Evolving Evolutoids

P. J. Giblin and J. P. Warder

Abstract

The envelope of straight lines normal to a plane curve C is its evolute; the envelope of lines tangent to C is the original curve, together with the entire tangent line at each inflexion of C. We introduce some standard techniques of singularity theory and use them to explain how the first of these envelopes turns into the second, as the (constant) angle between the set of lines forming the envelope and the set of tangents to C changes from $\frac{1}{2}\pi$ to 0. In particular we explain how cusps disappear and what happens at inflexions, where the evolute goes to infinity. We also study the family of "wavefronts" or "parallels" associated with these envelopes.
MSC2010 Classification: 53A04, 53A05, 57R45, 58K05.

1 Introduction

Let σ be a plane curve, which we shall often assume is closed, and always assume is free from singularities (such as cusps) and self-intersections. The family of tangent lines to σ and the family of normal lines to σ each have an *envelope*. We make the definitions precise in §2 but the general idea is that the envelope of a family of lines in the plane is a curve tangent to all of them. Unsurprisingly, the envelope of the tangent lines to σ contains—at least—σ itself. The envelope of normal lines is called the *evolute* of σ, and this has cusps corresponding to the curvature extrema of σ. The evolute also "goes to infinity" corresponding to inflexions—zeros of curvature—of σ.

It is natural to ask what lies in between the envelope of tangents and the envelope of normals. Let us fix an angle α and consider straight lines L obtained by rotating each tangent to σ at a point $p \in \sigma$ about p counter-clockwise through α, denoting the envelope of the lines L by τ_α. See Figure 1. Thus τ_0 is the envelope of tangent lines and $\tau_{\pi/2}$ is the envelope of normal lines. For other values of α, τ_α is a so-called *evolutoid* of σ. The geometry of these envelopes has been studied since Réaumur in 1709. For a modern reference see [10]; this, like most studies of evolutoids, restricts attention to the case when σ is an *oval*, that is a closed curve without inflexions and hence strictly convex, such as an ellipse. We relax this condition here and consider curves with inflexions. In [1] the authors study the opposite situation of generalized *involutes* of plane curves: σ is an involute of τ if τ is the evolute of σ.

What happens when α moves from 0 to $\frac{1}{2}\pi$, so that τ_α evolves from the envelope of tangent lines to a many-cusped envolute? The object of this article is to explain how some basic techniques of singularity theory enable us to say *exactly* what happens, in the sense of showing that in precisely defined conditions the cusps appear and disappear in a fixed manner and, crucially, explaining the contribution of inflexions. These are important since—as is well known, though we prove it in Proposition 2.3 below—the envelope of tangent lines to a plane curve σ contains, besides σ itself, also the entire tangent line at each inflexion of σ. Figure 2 shows the envelope of normals ($\alpha = \frac{1}{2}\pi$) and the envelope of tangents ($\alpha = 0$) to a closed curve with two inflexions. Later we investigate what happens for α close to 0; see §5, and Figure 9 for an illustration.

The key ingredients of singularity theory which we call on are the theory of *unfoldings and discriminants* and the theory of *functions on discriminants*. We cannot (alas) present all the details of these theories here but we hope that enough is said to show how powerful abstract techniques yield highly concrete geometrical results. (For details of most of the techniques, and other geometrical applications, see [6].)

The article is organized as follows. In §2 we firm up the definitions and give an explicit formula for the envelope τ_α, recalling some basic facts about plane parametrized curves. In §3 we study the cusps of τ_α, introducing the first ideas from singularity theory and the classification of functions. In §4 we

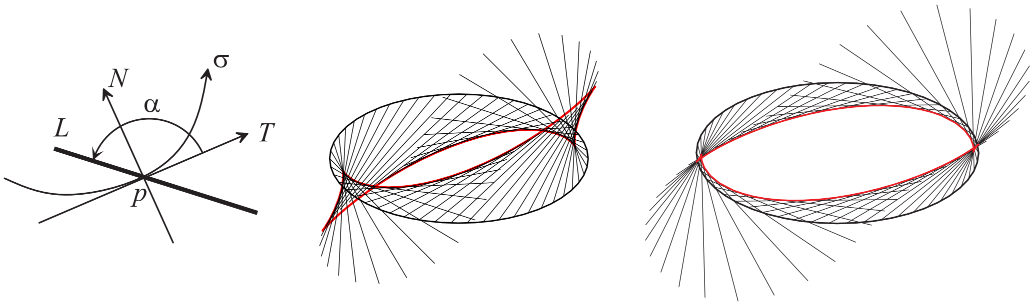

Figure 1: Left: we consider the envelope τ_α of lines L such that the counter-clockwise angle α between the tangent T to σ and L is constant. Center and right: an ellipse $\sigma(t) = (2\cos t, \sin t)$ and the envelope with $\alpha = \frac{1}{4}\pi = 0.785...$, clearly showing four cusps, and $\alpha = 0.5$, where the cusps have almost disappeared and the envelope is more closely approximating the ellipse itself. For clarity, the lines L are drawn only in the "forward" direction at each point. See also Example 3.2.

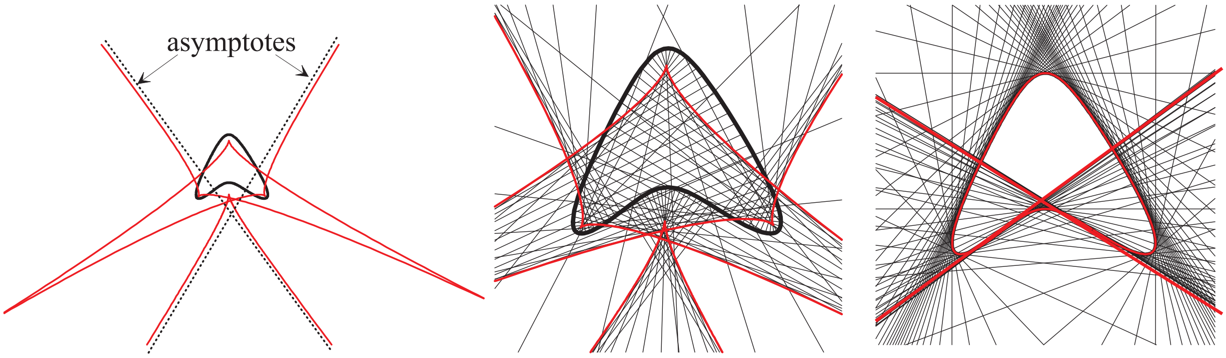

Figure 2: A curve with two inflexions—the curve appears as the dark line on the left. Left and middle: the envelope of normals, which has six cusps and two asymptotes, shown in full on the left. The center diagram shows the normals themselves; for clarity they are drawn only in the direction in which they "focus" on the envelope. Right: the envelope of tangents, this time drawn in both directions. Here, the envelope includes the original curve and the whole of the tangent lines at the inflexions.

set the study of envelopes in the general context of *discriminants* and *functions on discriminants*. In §5 we state the necessary results from singularity theory and apply them to the example of evolutoids, giving the main results as Corollary 5.7 and Theorem 5.9. In a nutshell the results say that as α varies the local appearance of the evolutoids changes in one of only two ways, a "swallowtail transition" as in Figures 5 and 6, or a "beaks" transition as in Figures 4 and 9.

In §6 we study the wavefronts associated to a given value of α; the cusps on these wavefronts trace out the envelope τ_α but in general the wavefronts are not closed curves. Finally in §7 and §8 we draw things together and offer some more details of proofs.

2 Envelopes

We shall mildly abuse notation by using the same symbol σ to denote a plane curve and a parametrization. For us, the most interesting examples occur when σ is a closed curve, such as an ellipse $\sigma(t) = (a\cos t, b\sin t)$, $a > 0$, $b > 0$, $0 \le t \le 2\pi$, and we shall give such examples below. Since the line L is not itself oriented the angle α in Figure 1 can be considered modulo π. Parametrizing σ by t automatically gives it an orientation (increasing t) but if we reverse this orientation then α gives *the same* line L.

Throughout this article we assume that $\sigma(t) = (X(t), Y(t))$ is a regular parametrized curve, that is X and Y are smooth functions of t and (using $'$ for d/dt), $X'(t) = Y'(t) = 0$ never happens, so that the speed $||\sigma'(t)||$ is always nonzero. We use the standard notation T, or $T(t)$, for the unit tangent $\sigma'(t)/||\sigma'(t)||$ and N, or $N(t)$, for the unit normal, obtained by rotating T counter-clockwise through $\frac{1}{2}\pi$. The *curvature* $\kappa(t)$ is given by $T'(t) = \kappa(t)||\sigma'(t)||N(t)$, and in terms of X and Y has the rather unattractive formula (omitting the variable t) $\kappa = (X'Y'' - X''Y')/(X'^2 + Y'^2)^{3/2}$.

Finally we make the following

Assumption 2.1 *The curve σ is* generic *in the precise sense that all inflexions (zeros of curvature) and vertices (extrema of curvature) are* ordinary*. This means that, for σ, inflexions are simple zeros and vertices are simple extrema: $\kappa = 0, \kappa' \neq 0$ at inflexions, and $\kappa \neq 0, \kappa' = 0, \kappa'' \neq 0$ at vertices*[1]*.*

We are interested in the line L obtained by rotating the tangent a fixed angle α; the direction of L is therefore $T(t)\cos\alpha + N(t)\sin\alpha$. Hence $T(t)\sin\alpha - N(t)\cos\alpha$ is perpendicular to L and a vector equation of the line L is $F(\mathbf{x}, t) = 0$ where

$$F(\mathbf{x}, t) = (\mathbf{x} - \sigma(t)) \cdot (T(t)\sin\alpha - N(t)\cos\alpha). \tag{1}$$

Here, $\cdot$ is the usual scalar product of vectors and $\mathbf{x} = (x, y) \in \mathbb{R}^2$. Thus we regard α as *fixed*; then $F(\mathbf{x}, t) = 0$ represents a *family of lines*: each t gives a line and as t varies the line moves in the (x, y)-plane.

There is a simple method for finding the envelope τ_α of these lines. Visually, the envelope is a curve tangent to all the lines, or the "limit of intersection points of consecutive lines"; see Figure 3. In fact, both these latter definitions, when made precise, give curves which are always *subsets* of τ_α, as defined below. This is shown in [6, pp. 107–109].

Figure 3: Envelopes of families of lines are intuitively formed by (left) intersections of "consecutive" lines or (right) a curve tangent to all the lines. Generally these definitions coincide with Definition 2.2 used here.

Definition 2.2 The envelope τ_α of the family of lines given by (1), for a fixed α, is the set of points $\mathbf{x} = (x, y)$ in the plane for which there exists t with

$$F(\mathbf{x}, t) = \frac{\partial F}{\partial t}(\mathbf{x}, t) = 0.$$

To solve these equations we use the standard *Serret-Frenet formulae*:

$$T'(t) = \kappa(t)N(t)||\sigma'(t)||, \quad N'(t) = -\kappa(t)T(t)||\sigma'(t)||, \tag{2}$$

which relate the derivatives of T and N to T and N themselves, the curvature κ and the speed $||\sigma'||$ of the curve σ. (See any book on differential geometry, or alternatively [6, Ch.2].)

Using the fact that α is *constant* this gives

$$\frac{\partial F}{\partial t} = -\sin\alpha||\sigma'(t)|| + (\mathbf{x} - \sigma(t)) \cdot (\kappa(t)\sin\alpha\ N(t)||\sigma'(t)|| + \kappa(t)\cos\alpha\ T(t)||\sigma'(t))||)\,.$$

Now any vector is a linear combination of the form $\lambda T(t) + \mu N(t)$ and applying this to the vector $\mathbf{x} - \sigma(t)$ and substituting in $F = \frac{\partial F}{\partial t} = 0$ we obtain two equations for λ and μ:

$$\begin{aligned} \lambda\sin\alpha - \mu\cos\alpha &= 0, \\ \lambda\kappa(t)\cos\alpha + \mu\kappa(t)\sin\alpha &= \sin\alpha. \end{aligned} \tag{3}$$

[1] It can be shown that in a precise sense "almost all" closed curves are generic in the sense used here. See [6, Ch.9].

Solving these gives a parametrization of the envelope, for a fixed α:

$$\mathbf{x}(t) = \sigma(t) + \frac{\sin\alpha\cos\alpha}{\kappa(t)}T(t) + \frac{\sin^2\alpha}{\kappa(t)}N(t), \tag{4}$$

provided $\kappa(t) \neq 0$. Thus if $\alpha = \pm\frac{1}{2}\pi$ the lines $F = 0$ are the normals to σ and the envelope is the usual evolute, namely the set of points $\sigma(t) + \frac{1}{\kappa(t)}N(t)$; these points are also called the *centers of curvature* of σ. If $\alpha = 0$ (or π) then the lines $F = 0$ are the tangents to σ and the envelope, away from inflexions where $\kappa(t) = 0$, is the original curve σ.

Note that when $\alpha = 0$ and $\kappa(t) = 0$ then using (3) directly we have $\mu = 0$, λ arbitrary so, as mentioned in the Introduction,

Proposition 2.3 *The envelope of tangent lines consists of the original curve σ together with the whole tangent line at inflexion points.* □

Our aim in what follows is to describe exactly how, as α moves from $\frac{1}{2}\pi$ to 0, the evolute of σ turns into the curve σ itself, and in particular to explain what happens at inflexions, where $\kappa(t) = 0$.

Remarks 2.4 There are numerous attractive properties of the envelope τ_α. Here are two, the first pointed out to us by the referees, and the second proved in [10, Prop.5].

(i) The line joining the center of curvature $\sigma(t)+(1/\kappa(t))N(t)$ of the curve σ at the point with parameter t to the envelope point as in (4) has the direction $\sin\alpha T(t) - \cos\alpha N(t)$, which is perpendicular to the line L. This has the following interpretation, using "angle in a semicircle": the envelope point (4) is the intersection of the line L with the circle tangent to σ at $\sigma(t)$ and passing through the center of curvature. When $\alpha = \frac{1}{2}\pi$ this point *is* the center of curvature. When $\alpha = 0$ the result holds—-the envelope point is $\sigma(t)$ itself—except of course at inflexions, where $\kappa(t) = 0$.

(ii) This relates the evolutoid τ_α with the *center symmetry set* (CSS) or *Minkowski set* of σ (it is given yet a third name in [10], namely the *midenvelope* of σ). This is the envelope of all lines joining pairs of distinct points of σ at which *the tangent lines are parallel*. The CSS is the subject of several investigations, both in the plane, and through generalizations (using tangent planes rather than tangent lines) in higher dimensions, employing more advanced techniques of singularity theory. See [8, §5] for an introduction and [9] for a more technical discussion. Now consider the points P, Q of the evolutoid τ_α corresponding with two distinct points p, q of σ at which the tangent lines are parallel. Let R be the point where the line PQ meets the line pq. Then R is the CSS point on pq. It is not clear to us whether there are sensible generalizations of evolutoids to higher dimensions, and hence a generalization of this result. Note that the CSS, unlike the evolutoid, is invariant under linear transformations of the plane, since such a transformation preserves parallel lines.

3 Cusps on the envelope

Consider the envelope curve given by (4): when will this curve not be regular? The condition is that the speed is zero, that is to say the derivative of $\mathbf{x}$ with respect to t is the zero vector. Again using the Serret-Frenet formulae (2) this derivative is, assuming $\kappa(t) \neq 0$ and omitting now the variable name t

$$\begin{aligned}\mathbf{x}' &= \left(||\sigma'|| + \frac{\kappa'}{\kappa^2}\sin\alpha\cos\alpha - ||\sigma'||\sin^2\alpha\right)T + \left(||\sigma'||\sin\alpha\cos\alpha - \frac{\kappa'}{\kappa^2}\sin^2\alpha\right)N \\ &= \left(||\sigma'||\cos\alpha - \frac{\kappa'}{\kappa^2}\sin\alpha\right)(\cos\alpha\ T + \sin\alpha\ N).\end{aligned}$$

This is zero if and only if $\kappa^2||\sigma'||\cos\alpha - \kappa'\sin\alpha = 0$. Writing s for the arclength parameter on σ, $\frac{d\kappa}{dt} = \frac{d\kappa}{ds}\frac{ds}{dt} = \frac{d\kappa}{ds}||\sigma'(t)||$; hence:

Proposition 3.1 *The envelope* (4) *(still assuming $\kappa \neq 0$) is not regular if and only if $\kappa^2\cos\alpha - \kappa_s\sin\alpha = 0$, where κ_s is the derivative of curvature with respect to arclength s on σ. This condition can also be written in terms of the radius of curvature $\rho = 1/\kappa$: $\rho_s\sin\alpha + \cos\alpha = 0$.* □

Note that κ^2 and κ_s are independent of the direction of orientation of σ, and that $\alpha > 0$, say, always means a counter-clockwise rotation of the oriented tangent to σ and so gives the same line whichever orientation of σ is used. That is why the equation in the Proposition is unaltered when the orientation of σ is reversed.

For $\alpha = \pm\frac{1}{2}\pi$, that is for the envelope of normals (the evolute of σ), the Proposition gives the familiar condition $\kappa' = 0$ (this can be the derivative with respect to any regular parameter), which says that σ has an extremum of curvature, that is a *vertex*. For $\alpha = 0$, the envelope of tangents, it says that, away from inflexions, the envelope has no cusps—but the envelope is σ itself so this is nothing new.

Example 3.2 In the special case where $\sigma(t) = (r\cos t, r\sin t)$, a circle of radius $r > 0$, then κ is $\frac{1}{r}$, a non-zero constant. The Proposition shows that there are no cusps at all, the only exception being $\cos\alpha = 0$, when all the lines are radii and pass through the center of the circle so the evelope degenerates to a point. In fact for other values of α we have $||\mathbf{x}(t)||^2 = r^2\cos^2\alpha$, so that the envelope is a concentric circle of radius $r|\cos\alpha|$. The reader may enjoy showing from the Proposition that, for an ellipse $\sigma(t) = (a\cos t, b\sin t),\ a > b > 0$, the value of α at which the four cusps appear (are "born") on τ_α, starting from $\alpha = 0$, is $\alpha_0 = \arctan\big(2ab/3(a^2 - b^2)\big)$. For $a = 2, b = 1$, as in Figure 1, this comes to about $\alpha = 0.418$ radians, or $24°$.

When a plane curve such as this envelope curve is not regular then in general we expect it to have an "ordinary cusp", that is a singular point which is "like" the cusp at the origin on the curve (t^2, t^3). More formally a local diffeomorphism of the plane should take the given curve to this standard cusp[2]. The condition for an ordinary cusp is in fact that the second and third derivatives of σ (with respect to any regular parameter), evaluated at the cusp point, should be *independent vectors*[3]. Note that for (t^2, t^3) at $t = 0$ these vectors are $(2, 0)$ and $(0, 6)$, hence certainly independent. Some modest calculation (see §8.1) reveals the following.

Proposition 3.3 *Assume as before that κ is nonzero, and also that $\sin\alpha \neq 0$, that is the lines forming the envelope are not the tangent lines to σ. Then the cusp as in Proposition 3.1 is an ordinary cusp if and only if $2\kappa_s^2 - \kappa\kappa_{ss} \neq 0$, where the derivatives, with respect to arclength s, are evaluated at the cusp point. This can also be written as $\rho_{ss} \neq 0$ where $\rho = 1/\kappa$ is the radius of curvature of σ.* □

Example 3.4 When $\alpha = \frac{1}{2}\pi$ the conditions for an ordinary cusp reduce to $\kappa \neq 0, \kappa' = 0, \kappa'' \neq 0$, and by Assumption 2.1 we know that all points of σ where the first two of these conditions hold also satisfy the third condition. So all cusps of $\tau_{\pi/2}$ (the evolute of σ) are ordinary cusps. But the Proposition does not guarantee that for other values of α the cusps on τ_α are ordinary. Indeed in Example 3.2, where σ is an ellipse, the cusps are not ordinary at the moment of "birth", when $\alpha = \alpha_0$. By suitably choosing a and b we can make α_0 take any value in $0 < \alpha_0 < \frac{1}{2}\pi$.

Our object in this article is not to study a single value of α but to study what happens to τ_α as α varies. For this we put the investigation in a wider context.

4 Discriminants

The function $F(\mathbf{x}, t)$ in (1), for a fixed α, is an example of a *family of functions of one variable t with two parameters $(x, y) = \mathbf{x}$*. We can include α in the parameters as well as x, y:

$$\mathcal{F}(\mathbf{x}, \alpha, t) = (\mathbf{x} - \sigma(t)) \cdot (T(t)\sin\alpha - N(t)\cos\alpha) : \tag{5}$$

[2] A local diffeomorphism here is a smooth map with smooth inverse, from a neighbourhood of the cusp point in the plane to a neighbourhood of the origin, taking the one curve to the other. Similar definitions apply to higher dimensions.

[3] Starting with $(x, y) = (at^2 + bt^3 + \dots,\ ct^2 + dt^3 + \dots)$ this means that $ad - bc \neq 0$. In fact an invertible linear transformation $x = aX + bY,\ y = cX + dY$ transforms the curve into $(X, Y) = (t^2 + \dots,\ t^3 + \dots)$, and a reparametrization easily turns this into $(u^2, u^3 + \dots)$ for a new parameter u. The final step to make a smooth and invertible change of coordinates in the plane turning this into "normal form" (u^2, u^3) takes something more substantial and it is usual to invoke the *Preparation Theorem.* This *reduction to normal form* is the very stuff of singularity theory; see for example [4] or, more technically, [7].

this is a three-parameter family, where we use a script $\mathcal{F}$ to emphasize this.

Any family $G(\mathbf{X},t) = G(x_1,x_2,\ldots,x_k,t)$ with k parameters (we shall always have $k=2$ or 3) has a *discriminant* $\mathcal{D}_G$, as follows, where we use a subscript as in G_t to denote partial differentiation. We will use $\mathbf{X}$ to denote $(x_1,x_2,\ldots,x_k)$, and $\mathbf{X}_0$ to denote a fixed value of $\mathbf{X}$, to avoid confusion with $\mathbf{x}$ which denotes (x,y) in our discussion of envelopes. (For contrasting discussions of discriminants see [6, Ch.6],[11].)

$$\mathcal{D}_G = \{\mathbf{X} = (x_1,x_2,\ldots x_k) : \text{ for some } t,\ G = G_t = 0 \text{ at } (\mathbf{X},t)\}\,. \tag{6}$$

Examples 4.1 (i) Cusp Let $G(\mathbf{X},t) = G(x_1,x_2,t) = t^3+x_1t+x_2$. For a fixed $\mathbf{X}$ this is a (reduced) polynomial of degree 3 in t, and $\mathcal{D}_G$ consists exactly of those polynomials with a repeated root: it is the curve $\mathbf{X} = (-3t^2,\ 2t^3)$ parametrized by t with an ordinary cusp at the origin.
(ii) Cuspidal edge surface Slightly more bizarrely, let $G(\mathbf{X},t) = G(x_1,x_2,x_3,t) = t^3+x_1t+x_2$, where x_3 plays no role on the right-side. The discriminant is called a *(standard) cuspidal edge surface*: it is the product of an ordinary cusp with a line, as in Figure 4, left, and is parametrized by x_3 and t, namely

$$(x_3,t) \mapsto (-3t^2,\ 2t^3,\ x_3) \tag{7}$$

The product of the cusp point itself with this line (the x_3-axis here) is called the *line of cusps*. This surface has an important property.

> *Consider a point* $\mathbf{p}$ *of the line of cusps, and the tangent* $\mathbf{T}$ *to the line of cusps at* $\mathbf{p}$*. Then the tangent planes at points away from the line of cusps but with limit* $\mathbf{p}$ *have a limit which contains* $\mathbf{T}$.

For the surface as in (7) a normal is in direction $(t,1,0)$ at points away from the line of cusps, and this has limit $(0,1,0)$ as $t \to 0$, so the limiting tangent plane is $x_2 = 0$. Of course in general when we encounter a cuspidal edge surface it will not be so "straight up and down" (see Figures 7 and 8)—it will be locally diffeomorphic to the standard surface in a neighbourhood of a point $\mathbf{p}$ as above—but the property stated will still be true locally.
(iii) Swallowtail surface Let $G(\mathbf{X},t) = G(x_1,x_2,x_3,t) = t^4+x_1t^2+x_2t+x_3$. For a fixed $\mathbf{X}$ this is a (reduced) polynomial of degree 4 in t: these polynomials fill a 3-dimensional space with coordinates (x_1,x_2,x_3). The discriminant of G consists exactly of those polynomials which have a repeated root. It is a surface known as a *(standard) swallowtail*[4] *surface* and is illustrated in Figure 5. Solving for x_2 and x_3 the surface is parametrized by x_1 and t:

$$(x_1,t) \mapsto (x_1,\ -4t^3-2x_1t,\ 3t^4+x_1t^2). \tag{8}$$

The origin is then called the *swallowtail point* and there are two lines of cusps through the origin, given by $G = G_t = G_{tt} = 0$ and parametrized by $(-6t^2, 8t^3, -3t^4)$, $t>0$ and $t<0$: these represent polynomials with a *triple root*. There is also a curve of self-intersection, parametrized by $(-2t^2,0,t^4)$ (a half-parabola), representing polynomials with *two double roots*. The origin represents the polynomial t^4 with a *fourfold root*. One important property of this surface is the following.

> *The tangents to the lines of cusps, and the tangent to the self-intersection curve, all have the same limit at the origin (here the limit is the* x_1*-axis). Furthermore the tangent planes to the swallowtail surface at points away from these curves all have the same limit (here the* x_1x_2*-plane), which contains the above limit of tangent lines.*

(A surface normal at such a point, with parameters x_1,t as in (8) is $(t^2,t,1)$ which has limit $(0,0,1)$ as $t \to 0$.) Any surface locally diffeomorphic to the standard swallowtail, in a neighbourhood of the swallowtail point, will also satisfy the stated property locally.
(iv) Of course there is the example at the heart of this article, given by (1), where the discriminant $\mathcal{D}_F$ is the envelope τ_α for a fixed α. Using (5) instead, x_1,x_2,x_3 become x,y,α respectively and the discriminant $\mathcal{D}_{\mathcal{F}}$, in (x,y,α)-space is

$$\mathcal{D}_{\mathcal{F}} = \{(x,y,\alpha) : \text{ there exists } t \text{ such that } \mathcal{F}(x,y,\alpha,t) = \mathcal{F}_t(x,y,\alpha,t) = 0\}\,. \tag{9}$$

[4]Originally *queue d'aronde* in the French of the great mathematician René Thom (1923–2002), one of the founders of singularity theory.

This is the union of all the envelopes, for all α: they are spread out in the α-direction. (Figures 6 and 7 below illustrate this discriminant.)

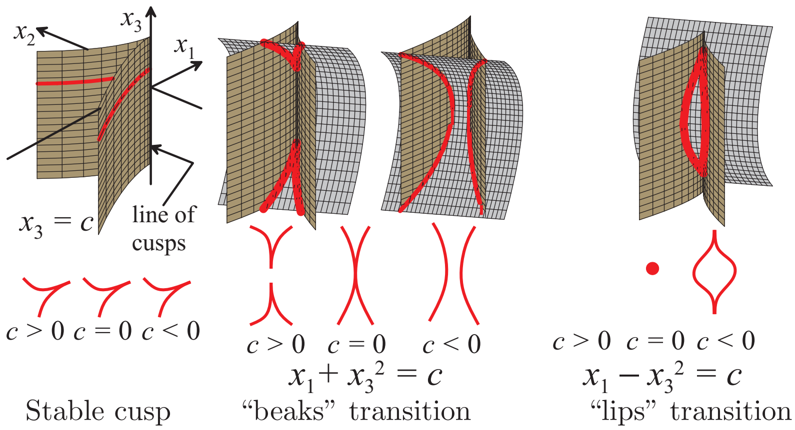

Figure 4: Left: the level sets $x_3 = c$ of the function $h_1(x_1, x_2, x_3) = x_3$ in the cuspidal edge surface are all curves with a cusp. Center and right: the level sets defined by functions $h_2(x_1, x_2, x_3) = x_1 + x_3^2$ and $h_3(x_1, x_2, x_3) = x_1 - x_3^2$. Below each figure is drawn a schematic diagrams of the transitions undergone by these level sets as c changes. The transition for h_2 is called a "beaks" or "bec-à-bec", and that for h_3 a "lips", where for $c > 0$ the level set is empty and for $c = 0$ it is a single point.

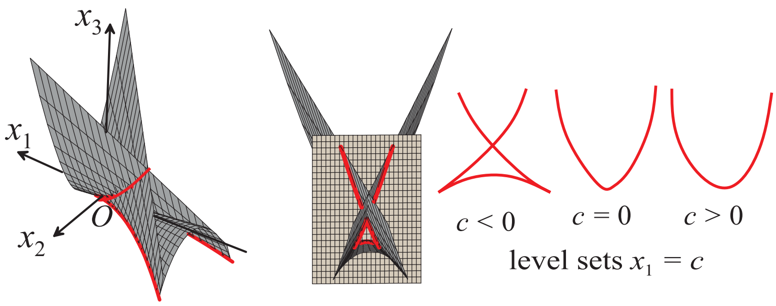

Figure 5: Left: a swallowtail surface with the curved lines of cusps and self-intersection curve marked. These all pass through the origin O. Right: a planar section $x_1 =$ constant $c < 0$ of a swallowtail surface, and the "swallowtail transition" which these sections—level sets of the function $h_0(x_1, x_2, x_3) = x_1$—undergo as the constant c moves through 0.

Singularity theory includes extensive investigations of discriminants and of *functions on discriminants*. To illustrate the latter consider examples **(ii)** and **(iii)** above, the three functions

$$h_1(\mathbf{X}) = x_3, \quad h_2(\mathbf{X}) = x_1 + x_3^2, \quad h_3(\mathbf{X}) = x_1 - x_3^2$$

for the cuspidal edge surface in **(ii)** and the function $h_0(\mathbf{X}) = x_1$ for the swallowtail surface in **(iii)**. These are illustrated in Figures 4 and 5 by means of their *level sets*, that is the sets of points of the cuspidal edge or swallowtail surface for which $h_i = c$, for values of c passing through 0. We are interested in the answers to two questions here:

Qu.1 How can we recognize, in a given situation, such as that of $\mathcal{D}_{\mathcal{F}}$, that a discriminant is, up to a local diffeomorphism of $\mathbb{R}^3$, a cuspidal edge or a swallowtail surface? See §5.1.

Qu.2 How can we recognize that a given function, e.g. $\alpha : \mathcal{D}_{\mathcal{F}} \to \mathbb{R}$, has level sets which undergo an evolution (or transition or "perestroika"[5]) in one of the "standard" ways of Figures 5 and 4? See

[5] "Perestroika" in Russian means approximately the same as "restructuring" in English, and the Western World heard a great deal about it in the 1980s and 1990s during the Gorbachev era in the Soviet Union and then the Russian Federation. Its use in a mathematical context was popularized by the great Russian mathematician Vladimir Igorevich Arnol'd (1937–2010).

§5.2.

In the case of when the discriminant is $\mathcal{D}_{\mathcal{F}}$ and the function is $h(x,y,\alpha)=\alpha$, the level sets $h=$ constant are of course the individual envelopes of the family, and we want to study precisely how these change as α changes.

Example 4.2 Let σ be an ellipse. The surface $\mathcal{D}_{\mathcal{F}}$ is illustrated in Figure 6. The surface appears to have the structure of (curved) cuspidal edge and swallowtail surfaces and the function $h(x,y,\alpha)=\alpha$ appears to have level sets which undergo a swallowtail transition for certain values of α. How can these observations be verified? Read on!

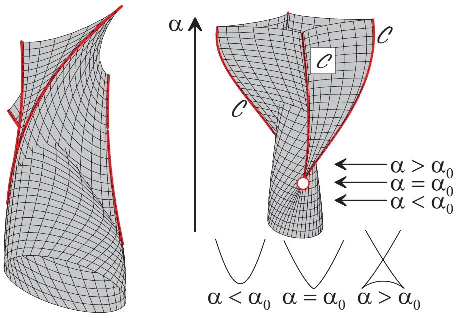

Figure 6: See Example 4.2. Two views of the discriminant surface $\mathcal{D}_{\mathcal{F}}$, as in (9), for σ an ellipse. The α-axis is vertical in the right-hand figure and $\alpha=0$ is at the bottom (the envelope is the original ellipse) and $\alpha=\frac{1}{2}\pi$ is at the top (the envelope of normals has four cusps). The surface appears to have (curved) cuspidal edges, with (curved) lines of cusps $\mathcal{C}$ marked, and the *horizontal* sections—the level sets of α close to the marked swallowtail point, where $\alpha=\alpha_0$ say—appear to undergo a swallowtail transition.

5 Applying results from singularity theory

In this section we shall state and then apply the results from singularity theory which allow us to make precise statements about the way in which the envelopes τ_α evolve as α changes. Details of the results in §5.1 are in [6] while those in §5.2 are found in various places, such as [2, 5].

5.1 How to recognize a discriminant surface

We consider a discriminant as in (6), and restrict to the case $k=3$, so that the general form will have a family of functions $G(\mathbf{X},t)=G(x_1,x_2,x_3,t)$, and

$$\mathcal{D}_G=\{\mathbf{X}: \text{ there exists } t \text{ such that } G=G_t=0 \text{ at } (\mathbf{X},t)\}.$$

Definition 5.1 For $\mathbf{X}=\mathbf{X}_0$ the function $g(t)=G(\mathbf{X}_0,t)$ has
(i) *type* A_2 at $t=t_0$ if $g'(t_0)=g''(t_0)=0, g'''(t_0)\neq 0$,
(ii) *type* A_3 at $t=t_0$ if $g'(t_0)=g''(t_0)=g'''(t_0)=0, g^{(4)}(t_0)\neq 0$.
We also say g has an "A_2 or A_3 singularity" at $t=t_0$.

Thus the type measures how many partial derivatives of G with the x_i parameters held fixed vanish at t_0. In the special case of the family $\mathcal{F}(x,y,\alpha,t)$ in (5) defining the envelopes τ_α, $\mathbf{X}$ is replaced by (x,y,α). The following Proposition gives the conditions for A_2 and A_3 singularities, expressed in terms of the curvature κ and its derivatives. Part (i) is a routine and not very interesting calculation using the Serret-Frenet formulas (2). Part (ii) deals with the case of an inflexion and we shall verify this since it is slightly more surprising. Recall from Prop. 2.3 that the envelope of tangents to a curve σ, that is τ_α with $\alpha=0$, when σ has inflexions, consists of σ and the whole tangent line at inflexion points. We shall be interested in the discriminant $\mathcal{D}_{\mathcal{F}}$ close to a point $(\mathbf{x}_0,0)$ where $\mathbf{x}_0=(x_0,y_0)$ is the inflexion point itself. It will turn out that this discriminant is locally diffeomorphic to a cuspidal edge surface, as in Figure 7, and we need the result of (ii) to show this.

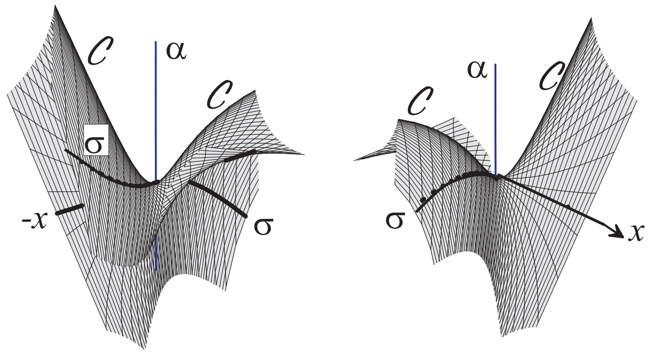

Figure 7: Two views of the cuspidal edge discriminant $\mathcal{D}_\mathcal{F}$ close to an ordinary inflexion of a curve and close to $\alpha = 0$, looking from below the plane $\alpha = 0$ (left) and from above (right). The lines of cusps $\mathcal{C}$ are labelled as are the parts of the curve σ in the plane $\alpha = 0$ which are visible; also the x-axis, which is the inflexional tangent to σ at the origin. The plane sections $\alpha =$ constant of $\mathcal{D}_\mathcal{F}$ evolve through $\alpha = 0$ by a "beaks" transition, as shown in Figure 9.

Proposition 5.2 *Let the point* $(\mathbf{x}_0, \alpha_0, t_0) = (x_0, y_0, \alpha_0, t_0)$ *satisfy* $\mathcal{F} = \mathcal{F}_t = 0$, *so that* $(\mathbf{x}_0, \alpha_0) \in \mathcal{D}_\mathcal{F}$.
(i) *Let s denote the arclength function on σ, as in §3. Suppose that $\kappa(t_0) \neq 0$ so that $\mathbf{x}_0$ is given by (4). Then $f(t) = \mathcal{F}(\mathbf{x}_0, \alpha_0, t)$ has, at $t = t_0$*

(a) *type A_2 provided, at t_0,*

$$\kappa^2 \cos\alpha - \kappa_s \sin\alpha = 0, \quad 2\kappa_s^2 - \kappa\kappa_{ss} \neq 0,$$

(b) *type A_3 provided, at t_0,*

$$\kappa^2 \cos\alpha - \kappa_s \sin\alpha = 0, \quad 2\kappa_s^2 - \kappa\kappa_{ss} = 0, \quad 6\kappa_s^3 - \kappa^2\kappa_{sss} \neq 0$$

(ii) *Suppose that $\kappa(t_0) = 0, \kappa'(t_0) \neq 0$, so that σ has an ordinary inflexion at $t = t_0$. Then, setting $\alpha_0 = 0$, the only $\mathbf{x}$ close to $\sigma(t_0)$ for which $\mathcal{F}_{tt}(\mathbf{x}, 0, t_0) = 0$ is $\mathbf{x} = \sigma(t_0)$ itself and $f(t) = \mathcal{F}(\sigma(t_0), 0, t)$ has type A_2 at t_0.*

Note that the A_2 condition in (i)(a) is the same as that for an ordinary cusp given in Proposition 3.3.

Proof of (ii) The calculations are marginally simplified by taking the parameter t to be itself arclength, so that Serret-Frenet formulae become $T' = \kappa N,\ N' = -\kappa T$. (But we shall still write t for the parameter.) Let us take $t_0 = 0$. We know that, locally, $\mathcal{F} = \mathcal{F}_t = 0$ at $(\mathbf{x}, 0, t)$ implies that either $\mathbf{x} = \sigma(t)$ or $\mathbf{x}$ is on the tangent line at the inflexion $\sigma(0)$. Using the formula (5), we differentiate twice and put $\alpha = t = 0$: $\mathcal{F}_{tt}(\mathbf{x}, 0, 0) = (\mathbf{x} - \sigma(0)) \cdot (\kappa'(0)T(0))$, which can only be zero, for $\mathbf{x}$ close to the inflexion point, when $\mathbf{x}$ coincides with that point. (This actually implies that at other points of the tangent line at the inflexion, $\mathcal{D}_\mathcal{F}$ is a nonsingular surface.) Furthermore $\mathcal{F}_{ttt}(\mathbf{0}, 0, 0) = -2\kappa'(0) \neq 0$ so $\mathcal{F}(\mathbf{0}, 0, t)$ has exactly an A_2 singularity at $t = 0$. □

Now let us return to a general family G as above. We will continue to use $\mathbf{X}$ to denote (x_1, x_2, x_3), and $\mathbf{X}_0$ to denote a fixed value of $\mathbf{X}$. Suppose that $G = G_t = 0$ at $(\mathbf{X}_0, t_0)$, where the function $g(t) = G(\mathbf{X}_0, t)$ has type $A_r, r = 2$ or 3, at t_0. It is *almost* true that $\mathcal{D}_G$, in a small neighbourhood of $\mathbf{X}_0$, is locally diffeomorphic to the standard discriminant of Examples 4.1. In fact there is an additional "genericity" condition to verify which, in the case $G = \mathcal{F}$, we shall show is automatic from our assumption 2.1.

Example 5.3 Consider the family $G(\mathbf{X}, t) = 2t^3 + x_1t^2 + x_2$ (independent of x_3), for which $\mathcal{D}_G = \{(x_1, 0, x_3)\} \cup \{(-3t, t^3, x_3)\}$. Then $g(t) = G(0, 0, 0, t) = 2t^3$ has, at $t_0 = 0$, an A_2 singularity. But $\mathcal{D}_G$ is not a cuspidal edge surface near the origin; it is the product of the curve $x_1^3 + 27x_2 = 0$, together with its inflexional tangent $x_2 = 0$, by the x_3-axis.

The additional condition which is needed, besides the A_2 or A_3 singularity, is as follows. It essentially says that the parameters x_i perturb the singularity in a "sufficiently general" way; the technical term is that they "unfold" the singularity in a (uni)versal manner.

Definition 5.4 Suppose that $G = G_t = 0$ at $(\mathbf{X}_0, t_0)$ and $g(t) = G(\mathbf{X}_0, t)$ has an A_r singularity at t_0. Consider the partial derivatives $G_{x_1}, G_{x_2}, G_{x_3}$ with respect to the parameters x_i, evaluated at $\mathbf{X}_0$, and in particular their Taylor polynomials $\mathcal{T}_i$ up to degree $r-1$, expanded about t_0 (so these have r terms). The family $G(\mathbf{X}, t)$ is called a *versal unfolding* of g at t_0 if the $\mathcal{T}_i$ span a vector space of dimension r. Thus, if the coefficients in the $\mathcal{T}_i$ are placed as the columns of an $r \times 3$ matrix, the rank is r. Clearly this is possible only for $r \leq 3$.

Examples 5.5 (i) For the G, and $\mathbf{X}_0 = (0,0,0), t_0 = 0$ in Example 5.3, where $r = 2$, the derivatives with respect to x_i are $t^2, t, 0$ respectively (independently of $\mathbf{X}_0$ in fact). The Taylor polynomials up to degree 1, expanded about 0, are $0, t, 0$ respectively so the criterion of Definition 5.4 does not hold. It is also clear that there is "something missing" from this family for, whatever the values of x_1, x_2, x_3, the function $g(t) = G(x_1, x_2, x_3, t)$ will always have a singularity at $t = 0$, in fact of type A_1 unless $x_1 = 0$. We cannot turn the graph of the function $g(t) = t^3$ into a graph without turning points by adjusting the x_i. On the other hand for Example 4.1(iii) the condition for a critical point is $3t^2 + x_1 = 0$ and there are no critical points if $x_1 > 0$.
(ii) For the G of Examples 4.1(i), (ii), (iii), the criterion is easily shown to hold. For example with 4.1(iii), where $r = 3$, the $\mathcal{T}_i$ are $t^2, t, 1$ respectively.

The key theorem then is as follows; for details see [6].

Theorem 5.6 *Suppose that G satisfies the criterion of Definiton 5.4. Then, in a neighbourhood of $\mathbf{X}_0 \in \mathcal{D}_G$, the discriminant is locally diffeomorphic to a standard cuspidal edge surface when $r = 2$ and a standard swallowtail surface when $r = 3$ (as in Examples 4.1(ii), (iii)).* □

The most important example for us is of course $\mathcal{D}_\mathcal{F}$ and there we must work a little harder to verify the criterion of Definition 5.4 when $G = \mathcal{F}$. We give some details in §8.2 and the result is then as follows.

Corollary 5.7 *The family $\mathcal{F}$, as in (5), satistifies the conditions of the above theorem. Thus, when $f(t) = \mathcal{F}(\mathbf{x}_0, t)$ has an A_r singularity at t_0, $r = 2$ or 3, in the cases covered by Proposition 5.2, the discriminant $\mathcal{D}_\mathcal{F}$, which is the union of the envelopes τ_α spread out in the α direction, is always locally diffeomorphic to a standard cuspidal edge ($r = 2$) or a standard swallowtail surface ($r = 3$) in a neighbourhood of $\mathbf{x}_0$.* □

So Figures 6 and 7 are not deceiving us.

5.2 How to recognize level sets of a function

There are two cases to consider, namely level sets of a function on a cuspidal edge surface and on a swallowtail surface in 3-space $\mathbb{R}^3$. Fortunately in both cases the conditions to realize the transitions of Figures 5 and 4 are intuitively very reasonable. We shall only state the answers here; there is more discussion of functions on discriminants in [2, 5]. For a smooth function $h : \mathbb{R}^3 \to \mathbb{R}$, defined in a neighbourhood of $\mathbf{X}_0 \in \mathcal{D}_G$, there will be a *kernel plane* K through $\mathbf{X}_0$. This is the plane tangent to the level set of h through $\mathbf{X}_0$ and has equation $(\mathbf{X} - \mathbf{X}_0) \cdot (h_{x_1}, h_{x_2}, h_{x_3}) = 0$, where the partial derivatives are evaluated at $\mathbf{X}_0$. We refer back to Examples 4.1(ii), (iii) for properties of tangent planes to a cusped edge and swallowtail surface.

Proposition 5.8 (i) *For a cuspidal edge surface, with $\mathbf{X}_0$ on the line of cusps, the level sets of h on $\mathcal{D}_G$ will all be cusped curves, as in Figure 4, left, provided the plane K does not contain the tangent to the line of cusps through $\mathbf{X}_0$. ("K is transverse to the line of cusps".)*
On the other hand, the levels sets undergo a "beaks" or "lips" transition, as in Figure 4, center and right, provided K does contain this tangent but does not coincide with the limiting tangent plane to the cuspidal edge surface at points approaching $\mathbf{X}_0$. ("K is transverse to this limiting tangent plane".)
(ii) *For a swallowtail surface, with $\mathbf{X}_0$ at the swallowtail point, the level sets on $\mathcal{D}_G$ undergo a swallowtail transition, as in Figure 5, with two cusps merging and disappearing, provided K does not*

contain the limiting tangent to the lines of cusps on $\mathcal{D}_G$ at $\mathbf{X}_0$. ("K is transverse to this limiting tangent line".)

See Figure 8 for examples where the conditions of (i) hold and also one where they do not hold. Similarly, level sets of a function on a swallowtail surface failing to satisfy the condition of (ii) will not resemble a swallowtail transition.

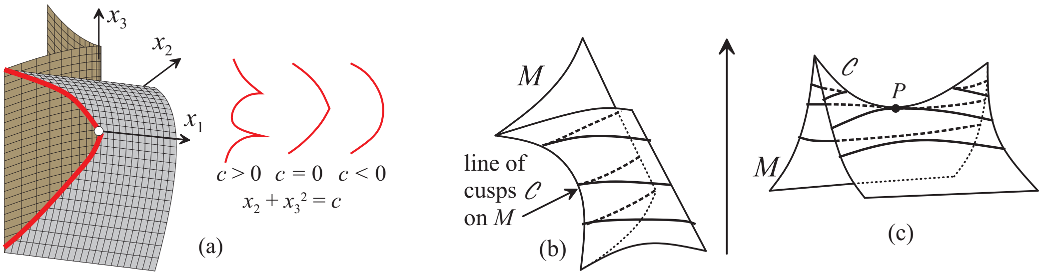

Figure 8: (a) An example of a function $h(\mathbf{X}) = x_2 + x_3^2$ on the cuspidal edge illustrated which does *not* satisfy the conditions of Proposition 5.8(i), since the level set for $c = 0$, appearing as the lighter colored surface on the left, has tangent plane at the origin equal to the x_1x_3-plane, which coincides with the limiting tangent planes to the cuspidal edge. The level sets where the surface $x_2 + x_3^2 = c$ meets the cuspidal edge evolve in the way illustrated. Clearly this is not anything like the transitions of Figure 4. In (b) and (c) the function on the curved cuspidal edge M is assumed to be height in the direction of the vertical arrow. For (b) the level set at any level is a horizontal plane, which is transverse to the line of cusps $\mathcal{C}$ and all the horizontal sections are cusps. For (c) the level set (horizontal plane) through P is tangent to $\mathcal{C}$ but does not coincide with the (vertical) limiting tangent planes to M, hence producing a "beaks" transition as in Proposition 5.8(i).

Remarkably, the conditions of Proposition 5.8 are *always* satisfied for the discriminant $\mathcal{D}_{\mathcal{F}}$, provided only that the original curve σ satisfies the Assumptions 2.1. We sketch the proof of this in §8.3 below. In particular the transition on the envelope τ_α through $\alpha = 0$, at an inflexion point of σ, is a "beaks" transition in which two cusps collide, momentarily giving the envelope the entire tangent line at the inflexion point, and then separate into two smooth branches. This is illustrated in Figure 9 for a closed curve with two inflexions, in fact the curve $\sigma(t) = ((\cos t + 2\cos(2t) + 10)\cos t,\ (12\sin t - 8)\sin t)$. (The transition cannot be of "lips" type as in Figure 4, right, since the envelope cannot become empty.) Thus we have the following, which completely describes the local changes in the envelopes τ_α for a curve σ satisfying the genericity assumption 2.1.

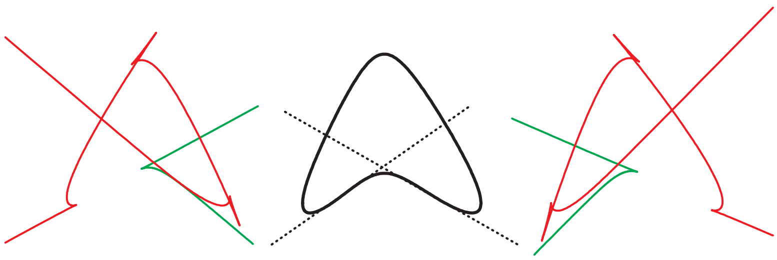

Figure 9: Center: a curve σ with two inflexions, and (left) the envelope τ_α for α small and negative; (right) for α small and positive. Two cusps approach and merge close to each inflexion, leaving smooth branches, in a "beaks" transition as in Figure 4, center. The envelope for $\alpha = 0$ is the original curve and the tangents at the inflexion points, which are drawn dashed in the center figure (see Proposition 2.3 and compare Figure 2, right). The figure also shows two swallowtail configurations on each envelope, which collapse in a swallowtail transition as $\alpha \to 0$. The envelope τ_α goes "to infinity" at points corresponding to the inflexions themselves, since the denominator in (4) vanishes.

Theorem 5.9 *The evolutoids τ_α evolve locally according to a stable cusp (Figures 4,left, 6 and 8(b)) at A_2 points where the curvature κ is nonzero; according to a swallowtail transition (Figure 6) at A_3*

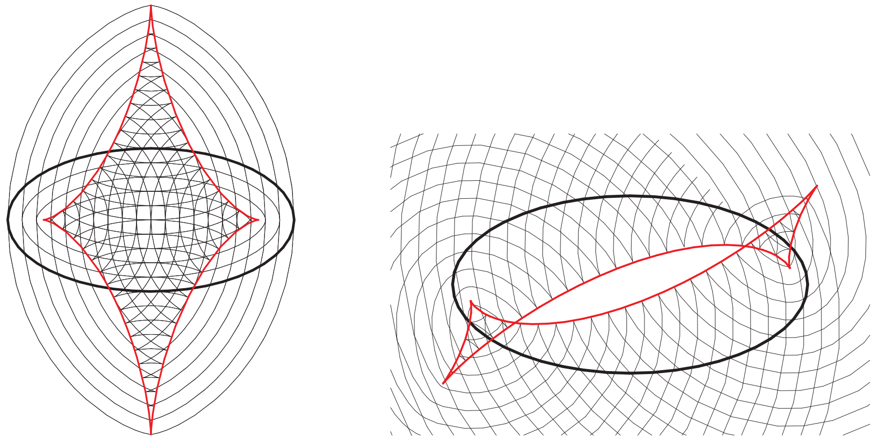

Figure 10: Left: the "ordinary" wavefronts of an ellipse σ, which are obtained by displacing σ a constant distance along its normals. The cusps on the wavefronts all lie on the envelope of normals, the 4-cusped curve which is also drawn. Right: non-closed wavefronts, given by (10), corresponding to the envelope of lines shown in Figure 1, center, for which $\alpha = \frac{1}{4}\pi$. Note that as the wavefronts (in either example) approach a cusp on the *envelope* two cusps on the wavefronts collapse together—this is in fact another example of a swallowtail transition.

points where $\kappa \neq 0$; and according to a "beaks" transition (Figures 4, center, and 8(c)) at points where $\kappa = 0$ and $\alpha = 0$. At all other points the envelope τ_α is a smooth curve.

For example, in Figure 1, the envelope is undergoing swallowtail transitions which are identical, up to a local diffeomorphism, with those of the standard swallowtail sections in Figure 5.

6 Wavefronts

For the envelope of normals $\tau_{\pi/2}$ to a curve σ there is associated a family of *wavefronts*, also called *parallels* or *offsets* which look like radiation emanating from σ into the surrounding space. See Figure 10, left. The cusps on the wavefronts trace out the envelope, which in this case is just the four-cusped evolute of σ. The wavefronts have the parametrization $\sigma(t) + wN(t)$ where w takes a constant value along each wavefront. Thus σ is displaced a constant distance w along the normals to σ. The general prescription for wavefronts from a family of lines (or curves) $F(\mathbf{x}, t)$ is as follows. We "integrate" F, that is we look for a family G satisfying $G_t = F$. Then the wavefronts are given by $F = 0$, $G =$ constant.

In the case of the family of normals, which is given by $F(\mathbf{x}, t) = (\mathbf{x} - \sigma(t)) \cdot T(t)$, it is easy to write down a suitable G. Let us take t to be the arclength parameter; then we can take $G(\mathbf{x}, t) = -\frac{1}{2}||\mathbf{x} - \sigma(t)||^2 = -\frac{1}{2}(\mathbf{x} - \sigma(t)) \cdot (\mathbf{x} - \sigma(t))$, since by (2) $\sigma'(t) = T(t)$, so that $G_t = F$. The solutions of $F = 0, G = w_1 =$ constant < 0 are the points of the normal ($F = 0$) for which the distance to the curve is $\pm\sqrt{-2w_1}$, that is the ordinary parallels (offsets) of σ.

For the family F in (1), with α fixed, we have to work a little harder but writing $F = F_1 \sin\alpha - F_2 \cos\alpha$ we can use the above solution for G_1 with $(G_1)_t = F_1$ and, for F_2 we need

$$G_2 = \int (\mathbf{x} - (X(t), Y(t))) \cdot (-Y'(t), X'(t)) \; dt,$$

where $\sigma(t) = (X(t), Y(t))$. Note that the integral is independent of the parametrization of σ and in fact it represents, up to an added constant, twice the area swept out by a line from $\mathbf{x}$ to $\sigma(t)$, as t travels from some arbitrary starting value t_0. When t is arclength $G_2 = \int (\mathbf{x} - \sigma(t)) \cdot N(t) \; dt$. In the special case of an ellipse $\sigma(t) = (a\cos t, b\sin t)$ we can write down the integral explicitly:

$$G_2(t) = ay\cos t - bx\sin t + abt \; (+ \text{ constant}).$$

Note that this contains t on its own, so that unless $\cos\alpha = 0$ in which case G_2 is not needed, the solution is not periodic: the wavefronts will not be closed curves, even though the ellipse σ is closed.

Returning to F for a fixed α and any regular curve σ, and using $G = G_1 \cos t + G_2 \sin t$, we have a prescription for finding the wavefronts. It is not hard to check that the following formula satisfies $F(\sigma_w(t), t) = 0, G(\sigma_w, t) = -\frac{1}{2}w^2 \sin\alpha$, which is constant for a given w and α.

$$\sigma_w(t) = \sigma(t) + (w - t\cos\alpha)(T(t)\cos\alpha + N(t)\sin\alpha). \tag{10}$$

This gives an explicit formula for the family of wavefronts corresponding to a given σ and α. Fixing w and letting t vary we get an individual wavefront, parametrized by t, corresponding to that w. It can be checked that the singular points (cusps) on the wavefronts are given by the additional condition $G_{tt} = 0$, that is $F_t = 0$, which says that the cusps lie on the envelope τ_α of lines given by $F = F_t = 0$. We say that the *cusps of the wavefronts sweep out the envelope* τ_α. The exception is $\sin\alpha = 0$, for which τ_α is the envelope of tangent lines to σ. In that case there are no cusps to sweep out anything.

If $\alpha = \frac{1}{2}\pi$ (the envelope of normals to σ) then $\sigma_w(t) = \sigma(t) + wN(t)$, which is the usual parallel or offset of σ, obtained by moving down the normals a distance w. Note that w and $-w$ give different parallels, but the same value of G: fixing the value of G gives *two* parallels, corresponding to values of w of opposite sign. If we use $\alpha = -\frac{1}{2}\pi$ instead then w and $-w$ are interchanged.

For a general $\alpha \neq \pm\frac{1}{2}\pi$ the factor $w + t\sin\alpha$ in (10) tells us how far along the line given by t we must go to reach the wavefront point. For a closed curve σ, parametrized by $0 \leq t < 2\pi$ we can add any mutliple of 2π on to t and obtain the same point of σ and the same line of the family. Thus the wavefront meets the line corresponding to the value t infinitely often.

An example, using the ellipse, is shown in Figure 10, right.

7 Conclusion

We have shown how to investigate a *family* of envelopes, parametrized by an angle α, using some results from singularity theory. The family τ_α of envelopes interpolates between the a plane curve σ ($\alpha = 0$) and its evolute ($\alpha = \frac{1}{2}\pi$), with some additional complications connected with inflexions of σ. We can apply some general results about *discriminants*—cusp, cuspidal edge, swallowtail—together with real-valued functions α on discriminants and their associated level sets, that is the sets on which α is constant. It is the identification of envelopes with discriminants which is at the heart of the applications given in this article. We have also investigated the corresponding *wavefronts* whose singular points sweep out the envelopes τ_α, and found that, unlike the parallels associated with the evolute of a curve, these wavefronts are in general not closed curves. The same techniques—reduction to normal form, use of standard models, discriminants and functions on discriminants—give a great deal of information about the differential geometry of surfaces and higher dimensional manifolds, and there are many applications to areas of science such as control theory and dynamical systems. Some examples are given in the classic text [3], and applications to shape analysis are in [12].

8 Some proofs and additional notes

8.1 Proof of Proposition 3.3

Here is an indication of how to prove Proposition 3.3 without getting tangled in too much algebra. We will assume for this that $||\sigma'(t)|| = 1$ for all t, which says that σ is unit speed, or that $t = s$ is the arclength parameter. This is no loss of generality (see any book on differential geometry of curves, or alternatively [6, pp.27-8]) and saves a little writing. In fact write $\lambda = -\cos\alpha + (\kappa' \sin\alpha/\kappa^2)$ so that the condition for a cusp is $\lambda = 0$. Then the envelope in (4) has $\mathbf{x}' = -\lambda\cos\alpha\ T + \lambda\sin\alpha\ N$. Differentiating this twice, to give $\mathbf{x}''$ and $\mathbf{x}'''$, and putting $\lambda = 0$ (*after* differentiating!) quickly gives the condition for these vectors to be linearly dependent as $2\kappa\lambda'^2 = 0$, that is $\lambda' = 0$, and this gives the required formula provided $\sin\alpha \neq 0$. □

8.2 Sketch of a proof of Corollary 5.7

Let us write s for $\sin\alpha$ and c for $\cos\alpha$, $\sigma(t) = (X(t), Y(t))$ and assume that σ is unit speed, that is $||\sigma'(t)|| = 1$ for all t, and that $\kappa \neq 0$ (see below for the case $\kappa = 0$). Recall (5) that

$$\mathcal{F}(\mathbf{x}, \alpha, t) = (\mathbf{x} - \sigma) \cdot (Ts - Nc) = (x - X)(X's + Y'c) + (y - Y)(Y's - X't).$$

To show that $\mathcal{F}$ is "versal" we need to consider the Taylor series in t, including constant term, up to degree 2, of the derivatives $\mathcal{F}_x, \mathcal{F}_y, \mathcal{F}_\alpha$, and check that they are independent when evaluated at an A_3 (swallowtail) point, that is one at which the conditions of Proposition 5.2(i)(b) hold. So we need to take these three derivatives and differentiate them with respect to t twice to get hold of the necessary terms of the Taylor series. We are of course allowed to assume (4). For example,

$$\mathcal{F}_\alpha = (\mathbf{x} - \sigma) \cdot (Tc + Ns), \text{ so } \mathcal{F}_{\alpha t} = -T \cdot (Tc + Ns) + (\mathbf{x} - \sigma) \cdot (\kappa Nc - \kappa Ts),$$

but using (4) this boils down to simply $-\cos\alpha$. Differentiating again and substituting from (4) shows $\mathcal{F}_{\alpha tt} = 0$. Let us write $\mathbf{u} = (s, c)$; then for example $\mathcal{F}_x = T \cdot \mathbf{u}$ and $\mathcal{F}_y = -N \cdot \mathbf{u}$. Altogether we find that the 2-jets are the columns of the matrix J_1 below (where the binomial coefficient $\frac{1}{2}$ has been omitted)

$$J_1 = \begin{pmatrix} \mathcal{F}_x & \mathcal{F}_y & \mathcal{F}_\alpha \\ \mathcal{F}_{xt} & \mathcal{F}_{yt} & \mathcal{F}_{\alpha t} \\ \mathcal{F}_{xtt} & \mathcal{F}_{ytt} & \mathcal{F}_{\alpha tt} \end{pmatrix} = \begin{pmatrix} T \cdot \mathbf{u} & -N \cdot \mathbf{u} & \frac{s}{\kappa} \\ \kappa N \cdot \mathbf{u} & \kappa T \cdot \mathbf{u} & -c \\ (-\kappa^2 T + \kappa' N) \cdot \mathbf{u} & (\kappa^2 N + \kappa' T) \cdot \mathbf{u} & 0 \end{pmatrix}$$

The determinant of this matrix is $\kappa^2 \sin\alpha + \kappa' \cos\alpha$. If this is zero then, using Proposition 5.2(2) (where κ_s is our κ'), we have $\kappa^4 + \kappa'^2 = 0$ so that $\kappa = 0$, contrary to assumption. This proves the required independence. See [6, Ch.6] for full details of this method.

In the case of an A_2 singularity where $\kappa \neq 0$ the versal unfolding condition is simply that the top left 2×2 minor of J_1 is nonsingular, and this amounts to $\kappa \neq 0$.

Finally when considering an (ordinary) inflexion, so that $\kappa = 0$, and working at $\alpha = 0$, it is easy to check that $\mathcal{F}$ has exactly an A_2 singularity. Let us take the unit tangent at the inflexion to be $(1, 0)$ so that the unit normal is $(0, 1)$. The vector $\mathbf{u}$ is $(0, 1)$ here, and the Taylor expansions to degree 1 of $\mathcal{F}_x, \mathcal{F}_y, \mathcal{F}_\alpha$ come to $0, -1$ and $-t$ respectively. These span polynomials of degree 1 in t. □

8.3 Verifying the conditions needed to apply Proposition 5.8 to the discriminant $\mathcal{D}_\mathcal{F}$

First, consider the case where the curve σ does not have an inflexion at $\sigma(t_0)$, but $f(t) = \mathcal{F}(\mathbf{x}_0, \alpha_0, t)$ has an A_2 or A_3 singularity at $t = t_0$. In the A_2 case, by Corollary 5.7, $\mathcal{D}_\mathcal{F}$ is locally diffeomorphic to a cuspidal edge surface close to $(\mathbf{x}_0, \alpha_0)$. This cuspidal edge is given by $\mathcal{F} = \mathcal{F}_t = \mathcal{F}_{tt} = 0$, that is 3 equations in the four variables x, y, α, t, and the solutions are then projected to (x, y, α)-space, where $\mathcal{D}_\mathcal{F}$ lies. A standard technique for calculating tangent vectors (the *implicit function theorem*, which is covered in books of advanced calculus, or see [6, Ch.4]) says that we look for non-zero kernel vectors of the 3×4 matrix J_2 of partial derivatives of $\mathcal{F}, \mathcal{F}_t, \mathcal{F}_{tt}$ with respect to the four variables (x, y, α, t), evaluated at $(\mathbf{x}_0, \alpha_0, t_0)$. Note that the first three columns of J_2 are the same as the three columns of J_1 in the previous section, while the fourth column is $(0, 0, \mathcal{F}_{ttt} \neq 0)^\top$ at an A_2 point and $(0, 0, 0)^\top$ at an A_3 point. Thus for an A_2 point we can always find a kernel vector $(\overline{x}, \overline{y}, \overline{\alpha}, \overline{t})$ say, whose first three components are not all zero, using the first two rows of J_2, and then determine $\overline{t}$ using the third row of J_2, since $\mathcal{F}_{ttt} \neq 0$. Then $(\overline{x}, \overline{y}, \overline{\alpha})$ is a nonzero tangent vector to the line of cusps $\mathcal{C}$ on $\mathcal{D}_\mathcal{F}$ in (x, y, α)-space. However this cannot be done with $\overline{\alpha} = 0$ in view of the nonsingularity of the top left 2×2 submatrix of J_2 (or J_1). So a tangent vector to $\mathcal{C}$ will never be horizontal and changing α to nearby values gives a stable cusp as in Figure 8(b) and not (c).

There is clearly a problem with this argument at an A_3 point $(\mathbf{x}_0, \alpha_0)$, where $F_{ttt} = 0$, since in view of the nonsingularity of J_1, all kernel vectors of J_2 have the form $(0, 0, 0, \overline{t})$. This simply says that, in (x, y, α)-space the curve $\mathcal{C}$ on $\mathcal{D}_\mathcal{F}$ is singular at $(\mathbf{x}_0, \alpha_0)$—not surprising since $\mathcal{D}_\mathcal{F}$ is a swallowtail surface and the space curve $\mathcal{C}$ itself has a cusp at $(\mathbf{x}_0, \alpha_0)$. However the above argument still applies, by taking say a unit tangent vector $(\overline{x}, \overline{y}, \overline{\alpha})$ and moving towards $\mathbf{x}_0, \alpha_0)$ along $\mathcal{C}$: the last component cannot tend to 0 without the other two tending to 0 as well, which is a contradiction. In the present case we can be more explicit: a tangent vector (of length $\sqrt{1 + \kappa^2} > 1$) to $\mathcal{C}$, obtained from the first two rows of J_2 is $((\sin(2\alpha), \cos(2\alpha)) \cdot N,\ (\sin(2\alpha), \cos(2\alpha)) \cdot T, \kappa)$. It is plain that this cannot have a limit in which the third component is 0. This is certainly visible in Figure 6, where the limiting tangent to the lines of cusps is far from horizontal.

Finally consider the case when σ has an inflexion at $t_0 = 0$ say, and $\mathbf{x}_0 = \sigma(t_0), \alpha_0 = 0$. Then $\mathcal{D}_\mathcal{F}$ is locally diffeomorphic to a cuspidal edge surface by Corollary 5.7. In this case it is easier to do a calculation in local coordinates, taking $\sigma(t) = (t, at^3 + bt^4 + ct^5 + \ldots)$ say, where $a \neq 0$ since there is an ordinary inflexion at the origin, and expanding everything about $(x, y, \alpha, t) = (0, 0, 0, 0)$. Then an explicit calculation shows that $\mathcal{D}_\mathcal{F}$ is locally parametrized by (x, t) and the line of cusps by t:

$$\begin{aligned} \mathcal{D}_\mathcal{F} : (x,t) &\mapsto (x,\ 6ax^2t - 9axt^2 + 4at^3 + \ldots,\ 6axt - 6at^2 + \ldots); \\ \mathcal{C} : t &\mapsto \left(2t + \ldots,\ \frac{8b(24b - 35ac)}{a^2}t^4 + \ldots,\ -40ct^5 + \ldots\right). \end{aligned}$$

It is clear that the limiting tangent to $\mathcal{C}$ is in the direction $(1, 0, 0)$ which is in the plane $\alpha = 0$. Since we know $\mathcal{D}_\mathcal{F}$ is a cuspidal edge surface, we can find the limiting tangent plane to $\mathcal{D}_\mathcal{F}$ at points away from $\mathcal{C}$ by taking any path on $\mathcal{D}_\mathcal{F}$ which avoids $\mathcal{C}$ (apart from at $(\mathbf{x}, \alpha) = (0, 0, 0)$), such as the path given by $t = 0$. The normal to $\mathcal{D}_\mathcal{F}$ then comes to $(0, -6ax + \ldots, 6ax^2 + \ldots)$ which has limit $(0, 1, 0)$ so that the limiting tangent plane is the plane $y = 0$ and hence does not coincide with the plane $\alpha = 0$, as required. □

Acknowledgements The second author is grateful for financial support from the Engineering and Physical Sciences Research Council in the UK during his PhD programme; the present article derives from part of Chapter 3 of [13]. Both authors thank Joel Haddley for suggesting that they consider wavefronts corresponding to the envelopes studied, Joel and Victor Goryunov for insisting that the inflexion case is also a beaks transition, and the referees who encouraged us to improve our exposition and gave us earlier references to the literature on evolutoids.

References

[1] T.M.Apostol and M.A.Mnatsakanian, 'Tanvolutes: generalized involutes', *Amer. Math. Monthly* 117 (2010), 701–713.

[2] V.I.Arnol'd, 'Wave front evolution and equivariant Morse lemma', *Comm. Pure Appl. Math.* 29 (1976), 557–582.

[3] V.I.Arnol'd, *Catastrophe Theory*, Third edition, Springer-Verlag 1992. ISBN13: 978-3540548119.

[4] Th. Bröcker and L.Lander, *Differentiable Germs and Catastrophes*, London Mathematical Society Lecture Note Series 17, Cambridge University Press 1975. ISBN13: 978-0521206815.

[5] J.W.Bruce, 'Functions on discriminants', J. London Math. Soc. (2) 30 (1984) 551–567.

[6] J.W.Bruce and P.J.Giblin, *Curves and Singularities*, Second Edition, Cambridge University Press 1992. ISBN13: 978-0521429993.

[7] J.W.Bruce, N.P.Kirk and A.du Plessis, 'Complete transversals and the classification of singularities', *Nonlinearity* 10 (1997), 253–275.

[8] Peter Giblin, 'Affinely invariant symmetry sets', *Caustics '06, Banach Center Publications* 82 (2008), 71–84,
http://www.liv.ac.uk/~pjgiblin/papers/caustics06_giblin_revised.pdf

[9] P.J.Giblin and V.M.Zakalyukin, 'Singularities of centre symmetry sets', *Proceedings of the London Mathematical Society* 90 (2005), 132–166.

[10] M.Hamman, 'A note on ovals and their evolutoids', *Beiträge zur Algebra und Geometrie* 50 (2009), 433–441.

[11] G. Katz, 'How tangents solve algebraic equations, or a remarkable geometry of discriminant varieties', *Expositiones Mathematicae* 21 (2003), 219–261.

[12] S.M.Pizer and K.Siddiqi (eds.) *Medial Representations*, Springer-Verlag 2008. ISBN13: 978-1402086571.

[13] J.P.Warder, *Symmetries of Curves and Surfaces*, PhD thesis, University of Liverpool, 2009. Available at www.liv.ac.uk/~pjgiblin

Peter J. Giblin and J. Paul Warder,
Department of Mathematical Sciences,
The University of Liverpool,
Liverpool L69 7ZL, UK,
pjgiblin@liv.ac.uk, paul.warder@gmail.com