_3^2 - 4\xi y_3^3$$

Hence we have

$$- 36y_1^5 + 6y_3^5 + 90y_1^4 y_3 - 60y_1^3 y_3^2$$

$$= \xi(4y_1^3 - 12y_1^2 y_3 + 12y_1 y_3^2 - 4y_3^3)$$

$$= 4\xi(y_1 - y_3)^3$$

Hence $\quad 4\xi = -36y_1^2 - 18y_1 y_3 - 6y_3^2$

i.e. $\quad \xi = \dfrac{-18y_1^2 - 9y_1 y_3 - 3y_3^2}{2}$

Thus by substitution $\quad \beta = -36y_1^5 - 4\left(\dfrac{-18y_1^2 - 9y_1 y_3 - 3y_3^2}{2}\right) y_1^3$

$$= 18y_1^4 y_3 + 6y_1^3 y_3^2$$

and $\quad \gamma = 45y_1^4 + 6\left(\dfrac{-18y_1^2 - 9y_1 y_3 - 3y_3^2}{2}\right) y_1^3$

$$= -9y_1^4 - 27y_1^3 y_3 - 9y_1^2 y_3^2$$

$$\delta = -20y_1^3 - 4\left(\frac{-18y_1^2 - 9y_1 y_3 - 3y_3^2}{2}\right) y_1$$

$$= 16y_1^3 + 18y_1^2 y_3 + 6y_1 y_3^2$$

So when we substitute into (4) we get, after eliminating the factor $(y_1 - y_3)$,

$$2(y_1^5 + y_1^4 y_3 + y_1^3 y_3^2 + y_1^2 y_3^3 + y_1 y_3^4 + y_3^5) + 2(18y_1^4 y_3 + 6y_1^3 y_3^2)$$

$$+\ 2(-9y_1^4-27y_1^3y_3-9y_1^2y_3^2)(y_1+y_3)$$

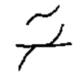
$$+\ 2(16y_1^3+18y_1^2y_3+6y_1y_3^2)(y_1^2+y_1y_3+y_3^2)$$

$$+\ (-18y_1^2-9y_1y_3-3y_3^2)(y_1^3+y_1^2y_3+y_1y_3^2+y_3^3)\ =\ 0$$

then we have

$$-\ 2y_1^5+7y_1^4y_3-8y_1^3y_3^2+2y_1^2y_3^3+2y_1y_3^4-y_3^5\ =\ 0$$

i.e. $(y_1-y_3)^4(2y_1+y_3)\ =\ 0$

Since $y_1 \neq y_3$, the curve is simply $2y_1+y_3 = 0$. Thus the A_1A_3's near A_5(irr) are in fact a single connected curve. This holds for each value of the modulus of the normal form and clearly the curve will vary continuously as the modulus varies. Hence the whole collection of A_1A_3's in the transversal is connected. But of the two types of A_1A_3 neither specializes to the other, hence only one type of A_1A_3 can occur near A_5  . Since we know A_1A_3  $\to A_5$  is true, it follows that A_1A_3(irr) $\not\to$ A_5  .

Check (9) A_1^3(irr) $\not\to$ A_1A_3  (ref. Check 8).
If we can check that for $A_1^2 \to A_3$, the collection of A_1^2 near A_3 in the transversal is in only one connected family, then we can say A_1^3 near A_1A_3 would also be in only one connected family. Let us look at the universal unfolding for A_3.

$$f(x,y)\ =\ x^2\ +\ y^4\ +\ \alpha\ +\ \beta y\ +\ \gamma y^2$$

$$f_x = 2x = 0 \Rightarrow x = 0 \qquad (1)$$

$$f_y = 4y^3 + \beta + 2\gamma y = 0 \Rightarrow \beta = -4y^3 - 2\gamma y \qquad (2)$$

By substituting (1) into f = 0, we have

$$y^4 + \alpha + \beta y + \gamma y^2 = 0 \qquad (3)$$

Two simple roots of (2), y_1 and y_2 say $(y_1 \neq y_2)$ must both satisfy (3), hence

$$y_1^4 - y_2^4 + \beta(y_1 - y_2) + \gamma(y_1^2 - y_2^2) = 0 \qquad (4)$$

Also both y_1 and y_2 satisfy (2). Hence

$$4y_1^3 - 4y_2^3 + 2\gamma y_1 - 2\gamma y_2 = 0$$

i.e. $$\gamma = \frac{-2(y_1^3 - y_2^3)}{(y_1 - y_2)} \qquad (5)$$

Substitute $\beta = -4y_1^3 - 2\gamma y_1$ and (5) into (4)

$$y_1^4 - y_2^4 + (-4y_1^3)(y_1 - y_2) + \frac{4(y_1^3 - y_2^3)}{(y_1 - y_2)}(y_1 - y_2)y_1$$

$$- \frac{2(y_1^3 - y_2^3)}{(y_1 - y_2)}(y_1^2 - y_2) = 0$$

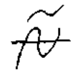
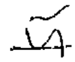
$$\therefore \quad y_1^4 - y_2^4 - 4y_1^4 + 4y_1^3 y_2 + 4y_1^4 - 4y_1 y_2^3$$

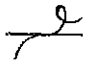
$$-2(y_1^4 + y_1^3 y_2 - y_1 y_2^3 - y_2^4) = 0$$

Hence, $$-y_1^4 + y_2^4 + 2y_1^3 y_2 - 2y_1 y_2^3 = 0$$

$$(y_1^2 - y_2^2)(y_1^2 + y_2^2) - 2y_1y_2(y_1^2 - y_2^2) = 0$$

i.e. $$(y_1 - y_2)^3 (y_1 + y_2) = 0$$

Therefore the only connected family is $y_1+y_2 = 0$. For the same reason, as in Check (8), since we have A_1^3  $\longrightarrow$ A_1A_3  , it follows that $A_1^3(\text{irr}) \not\to A_1A_3$  .

Check (10) $A_5(\text{irr}) \not\to A_1A_5$ 

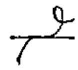
Let us look at the transversal for $A_1A_5(\text{irr})$

$$x(z^3+xy^2+xyz) + \alpha x^4+\beta y^4+\gamma y^3z+\delta y^2z^2+\varepsilon yz^3+\xi z^4 = 0$$

The condition for A_5 to remain is

$$\beta = \gamma = \delta = \varepsilon = \xi = 0$$

But then the resulting curve must still have the line $x = 0$ as a component, i.e. still reducible. Hence

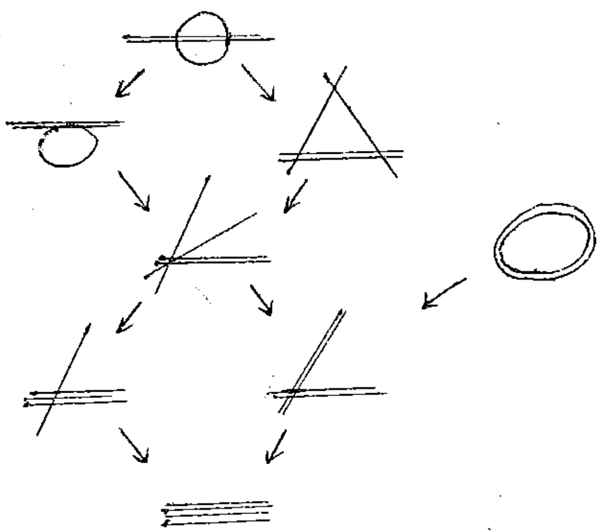
$A_5(\text{irr}) \not\to A_1A_5$  .

7.5 Specialization to Quartics with Repeated Components

The specializations concerning all cases where the lower stratum Σ has isolated singularities have already been covered in the previous sections. And the argument shows that if $\Sigma'\to\Sigma$ the whole of $\Sigma \subset$ closure of Σ'. Now we are left with two other types of specializations (I) Cases where Σ' has isolated singularities and Σ has non-isolated singularities (i.e. repeated components) (II) Cases where Σ' and Σ both have non-isolated singularities.

For (II), it is easy to deduce the specializations from the geometric structure. The following is a complete diagram of specializations of the repeated component strata amongst themselves.

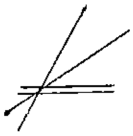
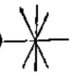
Diagram 7.5.1.

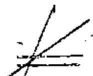


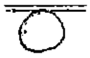
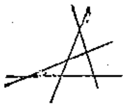
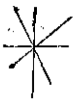
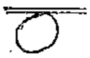
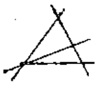
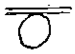
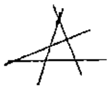
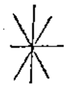
Note that all these are orbits, so it follows immediately that the lower orbit is wholly in the closure of an upper orbit.

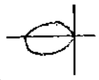
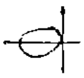
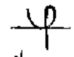
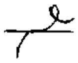
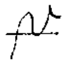
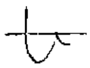
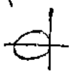
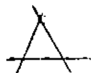
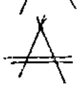
As for (I), we need to discuss each case individually. Let χ be a singularity type without repeated component.

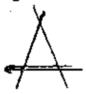
The situations are

(1) $\chi \to$  is true for $\chi = X_9(\tilde{E}_7)$ 

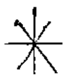
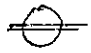
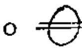
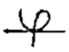
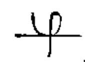
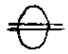
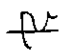
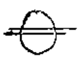
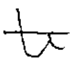
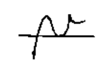
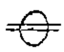
(Obvious by geometric observation). Since all χ without repeated components specialize to X_9 and  is an orbit, we can deduce that all χ without repeated components specialize to  and those below it in Diagram 7.5.1.

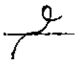
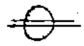
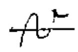
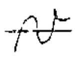
(2) $\chi \to$  is true for all χ except $A_1^3 D_4$ , A_1^6  and X_9 . This is because it is obvious by geometric appearance that all singularity type with $\Sigma\mu = 7$ specialize into  except $A_1^3 D_4$ . But A_5  specialize into  by similar reasons. That leaves us with $A_1^3 D_4$ , A_1^6  and X_9 , because lines can never degenerate into a conic.

(3) $\chi \to$  is true for $\chi = A_1 D_6$  since the normal form for $A_1 D_6$  is $xz(ax^2+yz) = 0$ and when $a \to 0$, it becomes . Hence for all strata with $\Sigma\mu \leq 5$ since they all specialize to $A_1 D_6$ . Also it is true for $\chi = D_6$ , $A_1 A_5$ , $A_1 A_5$ , $A_1 A_2 A_3$ , $A_1 D_5$ , $A_1^3 A_3$ , $A_1^2 D_4$ , since they all specialize into $A_1 D_6$ . Also it is true for $\chi = A_1 A_3^2$ , $A_1^3 D_4$  for obvious geometric reasons. For $\chi = A_7$, specialization does not occur; this follows because for $\chi = A_6$, it does not occur (see below). Since a cuspidal cubic does not specialize to , we have $A_2 A_5$ , E_7  do not specialize to . Also it is obvious that X_9  does not specialize to . This leaves us

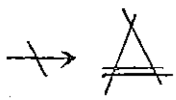
with $\chi = A_6$, E_6, A_2A_4, A_2^3 being doubtful. However, E_6 is a triple point with 3 coincident tangents, so there is no point of  which could be a limit of E_6 singularities. The case $\chi = A_6$ will be done later in some detail, we found that specialization does not occur, but for $\chi = A_2A_4$ and A_2^3, it still remains doubtful. The difficulties we meet with in these cases will be discussed.

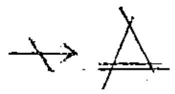
(4) $\chi \to$  is clear one way or the other by obvious geometric reasons except for $\chi = D_4$, D_5, E_6 and A_2^3. For example with $\Sigma\mu = 7$, we have A_7  $\to$ , but all the others don't. For the triple points $\chi = D_4$, D_5, E_6, it is clear that they do not specialize to a repeated conic since a limit of triple points must be at least a triple point, and every point of the repeated conic is double. This leaves only A_2^3 doubtful.

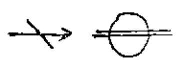
(5) $\chi \to$ . Neither $X_9(\tilde{E}_7)$  nor any of the ($\Sigma\mu = 7$) curves could specialize to  due to their geometric structure. Taking those χ with ($\Sigma\mu = 6$), it is obvious that A_3^2 , $A_1^2D_4$  .. . do specialize to . Also let us consider D_6  with the form $x(x^3+y^3-xyz)$, taking the transformation $x \to x$, $y \to \lambda y$, $z \to \frac{1}{\lambda}z$, the form becomes $x(x^3+\lambda^3y^3-xyz)$. Let $\lambda \to 0$, the curve specialize to $x^2(x^2-yz)$ which is . Hence we have D_6  $\to$ . For one reason or another, we can see that A_1^6 , $A_1^3A_3$ , A_1D_5 , $A_1A_2A_3$  and A_1A_5  don't specialize to . For example A_1D_5  and $A_1A_2A_3$  do not specialize to  because a cuspidal cubic can only specialize

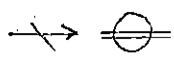
to a conic and a tangent. For the rest of the ($\Sigma\mu = 6$)-cases, A_6, E_6 and A_1A_5  will be shown to be not specializing to . (p. 250) and the cases A_2^3, A_2A_4 still remain doubtful. For ($\Sigma\mu = 5$) cases, those specializing to A_3^2 , $A_1^2D_4$  and D_6  will obviously specialize into . That leaves us with the cases $A_1A_2^2$, A_1A_4 and $A_1^3A_2$ . But since a cuspidal cubic can only specialize into a conic and a tangent, $A_1^3A_2$  is ruled out. The two cases $A_1A_2^2$ and A_1A_4 still remain doubtful. For ($\Sigma\mu = 4$) cases, every χ specializes.

Therefore after checking through (1) to (5), we are left with the following cases. We will check S(1) and S(6) of the following in detail. And S(2), S(3) and S(4) follow from S(1); S(5) and S(7) follow from S(6). Q(1) to Q(5) are the cases which still remain doubtful.

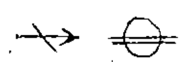
S(1) A_6 

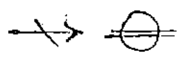
S(2) E_6 

S(3) A_6 

S(4) E_6 

S(5) A_1A_5 

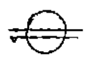
S(6) A_1A_4 

S(7) A_2A_4 

Doubtful Cases*

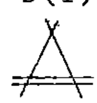
Q(1) $A_2A_4 \longrightarrow$ 

Q(2) $A_2^3 \longrightarrow$ 

Q(3) $A_2^3 \longrightarrow$ 

Q(4) $A_1A_2^2 \longrightarrow$ 

Q(5) $A_2^3 \longrightarrow$ 

S(1) can be done in the following way. Suppose we have  in the closure of the stratum χ containing an A_k

* all these five cases have been ruled out by a different method in [Bruce & Giblin].

singularity of type $k \geqslant 4$. We concentrate on the A_k' singularity P of the curve in the upper stratum and follow this curve down a path into the lower stratum . For , we take $x^2yz = 0$. The limiting position of the singularity P must be on the line $x = 0$ (since the other singularity at X is only a node). We have two cases (i) $P \to Z$ on $x = 0$, (ii) $P \to$ some point on $x = 0$ besides Y or Z, in this case, we can take $P \to (0,1,1)$ by a change of coordinate.

For case (i), suppose the sequence of curves with limit $x^2yz = 0$ is x^2yz + terms in the 15 monomials of quartics with small coefficients (i.e. coefficients $\to 0$ in the sequence).

Also suppose P is at $(\alpha,\beta,1)$ where $\alpha,\beta \to 0$ as $P \to Z$. Substitute $x \to x+\alpha z$, $y \to y+\beta z$, $z \to z$, into the equation, the effect is to take P to Z and to make small changes in the coefficients. Thus we have

$(1+\delta)x^2yz$ + small terms in other monomials; $\delta \to 0$.

By doing the following substitution,

$$x \to (1+\delta)^{-\frac{1}{2}}x, \; y \to y, \; z \to z,$$

we can turn the coefficients of x^2yz back to 1 again without altering the fact that all the other coefficients are small.

Now consider the direction of the unique tangent line to the A_k singularity. By passing to a subsequence we can assume that this line has a limiting direction, say $my = nx$. If $m \neq 0$, then we can take the limiting direction as $y = nx$. By a similar argument to the above we can assume that all curves in the sequence actually have tangent line equal to $y = nx$.

Suppose $\exists$ equi-singularity family f_t with limit $x^2yz = 0$ and having A_k at Z ($k \geqslant 2$) with limiting tangent direction $y = nx$. As above we may assume the tangent direction is fixed at $y = nx$.

$f_t \equiv x^2yz$ + small terms in the 15 monomials = 0.

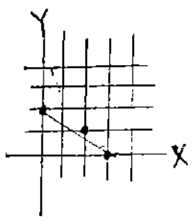
Now substitute $y \to y+nx$, then the new family g_t

g_t: $x^2(y+nx)z$ + still small term .... = 0 has limit $x^2yz + nx^3z = 0$ and has A_k at Z with limiting tangent direction $y = 0$ (Note that the family we had for the fixed tangent direction at $y = nx$ is projectively equivalent to g_t).

So we are looking for

$x^2yz + nx^3z$ + small terms in other 13 monomials

with A_k, $k \geqslant 2$ at Z, tangent direction $y = 0$. Having such an $A_k (k \geqslant 2)$ at Z implies terms z^4, xz^3, yz^3, x^2z^2 and xyz^2 must be missing from the equation and the term y^2z^2 must be present (double point, not triple point). Hence considering the leading terms in the resulting equation



$$x^2yz + nx^3z + \alpha y^2z^2 + \ldots\ldots = 0 \qquad (\alpha \neq 0)$$

we have A_2 at Z if $n \neq 0$

A_3 at Z if $n = 0$.

Therefore if $m \neq 0$, at worst, we can only have A_3 at Z.

When $m = 0$, that is the case when the tangent direction is $x = 0$, we must have for A_k, $k \geqslant 4$ at Z, the equation with the terms in z^4, xz^3, yz^3, y^2z^2, xyz^2 and y^3z missing, and the terms in x^2z^2, xy^2z and y^4 forming a perfect square.

Hence we have (at least)

$$x^2yz + ax^2z^2 + bxy^2z + cy^4 + dxy^3 + ex^2y^2 + fx^3z$$

$$+ gx^3y + hx^4 = 0 \text{ where } b^2-4ac = 0,\ a \neq 0,\ c \neq 0.$$

Consider the equation in non-homogeneous coordinates at Z.

$$X^2Y + a(X+\frac{b}{2a}Y^2)^2 + dXY^3 + eX^2Y^2 + fX^3 + gX^3Y + hX^4 = 0.$$

Take the transformation $X \to X - \frac{b}{2a}Y^2$, we have $(X - \frac{b}{2a}Y^2)^2Y+a(X)^2 + d(X - \frac{b}{2a}Y^2)Y^3 + e(X - \frac{b}{2a}Y^2)^2Y^2 + f(X - \frac{b}{2a}Y^2)^3 + g(X - \frac{b}{2a}Y^2)^3Y + h(X - \frac{b}{2a}Y^2)^4 = 0.$

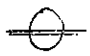
Hence leading terms are

$$aX^2 + \frac{b^2}{4a^2}Y^5 - d(\frac{b}{2a})Y^5 - \frac{b}{a}XY^3 + dXY^3 + \ldots = 0$$

If $d - \frac{b}{2a} \neq 0$, then $\mu = (2-1)(5-1) = 4$,

if $d = \frac{b}{2a}$, then $\mu = (3-1)2 + 1 = 5$

Therefore the milnor number for the singularity at Z is always $\leqslant 5$. Therefore we can't have an A_6. We can also see that similar situation holds for E_6.

For S(6), we check it in the following way. We take $x^2(x^2-yz) = 0$ for  . Suppose $\exists$ equi-singularity family f_t with limit $x^2(x^2-yz) = 0$ and having $A_k (k \geqslant 4)$ at Z.

$$f_t \equiv x^4 - x^2yz + \alpha xy^2z + \beta x^2z^2 + \gamma y^4 + \delta xy^3 + \xi x^2y^2 + \zeta x^3y + \eta x^3z = 0$$

where $\alpha, \beta, \gamma, \delta, \xi, \zeta$ and η are all small and $\alpha^2 = 4\beta\gamma$

$\beta \neq 0$ (since not triple point).

At Z, we have

$$X^4 - X^2Y + \alpha XY^2 + \beta X^2 + \gamma Y^4 + \delta XY^3 + \xi X^2Y^2 + \zeta X^3Y + \eta X^3 = 0$$

Take the transformation $X \to X - \frac{\alpha}{2\beta}Y^2$, the leading terms are

$$\beta X^2 + \left[\delta\left(-\frac{\alpha}{2\beta}\right) - \frac{\alpha^2}{4\beta^2}\right]Y^5 + \left[\delta + 2\left(\frac{\alpha}{2\beta}\right)\right]XY^3 + \ldots = 0$$

So the condition for A_4 is $\delta\left(-\frac{\alpha}{2\beta} - \frac{\alpha^2}{4\beta^2}\right) \neq 0$

$$\text{i.e. } \delta + \frac{\alpha}{2\beta} \neq 0 \text{ and } \alpha^2 = 4\beta\gamma$$

Now we want to see whether we can have A_4 together with another singularity (with tangent direction tending to $x = 0$). Let us check for the other singularity near Z.

$$\frac{\partial f_t}{\partial x} = 4x^3 - 2xyz + \alpha y^2z + 2\beta xz^2 + \delta y^3 + 2\xi xy^2 + 3\zeta x^2y + 3\eta x^2z = 0 \qquad \text{(i)}$$

$$\frac{\partial f_t}{\partial y} = -x^2z + 2\alpha xyz + 4\gamma y^3 + 3\delta xy^2 + 2\xi x^2y + \zeta x^3 = 0 \qquad \text{(ii)}$$

$$\frac{\partial f_t}{\partial z} = -x^2y + \alpha xy^2 + 2\beta x^2z + \eta x^3 = 0 \qquad \text{(iii)}$$

If $x = 0$, we have by $\frac{\partial f_t}{\partial y} = 0$, $y = 0$, assuming $\gamma \neq 0$. For $\gamma = 0$, by $\alpha^2 = 4\beta\gamma$ we have $\alpha = 0$, then $\delta \neq 0$ for isolated singularity, but by $\frac{\partial f_t}{\partial x}$ we can again have $y = 0$ if $x = 0$. So we can assume $x \neq 0$.

If $y = 0$, we have by (i), (ii), (iii).

$$4x^2+2\beta z^2+3\eta xz = 0 \qquad \text{(iv)}$$

$$-z + \zeta x = 0 \qquad \text{(v)}$$

$$2\beta z + \eta x = 0 \qquad \text{(vi)}$$

Substitute (v) and (vi) into (iv) we have

$$4 + 3\eta\zeta + 2\beta\zeta^2 = 0$$

which is a contradiction since β, η, ζ are small. So we can assume $y \neq 0$.

For singularity near Z, we can assume $z \neq 0$. Subtracting (iii)x(z) from (i) we have $4x^3-xyz+\delta y^3+2\xi xy^2+3\zeta x^2y+2\eta x^2z=0$ (vii)

By (iii) we have $z = \frac{xy-\alpha y^2-\eta x^2}{2\beta x}$ (viii)

Substitute (viii) into (ii) and (vii), we have

$$x^3(8\beta-2\eta^2)+x^2y(3\eta+6\beta\zeta)+xy^2(-1+4\beta\xi-2\alpha\eta)+y^3(\alpha+2\beta\delta) = 0 \qquad \text{(ix)}$$

$$x^3(\eta+2\beta\zeta)+x^2y(-1+4\beta\xi-2\alpha\eta)+xy^2(3\alpha+6\beta\delta)+y^3(-2\alpha^2+8\beta\gamma) = 0 \qquad \text{(x)}$$

But since $\alpha^2 = 4\beta\gamma$ for A_4, we have the term y^3 vanishes in equation (x).

Hence we have $x^2(\eta+2\beta\zeta)+xy(-1+4\beta\xi-2\alpha\eta)+y^2(3\alpha+6\beta\delta) = 0$ (xi)

Now (3)x(ix) - (y)x(xi) gives

$$x^2(24\beta-6\eta^2)+xy(8\eta+16\beta\zeta)+y^2(-2+8\beta\xi-4\alpha\eta) = 0 \quad \text{(xii)}$$

Now $-(-2+8\beta\xi-4\alpha\eta)$x(xi) + $(3\alpha+6\beta\delta)$x(xii) gives

$$x\left[(3\alpha+6\beta\delta)(24\beta-6\eta^2)-(-2+8\beta\xi-4\alpha\eta)(\eta+2\beta\zeta)\right]$$

$$+y\left[(3\alpha+6\beta\delta)(8\eta+16\beta\zeta)-2(-1+4\beta\xi-2\alpha\eta)^2\right] = 0 \quad \text{(xiii)}$$

Writing (xiii) as $-Ax+By = 0$, then $y = +\frac{A}{B}x$

where $A = -\left[(3\alpha+6\beta\delta)(24\beta-6\eta^2)-(-2+8\beta\xi-4\alpha\eta)(\eta+2\beta\zeta)\right]$

$B = \left[(3\alpha+6\beta\delta)(8\eta+16\beta\zeta)-2(-1+4\beta\xi-2\alpha\eta)^2\right]$

It is clear that $A \to 0$, $B \to -2$ as all the parameter approach 0. Hence if we put $x = 1$, y will approach zero, so we must have $z \to \infty$ (otherwise limiting position of singularity could not satisfy $x = 0$).

Now substituting $y = \frac{A}{B}x$ into (viii), $z = \frac{AB-\alpha A^2-\eta B^2}{B^2 2\beta}x$

and since $B^2 \to 4$, we must have $\frac{AB-\alpha A^2-\eta B^2}{2\beta} \to \infty$.

But now A = multiple of $\beta - 2\eta+14\alpha\eta^2$

$$B = \text{multiple of } \beta + 16\alpha\eta-2-8\alpha^2\eta^2$$

So $AB-\alpha A^2-\eta B^2 = \text{multiple of } \beta - 24\alpha^2\eta^3-52\alpha^3\eta^4-64\alpha^4\eta^5$

(Notice that terms in η, $\alpha\eta^2$ cancel!)

It follows that $\frac{\alpha^2\eta^3}{\beta} \to \infty$, so in particular $\frac{\alpha^2}{\beta} \to \infty$.

But $\frac{\alpha^2}{\beta} = 4\gamma \to 0$. This is a contradiction.

Hence we have shown that A_4 cannot occur with another singularity. i.e., $A_1A_4 \not\rightarrow \ominus$.

CHAPTER 8

$X_9(\tilde{E}_7)$

In this chapter we use a method due to J.W. Bruce to show that certain of the strata low down on the specialization diagram are regular over $\tilde{E}_7$. Several technical details are explained in Bruce's thesis (1978) and we shall not go into these. We consider the transversal to $\tilde{E}_7$ given by

$$\mathcal{J} = x^4+a_0x^2y^2+y^4+ax^2y^2$$
$$+ z(bx^2y+cxy^2)$$
$$+ z^2(dx^2+exy+fy^2)$$
$$+ z^3(gx+hy)$$
$$+ z^4(k) \qquad (a_0^2\neq 4)$$

It is enough to show that regularity holds along analytic paths $a = a(t)$, ..., $k = k(t)$, so we assume that a, ..., k are analytic functions of a complex variable t with value 0 for $t = 0$, and that $\mathcal{J}$ as above lies in some fixed stratum Σ of curves with isolated singularties. Let r = minimum order of a, ...,k. (We write $O(a)$ for order of a, etc.).

Lemma 8.1 Suppose that the order of d,e,f,g,h,k are all $>r$. Then regularity holds along the path given above.

Let T_t be the tangent space to Σ at F_t. It can be shown that $T_o = \lim_{t\to 0} T_t$ automatically exists. We have to prove that $T_o \supset T$ where T is the tangent space to the $\tilde{E}_7$ stratum at $x^4+a_o x^2y^2+y^4$. Now by the local product structure of orbits, we know that T_o will contain the tangent space to the orbit through $\mathcal{J}_o = x^4+a_o x^2y^2+y^4$. This is spanned by $x\frac{\partial \mathcal{J}_o}{\partial x}, \ldots z\frac{\partial \mathcal{J}_o}{\partial z}$ so that the vector

$$2x^4+a_o x^2y^2,\ 2x^3y+a_o 2y^3,\ 2xy^3+a_o x^3y,\ 2y^4+a_o x^2y^2$$

will all belong to T_o. It follows, using $a_o^2 \neq 4$, that x^3y and xy^3 belong to T_o. We shall use this fact below. Further, the tangent space to $\tilde{E}_7$ at $\mathcal{J}_o$ is the direct sum of the tangent space to the orbit through $\mathcal{J}_o$ and the "modulus direction" spanned by x^2y^2. Therefore we have only to prove $x^2y^2 \in T_o$.

Proof of Lemma 8.1 First we consider A-regularity

$$\frac{1}{t^r}\, y\, \frac{\partial \mathcal{J}_t}{\partial z} = \frac{bx^2y^2+cxy^3}{t^r} + \text{terms which} \to 0 \text{ as } t \to 0.$$

This is a tangent vector to the orbit through $\mathcal{J}_t = \mathcal{J}_{a(t)}, \ldots, k(t)$ at this point and we are interested in the limiting direction of this tangent vector as $t \to 0$.

If $O(b) = r$, $O(c) \geqslant r$ then the limit is

$$b_o x^2y^2+c_o xy^3 \in \mathcal{J}_o$$

where $b_o = \lim \frac{b(t)}{t^r} \neq 0$, $c_o = \lim \frac{c(t)}{t^r}$.

However $xy^3 \in \mathcal{J}_o$, so this proves $x^2y^2 \in T_o$, as required.

If $O(b) \geq r$, $O(c) = r$ then we use $\frac{1}{t^r} x \frac{\partial \mathcal{J}_t}{\partial z}$

If finally $O(b) > r$, $O(c) > r$ then by the hypothesis of the lemma $O(a) = r$. In that case the tangent vector at $\mathcal{J}_t$ to the path in Σ given by $\mathcal{J}_t$ is

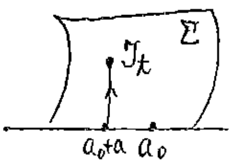
$$\frac{a'(t)}{t^{r-1}} x^2y^2 + \text{terms which} \to 0 \text{ as } t \to 0$$

This vector belongs to T_t so its limit, a nonzero multiple of x^2y^2, belongs to T_o.

This establishes A-regularity.

Now we turn to B-regularity. The vector perpendicular to the "modulus direction", i.e. a-axis, through $\mathcal{J}_t$ is

$$z(bx^2y+cxy^2) + \ldots\ldots + z^4(k) \qquad (1)$$



We have to show that the limiting direction of this vector (which always exists) is in T_o.

Consider the vector in T_t given by

$$\frac{1}{t^r} z \frac{\partial \mathcal{J}_t}{\partial z} = \frac{z(bx^2y+cxy^2)}{t^r} + \text{terms} \to 0 \text{ as } t \to 0$$

The limiting direction of this vector, is

$z(b_ox^2y+c_oxy^2)$, so this latter vector is in T_o.

But (1) has the same limiting direction so the result is proved.

Lemma 8.2 Suppose that Σ is a stratum which does not specialize to any stratum other than $\tilde{E}_7$ containing curves of the form

$$x^4+a_o x^2y^2+y^4+z^2\ Q(x,y,z) = 0$$

where Q is quadratic in x,y,z. Then the orders of d,e,...,k are all >r.

Proof. Let $n_1 = \min\,(O(b),\ O(c))$

$$n_2 = \min\,(O(d),\ O(e),\ O(f))$$

$$n_3 = \min\,(O(g),\ O(h))$$

$$n_4 = O(k)$$

Thus each $n_i \geqslant r$

We make the substitution in $\mathcal{J}_t$ given by $z \to zt^{-\alpha}$. This changes $\mathcal{J}_t$ to $\mathcal{J}'_t$ say, but $\mathcal{J}'_t, \mathcal{J}_t$ are in the stratum Σ (indeed in the same orbit). Each term of $\mathcal{J}'_t$ will have a finite limit provided

$$\begin{aligned} n_1 - \alpha &\geqslant 0 \\ n_2 - 2\alpha &\geqslant 0 \\ n_3 - 3\alpha &\geqslant 0 \\ n_4 - 4\alpha &\geqslant 0 \end{aligned} \qquad (2)$$

We shall use the following easily verified facts:

If $n_2 < 2r$ and $\alpha = \frac{n_2}{2}$ then $r-\alpha > 0$

If $n_3 < 3r$ and $\alpha = \frac{n_3}{3}$ then $r-\alpha > 0$

If $n_4 < 4r$ and $\alpha = \frac{n_4}{4}$ then $r-\alpha > 0$

Now suppose min $\{\frac{n_2}{2}, \frac{n_3}{3}, \frac{n_4}{4}\} < r$

Case I. $\frac{n_2}{2} \leqslant \frac{n_3}{3}$ and $\frac{n_4}{4}$. Put $\alpha = \frac{n_2}{2}$

Then $n_4 - 4\alpha \geqslant 2n_2 - 4\alpha = 0$

Similarly $n_3 - 3\alpha = 0$ and $n_2 - 2\alpha = 0$. Also $n_1 - \alpha \geqslant r - \alpha > 0$ from the above facts. Hence (2) holds.

Case II. $\frac{n_3}{3} \leqslant \frac{n_2}{2}$ and $\frac{n_4}{4}$. Put $\alpha = \frac{n_3}{3}$

Then as before we find (2) holds and $n_1 > \alpha$.

Case III. $\frac{n_4}{4} \leqslant \frac{n_3}{3}$ and $\frac{n_2}{2}$. Put $\alpha = \frac{n_4}{4}$

Then as before we find that (2) holds and $n_1 > \alpha$. Thus by choosing α we can ensure that $\lim_{t\to 0} \mathcal{J}'_t$ exists and has no terms involving (z)x(cubic in x,y), so that Σ does specialize to a stratum containing curves of the form $x^4 + a_o x^2 y^2 + y^4 + z^2 Q(x,y,z) = 0$ with $Q \not\equiv 0$.

Turning this round we have proved that if no such specialization exists then

$$\min\{\frac{n_2}{2}, \frac{n_3}{3}, \frac{n_4}{4}\} \geqslant r$$

which certainly implies that n_2, n_3, n_4 are all $> r$ (since $r > 0$).

Putting together Lemmas 1 and 2 we have:

Prop. 8.3. Supose Σ is a stratum specializing to $\tilde{E}_7$ but not to any other stratum containing curves of the form

$$x^4+a_o x^2 y^2+y^4+z^2 Q(x,y,z) = 0 \qquad (3)$$

Then Σ is A and B regular over $\tilde{E}_7$.

It remains to enumerate the strata which contain any curves of the special form above.

Lemma 8.4 A quartic curve can be put in the form (3) by a projective transformation if and only if there exists a line meeting the curve in 4 distinct points at which the tangents are concurrent.

Proof. (This result was pointed out by J.A. Tyrrell). The line $z = 0$ meet (3) in 4 distinct points since $a_o^2 \neq 4$. The polar cubic of (0,0,1) clearly contains the line $z = 0$, and it follows that the tangents at these points pass through (0,0,1).

Conversely if such a line exist choose it to be $z = 0$ and choose the point of concurrence to be (0,0,1). Then the curve must take the form $\wp + z^2 Q(x,y,z) = 0$ for some quartic $\wp$ in x,y with distinct roots. But a change of variable involving only x and y will now bring the curve to the form (3).

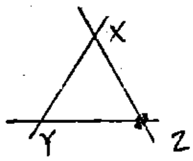
Lemma 8.5 No curve of any of the following types can be put in the form (3):
A_6, A_2A_4, A_2^3, D_6, A_1A_5 (two types), $A_1A_2A_3$, $A_1^3A_3$, $A_1^2A_4$, A_1^6, A_7, E_7, A_1D_6, A_2A_5, $A_1A_3^2$, $A_1^3D_4$.

Remark 8.6 It can be shown that at least some curves of any other type can be put in the form (3). It is a remarkable (and so far unexplained) fact that the types listed above are precisely the quartics with isolated singularities whose strata are orbits. Of course no quartic with a repeated component can be put in the form (3) since every line will meet it in at most 3 distinct points.

Proof. We illustrate the proof by some examples.

(i) A_6 $\quad (zx+y^2)^2+x^3y=0$

Let $f(x,y,z)=x^2z^2+2xy^2z+y^4+x^3y$



If there exists 4 distinct collinear points at which the tangents are concurrent at (a,b,c), then the polar P with respect to (a,b,c,) is $\quad P\equiv a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}+c\frac{\partial f}{\partial z}$

$$\therefore P\equiv a(2xz^2+2y^2z+3x^2y)$$
$$+b(4xyz+4y^3+x^3)$$
$$+c(2x^2z+2xy^2)$$

The line should not be passing through (0,0, 1) because it would cut cubic at coincident points.

A general line not passing through (0,0,1) is of the form $z=\alpha x+\beta y$. We want the condition for it to be a component of the polar cubic P.

Substitute into the cubic , we have

$$a\left[2x(\alpha x+\beta y)^2+2y^2(\alpha x+\beta y)+3x^2y\right]$$
$$+\,b\left[4xy(\alpha x+\beta y)+4y^3+x^3\right]$$
$$+\,c\left[2x^2(\alpha x+\beta y)+2xy^2\right]=0$$

That is $\quad [2a\alpha^2+b+2\alpha c]x^3+[4a\alpha\beta+3a+4b\alpha+2c\beta]x^2y$
$$+[2a\beta^2+2a\alpha+4b\beta+2c]xy^2+[2a\beta+4b]y^3=0$$

Hence we have four equations

$$2a\alpha^2 + b + 2\alpha c = 0 \qquad (i)$$
$$4a\alpha\beta + 3a + 4b\alpha + 2c\beta = 0 \qquad (ii)$$
$$2a\beta^2 + 2a\alpha + 4b\beta + 2c = 0 \qquad (iii)$$
$$2a\beta + 4b = 0 \qquad (iv)$$

If we have a = 0, by (iv), b = 0. Also by (iii) we have c = 0. This is a contradiction. Hence we can assume $a \neq 0$, then by (iv) $\beta = \frac{-2b}{a}$

Substitute into (iii), $2a\left(-\frac{2b}{a}\right)^2 + 2a\alpha + 4b\left(-\frac{2b}{a}\right) + 2c = 0$

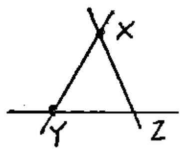
$$\therefore\ 2a\alpha + 2c = 0$$

$$\therefore\ \alpha = -\frac{c}{a}$$

Substitute into (ii), $4a\left(-\frac{c}{a}\right)\left(-\frac{2b}{a}\right) + 3a + 4b\left(-\frac{c}{a}\right) + 2c\left(-\frac{2b}{a}\right) = 0$

$\therefore\ a = 0$ Contradiction.

(ii) A_1A_5 $\oslash$ $(z^2+xy)(z^2+yz+xy)=0$.



Let $f(x,y,z) = z^4 + yz^3 + 2xyz^2 + xy^2z + x^2y^2$

If there exists 4 distinct collinear points at which the tangents are concurrent at (a,b,c), then the polar P with respect to (a,b,c), then the polar P with respect to (a,b,c) is $P \equiv a(2yz^2 + y^2z + 2xy^2)$

$$+ b(z^3 + 2xz^2 + 2xyz + 2x^2y)$$
$$+ c(4z^3 + 3yz^2 + 4xyz + xy^2)$$

The line should not pass through (1,0,0) and 0,1,0) A general line not passing through (1,0,0) and (0,1,0) is of the form $x = y + \alpha z$. We want the condition for it to be a component of the polar cubic P.

Substitute into the Polar cubic

$$a\left[\alpha y z^2 + y^2 z + 2(y+\alpha z) y^2\right]$$

$$+ b\left[z^3 + 2(y+\alpha z)z^2 + 2(y+\alpha z)yz + 2(y+\alpha z)^2 y\right]$$

$$+ c\left[4z^3 + 3yz^2 + 4(y+\alpha z)yz + (y+\alpha z)y^2\right] = 0$$

That is, $[b + 2b\alpha + 4c]z^3$

$$+ [2a + 2b + 2b\alpha + 2b\alpha^2 + 3c + 4c\alpha]yz^2$$

$$+ [a + 2a\alpha + 2b + 4b\alpha + 4c + c\alpha]y^2 z$$

$$+ [2a + 2b + c]y^3 = 0$$

Hence we have four equations

$$b + 2b\alpha + 4c = 0 \qquad \text{(i)}$$

$$2a + 2b + 2b\alpha + 2b\alpha^2 + 3c + 4c\alpha = 0 \qquad \text{(ii)}$$

$$a + 2a\alpha + 2b + 4b\alpha + 4c + c\alpha = 0 \qquad \text{(iii)}$$

$$2a + 2b + c = 0 \qquad \text{(iv)}$$

If $b = 0$, by (i) $c = 0$. And by (iv) we have $a = 0$, This is a contradiction.

Hence we can assume $b \neq 0$

By (iv) we have $c = -2a - 2b$ (v)

Substitute (v) into (i), we have

$$(2\alpha - 7)b - 8a = 0 \qquad \text{(vi)}$$

Substitute (v) into (iii), we have

$$-7a + (2\alpha - 6)b = 0 \qquad (vii)$$

Substitute (v) into (ii)

$$(-8\alpha - 4)a + (-6\alpha - 4 + 2\alpha^2)b = 0 \qquad (viii)$$

By (vi), we have $a = \frac{(2\alpha-7)b}{8}$. Substitute into (vii), since $b \neq 0$, we have

$$\frac{-7(2\alpha-7)}{8} + (2\alpha - 6) = 0 .$$

$$\therefore\ [2\alpha + 1] = 0$$

$$\therefore\ \alpha = -\frac{1}{2} .$$

But by substituting $\alpha = -\frac{1}{2}$ into (viii), we have

$$[-8(-\tfrac{1}{2}) - 4]a + [-6(-\tfrac{1}{2}) - 4 + 2(-\tfrac{1}{2})^2]b = 0$$

$$\therefore\ [-1 + \tfrac{1}{2}]b = 0$$

$$\therefore\ b = 0$$

This is a contradiction.

The rest of the cases in the list can be proved the same way as we have done in these two examples.

Theorem 8.7 The strata listed in Lemma 8.5 are regular over $\tilde{E}_7$.

Proof. Let Σ be one of the strata listed in Lemma 8.5. By Prop. 8.3 we need only verify that Σ does not specialize to any stratum other than $\tilde{E}_7$ containing curves of the form (3). But from the specialization diagram, Σ specialize only to strata on the list, to $\tilde{E}_7$ and to strata with repeated components. The result follows.

APPENDIX

Calculation of Milnor numbers

First we have to introduce the notion of quasihomogeneous and semiquasihomogeneous functions. The following definitions and results are from [Arnold 1974].

Def. 1. We consider the arithmetical space $\mathbb{C}^n$ with fixed coordinates $x_1 \dots x_n$. A function $f: (\mathbb{C}^n,0) \to (\mathbb{C},0)$ is said to be quasihomogeneous of degree 1 with exponents $\alpha_1, \dots, \alpha_n$ (rational numbers) if

$$f(\lambda^{\alpha_1}x_1, \dots, \lambda^{\alpha_n}x_n) = \lambda\, f(x_1 \dots x_n) \text{ for all } \lambda \in \mathbb{C}$$

In term of the Taylor series $f = \Sigma f_k x^k$, quasihomogeneity of degree 1 means that all the exponents of non-zero terms lie on the hyperplane

$$\Gamma = \{k: \alpha_1 k_1 + \dots\dots + \alpha_n k_n = 1\}.$$

We call this hyperplane Γ the diagonal.

Def. 2. A quasihomogeneous function f is said to be non-degenerate if 0 is an isolated critical point.

Def. 3. A monomial $x^k = x_1^{k_1} \dots\dots x_n^{k_n}$ (generalized) degree d if $\langle\alpha_1 k\rangle = \alpha_1 k_1 + \dots\dots + \alpha_n k_n = d$.

Def. 4. A polynomial has filtration d if all its monomials are of degree d or higher: when the degree of all monomials is d, we call d the (generalized) degree of the polynomial.

Def. 5. A polynomial is said to be semi-quasihomogeneous of degree 1 with exponents $\alpha = \alpha_1, \ldots, \alpha_s$ if it is of the form $f = f_o + f'$ where f_o is a non-degenerate quasihomogeneous polynomial of degree 1 with exponent α, and f' is a polynomial of filtration strictly greater than 1.

In our context, we deal with polynomials of (ordinary) degree four in three variables. Let us consider one of our normal forms, say the normal form of A_1A_3(irr) (p.16).

$$y^4+x^2z^2+x^2yz+\alpha xy^2z+\beta xy^3 = 0 \qquad \alpha \neq \pm 2$$
$$\beta^2-\alpha\beta+1\neq 0$$

(The α here is of course a modulus and not a quasihomogeneous exponent).

Locally at one of the singularities, which is in most cases at one of the vertices of the triangle of reference, say X, the normal form becomes (letting x = 1 in the equation) a semi-quasihomogeneous function of two variables y and z. In the above case this gives

$$Y^4+Z^2+YZ+\alpha YZ^2+\beta Y^3$$

The non-degenerate quasihomogeneous part can be found by using the Newton diagram. The Newton Diagram can be drawn by representing each of the monomials Y^pZ^q of the function by the point (p,q) in the Y×Z-plane. The Newton polygon consists of the segments joining monomials on the Newton diagram which have the property that every other monomials on the diagram is on or above such a segment. Or, in other words, it is the boundary of the convex hull of the monomials in the Newton Diagram.

Each segment of the Newton Diagram defines a quasi-homogeneous type (α_1,α_2) if the monomials on the segment are on the diagonal $\Gamma = \{k: \alpha_1 k_1 + \alpha_2 k_2 = 1\}$. And if the monomials on that segment define a nondegenerate function then the whole function is semi-quasihomogeneous. [Sometimes, though, they define a degenerate function and then we need the idea of piecewise quasihomogeneous function or the "magic formula" (see later) e.g. for $Y^5+Y^2Z^2+Z^4$ both $Y^5+Y^2Z^2$ and $Y^2Z^2+Z^4$ are degenerate; notice that this $Y^5+Y^2Z^2+Z^4$ does not actually occur on quartic curves].

In our example, Z^2, YZ and Y^3 are the terms on the Newton polygon but only Z^2 and YZ are on the diagonal $\Gamma = \{k: \alpha_1 k_1+\alpha_2 k_2 = 1\}$ where $(\alpha_1,\alpha_2) = (\frac{1}{2},\frac{1}{2})$ and the rest of the terms have degree > 1.

According to Arnold's result [Arnold 1974 Th. 3.1]

$$\mu(f_o+f') = \mu(f)$$

where f_o+f' is a semiquasihomogeneous function with f its non-degenerate quasihomogeneous part. And by [Arnold 1974

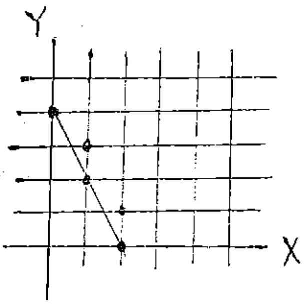
Th 10.2], we can find the Milnor number μ of a quasihomogeneous and nondegenerate function f by the following formula

$$\mu = \pi(\frac{1}{\alpha_i} - 1).$$

where (α_1, α_2) is the quasihomogeneous type of the function.

In our case, the Milnor number for $ZY-Y^2$ is $(2-1)(2-1)=1$ i.e. $\mu = 1$.

Let us look at the other singularity at Z. Letting z = 1 in the normal form, we have a semi-quasihomogeneous function of two variables in x and y



$$Y^4+X^2+X^2Y+\alpha XY^2+\beta XY^3$$

By the Newton diagram, the lowest terms are X^2, XY^2 and Y^4. All of these are on the diagonal Γ with $(\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{4})$ and the other terms are having degrees > 1. Since $\alpha \neq \pm 2$, the quasihomogeneous part is non-degenerate.

So $\mu = (2-1)(4-1) = 3$.

Let us see what happens when $\alpha = 2$ (this α is the parameter in the moduli space of the normal form). Then the lowest terms form a degenerate function $(X+Y^2)^2+X^2Y+\beta XY^3$. So in order to obtain a non-degenerate quasihomogeneous part for the function, we have to take it through a transformation, namely,

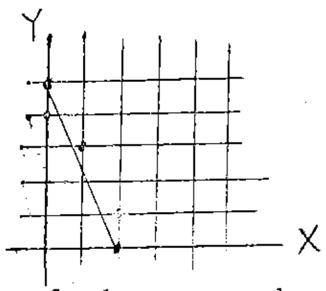
$$X \to X-Y^2$$
$$Y \to Y$$
$$Z \to Z$$

Hence, the function becomes

$$X^2+(X-Y^2)^2Y+\beta(X-Y^2)Y^3$$

i.e. $$X^2+X^2Y-(2-\beta)XY^3+(1-\beta)Y^5$$

Assuming $\beta \neq 1$, the coefficient of Y^5 is nonzero. The



Newton polygon consists of one segment, joining X^2 and Y^5 and they are both on the diagonal Γ with $(\alpha_1,\alpha_2) = (\frac{1}{2},\frac{1}{5})$, and the rest of the terms have degree >1. So the Milnor number μ can be obtained by

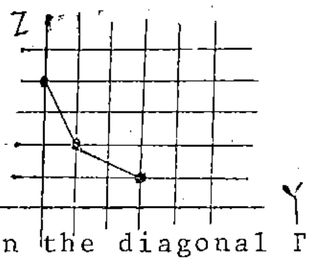
$$\mu = (2-1)(5-1) = 4$$

So when $\alpha = 2$, $\beta \neq 1$ the equation represents the stratum A_1A_4. (The singularity is A_4 and not D_4 since it is a double point).

Let us look at some more examples.

(i) D_6 with normal form $z(y^3+z^3+xyz) = 0$.

The singularity is at X, so letting $x = 1$, we have

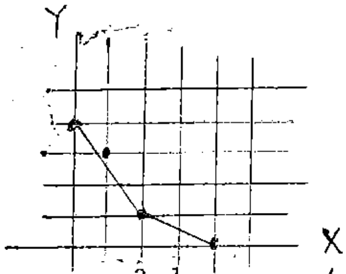


$$Y^3Z + Z^4 + YZ^2.$$

The Newton polygon joins all 3 monomials, but only Y^3Z and YZ^2 are on the diagonal Γ with $(\alpha_1,\alpha_2) = (\frac{1}{5},\frac{2}{5})$ and z^4 with degree > 1. Also $Y^3Z + YZ^2$ is nondegenerate. Hence the Milnor number $\mu = (5-1)(\frac{5}{2}-1) = 6$

(ii) D_5 $\quad x^2yz = x^4+\alpha xy^3+y^4$

The singularity is at Z, so letting $z = 1$, the function becomes



$$X^2Y - X^4 - \alpha XY^3 - Y^4.$$

The terms on the Newton polygon are X^4, X^2Y and Y^4, but only X^2Y and Y^4 are on the diagonal Γ with $(\alpha_1,\alpha_2) = (\frac{3}{8},\frac{1}{4})$ and X^4, XY^3 have degrees > 1. Also X^2Y-Y^4 is nondegenerate. Hence $\mu = (\frac{8}{3}-1)(4-1) = 5$.

(iii) A_6 $\quad (zx-y^2) = x^3y$

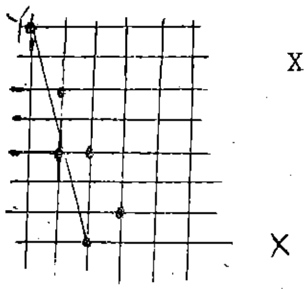
The singularity is at Z, so letting $z = 1$, we have

$$(X-Y^2)^2 - X^3Y$$

Taking the transformation

$$X \to X+Y^2$$
$$Y \to Y$$
$$Z \to Z$$

we have $\quad X^2 - (X+Y^2)^3Y$



$$X^2-X^3Y-3X^2Y^3-3XY^5-Y^7$$

The terms on the Newton polygon are X^2 and Y^7 and they are both on the diagonal Γ with $(\alpha_1,\alpha_2) = (\frac{1}{2},\frac{1}{7})$, and the rest of the terms have degrees > 1. Also X^2-Y^7 is nondegenerate. Hence $\mu = (2-1)(7-1) = 6$.

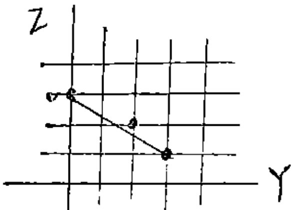
Apart from the above technique, we also have a magic formula from [A.G. Koshireuko 1976] for calculating μ for non-degenerate quasihomogeneous functions. Actually this formula is not just for non-degenerate quasihomogeneous function. It is suitable for any function for which the terms on the Newton polygon give a nondegenerate function (see example (iii)). The "magic formula" is

$$\mu(f) = 2V_2 - V_1 + 1$$

where V_2 the area of the polygon $\Gamma_-(f)$ (i.e. below the Newton Polygon and above the axes in the Newton Diagram) and V_1 is the length of the intersection of the polygon $\Gamma_-(f)$ with the coordinate axes.

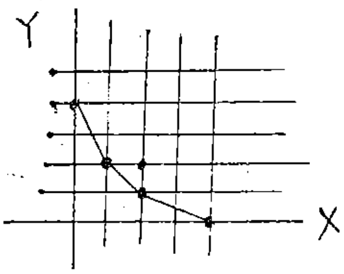
Examples

(i) E_7 $\quad z(y^3+xz^2+y^2z) = 0$



The singularity is at X, so letting x = 1, we have $Y^3Z+Z^3+Y^2Z^2$ (by previous technique $\mu = (\frac{9}{2}-1)(3-1) = 7$). According to the diagram, $V_2 = 6$, $V_1 = 6$. Hence $\mu(f) = 2\times6-6+1 = 7$.

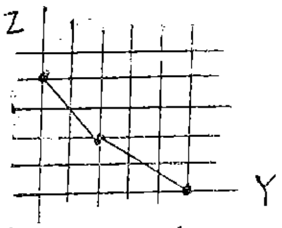
(ii) D_4 $\quad xyz(x+y) = x^4+\alpha x^2y^2+\beta y^4$ (by previous technique we have $\mu = 2\times2 = 4$). According to the diagram, $V_2 = 5\frac{1}{2}$, $V_1 = 8$. $\mu(f) = 2\cdot\frac{11}{2} - 8 + 1 = 4$.



(iii) To illustrate the point that the formula is suitable for any functions for which the terms on the Newton Polygon give a nondegenerate function, we choose the following example. Note that it does not occur in quartic curves.

E.g. $Y^5 + Y^2Z^2 + Z^4$

It is not quasihomogeneous and since both $Y^5+Y^2Z^2$ and $Y^2Z^2+Z^4$ are degenerate it is not even semi-quasihomogeneous. Hence we cannot use the quasi-homogeneous formula. But by



the magic formula, we can easily find out the Milnor number is $\mu = 2\times 9 - 9 + 1 = 10$.

If in case the terms on the Newton Polygon give a degenerate function, then we have to do coordinates changes to get a non-degenerate function before we can apply the "magic formula".

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