

# Singularities of centre symmetry sets

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## 1 Introduction

We study the singularities of envelopes of families of chords (special straight lines) intrinsically related to a hypersurface  $M \subset \mathbf{R}^n$  embedded into an affine space.

The origin of this investigation is the paper [10] of Janeczko. He described a generalization of central symmetry in which a single point—the centre of symmetry—is replaced by the bifurcation set of a certain family of ratios. In [6] Giblin and Holtom gave an alternative description, as follows.

For a hypersurface  $M \subset \mathbf{R}^n$  we consider pairs of points at which the tangent hyperplanes are parallel, and in particular take the family of chords (regarded as infinite lines) joining these pairs. For  $n = 2$  these chords will always have an envelope, and this envelope is called the *centre symmetry set (CSS)* of the curve  $M$ . When  $M$  is convex (the case considered in [10]) the envelope is quite simple to describe, and has cusps as its only generic singularities.

In [6] the case when  $M$  is not convex is considered, and there the envelope acquires extra components, and singularities resembling boundary singularities of Arnold. This is because, arbitrarily close to an ordinary inflexion, there are pairs of points on the curve with parallel tangents. The corresponding chords have an envelope with a limit point at the inflexion itself.

When  $n = 3$  we obtain a 2-parameter family of lines in  $\mathbf{R}^3$ , which may or may not have a (real) envelope. The real part of the envelope is again called the CSS of  $M$ . See Figure 1. The singularities of the CSS are closely related to Lagrangian and Legendre singularities [2, 12]: they are generalizations of singularities of families of normals to a hypersurface in Euclidean space. This is because, given a surface germ and a parallel (equidistant) surface germ, the common normals form a family of chords through points with parallel tangent planes; it follows that, locally, focal sets of surfaces are a special case of our construction and all the usual singularities of focal sets will occur. The same applies, of course, in any dimension  $n$ .

Note that the CSS is, unlike the focal set, an *affine* invariant of a hypersurface  $M$  in an affine space  $\mathbf{R}^n$ . This is of interest [1, 4] in various applications in physics and differential equations. For example, a Hamilton function defined on the cotangent bundle to a manifold determines a collection of hypersurfaces in affine space which are the intersection of the level hypersurfaces with the fibres (the latter possess a well defined affine structure). The affine geometry of these hypersurfaces is important in Hamiltonian mechanics and in optimization theory [1].

In this paper we present a general method for analysing the local structure of the envelope of chords, assuming that it is real. We construct a generating function depending

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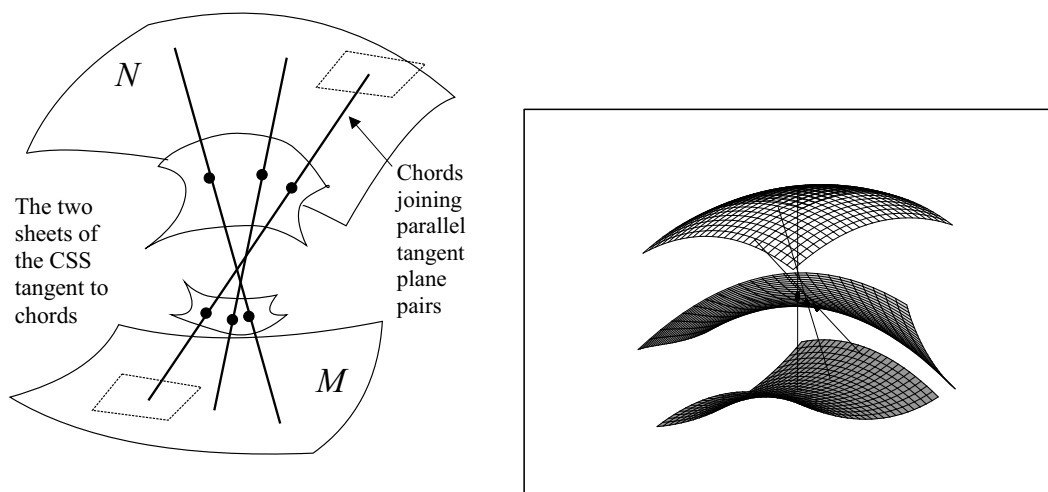


Figure 1: Left: A schematic drawing of the Centre Symmetry Set (CSS) defined locally by two pieces of surface  $M$  and  $N$ . The CSS is a surface tangent to all the chords from  $M$  to  $N$  where the tangent planes at the ends of the chords are parallel. In this figure the chords are tangent to two real sheets of the CSS. It is also possible for the two sheets to come together along a curve (see §3) and for the sheets to be imaginary. Right: a computer generated drawing of one sheet of a CSS, with three chords drawn tangent to the sheet.

on parameters whose bifurcation set, when real, is tangent to all these chords. We are then able to apply methods of [12] to analyse the structure of the CSS both for curves and surfaces, and to a lesser extent in higher dimensions. Some initial steps in the application of this method were taken by Holtom in his MSc dissertation [9].

In this paper we concentrate on the theory; in a subsequent paper we shall examine in detail the geometry of the cases  $n = 2$  and  $n = 3$ , also providing computer-drawn figures to illustrate the different forms that the CSS can take. Some of the results of the present article were announced in [7].

A different application of our constructions is to a description of generic singularities of families of equidistants in Finsler geometry. Consider a translation invariant Finsler metric on an affine space  $A$  which is a function on the cotangent space  $T^*A$  invariant under translations. The nonsingular level set of this function considered as a Hamilton function (e.g. the Riemannian metric determines a Hamiltonian which is a quadratic form with respect to fibres) determines a system of Finsler geodesics. Since this Hamiltonian is translation invariant, the geodesics are straight lines. Given an initial data hypersurface  $I \subset A$  the family of Finsler equidistants  $I_t$  arise. For a translation invariant metric the generic singularities of these families coincide with those of a family of chords determined by the hypersurface  $I$  and by the Finsler indicatrix of admissible velocities (see §2.)

We shall in general distinguish the following three cases which occur in the generic setting. We consider two hypersurface germs  $M, a_0$  and  $N, b_0$  in  $\mathbf{R}^n$ , where the tangent hyperplanes at the two base points  $a_0, b_0$  are parallel.

1. **Transversal case (N)**. The base points are distinct and the chord through them is transversal to both  $M$  and  $N$ .

There are two subcases :

- (NN)  $M$  and  $N$  are non-parabolic at these points. (For  $n = 1$ , this means that neither has an inflexion; for  $n \geq 2$  that all the eigenvalues of the second fundamental form are nonzero.)
- (ND) At least one of these points is parabolic.

2. **Tangential case (T)**. The points  $a_0, b_0$  are distinct but the tangent hyperplanes to  $M$  and  $N$  at these points coincide—the case of a double tangent hyperplane. This case will be considered only in  $\mathbf{R}^2$  and in  $\mathbf{R}^3$  since there arise functional moduli in higher dimensions. Such generic pairs of points of bitangency with planes form smooth curves on a generic pair of surfaces  $M$  and  $N$  in  $\mathbf{R}^3$ .

Again we treat separately the following subcases :

- (TN) The points  $a_0$  and  $b_0$  are not parabolic.
- (TD) At least one of these points is parabolic.

These two cases cover all local singularities of generic systems of chords joining the points from a pair of distinct manifolds, since generic hypersurfaces are nowhere tangent to each other. However since we aim to describe generic singularities of families of chords joining the points of one and the same hypersurface the following case has to be included:

3. **Inflexion case (I)**. Arbitrarily close to a parabolic point  $a_0$  on a hypersurface  $M$  there are pairs of points  $a, b$  with parallel tangent hyperplanes to  $M$ . We consider the family of chords joining these pairs of points. Here again we consider only  $n = 2, 3$ .

In this paper all constructions are local and all objects are assumed to be  $C^\infty$ -smooth. We are mainly interested in the case of surfaces in 3-space but for completeness we start with the case of curves in the plane. In Section 2 we give the necessary singularity theory background and formulate the main results on the normal forms and stability of generic singularities of the CSS. Following are the main results.

### 1. Transversal case, §3, Theorems 3.3, 3.7.

The generic singularities of the CSS of (N) type for  $n \leq 5$  are germs of irreducible (singular) hypersurfaces diffeomorphic to standard caustics of generic Lagrangian mappings of  $\mathbf{R}^n \rightarrow \mathbf{R}^n$ ; equivalently, they are diffeomorphic to bifurcation sets of the versal deformations of function singularities of types  $A, D, E$ . They are stable and simple and all the singularity classes are realised. In particular for  $n = 3$  the CSS besides regular points can have cuspidal edges, swallowtails and pyramids ( $D_4^-$  type) or purses ( $D_4^+$  type). For the (NN) case the germs of caustics are disjoint from the germs of  $M$  and  $N$ . If  $a \in M$  is a parabolic point ((ND) case) the CSS intersects the hypersurface  $N$  ‘transversally’ at the point  $b$ .

## 2. Tangential case, §4, Theorems 4.2,4.3.

The germ at a point on a bitangent chord of the CSS consists of two irreducible components: the closure of the union of all nearby bitangent lines (*envelope*) and a *caustic*. The germs of these reducible varieties for  $n \leq 3$  are diffeomorphic to one of the standard bifurcation sets of the types  $B_2, B_3, C_3, C_4, F_4$  from the V. Goryunov's list of simple singularities of projections of complete intersections onto a line (see [8, 12]). They are stable and simple. In the (TD) case they intersect the initial surfaces  $M$  or  $N$ . An additional simple singularity, similar to  $C_4$ , arises.

## 3. Inflexion case, §5, Theorem 5.3.

The germ of the CSS at an inflexion point of a curve in the plane ( $n = 2$ ) is diffeomorphic to a union of smooth line and a branch of a parabola tangent to the line at the origin. In 3-space there are several generic types of singularities. Some of them are not simple (diffeomorphic types depend on functional invariants) and contains two components. The geometric description and normal forms are discussed in §5.

Finally in §6 we consider the question: which 2-parameter families of lines in  $\mathbf{R}^3$  can arise locally as chords joining pairs of points of two surface germs where the tangent planes are parallel? We find that all generic families can occur in this way. However the degenerate families of chords carry special Lagrange singularities different from singularities of general mappings. Here the situation is similar to comparing Hamiltonian and general vector fields. Near non-singular points they have the same diffeomorphic type, while near singular points their classifications diverge.

# 2 Affine generating families

Let  $M, a_0$  and  $N, b_0$  be two germs at points  $a_0$  and  $b_0$  of smooth hypersurfaces in an affine space  $\mathbf{R}^n$ . Let  $\mathbf{r}_i : U_i^{n-1} \rightarrow \mathbf{R}^n \quad i = 1, 2$  be local regular parametrizations of  $M$  and  $N$ , where  $U_i$  are neighbourhoods of the origin in  $\mathbf{R}^{n-1}$  with local coordinates  $x$  and  $y$  respectively,  $\mathbf{r}_1(0) = a, \mathbf{r}_2(0) = b$ .

### Definitions related to the Centre Symmetry Set

A *parallel pair* is a pair of points  $a \in M, b \in N, a \neq b$  such that the hyperplane  $T_a M$  which is tangent to  $M$  at  $a$  is parallel to the tangent hyperplane  $T_b N$ . In the sequel we will always suppose the distinguished pair  $a_0, b_0$  is a parallel one.

A *chord* is the straight line  $l(a, b)$  passing through a parallel pair:

$$l(a, b) = \{q \in \mathbf{R}^n \mid q = \lambda a + \mu b, \lambda \in \mathbf{R}, \mu \in \mathbf{R}, \lambda + \mu = 1\}.$$

An *affine*  $(\lambda, \mu)$ -*equidistant*  $E_\lambda$  of the pair  $(M, N)$  is the set of all  $q \in \mathbf{R}^n$  such that  $q = \lambda a + \mu b$  for given  $\lambda \in \mathbf{R}, \mu \in \mathbf{R}, \lambda + \mu = 1$  and all parallel pairs  $a, b$  (close to  $a_0, b_0$ ). Note that  $E_0$  is the germ of  $M$  at  $a_0$  and  $E_1$  is the germ of  $N$  at  $b_0$ .

The *extended* affine space is the space  $\mathbf{R}_e^{n+1} = \mathbf{R} \times \mathbf{R}^n$  with barycentric coordinate  $\lambda \in \mathbf{R}, \mu \in \mathbf{R}, \lambda + \mu = 1$  on the first factor (called *affine time*). We denote by  $\pi : w = (\lambda, q) \mapsto q$  the projection of  $\mathbf{R}_e^{n+1}$  to the second factor.

An *affine extended wave front*  $W(M, N)$  of the pair  $(M, N)$  is the union of all affine equidistants each embedded into its own slice of the extended affine space:  $W(M, N) = \{(\lambda, E_\lambda)\} \subset \mathbf{R}_e^{n+1}$ .

The *bifurcation set*  $B(M, N)$  of a family of affine equidistants (or of the family of chords) of the pair  $M, N$  is the image under  $\pi$  of the locus of the critical points of the restriction  $\pi_r = \pi|_{W(M, N)}$ . A point is *critical* if  $\pi_r$  at this point fails to be a regular projection of a smooth submanifold.

In general  $B(M, N)$  consists of two components: the *caustic*  $\Sigma$  being the projection of the singular locus of the extended wave front  $W(M, N)$  and the *crimulant*  $\Delta$  being the (closure of) the image under  $\pi_r$  of the set of regular points of  $W(M, N)$  which are the critical points of the projection  $\pi$  restricted to the regular part of  $W(M, N)$ . The caustic consists of the singular points of momentary equidistants  $E_\lambda$  while the envelope is the envelope of the family of regular parts of momentary equidistants.

Besides being swept out by the momentary equidistants, the affine wave front is swept out by the liftings to  $\mathbf{R}_e^{n+1}$  of chords. Each of them has regular projection to the configuration space  $\mathbf{R}^n$ . Hence the bifurcation set  $B(M, N)$  is essentially the envelope of the family of chords, that is the CSS of the pair  $M, N$ . From now on we shall in fact use  $B(M, N)$  as our model for the CSS.

### Singularity theory

We now recall some standard definitions and results (see e.g. [2, 12, 8]) on the singularities of families of functions depending on two groups of parameters (space-time unfoldings), which will be used below.

A germ of a family  $F(z, w)$  of functions in  $z \in \mathbf{R}^k$  with parameters  $w = (t, q) \in \mathbf{R}_e^{n+1}$  where  $t \in \mathbf{R}$  and  $q \in \mathbf{R}^n$  determines the following collection of varieties:

- The *fibrewise critical set* is the set  $\mathcal{C}_F \subset \mathbf{R}^k \times \mathbf{R} \times \mathbf{R}^n$  of solutions  $(z, w)$  of the so-called *Legendre equations*:

$$F(z, w) = 0, \quad \frac{\partial F}{\partial z}(z, w) = 0.$$

- The *big wave front* (or *discriminant*) is  $W(F) = \{w = (t, q) \mid \exists z : (z, w) \in \mathcal{C}_F\}$ .
- The intersection of the (big) wave front with  $t = \text{const}$  is called the *momentary wave front*  $W_t(F)$ .
- The bifurcation set  $B(F)$  is the image under the projection  $\pi : (t, q) \mapsto q$  of the points of  $W(F)$  where the restriction  $\pi|_{W(F)}$  fails to be a regular projection of a smooth submanifold. Projections of singular points of  $W(F)$  form the *caustic*  $\Sigma(F)$ , and the closure of the set of singular projections of regular points of  $W(F)$  determine the *crimulant*  $\Delta(F)$ .
- The Legendre subvariety  $\Lambda(F)$  is a subvariety of the projectivised cotangent bundle  $PT^*(\mathbf{R}^{n+1})$ :

$$\Lambda(F) = \left\{ (w, \bar{w}) \mid \exists z, (z, w) \in \mathcal{C}_F, \bar{w} = \left[ \frac{\partial F}{\partial w} \right] \right\}$$

Here  $[ \ ]$  stands for the projectivisation of a vector.

The family  $F$  is called a *generating* family for  $\Lambda(F)$ .

The germ of  $\Lambda(F)$  is smooth provided that the Legendre equations are locally regular, i.e. that the standard Morse conditions are fulfilled (see e.g. [2]).

The following equivalence relations will be used below. Two germs of families  $F_i$ ,  $i = 1, 2$  are

- *contact-equivalent* (c-equivalent for short) if there exist a non-zero function  $\phi(z, w)$  and a diffeomorphism  $\theta : \mathbf{R}^k \times \mathbf{R}^{n+1} \rightarrow \mathbf{R}^k \times \mathbf{R}^{n+1}$ , of the form  $\theta : (z, w) \mapsto (X(z, w), W(w))$  such that  $\phi F_1 = F_2 \circ \theta$ .
- *right + equivalent* ( $\mathbf{R}^+$ -equivalent for short) if there exist a function  $\rho(w)$  and a diffeomorphism  $\theta : \mathbf{R}^k \times \mathbf{R}^{n+1} \rightarrow \mathbf{R}^k \times \mathbf{R}^{n+1}$  of the form  $\theta : (z, w) \mapsto (X(z, w), W(w))$  such that  $\rho + F_1 = F_2 \circ \theta$ .

If  $\rho(w) = 0$  the families are called *right* ( $\mathbf{R}$ )-equivalent.

- *space-time-contact-equivalent* (v-equivalent for short) if there exist a non-zero function  $\phi(z, t, q)$  and a diffeomorphism  $\theta : \mathbf{R}^k \times \mathbf{R}^{n+1} \rightarrow \mathbf{R}^k \times \mathbf{R}^{n+1}$ , of the form

$$\theta : (z, t, q) \mapsto (X(z, t, q), T(t, q), Q(q))$$

such that  $\phi F_1 = F_2 \circ \theta$ .

- *space-time-shift-equivalent* (r-equivalent for short) if there exist a non-zero function  $\phi(z, t, q)$  and a diffeomorphism  $\theta : \mathbf{R}^k \times \mathbf{R}^{n+1} \rightarrow \mathbf{R}^k \times \mathbf{R}^{n+1}$ , of the form  $\theta : (z, t, q) \mapsto (X(z, t, q), t + \psi(q), Q(q))$  with some smooth function  $\psi$  such that  $\phi F_1 = F_2 \circ \theta$ .

The sum of the family  $F(z, t, q)$  with a non-degenerate quadratic form in extra variables  $y_1, \dots, y_m$  is called a *stabilization* of  $F$ . Two germs of families are *stably*  $*$ -equivalent if they are  $*$ -equivalent ( $*$  stands for one of the above c, v or r equivalences), to stabilizations of one and the same family in fewer variables.

We now recall some standard facts.

- The discriminants of stably c-equivalent families are diffeomorphic, and the bifurcation sets of stably v-equivalent (r-equivalent) families are diffeomorphic.
- Legendre submanifolds  $L_F$  of stably c-equivalent families are Legendre equivalent: the germ of  $\theta$  determines a contactomorphism of the projectivised cotangent bundle  $PT^*\mathbf{R}^{n+1}$  which preserves the fibres and maps one Legendre submanifold onto the other.
- Moreover the big wave front  $W(F)$  determines uniquely the stable c-equivalence class of its generating family  $F$  provided that the regular points form a dense subset of  $W(F)$ .

The critical points of the projection  $\pi|_{W(F)}$  satisfy the equation:

$$\det \begin{vmatrix} \frac{\partial F}{\partial z} & \frac{\partial F}{\partial t} \\ \frac{\partial^2 F}{\partial z^2} & \frac{\partial^2 F}{\partial t \partial z} \end{vmatrix} = 0.$$

Since at a point of  $W(F)$  the first  $k$  entries  $\frac{\partial F}{\partial z}$  of the first row vanish, the determinant factorises. Hence the bifurcation set  $B(F)$  splits into two components. One of them (which is the *criminant*  $\Delta(F)$ ) is the image under the projection  $(z, t, q) \mapsto q$  of the subvariety of  $\mathcal{C}_F$  in  $(z, t, q)$ -space determined by the equation  $\frac{\partial F}{\partial t} = 0$ . The other one (which is the *caustic*  $\Sigma(F)$ ) is the image under the projection  $(z, t, q) \mapsto q$  of the subvariety of  $\mathcal{C}_F$  in  $(z, t, q)$ -space determined by the equation  $\det \left( \frac{\partial^2 F}{\partial z^2} \right) = 0$ .

The following version of Huygens' principle holds: the criminant (envelope) coincides with the wave front of  $F$ , considered as a family in variables  $z$  and  $t$  with parameters  $q$  only.

Two families with diffeomorphic caustics are called *weakly* equivalent.

Standard arguments of singularity theory (see e.g. [2]) imply that versality and infinitesimal versality conditions for  $c$  or  $v$  groups yield the stability of wavefronts or bifurcation sets (small perturbation of a versal family produces a wavefront or bifurcation set diffeomorphic to the unperturbed one).

In particular, we will use the following standard result (see [2]) on straightening of vector fields transversal to stable wave fronts. Denote by  $\mathcal{O}_x$  the algebra of germs at the origin of smooth functions in variables  $x$  and by  $\mathcal{M}_x$  the maximal ideal of  $\mathcal{O}_x$ . If the germ of an unfolding  $F(z, t, q) \in \mathcal{M}_{z,t,q}$  of functions in  $z$  with parameters  $t, q$  is versal with respect to right equivalence and  $\frac{\partial F}{\partial t} \neq 0$  then the family is  $r$ -stable.

## Generating family for the CSS

We now come to the link between singularity theory and the CSS: a generating family which describes the CSS and to which we can apply the above results. Let, as above,  $\lambda, \mu = 1 - \lambda$  be barycentric coordinates on  $\mathbf{R}$ ; let  $\langle \cdot, \cdot \rangle$  be the standard pairing of vectors from  $\mathbf{R}^n$  and covectors  $p$  from the dual space  $(\mathbf{R}^n)^\wedge$ . Let  $\mathbf{r}_1, \mathbf{r}_2$  be local parametrizations of  $M, N$  close to  $a_0, b_0$  respectively, defined on neighbourhoods  $U, V$  of the origin in  $\mathbf{R}^{n-1}$ , with coordinates  $u, v$  and satisfying  $\mathbf{r}_1(0) = a_0, \mathbf{r}_2(0) = b_0$ .

**Definition 2.1** An *affine generating family*  $\mathcal{F}$  of a pair  $M, N$  is a family of functions in  $x, y, p \in U \times V \times ((\mathbf{R}^n)^\wedge \setminus \{0\}, 0)$  with parameters  $\lambda, q \in \mathbf{R} \times \mathbf{R}^n$  of the form

$$\mathcal{F}(x, y, p, \lambda, q) = \lambda \langle \mathbf{r}_1(x) - q, p \rangle + \mu \langle \mathbf{r}_2(y) - q, p \rangle.$$

**Proposition 2.2** *The wave front  $W(\mathcal{F})$  coincides with the affine extended wavefront  $W(M, N)$  and the bifurcation set  $B(\mathcal{F})$  coincides with the CSS of the pair  $M, N$ .*

**Proof.** The equations  $\frac{\partial \mathcal{F}}{\partial p} = 0$  clearly imply that  $\lambda \mathbf{r}_1(x) + \mu \mathbf{r}_2(y) - q = 0$ . Note that as  $\mathcal{F}$  is homogeneous of degree 1 in  $p$ , it is equal to an appropriate linear combination of its derivatives with respect to  $p$ . So the equations  $\frac{\partial \mathcal{F}}{\partial p} = 0$  also imply the equation  $\mathcal{F} = 0$ .

The conditions  $\frac{\partial \mathcal{F}}{\partial x} = 0$  imply that  $\langle \lambda \frac{\partial \mathbf{r}_1}{\partial x}, p \rangle = 0$  or equivalently either  $\lambda = 0$  or  $\langle \frac{\partial \mathbf{r}_1}{\partial x}, p \rangle = 0$ . The latter means that the hyperplane  $\langle \cdot, p \rangle = 0$  annihilates all tangent vectors to  $M$  at  $\mathbf{r}_1(x)$ . Together with the equations  $\frac{\partial \mathcal{F}}{\partial y} = 0$  this implies that either  $p$  is a common conormal to the tangent planes to  $M$  and  $N$  at corresponding points, or  $\lambda = 0$ , and  $p$  is conormal to  $N$  which gives  $q = \mathbf{r}_2(y)$  for the corresponding  $x$ , or  $\mu = 0$  and  $p$  is conormal to  $N$  which gives  $q = \mathbf{r}_1(x)$ .

Hence  $W(\mathcal{F})$  is reducible and consists of three components: the germ of  $N : \lambda = 0, q = \mathbf{r}_2(y)$ ; the germ of  $M : \mu = 0, q = \mathbf{r}_1(x)$ ; and the germ of the set of  $(\lambda, q)$  such that  $\lambda \neq 0, \mu \neq 0$ , and  $q$  is a point on a chord joining the parallel pair  $\mathbf{r}_1(x), \mathbf{r}_2(y)$ . The other claim of the proposition follows.  $\square$

### Remarks and Notation

1. The closure of the third component  $W_*(\mathcal{F})$  of  $W(\mathcal{F})$  coincides with union of all three and therefore coincides exactly with the set  $W(M, N)$ . However, formally speaking the wave front  $W(\mathcal{F})$  contains also the other two exceptional divisors, and the Legendre variety  $L(F)$  has three irreducible components.
2. The proof above shows that the germ of  $W(\mathcal{F})$  at any point  $(\lambda_0, q_0)$  where  $q_0 = \lambda_0 a_0 + (1 - \lambda_0) b_0$ , corresponding to  $x = 0, y = 0, [p] = [d\mathbf{r}_1|_{a_0}] = [d\mathbf{r}_2|_{b_0}]$ , coincides with the germ of the extended wavefront  $W(M, N)$  at this point.
3. In the sequel we deal with the component  $W_*(\mathcal{F})$  and with the corresponding component of the critical locus  $\mathcal{C}_{\mathcal{F}}$ . When investigating the germ of the CSS we will ignore the redundant components corresponding to  $\lambda\mu = 0$ , which are merely the germs of the two surfaces  $M$  and  $N$ .

### Chords as extremals

The system of chords determined by a pair of (germs of) hypersurfaces  $M, a$  and  $N, b$  arises naturally in the following variational problem.

In the tangent space  $T_q \mathbf{R}^n$  (which we identify with  $\mathbf{R}^n$  itself) at any point  $q \in \mathbf{R}^n$  choose a set (indicatrix)  $I_q$  of admissible velocities  $\dot{q}$  of motion to be the set of vectors  $\mathbf{r}_2(y) - q$ . So, the directions of admissible motion at  $q$  are directed towards the hypersurface  $N$  parametrised by  $y$ . Let  $q(t)$  be an absolutely continuous (or piecewise smooth and continuous) trajectory of admissible motion (almost everywhere  $\dot{q} \in I_q$ ) issuing at  $t = 0$  from hypersurface  $M$ . For a fixed value  $t_0$  consider the end-point mapping  $\mathcal{E} : \mathcal{C} \rightarrow \mathbf{R}^n$  (here  $\mathcal{C}$  is the Banach manifold of all admissible trajectories defined on the segment  $[0, t_0]$ ), which associates the end-point  $q(t_0)$  to a trajectory  $q(t)$ .

As it is well known (this is a modification of Bellman or Pontryagin Maximum principle; for the details, see e.g. [1, 12]) the critical values of  $\mathcal{E}$  for all  $t_0$  trace the extremal trajectories, which, in our case, are projections to  $\mathbf{R}^n$  of the solutions of the associated Hamilton canonical equations in the cotangent bundle

$$\dot{q} = \frac{\partial H_*(p, q)}{\partial p}, \dot{p} = -\frac{\partial H_*(p, q)}{\partial q}$$

with the Hamiltonian function  $H_*(p, q)$  on the cotangent bundle  $T^* \mathbf{R}^n$  which in our case is the (multivalued, in general) restriction to the subset  $\{(p, q) | \exists y : \frac{\partial H_*(p, q)}{\partial y} = 0\}$  of the



function  $H = \langle r_2(y) - q, p \rangle$ , provided that the initial conditions  $(p_0, q_0)$  satisfy the relation  $\langle v, p_0 \rangle = 0$  for each  $v$  tangent to  $M$  at  $q_0$ .

Direct verification shows that chords, joining points of  $M$  and  $N$  are extremals in this problem. So CSS is the envelope of extremals (that is the union of singular points of sets of critical values of  $\mathcal{E}_{t_0}$  for all  $t_0$ ).

The case of a translation invariant Finsler metric corresponds to the indicatrix  $I_q = \mathbf{r}_2(y)$ , the extremals being straight lines issuing from points  $q_0$  of  $M$  with the constant velocity  $\dot{q}_0$  such that the tangent hyperplane to  $I$  at  $\dot{q}_0$  is parallel to the tangent hyperplane to  $M$  at  $q_0$ . In this case (which is the limiting case of systems of chords for  $N$  located “very far” from  $N$ ) the envelope of these extremals is the bifurcation set of family of functions  $\tilde{F} = \langle \mathbf{r}_1(x) + \mathbf{t}r_2(y) - q, p \rangle$  in variables  $p, x, y$  with parameters  $t, q$ . This is quite similar to  $\mathcal{F}$  defined above for systems of chords and produces the same singularities in a generic setting.

### 3 Transversal case

In the transversal case (NN and ND in the terminology of §1) we can choose affine coordinates  $q = (h, s_1, \dots, s_{n-1})$  in  $\mathbf{R}^n$  such that  $a_0 = (1/2, 0, \dots, 0)$ ,  $b_0 = (-1/2, 0, \dots, 0)$  and the hyperplanes tangent to  $M, a_0$  and  $N, b_0$  are parallel to the  $h = 0$  coordinate hyperplane.

Take local parametrizations of  $M$  and  $N$  in Monge form:

$$\mathbf{r}_1(x) = \left(\frac{1}{2} + f(x), x_1, \dots, x_{n-1}\right), \quad \mathbf{r}_2(y) = \left(-\frac{1}{2} + g(y), y_1, \dots, y_{n-1}\right).$$

Here  $x = (x_1, \dots, x_{n-1}) \in U$ ;  $y = (y_1, \dots, y_{n-1}) \in V$ , where  $U, V$  are neighbourhoods of the origin in  $\mathbf{R}^{n-1}$  and the smooth functions  $f, g$  have zero 1-jet at the origin:  $f \in \mathcal{M}_x^2, g \in \mathcal{M}_y^2$ .

**Lemma 3.1** *The germ of the family  $\mathcal{F}$  (see Definition 2.1) at the point  $x = 0, y = 0, p_0 = (1, 0, \dots, 0), \lambda = \lambda_0, q_0 = (h_0, 0, \dots, 0), h_0 = \frac{1}{2}(\lambda_0 - \mu_0)$  (which corresponds to the point  $q_0 = \lambda_0 a_0 + \mu_0 b_0$  on the chord  $l(a_0, b_0)$ ) is stably  $r$ -equivalent to the germ of the following family  $G$  of functions in  $z \in \mathbf{R}^{n-1}$  with parameters  $q = (h, s), \lambda$  at the point  $z = 0, \lambda = \lambda_0, q = q_0 = (h_0, 0)$ :*

$$G = -h + \lambda \left(\frac{1}{2} + f(s + \mu z)\right) + \mu \left(-\frac{1}{2} + g(s - \lambda z)\right).$$

**Proof.** The family  $\mathcal{F}$  differs only by a non-zero factor from its restriction  $\mathcal{F}_r$  to the subspace  $p = (1, p_1, \dots, p_{n-1})$  which is

$$\mathcal{F}_r = -h + \frac{1}{2}(\lambda - \mu) + \lambda f(x) + \mu g(y) + \sum_{i=1}^{n-1} (\lambda x_i + \mu y_i - s_i) p_i.$$

Let  $w_i = \lambda x_i + \mu y_i - s_i$  and  $z_i = x_i - y_i$  for  $i = 1, \dots, n-1$ . The determinant of the transformation  $x, y \mapsto z, w$  equals 1. In the new variables  $z, w$  the family takes the form  $\mathcal{F}_* = G_*(z, w, \lambda, q) + \sum w_i p_i$  where

$$G_* = -h + \frac{1}{2}(\lambda - \mu) + \lambda f(x(z, w)) + \mu g(y(z, w)).$$

By Hadamard's lemma  $\mathcal{F}_* = G_*(z, 0, \lambda, q) + \sum w_i (p_i + \phi_i(z, w, \lambda, q))$ , where  $\phi_i$  are smooth functions vanishing at  $w = z = 0$ ,  $\lambda = \lambda_0$ ,  $q = q_0$ . Hence  $\mathcal{F}_*$  is a stabilization of  $G_*(z, 0, \lambda, q)$ . Since the restriction to  $w = 0$  of the inverse mapping  $z, w \mapsto x, y$  yields  $x = s + \mu z$ ,  $y = s - \lambda z$  we obtain

$$G(z, \lambda, q) = G_*(z, 0, \lambda, q) = -h + \frac{1}{2}(\lambda - \mu) + \lambda f(s + \mu z) + \mu g(s - \lambda z). \quad \square$$

Given a point  $q_0 = (h_0, 0)$ ,  $h_0 = \frac{1}{2}(\lambda_0 - \mu_0)$  of the chord  $l(a_0, b_0)$ , the family  $G$  determines an unfolding in small parameters  $s_1, \dots, s_{n-1}$ ,  $\tilde{h} = h - h_0$ ,  $\varepsilon = \lambda - \lambda_0$  of the organising centre  $G_0 = G|_{\lambda=\lambda_0, q=q_0}$ , a function of  $z$  alone. Then the quadratic part of  $G_0$  is equal to

$$j_2 G_0(z) = \lambda_0 \mu_0^2 f_2(z) + \lambda_0^2 \mu_0 g_2(z),$$

where  $f_2, g_2$  are the quadratic parts of these functions. If this form is non-degenerate and  $\lambda_0 \neq 0 \neq \mu_0$  the corresponding germ of  $G$  is a versal unfolding of a Morse singularity. Its wave front is regular and the bifurcation set is void. This remains true also for the limit cases  $\lambda_0 = 0$  or  $\mu_0 = 0$ . To see this it is enough to consider the equations determining the caustic of the component  $W_*(G)$  corresponding to  $\lambda_0 \mu_0 \neq 0$  (see the Remarks following Proposition 2.2).

The equality  $\frac{\partial G}{\partial \lambda} = 1$  for  $z = 0$  implies that the discriminant  $\Delta(F)$  is always void. So the following holds.

**Corollary 3.2** *A point  $q_0$  belongs to  $B(M, N)$  if and only if the quadratic form  $\mu_0 f_2(z) + \lambda_0 g_2(z)$  is degenerate.*  $\square$

**Remark. Drawing the Centre Symmetry Set** The above corollary is the key to *drawing* the Centre Symmetry Set in dimensions 2 and 3. Consider two surfaces, say  $M = \{(x_1, x_2, f(x_1, x_2))\}$  and  $N := \{(y_1, y_2, g(y_1, y_2) + k)\}$  where  $f, g$  and their first partial derivatives vanish at  $(0, 0)$ , so that the tangent planes to  $M$  at  $(0, 0, 0)$  and  $N$  at  $(0, 0, k)$  are parallel. We need find the pairs of points  $a$  of  $M$  and  $b$  of  $N$  close to these two for which the tangent planes are parallel, parametrizing these by say  $(x_1, x_2)$ . Of course this may pose serious computational problems; many examples however can be drawn by choosing one of the surfaces so that the resulting equations  $f_1 = g_1$ ,  $f_2 = g_2$  are linear in two of the variables. Here subscripts denote differentiation with respect to the variables,  $f_1 = \partial f / \partial x_1$ , etc. For example we can often use  $g = -y_1^2 - y_2^2$ .

We then calculate the zeros of  $\det \mathcal{H}$  where

$$\mathcal{H} = \begin{pmatrix} f_{11} + \theta g_{11} & f_{12} + \theta g_{12} \\ f_{21} + \theta g_{21} & f_{22} + \theta g_{22} \end{pmatrix},$$

For each such (real) zero  $\theta$  we use the values  $\lambda = \theta / (1 + \theta)$ ,  $\mu = 1 / (1 + \theta)$  (thus  $\theta = \lambda / \mu$  and  $\lambda + \mu = 1$ ) to find the point  $\lambda a + \mu b$  on the CSS. An example showing just one of the two sheets of the CSS is drawn in Figure 1. A full discussion of the CSS in the surface case, with illustrations, will appear elsewhere.

**Remark.** The pencil  $\mu f_2(z) + \lambda g_2(z)$ ,  $\lambda \in \mathbf{R}$  of quadratic forms can meet the variety of degenerate quadratic forms at fewer than  $n - 1$  real points (some roots can merge or be complex). We describe the geometry of this for  $n = 3$  at the end of the current section.

Fix now domains  $U, V$  and the Monge parametrisation  $g(y), y \in V$  of  $N$ . Suppose that at any point the corresponding form  $g_2$  is non-degenerate, that is  $N$  does not have a parabolic point. Thus we are considering the *NN case here*, in the terminology of §1.

**Theorem 3.3** *Let  $n \leq 5$ . There is an open and dense subset of the space of hypersurfaces  $M : (f(x), x), x \in U$  with non-degenerate quadratic form at any point consisting of those  $M$  which together with  $N$  as above are such that the corresponding family  $G$  at each point is  $r$ -stably equivalent to a versal deformation of one of the germs of functions with a simple singularity of the types  $A_k, (1 \leq k \leq n+1), D_k (4 \leq k \leq n+1), E_6$ .*

**Remark.** Generic caustics as above are stable and simple. All the classes listed appear in examples.

For the proof of Theorem 3.3 we shall need several lemmas. If  $f_2$  is a non-degenerate quadratic form then the gradient mapping  $\chi_f : x \mapsto -\frac{\partial f}{\partial x}$  is invertible. Denote by  $f_*(\bar{p})$  the Legendre transform of  $f(x)$  :

$$f_*(\bar{p}) = \left( f(x) - x \frac{\partial f}{\partial x} \right) \Big|_{x=\chi_f^{-1}(\bar{p})}, \quad \bar{p} = (p_1, \dots, p_{n-1}).$$

**Lemma 3.4** *If both  $f_2$  and  $g_2$  are non-degenerate then the family  $\mathcal{F}$  (see Definition 2.1) is  $r$ -stably equivalent to a family  $H$  of functions in  $\bar{p} \in \mathbf{R}^{n-1}$  with parameters  $\lambda, \mu$ , of the following form.*

$$H = -h + \frac{1}{2}(\lambda - \mu) - \sum_{i=1}^{n-1} p_i q_i + \lambda f_*(\bar{p}) + \mu g_*(\bar{p})$$

**Proof of the lemma.** In the affine chart  $p_0 = 1$  of the dual space the Legendre conditions  $\frac{\partial \mathcal{F}}{\partial x} = 0$  for  $\lambda \neq 0 \neq \mu$  imply  $\frac{\partial f(x)}{\partial x} + p_i = 0, \frac{\partial g(y)}{\partial y} + p_i = 0$  for  $i = 1, \dots, n-1$ . Solving these equations for  $x$  and  $y$  and applying a stabilization procedure similar to that of Lemma 3.1 the family  $\mathcal{F}$  becomes stably equivalent to the required form

$$H = -h + \frac{1}{2}(\lambda - \mu) - \sum_{i=1}^{n-1} p_i q_i + \lambda \left( f(x) - x \frac{\partial f(x)}{\partial x} \right) \Big|_{x=\chi_f^{-1}(\bar{p})} + \mu \left( g(y) - y \frac{\partial g(y)}{\partial y} \right) \Big|_{y=\chi_g^{-1}(\bar{p})}. \square$$

Take an integer  $m$  large enough, say  $m > 7$ .

For  $n \leq 5$  the subspace  $\mathcal{M}_m^2 \subset J^m(n-1, 1)$  of  $m$ -jets at the origin of functions with zero 1-jet splits into finitely many orbits  $O_\alpha^c$  of the right action (of jets of diffeomorphisms of the source space  $\mathbf{R}^{n-1}$  preserving the origin) having codimension  $c \leq n$  (in  $\mathcal{M}_m^2$ ) and a closed algebraic subset  $\bar{O}^{n+1}$  which is the closure of all right orbits of codimension  $\geq n+1$ .

Take an orbit  $O_\alpha$  of positive codimension in  $\mathcal{M}_m^2$  (containing jets with degenerate quadratic part).

**Lemma 3.5** *Jets  $j^m f \in O_\alpha$  such that vector  $j^m g_*$  is tangent to  $O_\alpha$  at  $j^m f$  form an algebraic subset  $S_\alpha \subset O_\alpha$  of positive codimension in  $O_\alpha$ .*

**Remark.** We identify the tangent space to the affine space  $J^m(n-1, 1)$  at each point with this space itself.

**Proof of the lemma.** Clearly, subset  $S_\alpha$  is algebraic. To show that it has positive codimension take a jet  $j^m f \in O_\alpha$ . Let  $K$  be the kernel subspace of the quadratic form of  $j^2 f$ . The quadratic part of a tangent vector to  $O_\alpha$  at  $j^m f$  is a form whose restriction to  $K$  is degenerate. Linear transformations  $\tau : \mathbf{R}^{n-1} \rightarrow \mathbf{R}^{n-1}$  such that the form  $j^2 g_*$  is not degenerate when restricted to  $\tau(K)$  form a Zariski open set in  $GL(n-1, \mathbf{R})$ , since the whole quadratic form  $j^2 \varphi$  is non-degenerate.

So  $j^2 \varphi$  is transversal to  $O_\alpha$  at all  $j^2(f \circ \tau)$  for any  $\tau$  from this set. So the complement to  $S_\alpha$  is dense in  $O_\alpha$ .  $\square$

Denote by  $\mathcal{T}_{m+1}$  the subspace of  $J^{m+1}(\mathbf{R}^{n-1}, \mathbf{R})$  formed by jets of functions  $f(\bar{p})$  such that the mapping from  $\bar{p}, q$  space to  $J^m(\mathbf{R}^{n-1}, \mathbf{R})$  which sends  $\bar{p}, q$  to the  $m$ -jet at  $\varepsilon = 0$ ,  $\bar{p}$ ,  $q$  of the family

$$\tilde{H} = -h + \frac{1}{2}(\lambda - \mu) - \sum_{i=1}^{n-1} p_i q_i + f_*(\bar{p}) + \varepsilon g_*(\bar{p})$$

is transversal to  $r$ -orbit through  $j^m(f)$ , and hence, in particular, the respective germ of the family is  $r$ -versal.

**Lemma 3.6** *The complement  $\bar{S}^{m+1} = J^{m+1}(\mathbf{R}^{n-1}, \mathbf{R}) \setminus \mathcal{T}_{m+1}$  has codimension  $\geq n$  in  $J^{m+1}(\mathbf{R}^{n-1}, \mathbf{R})$ .*

**Proof.** Clearly  $\bar{S}^{m+1}$  is a semialgebraic subset, since the transversality to the orbit in  $m$ -jet space is determined by the  $(m+1)$ -jet of the family. Assume its codimension is less than  $n$ .

Split the space  $\mathcal{M}_m^2$  into the  $r$ -orbits  $O_\alpha$  of codimension  $\leq n-1$ , the sets  $O_\beta \setminus S_\beta$  for all  $r$ -orbits  $O_\beta$  of codimension  $n$  and the remaining part, which is the union of  $E^{n+1}$  and  $S_\beta$  (and, hence, has codimension  $\geq n+1$ ). The product of each stratum by the set of jets of affine functions provides a stratification of the space  $J^m(n-1, \mathbf{R})$  of jets at the origin. Such stratifications over each  $\bar{p}$  provide a stratification  $W$  (by semi-algebraic sets) of  $J^m(\mathbf{R}^{n-1}, \mathbf{R})$ . The Thom transversality theorem implies that the set of functions  $f$  whose jet extensions are transversal to  $W$  (this condition is equivalent to the transversality of the family  $\hat{H}$  to the orbit  $O_\alpha$  through  $f$ ) is dense in the space of all functions. For  $m$  large enough  $O_\alpha$  orbits are  $m$ -determined. So,  $m+1$ -jets of generic  $f$  do not meet  $\bar{S}^{m+1}$ . However this contradicts the assumption.  $\square$

We come now to the proof of Theorem 3.3. Consider the projection  $\rho : J^{m+1}(\mathbf{R}^{n-1}, \mathbf{R}) \times \{\mathbf{R} \setminus \{0\}\} \rightarrow J^{m+1}(\mathbf{R}^{n-1}, \mathbf{R})$  which sends the pair (jet of  $\varphi$ ;  $\lambda$ ) to the jet of  $\frac{1}{\lambda}(\varphi - \mu g)$ . Here, as usual,  $\lambda + \mu = 1$ . The inverse image  $\rho^{-1}(j^{m+1} f)$  of jet of  $f$  under this projection is a pencil of jets of  $\lambda f + \mu g$ . Since  $j^2 g$  is non-degenerate, any pencil intersects the subset  $\bar{S}^{m+1}$  in at most  $n-1$  points. Hence, the restriction of the projection  $\rho$  to  $\bar{S}^{m+1} \times \{\mathbf{R} \setminus \{0\}\}$  is a finite map and its image  $I_{m+1, n}$  is a semialgebraic subset (according to Tarski-Seidenberg theorem) of codimension  $\geq n$ .

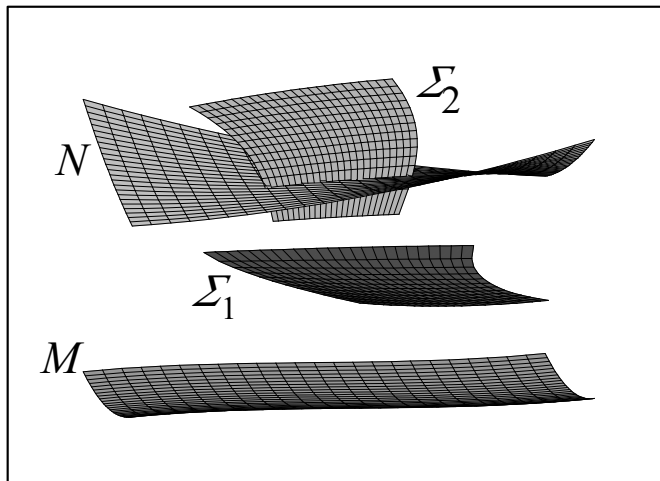


Figure 2: The simplest case of Theorem 3.7(ii) in which one parabolic surface  $M$  produces a CSS one sheet of which which,  $\Sigma_2$ , is the product of a line transverse to the other surface  $N$  and the caustic of, in this case, an  $A_2$  singularity. Thus the upper sheet  $\Sigma_2$  of the CSS is a smooth surface crossing  $N$  transversally. The next case would have  $\Sigma_2$  a cuspidal edge surface with the edge transverse to  $N$ .

Now, the Transversality Theorem implies that at any point  $\bar{p}, q, \lambda$  the jet  $j^{m+1}f$  for  $f$  from a open and dense subset of the space of functions endowed with the fine Whitney topology does not meet  $I_{m+1,n}$  and hence the family  $H$  is  $r$ -versal.  $\square$

We turn now to the case ND (§1), supposing that  $a_0$  is a parabolic point. Then  $b_0 \in N$  is a caustic point on the chord  $l(a_0, b_0)$ , corresponding to  $\lambda = 0$ .

Fix a generic function  $g : V \rightarrow \mathbf{R}$ .

**Theorem 3.7** *Let  $n \leq 5$ . There is an open and dense subset of the space of functions  $f(x), x \in U$  such that*

- (i) *for  $\lambda \neq 0 \neq \mu$  and the germ of the family  $\mathcal{F}$  is  $r$ -stable equivalent to a versal deformation of one of the germs of functions with simple singularities of the types  $A_k, (1 \leq k \leq n+1), D_k (4 \leq k \leq n+1), E_6$ ;*
- (ii) *for  $\lambda = 0$  the caustic is diffeomorphic to a cartesian product of a line which is transversal to the boundary  $N$  and a caustic of a simple singularity of type  $A_k, D_k, k \leq n$ .*

**Proof.** Let  $G$  be as defined in Lemma 3.1. Solving locally the equation  $G = 0$  for  $\lambda$  we get another useful transformation of the generating family.

A point  $\lambda_0, \mu_0$  on the affine time axis being fixed, the decomposition of  $G$  into power series in a small parameter  $\varepsilon = \lambda - \lambda_0$  provides:

$$G(z, \lambda_0 + \varepsilon, q) = -\tilde{h} + e_0(z, s) + e_1(z, s)\varepsilon + e_2(z, s)\frac{\varepsilon^2}{2} + e_3(z, s)\frac{\varepsilon^3}{6} + o(\varepsilon^4),$$

where

$$\begin{aligned}
\tilde{h} &= h + \frac{1}{2}(\mu_0 - \lambda_0) = h - h_0, \\
e_0 &= \lambda_0 f(x) + \mu_0 g(y), \\
e_1 &= 1 + f(x) - g(y) - \lambda_0 f_x(x)z - \mu_0 g_y(y)z, \\
e_2 &= -2f_x(x)z + 2g_y(y)z + \lambda_0 f_{xx}(x)z^2 + \mu_0 g_{yy}(y)z^2, \\
e_3 &= 3f_{xx}(x)z^2 - 3g_{yy}(y)z^2 - \lambda_0 f_{xxx}(x)z^3 - \mu_0 g_{yyy}(y)z^3,
\end{aligned}$$

$x = s + \mu_0 z$ ,  $y = s - \lambda_0 z$ ,  $f_x z$  is the scalar product at the point  $x = s + \mu_0 z$  of the gradient of  $f(x)$  with the vector  $z$ , similarly  $f_{xx} z^2$  is the value of the second differential form of  $f(x)$  (evaluated at  $x$ ) on the vector  $z$  etc.

Since  $e_1 \neq 0$  near the origin, the equation  $G(z, \varepsilon, q) = 0$  has the following solution for  $\varepsilon$ :

$$\varepsilon = -\frac{Q}{e_1} \left[ 1 + \frac{e_2}{2e_1^2} Q + \frac{3e_2^2 - e_1 e_3}{6e_1^4} Q^2 + \phi Q^3 \right],$$

where  $Q = e_0(z, s) - \tilde{h}$  and  $\phi$  is a smooth function in  $z, q$ . Equivalently, the family germ  $G$  multiplied by an appropriate a non-zero factor (depending on  $z, \varepsilon, q$ ) takes the form

$$G_r = \varepsilon + \frac{Q}{e_1} \left[ 1 + \frac{e_2}{2e_1^2} Q + \frac{3e_2^2 - e_1 e_3}{6e_1^4} Q^2 + \phi Q^3 \right].$$

Remembering  $z = x - y, 0 = w = \lambda x + \mu y - s$ ,

$$Q = \lambda_0 f(s + \mu_0 z) + \mu_0 g(s - \lambda_0 z) - \tilde{h} = \lambda_0 \left[ f(x) + \frac{\mu_0}{\lambda_0} g \left( \frac{s}{\mu_0} - \frac{\lambda_0}{\mu_0} x \right) \right].$$

a stratification similar to that constructed in the proof of Theorem 3.3 proves the following

**Lemma 3.8** *Let  $n \leq 5$  and let a generic function  $g$  be fixed. For generic  $f$  at any  $\lambda_0 \neq 0 \neq \mu_0$  the germ at the origin of the family  $Q$  of functions in  $z$  with parameters  $(\tilde{h}, s)$  is  $R^+$ -versal.*

It remains now to consider the boundary case  $\lambda_0 = 0, \mu_0 = 1$  (recall that we are assuming that  $b_0$ , given by  $\lambda = 0$ , is parabolic).

The function  $Q_0 = Q|_{\lambda_0=0} = g(s) - \tilde{h}$  does not depend on  $z$  and vanishes exactly on  $N = \{q : h = g(s) - \frac{1}{2}\}$ . To prove the infinitesimal r-stability of the family  $G_r$  it suffices to consider the coset of  $G_r$  modulo the ideal  $\mathcal{A} = \mathcal{O}_{x,q} \mathcal{M}_q^2$ .

The inclusions:  $Q_0 \in \mathcal{M}_q$ ,  $Q_0 + \tilde{h} \in \mathcal{M}_s^2$ , and equalities:

$$e_1 = 1 + f(z) + s f_x(z) - g_2(0) z s \pmod{\mathcal{A}}, \quad Q_0 e_2 = \tilde{h} [-2f_x(s+z)z + g_{2yy} z^2] \pmod{\mathcal{A}},$$

prove that

$$G_0 = G_r|_{\lambda_0=0} =$$

$$\varepsilon + Q_0 - Q_0 \left[ \frac{f(z)}{1+f(z)} + \frac{f_x(z)s - g_{2yy}(0)zs}{(1+f(z))^2} + \frac{-f_x(z)z + \frac{1}{2}g_{2yy}(0)zz}{(1+f(z))^3} \tilde{h} \right] \pmod{\mathcal{A}}.$$

The caustic  $\Sigma(G_0)$  splits into two components: a redundant component  $Q_0 = 0$  (which is the germ of underlying hypersurface  $N$ ) and the caustic of the family  $\Gamma$  being the expression from square brackets in the latter formula:

$$\Gamma = \frac{f(z)}{1+f(z)} + \frac{f_x(z)s - g_{2yy}(0)zs}{(1+f(z))^2} + \frac{-f_x(z)z + \frac{1}{2}g_{2yy}(0)zz}{(1+f(z))^3} \tilde{h}.$$

To show that for generic  $f$  (provided a generic  $g_2$  is fixed) the family  $\Gamma$  of functions in  $z$  with parameters  $q = (\tilde{h}, s)$  is  $R^+$ -versal we can assume that the germ of  $f$  is right equivalent to a quasihomogeneous one.

**Proposition 3.9** *The germ at the origin of  $\Gamma$  is  $R^+$ -versal if and only if the simplified family*

$$G_*(z, q) = G_*(z, \tilde{h}, s) = f(z) - g_{2yy}(0)zs + \frac{1}{2}\tilde{h}g_{2yy}(0)z^2$$

*is  $R^+$ -versal. (If so then they are  $R^+$  - equivalent).*

**Proof.** The assumption on  $f$  implies that the germ  $f$  belongs to the gradient ideal  $\mathcal{O}_z \left\{ \frac{\partial f}{\partial z} \right\}$ . Hence the local algebra  $\mathcal{A}_f = \mathcal{O}_z / \mathcal{O}_z \left\{ \frac{\partial f}{\partial z} \right\}$  coincides with the local algebra of the organizing center  $\Gamma_0 = \Gamma|_{s=0} = \frac{f(z)}{1+f(z)}$  and the coset of  $\frac{\partial \Gamma}{\partial q}$  in this algebra coincides with that of  $\frac{\partial G_*}{\partial q}$ .

**Remark.** Another simplification is possible. The  $R^+$ -versality of  $\tilde{G} = f(x) - g_{2yy}(0)zs$  implies the  $R^+$ -versality of  $G_*$ . The corresponding diffeomorphism of parameter space map the cylinder over the caustic of  $\tilde{G}$  in  $s$  space onto the caustic of  $G_*$  in  $q$  space, tangent vector to the generator of the cylinder being  $\frac{\partial}{\partial \tilde{h}}$ .

To complete the proof of theorem 3.7 we use arguments similar to that of theorem 3.3 constructing an appropriate Whitney regular stratification of the jet bundle and applying the Thom transversality theorem.

Due to the well defined affine structure in  $z$  space induced from  $\mathbf{R}^n$  the shift of the reference point  $z = 0$  to any  $z_0$  results only in subtraction of an affine form  $df|_{z_0}$  from the germ  $f, z_0$ . So it suffices to construct a stratification of the subspace  $\mathcal{M}_m^2$  of the fibre  $J_{z_0}^m(n-1, \mathbf{R})$  over each point and then multiply it by the subspace of all affine forms.

Since the right orbits of codimension  $\leq 5$  are the orbits of quasihomogeneous germs, Proposition 3.9 implies that they form the required strata over the points  $z_0$  where the quadratic form  $g_{2yy}(0)$  is non-degenerate.

Stratification of the base  $\mathbf{R}^{n-1}$  ( $z$ -space) by the degenerations of  $g_2$  and the corresponding refinements of the orbit stratification in the fibres provides a stratification of the jet bundle such that its transversality to the jet extension of  $f$  implies  $r$ -versality of the generating family.  $\square$

### Special Chords in 3-space

In the particular case of two parallel surfaces  $M$  and  $N$  in  $\mathbf{R}^3$ , where the chords joining points with parallel tangent planes are simply the common normals to each surface, the

CSS reduces to the common focal set of the two surfaces. In that case a generic normal meets the CSS at two distinct points, and these two CSS points coincide on a chord (normal) precisely for an umbilic on  $M$  (and therefore on  $N$ ). The CSS (focal set) then has the singular form  $D_4^\pm$ . A similar situation holds in the more general case of a system of geodesics normal to an initial hypersurface in a Riemannian manifold.

Turning now to the general CSS in  $\mathbf{R}^n$ ,  $n > 2$ , there can be fewer than  $n - 1$  real caustic points—that is, points of the CSS—on a generic chord. Recall from Corollary 3.2 that CSS points on the chord correspond with degenerate quadratic forms in a pencil determined by the 2-jets of the two surfaces. For example, if  $n = 3$  and both  $f_2(z)$  and  $g_2(z)$  are hyperbolic, the pencil of quadratic forms  $\mu_0 f_2(z) + \lambda_0 g_2(z)$  can pass outside the cone of degenerate forms in the 3-space of all quadratic forms in two variables and hence contain no caustic points: there are no real points on the CSS here. A pencil tangent to the cone contains a single caustic point. A pencil passing through a positively (negatively) definite form either meets the cone at two distinct points or passes through the origin (zero quadratic form).

In the Euclidean (or Riemannian) case a normal geodesic traces a line on the Legendre variety  $\Lambda$  of the family which is transversal to the inverse image  $I$  of the caustic under the Legendre projection of  $\Lambda$  to the base  $\mathbf{R}^n$ . This is called an *optical* condition (see e.g. [2]).

Now we will describe generic singularities of sets  $B_M$ ,  $B_N$  of points  $a \in M$  (respectively,  $b \in N$ ) such that the chord through  $a, b$  has exactly one caustic point. Such chords will be called *special*. Special chords just correspond to lines which are either non-transversal to  $I$  or pass through the singular locus of  $I$  which, in a generic setting, is the closure of the  $D_4$  stratum.

Using the generating family  $\mathcal{F}(z, w)$ , where  $z = (x, y, \bar{p})$ ,  $w = (\lambda, q)$ , any chord is the set of solutions of Legendre equations  $F(z, w) = 0$ ,  $\frac{\partial F}{\partial z}(z, w) = 0$  for certain fixed values of  $x, y, \bar{p}$ . Since ( $r-$ ,  $v-$ , etc.) equivalences used above do not preserve the fibration  $(z, w) \mapsto z$ , the diffeomorphic type of caustic does not determine the behaviour of special chords. We need to work with the initial family  $\mathcal{F}$  or outside the parabolic curves with  $H = -h + \frac{1}{2}(\lambda - \mu) - p_1 q_1 - p_2 q_2 + \lambda f_*(\bar{p}) + \mu g_*(\bar{p})$  using the conormals  $\bar{p}$ .

Let  $\bar{p} = (p_1, p_2)$  and

$$\mathcal{H} = \text{Hess}_{\bar{p}}(\lambda f_*(\bar{p}) + \mu g_*(\bar{p})) = \det \begin{pmatrix} \frac{\partial^2 H}{\partial p_1^2} & \frac{\partial^2 H}{\partial p_1 \partial p_2} \\ \frac{\partial^2 H}{\partial p_1 \partial p_2} & \frac{\partial^2 H}{\partial p_2^2} \end{pmatrix}$$

**Proposition 3.10** *Outside the parabolic locus germs of the sets  $B_M, B_N$  are diffeomorphic to germs of the set  $U \subset (\mathbf{R}^{n-1})^\wedge$*

$$B = \left\{ \bar{p} \mid \exists \lambda : \mathcal{H} = 0, \frac{\partial \mathcal{H}}{\partial \lambda} = 0 \right\}.$$

**Proof.** Since  $H$  is affine in parameters  $q$ , the Legendre conditions (determining the Legendre projection of the variables  $(\bar{p}, \lambda)$  which form a local coordinate system on  $\Lambda$ )



can be solved for  $q$ . The inverse image  $I$  of the caustic is formed by the solutions of the equation  $\text{Hess}_{\bar{p}}H = \mathcal{H} = 0$  which does not involve  $q$  explicitly.

Hence the chord through  $(\bar{p}, \lambda)$  is special if and only if  $\frac{\partial \mathcal{H}}{\partial \lambda} = 0$ .  $\square$

Fortunately, these conditions are Legendre equations for the family  $\mathcal{H}$  of functions in  $\lambda$  with parameters  $\bar{p}$ , the set  $B$  being the bifurcation set of this family.

**Theorem 3.11** *For a generic pair of surfaces  $M, N$  :*

- (i) *a special chord is isolated if it crosses either  $M$  or  $N$  at an elliptic point (the corresponding caustic germ is of  $D_4^\pm$  type);*
- (ii) *inside the hyperbolic regions of  $M$  and  $N$  the germ of  $B$  is either the germ of a smooth curve (the corresponding caustic points are of  $A_2$  and  $A_3$  types), or the germ of a pair of transversally intersecting smooth curves, or an isolated point (the corresponding chord passes through  $D_4^\pm$  singularity of the caustic);*
- (iii) *the germ of  $B_M$  at a special chord which crosses  $M$  at a parabolic point is the germ of a smooth curve belonging to the closure of the hyperbolic region and having second order tangency with the parabolic line. The points of intersection of these special chords trace a smooth curve on  $N$  which has fourth order tangency with the intersection of (the regular part of) the caustic with  $N$ .*

Figure 4 shows a computer picture of two hyperbolic surfaces with their associated CSS in the case where there is coincidence of caustic (CSS) points along a curve. Figure 3 shows the  $D_4$  case where both surfaces are elliptic.

**Proof** Using small parameters  $\bar{p}, \varepsilon = \lambda - \lambda_0$  the germ of  $\mathcal{H}$  at  $\bar{p} = 0, \lambda = \lambda_0$  takes the form  $\mathcal{H} = A\varepsilon^2 + B\varepsilon + C$  with

$$C = \text{Hess}_{\bar{p}}\varphi(\bar{p}), \quad A = \text{Hess}_{\bar{p}}\psi(\bar{p}), \quad B = \det \begin{pmatrix} \frac{\partial^2 \varphi}{\partial p_1^2} & \frac{\partial^2 \psi}{\partial p_1 \partial p_2} \\ \frac{\partial^2 \varphi}{\partial_1 \partial p_2} & \frac{\partial^2 \psi}{\partial p_2^2} \end{pmatrix} - \det \begin{pmatrix} \frac{\partial^2 \psi}{\partial p_1^2} & \frac{\partial^2 \varphi}{\partial p_1 \partial p_2} \\ \frac{\partial^2 \psi}{\partial_1 \partial p_2} & \frac{\partial^2 \varphi}{\partial p_2^2} \end{pmatrix}$$

where  $\varphi = \lambda_0 f_*(\bar{p}) + \mu_0 g_*(\bar{p})$  and  $\psi = f_*(\bar{p}) - g_*(\bar{p})$ .

Hence the equation of  $U$  is  $B^2 - AC = 0$ . Also at the origin  $C = B = 0$ .

At first consider the generic points of  $I$  of corank 1 (with the singularities  $A_{2,3,4}$  of the caustic). The equation  $C = 0$  determines a germ of a smooth curve in  $\bar{p}$  space tangent to  $I$ .

Note that vanishing of all three  $A, B, C$  at the origin provides a non-generic condition. In fact, vanishing of  $C$  and  $B$  (the rank of the Hessian matrix  $\hat{H}_0$  of  $\varphi$  at the origin being 1) implies that  $\psi$  vanishes on the kernel subspace of  $\hat{H}_0$  (in other words, if  $d^2\varphi = dp_1^2$ , then  $d^2\psi = adp_1^2 + bdp_1dp_2$ .) The extra condition  $A = 0$  then implies that  $b = 0$  so reference points on  $M$  and  $N$  both are parabolic with coinciding kernel directions.

So, generically,  $A \neq 0$  and the equation  $B^2 - 4AC = 0$  determines a smooth curve (tangent to  $I$ ).

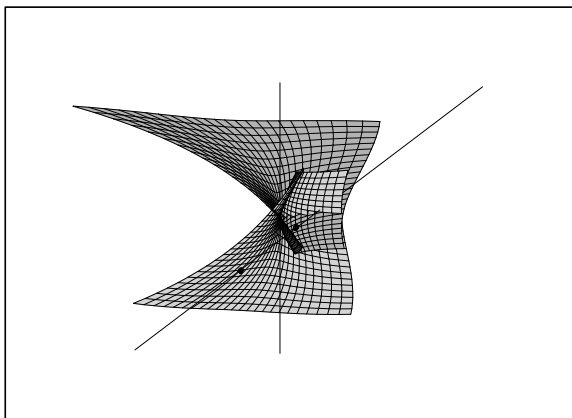


Figure 3: This illustrates Theorem 3.11(i). A part of the CSS in the  $D_4$  case where both pieces of surface (not shown here) are elliptic. There is a single chord which is tangent to the (singular) CSS once; this appears as vertical in the figure. All other chords, such as the one shown, are tangent to the CSS in two points.

Now consider the  $D_4$  case. The matrix  $\hat{H}_0$  totally vanishes, the function  $C$  has zero 1-jet at the origin and  $B(0) = 0$ .

To write down the 2-jet of the function  $B^2 - 4AC$  it suffices to retain second order terms in  $\psi$  and cubic terms in  $\varphi$ . By an appropriate affine transformation reduce  $j^3\varphi$  to standard form  $\frac{1}{6}(p_1^3 \pm p_2^3)$ , and take  $j^2\psi = \frac{1}{2}(ap_1^2 + 2bp_1p_2 + cp_2^2)$ . Then the form

$$j^2(B^2 - 4AC) = (\pm ap_2 - cp_1)^2 \mp 4(ac - b^2)p_1p_2$$

is positive definite if

$$a^2p_2^2 + c^2p_1^2 \mp (6ac - 4b^2)p_1p_2 = 16(ac - b^2)(2ac - b^2) > 0.$$

So if  $\frac{1}{2}b^2 < ac < b^2$  the form is hyperbolic and the function  $B^2 - 4AC$  has hyperbolic Morse singularity at the origin with zero level surface being the normal crossing of two smooth branches, otherwise the form is elliptic and  $B^2 - 4AC$  vanishes only at the origin.

The remaining parabolic case can be treated similarly using the family  $\mathcal{F}$ .  $\square$

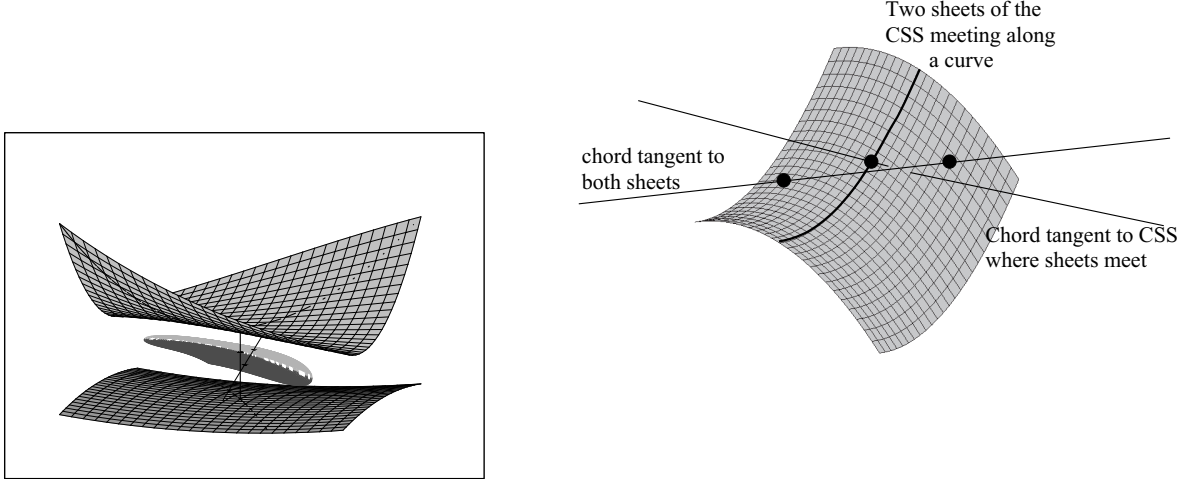


Figure 4: This illustrates Theorem 3.11(ii). Left: a computer figure of two hyperbolic pieces of surface with the CSS between them. In this case there is a curve of points on each surface (marked as a dashed curve) where the tangent planes at corresponding points are parallel and the chord joining these points is tangent exactly once to the CSS: the two sheets of the CSS come together along a curve. At other parallel tangent pairs the chord is tangent twice to the CSS. The figure shows one example of each type of chord. The slightly ragged appearance of the CSS where the two sheets join is due to computational errors. Right: a schematic figure of this situation, with two CSS sheets meeting along a curve which is drawn heavily.

## 4 Tangential case

### Preliminaries ( $n = 3$ )

For the present we work with  $n = 3$ ; the details of what follows are similar, and simpler, in the case  $n = 2$ . For higher dimensions the corresponding classification of v-orbits (see [12]) has functional invariants.

In  $\mathbf{R}^3 = \{q = (h, s, t)\}$  we can without loss of generality assume that  $a_0 = (0, -1/2, 0)$ ,  $b_0 = (0, 1/2, 0)$  and the common tangent plane to  $M, a_0$  and to  $N, b_0$  containing these points is defined by the equation  $h = 0$ . The surfaces  $M$  and  $N$  are parametrized by local coordinates  $x_1, x_2$  and  $y_1, y_2$  centred at the origin in  $\mathbf{R}^2$ . Thus

$$M = \left\{ \left( f(x_1, x_2), -\frac{1}{2} + x_1, x_2 \right) \right\}, \quad N = \left\{ \left( g(y_1, y_2), \frac{1}{2} + y_1, y_2 \right) \right\}.$$

We proceed as in Lemma 3.1 to replace  $p$  in the formula of Definition 2.1 for the generating family  $\mathcal{F}$  by  $(1, p_1, p_2)$  and then to eliminate variables by a stabilization procedure. Note that now the chord through  $a_0$  and  $b_0$  is along the  $s$ -axis and the base point on this chord will be  $\lambda_0 a_0 + \mu_0 b_0 = (0, \frac{1}{2}(\mu_0 - \lambda_0), 0)$ . Thus we write  $\lambda = \lambda_0 + \varepsilon$ ,  $\mu = \mu_0 - \varepsilon$ ,  $s = \bar{s} + \frac{1}{2}(\mu_0 - \lambda_0)$  where  $\varepsilon, \bar{s}$  are small. We then use the change of variables  $(x_1, x_2, y_1, y_2) \mapsto (z_1, z_2, w_1, w_2)$ , where  $z = x - y$ ,  $w_1 = \lambda x_1 + \mu y_1 - \varepsilon - \bar{s}$ ,  $w_2 = \lambda x_2 + \mu y_2 - t$ , and

eliminate the variables  $w$  and  $p$  by stabilization. In this way we can reduce  $\mathcal{F}$  to

$$F(z, \varepsilon, h, \bar{s}, t) = -h + \lambda f(\bar{s} + \varepsilon + \mu z_1, t + \mu z_2) + \mu g(\bar{s} + \varepsilon - \lambda z_1, t - \lambda z_2).$$

This implies the following

**Proposition 4.1** 1. *The caustic points on a bitangent chord correspond to the values  $\lambda_0, \mu_0$  such that the quadratic form  $\mu_0 f_2(z) + \lambda_0 g_2(z)$  is degenerate. Compare Corollary 3.2.*

2. *The organizing centre  $F_0 = F$  restricted to  $\{h = \bar{s} = t = 0\}$  has the form*

$$F_0 = (\lambda_0 + \varepsilon) f(\varepsilon + (\mu_0 - \varepsilon) z_1, (\mu_0 - \varepsilon) z_2) + (\mu_0 - \varepsilon) g(\varepsilon - (\lambda_0 + \varepsilon) z_1, -(\lambda_0 + \varepsilon) z_2).$$

3. *The quadratic part  $Q_F$  of the organizing centre  $F_0$  is*

$$Q_F = \lambda_0 \mu_0 \left[ f_2(z) \mu_0 + \lambda_0 g_2(z) + \left( \frac{\partial^2 f_2}{\partial x_1^2} - \frac{\partial^2 g_2}{\partial y_1^2} \right) \varepsilon z_1 + \left( \frac{\partial^2 f_2}{\partial x_1 \partial x_2} - \frac{\partial^2 g_2}{\partial y_1 \partial y_2} \right) \varepsilon z_2 \right] + \lambda_0 f_2(\varepsilon, 0) + \mu_0 g_2(\varepsilon, 0).$$

4. *Since  $F, \frac{\partial F}{\partial \lambda}$  and  $\frac{\partial F}{\partial z}$  all vanish at any point of the form  $(z, \varepsilon, h, \bar{s}, t) = (0, -\bar{s}, 0, \bar{s}, 0)$  the whole of the  $s$ -axis is contained in the discriminant  $\Delta$ . This is the chord passing through  $a_0$  and  $b_0$ .*

The classification of germs of functions in  $\mathcal{M}_{w,\varepsilon}^2$  with respect to stable v-equivalence (without parameters) starts with the following orbits [8, 12] ( $w \in \mathbf{R}$ ) :

$$B_k : \pm w^2 + \varepsilon^k; \quad C_k : w^k + \varepsilon w; \quad k = 2, 3, 4 \quad F_4 : w^3 + \varepsilon^2.$$

Their miniversal deformations in parameters  $q$  and adjacency diagram are the following:

$$\begin{aligned} B_k &: \pm w^2 + \varepsilon^k + q_{k-2} \varepsilon^{k-2} + \cdots + q_0; \\ C_k &: w^k + \varepsilon w + q_{k-2} w^{k-2} + \cdots + q_2 w^2 + q_1 \varepsilon + q_0; \\ F_4 &: w^3 + \varepsilon^2 + q_2 w \varepsilon + q_1 w + q_0. \end{aligned}$$

$$\begin{array}{ccccc} B_2(C_2) & \leftarrow & C_3 & \leftarrow & C_4 \\ \uparrow & & \uparrow & & \\ B_3 & & \leftarrow & & F_4 \\ \uparrow & & & & \\ B_4 & & & & \end{array}$$

The remaining classes form subset of codimension 3 in the space  $\mathcal{M}_{z,\varepsilon}^2$ .

**The planar case**

This case was first investigated by Paul Holtom [9]. In the plane ( $n = 2$ ) we are considering two germs of curves  $M$  and  $N$  tangent to the  $s$ -axis in the plane with coordinates  $q = (h, s)$ , at the points  $a_0 = (0, -\frac{1}{2})$ ,  $b_0 = (0, \frac{1}{2})$ . The two curves are parametrized by  $x$  and  $y$ , and we use  $z = x - y$ ,  $\bar{s} = s - \frac{1}{2}(\mu_0 - \lambda_0)$ . The generating function, after the reduction described above, is

$$F = -h + \lambda f(\bar{s} + \varepsilon + \mu z) + \mu g(\bar{s} + \varepsilon - \lambda z).$$

Generically bitangency occurs at isolated pairs of points with non-vanishing second order terms of the corresponding functions  $f(x) = a_2x^2 + a_3x^3 + \dots$  and  $g(y) = b_2y^2 + b_3y^3 + \dots$ , that is, with neither  $a_0$  nor  $b_0$  an inflexion. The point  $h = 0, \bar{s} = 0$  is a caustic point if and only if  $\mu_0a_2 + \lambda_0b_2 = 0$ .

**Theorem 4.2** *For generic pair of curves in a plane near a double tangent the criminant  $\Delta$  coincides with the double tangent. On this line there is at most one caustic point. The germ of the caustic  $\Sigma$  at this point is a smooth curve having inflexional contact with  $\Delta$ . The generating function at this point is  $v$ -equivalent to the  $C_3$  type family:*

$$C_3 : F(z, \varepsilon, q) = z^3 + z(\varepsilon + \bar{s}) + h.$$

**Proof.** (See Figure 5.) The quadratic part of the organizing centre  $F_0 = F|_{q=0}$  is

$$Q_F(z, \varepsilon) = \lambda_0\mu_0((\mu_0a_2 + \lambda_0b_2)z^2 + 2(a_2 - b_2)\lambda_0\mu_0\varepsilon z + (\lambda_0a_2 + \mu_0b_2)\varepsilon^2).$$

The determinant of this form can be converted (due to the relation  $\lambda_0 + \mu_0 = 1$ ) into a very simple expression  $\det Q_F = \lambda_0\mu_0a_2b_2$  which does not vanish at the intermediate points  $\lambda_0 \neq 0 \neq \mu_0$  at the chord.

Restricting the form  $Q_F$  to the subspace  $\varepsilon = 0$  we get  $Q_F^0 = Q_F|_{\varepsilon=0}$  with the determinant  $\det(Q_F^0) = \lambda_0\mu_0(a_2\mu_0 + b_2\lambda_0)z^2$ . So if  $a_2 \neq b_2$  there is a single point on a generic bitangent where this determinant vanishes. Generically at this point the cubic form in  $z$  ( $\varepsilon = 0$ ) does not vanish. So the organizing centre is of  $C_3$  type. Moreover generically the coefficient  $2(\lambda_0a_2 + \mu_0b_2)$  of the  $s\varepsilon$  term in the Taylor decomposition of  $F$  is non-zero—in fact this just requires that the absolute values of the curvatures of the two pieces of curve at  $a_0, b_0$  are not equal. Hence the family  $F$  is versal.

The remaining case  $\lambda_0 = 0$  can be treated similarly to that of transversal pair. At  $\lambda_0 = 0$  the family takes the form:

$$F = -h + \varepsilon f(s + \varepsilon + (1 - \varepsilon)z) + (1 - \varepsilon)g(s + \varepsilon - \varepsilon z) = -h + \varepsilon [a_2z^2 + 2a_2\varepsilon z + \varepsilon b_2 + \dots]$$

(where  $\dots$  stand for a function from  $\mathcal{M}_{z,\varepsilon}^3$ ) and so has no caustic (since apart from  $\varepsilon = 0$  there are no solutions of  $F = \frac{\partial F}{\partial z} = \frac{\partial^2 F}{\partial z^2} = 0$  near the origin).

We have  $F = \frac{\partial F}{\partial z} = \frac{\partial F}{\partial \varepsilon} = 0$  only when  $(z, \varepsilon, h, \bar{s}) = (0, -\bar{s}, 0, \bar{s})$  so that the germ of the bitangent line  $h = 0$  in the  $q$ -plane forms the envelope  $\Delta$ .  $\square$

Figure 5 shows a schematic diagram of the situation with a bitangent line.

### The space case

To describe generic singularities in  $\mathbf{R}^3$  we use notation  $q = (h, s, t) \in \mathbf{R}^3, z = (u, v) \in \mathbf{R}^2, \bar{s} = s - \frac{1}{2}(\mu_0 - \lambda_0)$  so that  $h, \bar{s}, t, u$  and  $v$  are all small quantities. We can always assume

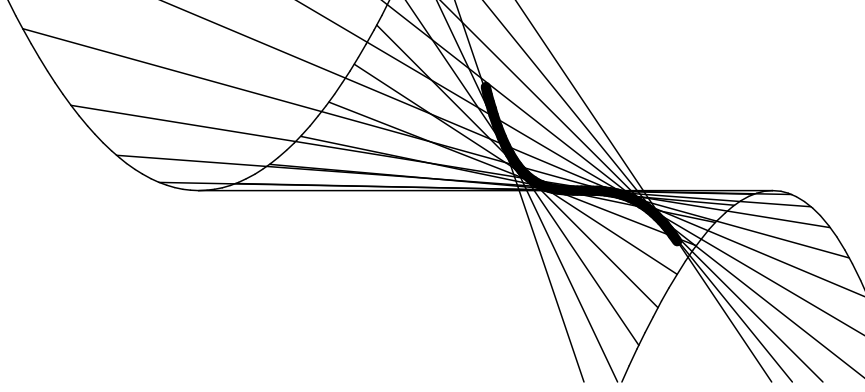


Figure 5: Two pieces of curve with a common tangent, and chords joining points with parallel tangents. The caustic  $\Sigma$  is drawn heavily and is the ‘true’ CSS, tangent to all the chords, and the criminant  $\Delta$  is the bitangent itself. Notice that  $\Sigma$  has inflexional contact with  $\Delta$ , as in Theorem 4.2.

that at least one of the quadratic forms  $f_2$  or  $g_2$  is non-degenerate and does not vanish on the bitangent chord. Suppose this is  $f_2$ . By an appropriate affine transformation it can be reduced to the normal form  $f_2 = x_1^2 \pm x_2^2$ , while the other one takes the general form  $g_2 = ay_1^2 + 2by_1y_2 + cy_2^2$ . The surfaces have basepoints  $(h, s, t) = (0, \pm\frac{1}{2}, 0)$ . When  $a = 0$  an asymptotic direction of  $N$  lies along the chord between basepoints; note that this is an affinely invariant condition.

The recognition of space-time singularities involves the quadratic part

$$Q_F(\varepsilon, u, v) = \lambda_0 ((\varepsilon + \mu_0 u)^2 \pm \mu_0^2 v^2) + \mu_0 (a(\varepsilon - \lambda_0 u)^2 - 2b(\varepsilon - \lambda_0 u)\lambda_0 v + c\lambda_0^2 v^2)$$

of the organizing center  $F_0$  and the quadratic part of its restriction  $Q_F^0$  to the subspace  $\varepsilon = 0$ :

$$Q_F^0(u, v) = \lambda_0 \mu_0 (\mu_0 (u^2 \pm v^2) + \lambda_0 (au^2 + 2buv + cv^2)).$$

Calculating their determinants a significant cancellation of terms, due to the relation  $\lambda_0 + \mu_0 = 1$ , yields (apart from a factor  $\lambda_0^2 \mu_0^2$  in each case)

$$\det(Q_F) = \pm a\mu_0 + (ac - b^2)\lambda_0, \quad \det(Q_F^0) = \lambda_0^2 b^2 - (\mu_0 + \lambda_0 a)(\pm\mu_0 + \lambda_0 c).$$

Hence generically on each bitangent chord (parametrized by  $\lambda_0$ ) there is one point where  $\det(Q_F) = 0$ , and at most two points where  $\det(Q_F^0) = 0$ . The value of  $\det(Q_F^0)$  at the root of  $\det(Q_F)$  is equal to  $\pm(ac - b^2)\lambda_0^2 b^2 a^{-2}$ . So the forms are both degenerate if (i)  $b = 0$  or (ii)  $ac - b^2 = 0$ . Note that the first of these means that the principal axes of the two surfaces at the basepoints coincide. This has an affinely invariant formulation: the pencil  $\lambda f_2 + \mu g_2$  contains a degenerate form whose kernel subspace is the bitangent direction. The condition (ii) means that the basepoint is a parabolic point on  $N$ .

**Theorem 4.3** *For fixed generic  $f$  and for  $g$  from an open dense subset in the space of functions on  $V$  the germ of  $B(F)$  at a point where  $\lambda_0 \neq 0 \neq \mu_0$  is diffeomorphic to one of the bifurcation sets of  $B_k$ ,  $C_k$ ,  $k = 2, 3, 4$ , or  $F_4$  versal deformations. In all cases the criminant coincides with the ruled surface swept by bitangent chords.*

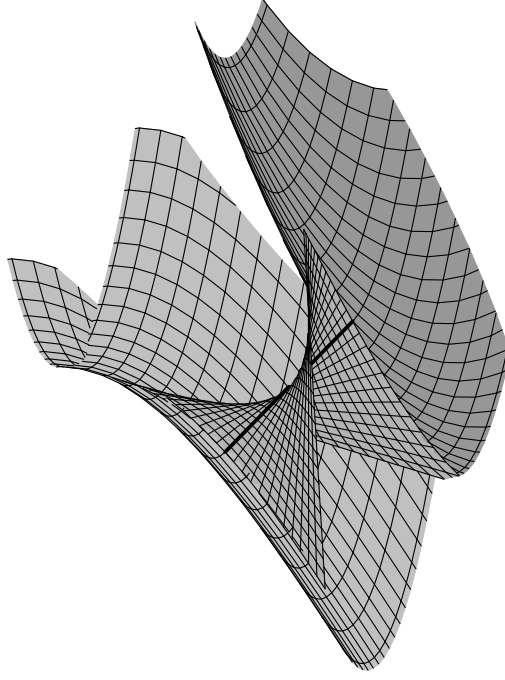


Figure 6: Two pieces of surface, upper and lower in the figure, together with the ruled surface  $\Delta$  of bitangent chords, which forms a ‘semi-cubic cylinder’ (a surface with a cuspidal edge), as in the case  $B_3$  of Theorem 4.3. The line through the base-points  $(0, \pm\frac{1}{2}, 0)$  on the two surface pieces is drawn heavily and, like all the bitangent chords, is tangent to the cuspidal edge, this line being tangent at the origin. The remaining part of the CSS (the caustic  $\Sigma$ ) is not local to the origin and is not shown.

The case  $B_3$  is illustrated in Figure 6.

The **proof** given below consists of the direct verification of the versality of the family  $F$  for generic  $g$ . Classes  $B_3, B_4$  occur when  $\text{rank}(Q_F^0) = 2$ ,  $\text{rank}(Q_F) = 2$ . Classes  $C_3, C_4$  correspond to the points where  $\text{rank}(Q_F^0) = 1$ ,  $\text{rank}(Q_F) = 3$ , and the class  $F_4$  corresponds to the points where  $\text{rank}(Q_F^0) = 1$ ,  $\text{rank}(Q_F) = 2$ .

The pairs  $(f, g)$  admitting a point  $\lambda$  on a bitangent chord where one or both of these forms has  $\text{corank} \geq 2$  are not generic (their jets form a subset of codimension  $> 2$  in the jet space).

**Case 1** Consider the situations where the singularities  $B_3$  and  $B_4$  can appear. Let  $\pm a\mu_0 + (ac - b^2)\lambda_0 = 0$  but  $\lambda_0 \neq 0 \neq \mu_0$ ,  $b^2 - ac \neq 0$   $b \neq 0$ . Note that since  $\lambda_0 + \mu_0 = 1$ , the values of  $\lambda_0$  and  $\mu_0$  are determined by  $a, b, c$ . They are  $\lambda_0 = \pm a/(ac - b^2 \mp a)$ ,  $\mu_0 = (ac - b^2)/(ac - b^2 \mp a)$ ; in particular the denominator in these expressions cannot be zero.

Since  $\text{rank}(Q_F^0) = 2$  the variables  $u$  and  $v$  can be eliminated by a stabilisation. Namely, solving the equations  $\frac{\partial F}{\partial u} = 0$ ,  $\frac{\partial F}{\partial v} = 0$  for  $u$  and  $v$  we find

$$\begin{aligned}
u &= -\frac{1}{\mu_0}(\varepsilon + \bar{s}) \mp \frac{a}{b\lambda_0\mu_0}t + \dots, \\
v &= \frac{a}{b\lambda_0\mu_0}(\varepsilon + \bar{s}) \mp \frac{\mu_0(1 \mp c) + a}{b^2\lambda_0\mu_0}at + \dots
\end{aligned}$$

where dots stand for functions of  $\varepsilon, \bar{s}, t$  in  $\mathcal{M}_{\varepsilon, \bar{s}, t}^2$ .

After substitution of these expressions into the family  $F$  we obtain:

$$\tilde{F}(\varepsilon, q) = -h + c_3\varepsilon^3 + c_4\varepsilon^4 + \bar{s}(c_{1,1}\varepsilon + c_{2,1}\varepsilon^2) + t(c_{1,2}\varepsilon + c_{2,2}\varepsilon^2) + \varphi(\varepsilon, \bar{s}, t),$$

where  $\varphi \in \mathcal{M}_{\varepsilon}^5 \cup \mathcal{M}_{\bar{s}, t}^2 \cup \mathcal{M}_{\varepsilon}^3 \mathcal{M}_{\bar{s}, t}$ . Let  $\alpha = \frac{\partial^3 f}{\partial x_2^3}$  evaluated at  $(0,0)$ . Then we find the following:

$$\begin{aligned}
c_{1,1} &= 0, \quad c_{1,2} = \frac{2b}{\pm c\lambda_0 + \mu_0} = \frac{2(\pm a - ac + b^2)}{b} \neq 0, \\
c_3 &= \mp \frac{a^2}{b^2\lambda_0^2\mu_0} + \frac{a^3}{b^3\lambda_0^2}\alpha + \frac{g_3(b, -a)}{b^3\mu_0^2}, \quad c_{2,1} = \mp \frac{2a^2}{b^2\lambda_0^2\mu_0} + \frac{3a^3}{b^3\lambda_0^2}\alpha + \frac{3g_3(b, -a)}{b^3\mu_0^2},
\end{aligned}$$

where  $g_3(b, -a)$  stands, as usual, for the result of substituting  $y_1 = b, y_2 = -a$  in the cubic form of the expansion of  $g$  about  $(0,0)$ . From these it is clear that, so long as  $a \neq 0$ , we cannot have both  $c_3$  and  $c_{2,1}$  equal to 0. The coefficient  $c_4$  involves the coefficients of the cubic and quartic forms from the Taylor decompositions of  $f$  and  $g$ .

Hence, if  $c_3 \neq 0$  the family  $F$  is a versal unfolding of a  $B_3$ -singularity (its caustic is void and the discriminant is a cylinder over semi-cubic parabola). If  $c_3 = 0$  then for generic  $f$  and  $g$  we have  $c_{2,1} \neq 0$  and  $c_4 \neq 0$ . In this case the family  $F$  is a versal unfolding of a  $B_4$  singularity (the caustic is void and the discriminant  $\Delta$  is an ordinary swallowtail).

**Case 2** Consider the case where singularities  $C_3, C_4$  can occur. So we suppose that  $\lambda_0 \neq 0 \neq \mu_0, b \neq 0 \neq b^2 - ac, \lambda_0^2(b^2 - ac) - \mu_0\lambda_0(c \pm a) \mp \mu_0^2 = 0$

The kernel vector  $K = -\lambda_0 b \frac{\partial}{\partial u} + (\mu_0 + a\lambda_0) \frac{\partial}{\partial v}$  of the quadratic form  $Q_F^0$  has both its components non-zero. The inequality  $\frac{\partial^2 F}{\partial \varepsilon \partial t} \Big|_0 = 2\mu_0 b \neq 0$  shows that provided the cubic form  $F_3(dz)^3$  of the function  $F|_{\varepsilon=q=0}$  does not vanish on  $K$  then the family  $F$  is a versal unfolding of a  $C_3$ -singularity.

For generic  $f$  and  $g$  at isolated points on chords joining points of contact of bitangent planes the form  $F_3(dz)^3$  can vanish on  $K$ . Consider such a point.

Since  $\text{rank}(Q_F^0) = 1$  there exists a diffeomorphism

$$\theta : (u, v, q) \mapsto (\rho(u, v, q), \tau(u, v, q), q)$$

which reduces the family  $F|_{\varepsilon=0}$  to a stabilization  $\tilde{F} = F \circ \theta^{-1} = \rho^2 + \psi(\tau, q)$ , of a family in one variable  $\tau$  with  $j^2(\psi(\tau, 0)) = 0$ . The vector  $K$  is tangent to the curve  $\gamma = \{q = 0, \rho = 0\}$ .

Denote by  $\alpha = \frac{\partial^3 f}{\partial x_1^3} \Big|_0$  and by  $\beta = \frac{\partial^3 g}{\partial y_1^3} \Big|_0$ . Since  $F|_{\varepsilon=0, q=0} = \lambda_0 f(\mu_0 z) + \mu_0 g(-\lambda_0 z)$ , the third order terms in  $j^3(\tilde{F}|_{\gamma})$  depend only on the components of  $K$ . Hence the equation



$j^3 \left( \tilde{F} |_\gamma \right) = 0$  splits into a sum of the terms which do not include the coefficients  $\alpha$  and  $\beta$  and the term  $T_1 = \frac{1}{6} \lambda_0 \mu_0 (\alpha \mu_0^2 - \beta \lambda_0^2) b^3$ .

The equation  $\frac{\partial^3 \tilde{F}}{\partial v \partial \tau^2} |_\gamma = 0$  does not include  $\alpha$  and  $\beta$ , while the equation  $\frac{\partial^3 \tilde{F}}{\partial \bar{s} \partial \tau^2} |_\gamma = 0$  splits into a sum of terms which do not depend on  $\alpha$  and  $\beta$  and the term  $T_2 = \frac{3}{2} \lambda_0 \mu_0 (\alpha \mu_0 + \beta \lambda_0) b^2$ .

The determinant of the mapping  $(\alpha, \beta) \mapsto (T_1, T_2)$  is non-zero since  $\lambda_0, \mu_0$  and  $b$  are non-zero. Hence, if for generic  $f, g$  we have  $j^3 \left( \tilde{F} |_\gamma \right) = 0$  then  $\frac{\partial^3 \tilde{F}}{\partial \bar{s} \partial \tau^2} |_\gamma \neq 0$  and (as it can be shown considering fourth order terms)  $j^4 \left( \tilde{F} |_\gamma \right) \neq 0$ . This implies that the germ of  $F$  is a versal unfolding of a  $C_4$ -singularity.

**Case 3** In the remaining generic case when both quadratic forms are degenerate and  $\lambda_0 \neq 0 \neq \mu_0$  the equations

$$\pm a \mu_0 + (ac - b^2) \lambda_0 = 0, \quad \lambda_0^2 b^2 (b^2 - ac) = 0$$

imply that  $b = 0$ ,  $b^2 - ac \neq 0$ ,  $\pm \mu_0 + c \lambda_0 = 0$  (since otherwise either  $\mu_0 = 0$ , or  $a = 0$  or  $c = \pm a$  which all provide an extra condition to be avoided in a generic setting). Geometrically in this case the pencil of quadratic forms  $f_2$  and  $g_2$  has the bitangent chord as a principal direction. The kernel direction of  $Q_F^0$  is  $K = \frac{\partial}{\partial v}$ .

Since the family  $F$  has the form

$$\begin{aligned} F = & -h + (\lambda_0 + \varepsilon) \left[ (s + \varepsilon + (\mu_0 - e)u)^2 \pm (t + (\mu_0 - \varepsilon)v)^2 + \right. \\ & \left. f_3(s + \varepsilon + (\mu_0 - e)u, t + (\mu_0 - \varepsilon)v) + \dots \right] + \\ & (\mu_0 - \varepsilon) \left[ a(s + \varepsilon + (\mu_0 - e)u)^2 + c(t + (\mu_0 - \varepsilon)v)^2 + \right. \\ & \left. g_3(s + \varepsilon + (\mu_0 - e)u, t + (\mu_0 - \varepsilon)v) + \dots \right] \end{aligned}$$

the third order form  $G_3(u, v)$  of the restricted organizing centre  $G = F |_{\varepsilon=0, q=0}$  when evaluated on the vector  $K$  is generically non-zero:

$$G_3(K) = \frac{1}{6} \lambda_0 \mu_0 \left( \mu_0^2 \frac{\partial^3 f}{\partial x_2^3} \Big|_0 - \lambda_0^2 \frac{\partial^3 g}{\partial y_2^3} \Big|_0 \right) \neq 0.$$

So the organizing centre  $F_0 = F |_{q=0}$  is equivalent to the  $F_4$  normal form  $\varepsilon^2 + z^3$ .

The Taylor decomposition of the derivative

$$\frac{\partial F}{\partial t} \Big|_{u=q=0} = 2 \lambda_0 \mu_0 (\pm 1 - c) v \pmod{\mathcal{M}_\varepsilon \cup \mathcal{M}_v^2}$$

starts with a generically non-vanishing term, which is one of the basic monomials in local algebra of the  $F_4$  miniversal deformation.

The other derivative  $\frac{\partial F}{\partial \bar{s}} |_{u=q=0}$  belongs to  $\mathcal{M}_\varepsilon \cup \mathcal{M}_v^2$ ; however it contains the other monomial  $\varepsilon v$  of the local algebra multiplied by a generically non-zero factor:

$$\frac{\partial F}{\partial \bar{s}} \Big|_{u=q=0} = \lambda_0 \mu_0 \varepsilon v \left( \frac{\partial^3 f}{\partial x_1^2 \partial x_2} \Big|_0 - \frac{\partial^3 g}{\partial y_1^2 \partial y_2} \Big|_0 \right) + \dots$$

where dots denote the terms which do not involve these components of the jets of the underlying functions  $f$  and  $g$ .

So, generically the deformation  $F$  is space-time equivalent to a standard deformation of an  $F_4$  singularity.  $\square$

To complete the current section, we describe the singularities arising at the reference points when either  $\lambda_0$  or  $\mu_0$  vanish at the bitangent chord.

**Theorem 4.4** *For generic pair of germs of surfaces:*

(i). *If  $a \neq 0$  then at  $\lambda_0 = 0$  the family  $F$  is equivalent to  $B_2$  normal form. It has no caustics, and the criminant coincides with the ruled surface swept by bitangent chords.*

(ii). *If  $a = 0$  (the form  $g_2$  vanishes on the bitangent chord, which means that an asymptotic direction of the surface  $N$  lies along the chord) then the germ of  $F$  at  $\lambda_0 = 0$  is equivalent to the  $B_3$  normal form. It has no caustic, the criminant is a surface with a cuspidal edge.*

(iii). *If the quadratic form  $g_2$  is non-degenerate then the germ of  $F$  at  $\mu_0 = 0$  is equivalent to the  $B_2$  normal form.*

(iv). *If  $b^2 - ac = 0$  (the quadratic form  $g_2$  is degenerate) the germ  $F$  at  $\mu_0 = 0$  is space-time stable equivalent to the normal form (similar to that of  $C_4$ ):*

$$\tilde{C}_4: \quad H = -\tilde{h} + \varepsilon (\varepsilon + ut + s + u^3 + v^2)$$

**Remark.** The caustic of  $\tilde{C}_4$  is diffeomorphic to a smooth surface (similar to the criminant of  $C_4$ ) with the redundant component  $\tilde{h} = 0$ . The criminant of  $\tilde{C}_4$  is diffeomorphic to the folded Whitney umbrella (that is to the caustic of  $C_4$ ).

**Proof.** At  $\lambda_0 = 0$  the family  $F$  takes the form

$$F = -h + \varepsilon f(s + \varepsilon + (1 - \varepsilon)u, t + (1 - \varepsilon)v) + (1 - \varepsilon)g(s + \varepsilon - \varepsilon u, t - \varepsilon v).$$

(i)+(ii). The decomposition of the organizing centre  $F_0 = F|_{q=0}$  starts with the terms

$$F_0 = \varepsilon (a\varepsilon + u^2 \pm v^2 + 2(1 - a)u\varepsilon - 2bv\varepsilon + \varepsilon^2(1 - a + g_{3,0}) + \varphi)$$

where  $g_{3,0}$  is the coefficient of  $y_1^3$  in the expansion of  $g(y_1, y_2)$  and the function  $\varphi \in \mathcal{M}_{\varepsilon, u, v}^3$ . If  $a \neq 0$  the function in brackets is right stable equivalent to the function  $a\varepsilon + \xi(\varepsilon, s, t)$  with  $\xi|_{s, t=0} \in \mathcal{M}_{\varepsilon}^2$ . So the family is right stable equivalent to  $\tilde{h} + \varepsilon^2$ , which is of  $B_2 \approx C_2$  type and has no caustic.

If  $a = 0$  then generically the organizing centre of the family has a decomposition which starts with the terms:

$$F_0 = \varepsilon(k\varepsilon^2 + 2u\varepsilon - 2bv\varepsilon + u^2 \pm v^2 + \dots)$$

with  $k \neq 0$ . This is stable right equivalent to the germ of  $\varepsilon^3$ .

The decompositions of the derivatives

$$\left. \frac{\partial F}{\partial s} \right|_{q=0} = 2a\varepsilon + 2\varepsilon^2(1 - a) + \dots, \quad \left. \frac{\partial F}{\partial t} \right|_{q=0} = 2b\varepsilon - 2b\varepsilon^2 \dots$$

prove (using  $b \neq 0$ ) that the family is stably  $v$ -equivalent to the  $B_3$ -normal form

$$F = \tilde{h} + t\varepsilon + s\varepsilon^2 + \varepsilon^3$$

with no caustic and the discriminant being a surface with cuspidal edge.

The case  $\mu_0 = 0$  and  $g_2$  non-degenerate coincides with the case (i).

Finally, if  $\mu_0 = 0$  and the form  $g_2$  is degenerate the organizing centre  $F_0 = F|_{q=0}$  of the family

$$F = -h + (1 + \varepsilon) [(s + \varepsilon - \varepsilon u)^2 \pm (t - \varepsilon v)^2 + f_3(s + \varepsilon - \varepsilon u, t - \varepsilon v) + \dots] - \varepsilon [a(s + \varepsilon - (1 + \varepsilon)u)^2 + 2b(s + \varepsilon - (1 + \varepsilon)u)(t - (1 + \varepsilon)v) + c(t - (1 + \varepsilon)v)^2 + g_3(s + \varepsilon - (1 + \varepsilon)u, t - (1 + \varepsilon)v) + \dots]$$

takes the form

$$F_0 = \varepsilon [\varepsilon - g_2(u, v) + g_3(u, v) + \psi(\varepsilon, u, v)]$$

with  $\psi \in \mathcal{M}_\varepsilon \mathcal{M}_{\varepsilon, u, v} \cup \mathcal{M}_{u, v}^4$ .

Again using the decompositions of the derivatives  $\frac{\partial F}{\partial s}$  and  $\frac{\partial F}{\partial t}$  and an appropriate right equivalence  $(u, v, \varepsilon, q) \mapsto (x(u, v, \varepsilon, q), y(u, v, \varepsilon, q), \varepsilon, q)$  we can reduce the family  $F$  (in a generic setting) to the form:

$$H(x, y, \varepsilon, q) = -\tilde{h} + \varepsilon [\varepsilon + k_1 x t + k_2 s + x^3 + y^2 + \psi]$$

with  $\psi \in \mathcal{M}_q \mathcal{M}_x^2 \cup \mathcal{M}_\varepsilon^2$  and non vanishing coefficients  $k_1, k_2$ .

**Lemma 4.5** *The family  $H$  is  $v$ -equivalent to the normal form*

$$H_0 = -\tilde{h} + \varepsilon [\varepsilon + x t + s + x^3 + y^2].$$

**Proof of the lemma.** The space  $\mathcal{H}$  of functions  $H$  is an  $\mathcal{O}_{x, y, \varepsilon, q}$ -module. The tangent space  $T_H$  at  $H$  to the  $v$ -orbit of the family  $H$  contains the sum of the following modules

$$\mathcal{O}_{x, y, \varepsilon, q} \left\{ H, \frac{\partial H}{\partial x} \right\} + \mathcal{O}_{\varepsilon, q} \left\{ \varepsilon \frac{\partial H}{\partial \varepsilon} \right\} + \mathcal{O}_q \left\{ \frac{\partial H}{\partial q} \right\}.$$

The latter coincides with the sum

$$\mathcal{O}_{x, y, \varepsilon, q} \{ \varepsilon x^2, \varepsilon y \} + \mathcal{O}_{\varepsilon, q} \{ \varepsilon^2 \} + \mathcal{O}_q \{ 1, \varepsilon x, \varepsilon \}.$$

Hence,  $T_H$  coincides with the total tangent space to  $\mathcal{H}$  at  $H$ . Using the Malgrange preparation theorem the infinitesimal  $v$ -versality of  $H$  holds and implies  $v$ -versality. So for all  $\psi$  the families  $H$  are equivalent.

## 5 Chords near inflections and parabolic points

Lemma 3.1 still holds for systems of chords through parallel pairs of points close to a parabolic point on a single hypersurface  $M \in \mathbf{R}^n$ . However the resulting affine generating family possesses more specific features. Since germs of  $M$  and  $N$  at the distinguished points  $a_0$  and  $b_0$  coincide the family  $G(z, \lambda, q)$  takes the form:

$$G = -h + \lambda f(s + \mu z) + \mu f(s - \lambda z),$$

where the germ of the function  $f$  (defining the hypersurface) has vanishing 1-jet at the origin and its second differential at the origin is degenerate.

This formula implies the following properties of the family  $G$ :

**Proposition 5.1** 1. The function  $G|_{z=0} = -h + f(s)$  vanishes exactly on the hypersurface  $M$ .

2. The family  $G$  contains no terms linear in  $z$ :  $\frac{\partial G}{\partial z}|_{z=0} = 0$ .

3. The family  $G$  is invariant under the symmetry  $\lambda \mapsto \mu$ ,  $z \mapsto -z$  which has a fixed point at  $\lambda = \mu = 1/2$ ,  $z = 0$ .

4. The component  $W_*(G)$  of the extended wavefront corresponding to  $\lambda\mu \neq 0$  (compare the Remarks following Proposition 2.2) contains the axis  $q = 0$ ,  $\lambda \in \mathbf{R}$ , which projects to the origin in  $q$  space. So the mapping  $\pi|_{W(F)}$  is not a proper map but is a kind of blowing-down map.

### 5.1 Planar inflection case

Starting with the planar case  $z \in \mathbf{R}$ ,  $s \in \mathbf{R}$  observe that up to an appropriate affine transformation a germ of a generic plane curve at an ordinary inflection point is the graph  $h = f(s)$  of a function  $f(z) = z^3 + c_4 z^4 + c_5 z^5 + \dots$  with  $c_4 \neq 0$ .

Hence

$$G = -h + f(s) + \lambda\mu [(\mu - \lambda)z^3 + (\mu^2 + \lambda^2 - \mu\lambda)c_4 z^4 + \dots + sz^2(3 + 6c_4 s + \dots) + (\mu - \lambda)sz^3(4c_4 + \dots)].$$

**Proposition 5.2** (i) For  $\lambda_0 \neq 0$ ,  $\frac{1}{2}$ , 1 the germ of  $G(z, \varepsilon + \lambda_0, s)$  at the origin is  $r$ -equivalent to the germ at the origin of

$$H = -\tilde{h} + (1 + \varepsilon)sz^2 + z^3,$$

The caustic and criminant here contain only the redundant component  $\tilde{h} = h - f(s) = 0$  (the curve itself).

(ii) For  $\lambda_0 = 0$  or  $\lambda_0 = 1$  the normal form is

$$H = -\tilde{h} + \varepsilon(sz^2 + z^3)$$

which has only the redundant component of the caustic and criminant.

(iii) For  $\lambda_0 = \frac{1}{2}$  the family is weakly equivalent (giving only diffeomorphism of caustics) to the normal form

$$H = -\tilde{h} + sz^2 + \varepsilon z^3 + z^4.$$

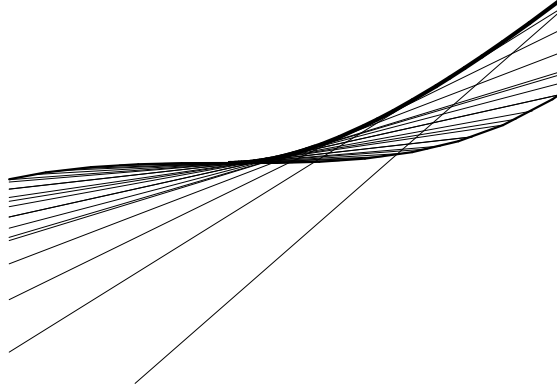


Figure 7: A plane curve with an inflection, and some chords drawn through points with parallel tangents local to the inflection. The upper curve tangent to the chords is the CSS in this case; see Proposition 5.2(iii).

In this case the discriminant is the redundant component  $\tilde{h} = 0$  and the caustic is the union of this with a  $B_2$  bifurcation diagram, in fact  $\{(\tilde{h}, s) = (\frac{1}{3}z^4, 2z^2) : z \in \mathbf{R}\}$ . See Figure 7.

**Proof.** (i) An appropriate rescaling of  $z, s$  and a diffeomorphism  $s \mapsto 3s + 6c_4s^2 + \dots$  reduce the family to the following one:

$$G = -\tilde{h} + sz^2(1 + A(\varepsilon)) + z^3(1 + B(z, \varepsilon, q))$$

with smooth functions  $A, B$  such that  $A(0) = 0$ ,  $\frac{\partial A}{\partial \varepsilon}|_0 \neq 0$  and  $B(0) = 0$ .

Applying the Moser homotopy method it is easy to prove that all such germs are  $r$ -equivalent. In fact, the tangent space  $T_G\mathcal{A}$  at  $G$  to the space  $\mathcal{A}$  of deformations of this form is an  $\mathcal{O}_{z, \varepsilon, q}$ -module.

The tangent space  $T_v(G)$  to the  $r$ -orbit through  $G$  contains a subspace

$$T_* = \mathcal{O}_{z, \varepsilon, q} \frac{\partial G}{\partial z} z + \mathcal{O}_{\varepsilon, q} \frac{\partial G}{\partial \varepsilon} \varepsilon + \mathcal{O}_q \frac{\partial G}{\partial s} s \subset T_v(G),$$

Evidently  $T_* = T_G\mathcal{A}$  and  $r$ -versality of  $G$  follows.

The proof of (ii) is similar. Now the space of deformations

$$\mathcal{A} = \{\varepsilon z^2 s A(s, \varepsilon) + \varepsilon z^3 B(z, \varepsilon, s)\}$$

is exhausted by sum of the subspaces of the right orbit generated (over the corresponding algebras of germs of functions) by:

$$z \frac{\partial G}{\partial z} = 2\varepsilon s A(\varepsilon, s) z^2 + 3\varepsilon z^3 \bar{B}(z, \varepsilon, s);$$

$$\varepsilon \frac{\partial G}{\partial \varepsilon} = \varepsilon s \hat{A}(\varepsilon, s) z^2 + \varepsilon z^3 \hat{B}(z, \varepsilon, s);$$

$$s \frac{\partial G}{\partial s} = \varepsilon s \tilde{A}(\varepsilon, s) z^2 + \varepsilon z^3 \tilde{B}(z, \varepsilon, s)$$

with smooth functions  $\tilde{A} \neq 0$ ,  $\hat{A} \neq 0$ ,  $\bar{B} \neq 0$  and  $\tilde{B}$ ,  $\hat{B}$ .

To prove (iii) observe that the equations  $\frac{\partial G}{\partial z} = 0$ ,  $\frac{\partial^2 G}{\partial z^2} = 0$  do not involve  $h$ . The infinitesimal  $r$ -versality condition holds for the family  $\frac{1}{z} \frac{\partial G}{\partial z} \approx s + \varepsilon z + z^2 + \dots$ . Hence there exists a diffeomorphism of  $(z, \varepsilon, s)$ -space  $\theta : (z, \varepsilon, s) \mapsto (Z(z, \varepsilon, s), \Upsilon(\varepsilon, s), S(s))$  which reduces the family  $\frac{1}{z} \frac{\partial G}{\partial z}$  to the form  $S + \Upsilon Z + Z^2$  and which is equivariant under the symmetry  $Z(-z, -\varepsilon, s) = -Z(z, \varepsilon, s)$ ,  $\Upsilon(-z, -\varepsilon, s) = -\Upsilon(z, \varepsilon, s)$ ,  $S(-z, -\varepsilon, s) = -S(z, \varepsilon, s)$ . In the new variables the caustic is determined by  $\Upsilon = -2Z$ ,  $s = Z^2$ .

The equation  $G = 0$  implies now that at the points of the caustic  $\tilde{h} = Z^4 \alpha(Z^2)$  for certain smooth function  $\alpha$  such that  $\alpha(0) \neq 0$ . Hence an appropriate transformation of the form  $\tilde{h} \mapsto \tilde{h} + \psi(s)$ , where the 2-jet at the origin of  $\psi$  vanishes reduces the caustic to the required form.

## 5.2 Space case

Let a germ of a generic surface  $M$  be the graph of a function  $h = f(s, t)$  where  $h, s, t \in \mathbf{R}$ . Let

$$f = f_2 + f_3 + \dots, \quad f_k = \sum_{i+j=k} a_{i,j} s^i t^j$$

be the Taylor decomposition of  $f$  into homogeneous forms.

On a generic surface  $M$  parabolic points form a smooth curve. At any of them the quadratic form  $f_2$  has rank 1. At a *generic* parabolic point on the parabolic curve the dual surface has an  $A_2$  singularity (cuspidal edge). By an appropriate affine transformation (of  $s, t$  plane) the 3-jet of  $f$  at such a point can be reduced to the form:

$$A_2 : f_3(s, t) = s^2 + t^3 + a_{2,1} t s^2 + a_{3,0} s^3 + \dots$$

After this normalisation of the 3-jet at some isolated points the 4-th order form  $f_4$  can vanish on the line  $s = 0$ . These special points will be called  $A_2^*$  points. The notation  $A_2$  remains for generic points with non-vanishing  $f_4|_{s=0}$ .

In these cases  $(A_2, A_2^*)$  the organizing centre of the affine generating family takes the form:

$$G_0 = \lambda \mu [x^2 + (\mu - \lambda)(y^3 + a_{2,1} x^2 y + a_{3,0} x^3) + (\mu^2 - \lambda \mu + \lambda^2) f_4 + \dots].$$

**Theorem 5.3 I.** *In the cases  $A_2, A_2^*$  if  $\lambda_0 \neq 0, \frac{1}{2}, 1$  the affine generating family is  $v$ -equivalent to the normal form*

$$G = -\tilde{h} + (1 + \varepsilon)(s y^2 + y^3)$$

*if  $\lambda_0 = 0, 1$  the affine generating family is  $v$ -equivalent to the normal form*

$$G = -\tilde{h} + \varepsilon(s y^2 + y^3)$$

*These families have only redundant components of the caustic and the discriminant - the surface itself  $\tilde{h} = 0$ .*

**II.** In the case  $A_2$  at  $\lambda_0 = \frac{1}{2}$  the germ of the generating family is weakly equivalent to the normal form :  $G = -\tilde{h} + sy^2 + \varepsilon y^3 + y^4$ . The caustic is a cylindrical smooth surface with a boundary, which coincides with the parabolic line. The caustic is tangent to  $M$  along the boundary. The criminant is only the redundant component  $\tilde{h} = 0$ .

**III.** In the case  $A_2^*$  at  $\lambda_0 = \frac{1}{2}$  the generating family is weak equivalent to the normal form

$$G = -\tilde{h} + (1 - \varepsilon^2) (sy^2 + \varepsilon y^3 + ty^4 + y^6),$$

Hence the caustic is diffeomorphic to the image of a half plane under the simple mapping  $\hat{A}_4$  (from the classification by D.Mond of mappings from  $\mathbf{R}^2$  to  $\mathbf{R}^3$ ):

$$(t, \tau) \mapsto (t, \tau^2, \tau^3 + t^5 \tau).$$

The criminant is  $\tilde{h} = 0$ .

**Proof.** The organising centre and terms linear in parameters  $\tilde{h}, s, t$  from the affine generating family are given by the formula:

$$G_1 = -h + \lambda\mu \left[ f_2(x, y) + \sum_{i>2} f_i(x, y)(\mu^{i-1} + (-1)^i \lambda^{i-1}) \right. \\ \left. + \sum_{i>2} \left( \frac{\partial f_i(x, y)}{\partial x} s + \frac{\partial f_i(x, y)}{\partial y} t \right) (\mu^{i-2} + (-1)^{i-1} \lambda^{i-2}) \right].$$

When  $\lambda_0 \mu_0 \neq 0$  (also when  $\lambda_0 = 0, 1$  but dealing with non-redundant components) the variable  $x$  can be eliminated by stabilization:

$$\frac{1}{\lambda\mu} \frac{\partial G_1}{\partial x} = 2x + 2a_{2,1}sy + 6a_{3,0}xs + 2a_{2,1}tx \\ + (\mu - \lambda) [2a_{2,1}xy + 3a_{3,0}x^2 + \dots] \\ + (\mu^2 + \lambda^2 - \mu\lambda) \left[ \sum_{i+j=3} a_{i,j}ix^{i-1}y^j + \dots \right] + \dots$$

Solving the latter for  $x$  and substituting the result into the expression of the family  $G$  gives a family in  $y, q, \lambda$  with only the following low degree terms

$$\hat{G} = -\tilde{h} + \mu\lambda [(\mu - \lambda)y^3 + (\mu^2 + \lambda^2 - \mu\lambda)a_{0,4}y^4 + 3ty^2 + \dots].$$

Clearly, the reduced family remains symmetric under the involution  $(s, t, \mu, \lambda, y) \mapsto (s, t, \lambda, \mu, -y)$ .

If  $a_{0,4} \neq 0$  or ( $a_{0,4} = 0$  but  $\lambda_0 \neq \mu_0$ ) the family has the form already considered in Proposition 5.2. Hence the result follows.

In the remaining case  $\mu = \frac{1}{2} - \varepsilon, \lambda = \frac{1}{2} + \varepsilon$  and  $a_{0,4} = 0$  the family (up to an appropriate right equivalence) is the following

$$\hat{G} = -\tilde{h} + \lambda\mu (ty^2 + \varepsilon A(\varepsilon^2, s, t)y^3 + sB(\varepsilon^2, s, t)y^4 + \varepsilon C(\varepsilon^2, s, t)y^5 + D(\varepsilon^2, s, t)y^6 + \dots).$$

Here ... stands for terms divisible by  $y^7$  and the functions  $A, B, C, D$  do not vanish at the origin generically. For example,  $D = a_{0,6} + \frac{1}{4}(\mu^2 + \lambda^2 - \lambda\mu)^2 a_{1,3}^2$ .

The following proposition completes the proof of the theorem.

**Lemma 5.4** *The caustic of  $\hat{G}$  is diffeomorphic to that of the family*

$$G_* = -\tilde{h} + \lambda\mu (ty^2 + \varepsilon y^3 + sy^4 + y^6)$$

**Proof.** Due to the invariance under the involution the family  $\hat{G}$  is a function of the basic invariants  $y^2, y\varepsilon, \varepsilon^2, s, t, \tilde{h}$ . Assign weights 1, 2, 3, 4, 6 to the variables  $y, s, \varepsilon, t, \tilde{h}$  respectively. The lowest weighted homogeneous part of  $\hat{G}$  up to rescaling of the variables equals

$$G_6 = -\tilde{h} + ty^2 + \varepsilon y^3 + sy^4 + y^6$$

and has weight 6. The explicit formulas determining the caustic provide

$$\varepsilon = -4sy - 10y^3, \quad t = -\frac{3}{2}\varepsilon y - 2sy^2 - 3y^4.$$

Together with the equation  $G_6 = 0$  they determine the caustic diffeomorphic to the image of the Legendre mapping

$$(t, \tau) \mapsto (t, \tau^2, \tau^3)$$

(semicubic cylinder) where  $\tau = y^2 - \frac{2}{9}s$ . All deformations  $\tilde{G}$  of  $G_6$  involving only terms of weights  $< 12$  do not contain invariant factors  $\varepsilon^2$  and thus are affine functions in  $\varepsilon$ . Solving the equation  $\tilde{G} = 0$  for  $\varepsilon$  we obtain antiinvariant rational function, whose caustic again is semicubic cylinder. However generic deformations involving terms of weight 12 (e.g., factors  $\varepsilon^2$  as in  $G_*$ ) fail to produce a Legendre parametrization of the caustic. The lowest weight terms in the parametrizations of the caustic give a simple  $\hat{A}_4$  D.Mond singularity

$$(t, \tau) \mapsto (t, \tau^2, \tau^3 + \tau t^5)$$

Since the 12-weighted jet of this mapping is sufficient the claim of the lemma follows.

On the parabolic line there are also special isolated points of  $A_3$  type. The dual surface has swallow tail ( $A_3$  singularity) at the corresponding dual point. The 4-jet of a generic surface at  $A_3$  point by an appropriate affine transformation can be reduced to the form:

$$A_3 : \quad f = s^2 + st^2 + a_{3,0}s^3 + a_{0,4}t^4 + s\varphi(s, t)$$

with  $a_{0,4} \neq 0$  and some cubic polynomial  $\varphi(s, t)$ . Generically at these points the term  $f_5$  restricted to the line  $s = 0$  does not vanish.

Explicit calculations of the affine generating family in this case results in the following:

**Theorem 5.5** *In the case  $A_3$  the germ of the preimage  $\Sigma_0$  of the caustic (germ of singular points of the extended wave front) with a generic value of  $\lambda$  different from  $\lambda_0 \neq 0, 1, \frac{1}{2}$  is diffeomorphic (in the space  $\tilde{h}, s, t, \varepsilon$ ) to the caustic of the family*

$$G = -\tilde{h} + sy^2 + ty^3 + y^4,$$



which is cylindrical along the  $\varepsilon$  axis. For  $\lambda_0 = \frac{1}{2}$  the caustic is diffeomorphic to that of the family

$$G = -\tilde{h} + sy^2 + \varepsilon ty^3 + y^4.$$

There also exist two values of  $\lambda_0$  symmetric with respect to  $\lambda_0 = \frac{1}{2}$  the germ of  $\Sigma_0$  at which is diffeomorphic to the germ at the origin of the caustic of the family

$$G = -\tilde{h} + sy^2 + ty^3 + \varepsilon y^4 + y^5.$$

**Remark.** The projection of  $S_0$  to the  $(\tilde{h}, s, t)$ -space maps the total coordinate axis  $\varepsilon$  to the origin. Hence the germ at the origin of the caustic  $\Sigma$  is the image of the germ at the total line of  $\Sigma_0$  and therefore generically is not simple.

## 6 Families of lines

In this final section we consider the question: Given a smooth 2-parameter family of lines in  $\mathbf{R}^3$ , can we find a surface  $S$  such that the lines join pairs of points of  $S$  where the tangent planes are parallel? This question amounts to asking whether the families of lines considered in this paper, namely those arising as chords joining pairs of parallel-tangent-plane points, are special, at any rate locally. In fact, in most cases we reduce the *local* construction of a surface  $S$  to the solution of a second order linear PDE, showing that generically the surface  $S$  can be constructed.

Suppose we are given a family of lines  $L(u, v)$  in  $\mathbf{R}^3$ , where the line passes through  $(u, v, 0)$  and has direction  $(a(u, v), b(u, v), 1)$ . A general point of one of these lines is  $(x, y, z) = (u + az, v + bz, z)$ . We shall work locally, that is assume  $(u, v)$  is any point in some neighbourhood of a base-point  $(u_0, v_0)$ .<sup>2</sup>

Consider now a distribution of 2-planes in  $\mathbf{R}^3$  given by  $\alpha = 0$ , where  $\alpha$  is the 1-form

$$\alpha = A(x, y, z)dx + B(x, y, z)dy + C(x, y, z)dz.$$

We shall seek to impose the condition that, at all points of  $L(u, v)$  for a fixed  $u, v$ , the 1-form  $\alpha$  is the same, that is, we want  $A, B$  and  $C$  to be functions of  $u, v$ . The object is to find such an *integrable* distribution  $\alpha$  and then to choose two integral surfaces  $S_1, S_2$ . It will follow that tangent planes to the  $S_i$  at points of intersection with the lines  $L$  will be parallel. This will provide the pair of local surfaces which we seek.

The general integrability condition is  $\alpha \wedge d\alpha = 0$ , but in fact we shall be able to find a solution with  $d\alpha = 0$ . We have

$$\begin{aligned} \alpha &= A(u, v)dx + B(u, v)dy + C(u, v)dz \\ &= A(u, v)(du + (da)z + adz) + B(u, v)(dv + (db)z + bdz) + Cdz \\ &= Adu + Bdv + z(Ada + Bdb) + (Aa + Bb + C)dz \\ &= \alpha_1 + z\alpha_2 + (Aa + Bb + C)dz, \quad \text{say,} \end{aligned}$$

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<sup>2</sup>Of course even this is problematic in some circumstances we have considered in this paper. For example if we consider the family of lines arising from a parabolic point of a single surface, as in §5, then the lines cannot be parametrized in this way.

where  $\alpha_1, \alpha_2$  are 1-forms in  $u$  and  $v$ . Hence

$$d\alpha = d\alpha_1 + zd\alpha_2 + (-\alpha_2 + d(Aa + Bb + C)) \wedge dz.$$

Assume now that  $d\alpha = 0$  (locally) as a 2-form in  $u, v, z$ . The only coefficients of  $du \wedge dz$  and  $dv \wedge dz$  come from the last term in  $d\alpha$ , so that  $-\alpha_2 + d(Aa + Bb + C) = 0$ . Then since  $z$  only occurs in  $zd\alpha_2$  we must also have  $d\alpha_2 = 0$  and finally  $d\alpha_1 = 0$ . Note that, conversely, if  $d\alpha_1 = d\alpha_2 = 0$ , then we can make  $d\alpha = 0$  by choosing  $C$  satisfying  $dC = \alpha_2 - d(Aa + Bb)$ .

Now  $d\alpha_1 = d(Adu + Bdv) = 0$  implies (locally) that  $A = \partial f / \partial u$ ,  $B = \partial f / \partial v$  for some function  $f$ . Consider  $d\alpha_2 = d(Ada + Bdb) = 0$ , which implies that  $\alpha_2 = dg$  for some function  $g(u, v)$ . That is,

$$\begin{aligned} g_u &= Aa_u + Bb_u = f_u a_u + f_v b_u \\ g_v &= Aa_v + Bb_v = f_u a_v + f_v b_v, \end{aligned}$$

which gives  $\frac{\partial}{\partial v}(f_u a_u + f_v b_u) = \frac{\partial}{\partial u}(f_u a_v + f_v b_v)$ , that is

$$f_{uu}a_v - f_{uv}(a_u - b_v) - f_{vv}b_u = 0 \quad (1)$$

Solutions  $f$  to this PDE determine functions  $A = \frac{\partial f}{\partial u}$ ,  $B = \frac{\partial f}{\partial v}$  and  $C$  as above, solving our problem of finding a suitable distribution of 2-planes.

According to the classic Cauchy-Kovalevskaya theorem equation (1) has a (local) solution provided that the coefficients of the three partial derivatives do not all vanish. We conclude with a geometrical interpretation of this condition, and some other remarks.

The equation (1) is hyperbolic if and only if  $(a_u - b_v)^2 + 4a_v b_u > 0$ . Note that the matrix

$$\Lambda = \begin{pmatrix} a_u & a_v \\ b_u & b_v \end{pmatrix}$$

has eigenvalues  $\lambda$  which are roots of the equation

$$\lambda^2 - \lambda(a_u + b_v) + (a_u b_v - a_v b_u) = 0.$$

Hence (1) is hyperbolic if and only if the eigenvalues of the matrix  $\Lambda$  are real and distinct.

Consider two nearby points  $(u, v, 0)$  and  $(u + \delta u, v + \delta v, 0)$  in the parameter plane, and the corresponding lines of the family, in directions  $(a, b, 1)$  and  $(a + \delta a, b + \delta b, 1)$ . The condition for these two lines to be *coplanar* is that the vectors along them, and  $(\delta u, \delta v, 0)$ , should be linearly dependent. This amounts to  $\delta u \delta b - \delta a \delta v = 0$ , or

$$a_v(\delta v)^2 + (a_u - b_v)\delta u \delta v - b_u(\delta u)^2 = 0.$$

The directions in the parameter plane which give ‘infinitesimally coplanar’ lines of the family are therefore the directions  $(\delta u, \delta v)$  satisfying this equation. These are easily checked to be the eigenvectors of  $\Lambda$ . When they are real, these directions can be called ‘principal directions’ of the family of lines at a particular point in the family.

For example when the family of lines forms the normals to a given surface  $M$ , the principal directions in the above sense always exist and are the principal directions on the parallel surface to  $M$  through a chosen point. At umbilic points of  $M$ , where the

eigenvalues are equal, all directions are principal, since  $\Lambda$  is symmetric in this case. The eigenvalues remain real over the whole surface  $M$ .

In the general case  $\Lambda$  will not be symmetric, and it is possible for the eigenvalues of  $\Lambda$  to coincide without all directions being principal. In fact, the condition for equal eigenvalues is

$$(a_u - b_v)^2 + 4a_v b_u = 0, \quad (2)$$

while the condition for all directions to be principal is

$$a_u = b_v, \quad a_v = b_u = 0, \quad (3)$$

which is much more special. This is the condition for all of the coefficients in (1) to vanish.

When the equation (1) is elliptic there are no real eigenvectors or eigenvalues, though the construction of the surfaces proceeds as before. In the parabolic case, the eigenvalues coincide. Compare §3, ‘Special chords’.

Continuing in the general case to consider principal directions, we can calculate the point  $(u + za, v + zb, z)$  at which lines of the family which are infinitesimally coplanar actually meet. A short calculation shows that  $z = -1/\lambda$  where  $\lambda$  is as above an eigenvalue of  $\Lambda$ . Let us remain within a region where the eigenvalues are real and distinct (hence (1) is a hyperbolic PDE) and non-zero (hence  $\Lambda$  is nonsingular). Then  $\lambda$  and  $z$  are smooth nonzero functions of  $u$  and  $v$ , and the point  $(u - a/\lambda, v - b/\lambda, -1/\lambda)$  traces out a (possibly singular) surface  $C$ . The tangent plane to this surface at the given smooth point contains the direction  $(a, b, 1)$ , that is, the surface is tangent to the corresponding line of the family. The surface  $C$  can be regarded as *the CSS of any two surfaces which are constructed, as above, to have parallel tangent planes at their points of intersection with the lines of the family*. We can refer to  $C$  also as the *envelope* of the family of lines.

What this amounts to saying is that the CSS can, at least in favourable cases, be regarded as the critical locus (image of the critical set) of a mapping of the form

$$F : \mathbf{R}^3 \rightarrow \mathbf{R}^3, \quad F(u, v, z) = (u + a(u, v)z, v + b(u, v)z, z), \quad z \neq 0,$$

the values of  $z$  giving points of the critical set being  $z = -1/\lambda$  where  $\lambda$  is an eigenvalue of  $\Lambda$ . There are a number of results on mappings  $\mathbf{R}^3 \rightarrow \mathbf{R}^3$ , for example [3, 11], and these can be used to analyse some of the cases treated in this paper.

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