Epipolar curves on surfaces

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The view lines associated with a family of profile curves of the projection of a surface onto the retina of a moving camera defines a multi-valued vector field on the surface. The integral curves of this field are called *epipolar curves*, and together with a parametrization of the profiles provide a parametrization of regions of the surface. We present an investigation of epipolar curves on the object surface and in a related 'spatio-temporal surface'. We also consider the epipolar constraint in the image and the resulting epipolar curves there. In particular, we make an exhaustive list of the circumstances where the epipolar parametrization breaks down. These results gives a systematic way of detecting the gaps left by reconstruction of a surface from profiles. They also suggest methods for filling in these gaps.

Keywords: surface reconstruction, epipolar constraint, epipolar curve, spatio-temporal surface

This paper is concerned with some aspects of the reconstruction of a smooth surface M from a sequence of profiles (also called *apparent contours*, *outlines* and *occluding contours*) where the motion of the observer is known. Such a reconstruction was introduced[1] for a simple class of motions, and was generalized[2–4] to arbitrary motion. For the general motion case, the *epipolar correspondence* played an important role in matching points from one profile to the next.

Given a smooth surface M and a curve $\mathbf{c}(t)$ of camera centres, we have, for each t, a *critical set* or *contour generator* Σ_t on M consisting of those points $\mathbf{r}$ where the 'visual ray' from $\mathbf{c}(t)$ to $\mathbf{r}$ is tangent to M. The *epipolar plane* is the plane spanned by this ray and the tangent to the curve $\mathbf{c}$ of centres, by analogy with the epipolar plane of stereo, which is spanned by a visual ray to one camera centre and the baseline connecting the two centres (Faugeras[5], p.170).

On the surface M there is an *epipolar curve* through $\mathbf{r}$ which has its tangent along the visual ray. As $\mathbf{c}$ moves with time, the visual ray slips along the epipolar curve. (We make a precise definition later using an 'epipolar field' on M.) In general, the critical sets and epipolar curves make a coordinate grid on M: a local parametrization $\mathbf{r}(t, u)$ can be found in which the critical sets are given by $t =$ constant and the epipolar curves by $u =$ constant (*Figure 1*). It is this 'epipolar parametrization' of M which is used by Blake and Cipolla[2,3] to reconstruct M from its profiles, which are the images of the critical sets in a viewplane or viewsphere. It is shown elsewhere[6] that reconstruction from the epipolar parametrization is readily transformable into an optimal estimation problem. The epipolar parametrization also has another very interesting property: the viewlines associated with points $\mathbf{r}(t, u)$ and $\mathbf{r}(t+\delta t, u)$ of M will (being lines in space) generally not intersect. However, for the epipolar parametrization, the point at which these lines come *closest* to one another is (as $\delta t \to 0$) on the surface M. So the intuitive idea of reconstruction via 'intersection of viewlines' is actually valid for the epipolar parametrization. (See below, where we show that this holds for the epipolar parametrization and only one other.)

The epipolar curves on M have the striking property that their osculating planes are precisely epipolar planes. The epipolar plane becomes tangent to M at points which we shall identify as 'frontier points', and it is precisely at such points that the epipolar curve becomes singular. The 'frontier' which is the locus of such points of M plays an important role in this paper, since it is along the frontier that the epipolar parametrization breaks down. To examine the situation at the frontier we introduce a 'spatio-temporal surface' (Definition 1).

Intersections of critical sets (such as Σ_0 and Σ_t in *Figure 1*) are described as 'centres of spin' by Rieger[7], and our frontier points can be considered as the limiting case as $t \to 0$ (Definition 5). However, the frontier points do not remain stationary as the camera moves, but slip along the frontier curve. Frontier points have also been used in special cases[8,9].

There are other places where the epipolar parametrization breaks down. First, there are points where the epipolar curve and critical set become tangent. This results in a profile with a cusp (or higher singularity)

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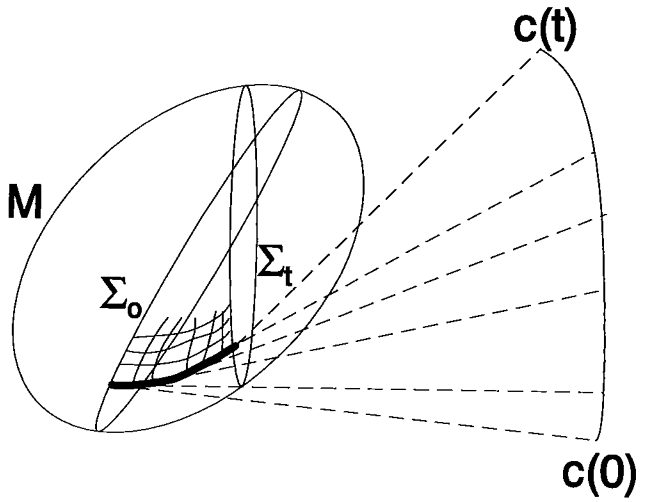

Figure 1 A surface M and segment of camera path from $\mathbf{c}(0)$ to $\mathbf{c}(t)$. Also shown are the two corresponding critical sets Σ_0, Σ_t, a segment of epipolar curve (drawn heavily), viewlines (dashed) tangent to the epipolar curve, and a local coordinate grid of critical sets and epipolar curves

and is also reflected in the geometry of the epipolar curve: it has a zero of curvature (Proposition 11). Second, the critical sets themselves can become singular (having an isolated point or a crossing).

In this paper, we analyse all the situations where the epipolar parametrization breaks down, showing the patterns of critical sets and epipolar curves near such points. Each possible breakdown has its characteristic pattern, which should make it possible to prescribe a systematic way of filling in the 'gaps' in the reconstruction of M. We analyse the critical sets, the epipolar curves and the various breakdowns.

Some of the results of this paper have appeared earlier[10].

CRITICAL SETS AND THE SPATIO-TEMPORAL SURFACE

Let M be a smooth surface in 3-space, without boundary, and let $\mathbf{c}(t)$ be a smooth path of camera centres which lies outside M. We use $\mathbf{r}$ to denote surface points, and $\mathbf{n}$ to denote the unit (say outward) normal at $\mathbf{r}$. Then the *critical set* or *contour generator* Σ_t corresponding to 'time' t is the set of points $\mathbf{r}$ of M satisfying:

$$(\mathbf{r} - \mathbf{c}(t)) \cdot \mathbf{n} = 0 \quad (1)$$

(see *Figure 1*). Writing $\mathbf{p}$ for the unit vector in the direction from $\mathbf{c}(t)$ to $\mathbf{r}$, we have:

$$\mathbf{r} = \mathbf{c} + \lambda \mathbf{p} \quad (2)$$

for some (positive) number λ which represents the 'depth' or 'distance' of $\mathbf{r}$ from $\mathbf{c}(t)$. We follow Blake and Cipolla[2,3] in taking an 'image sphere' which is a unit sphere centred at $\mathbf{c}(t)$. For image coordinates we use $\mathbf{p}$, regarded as a point of the unit sphere centred at the origin. We can also allow for camera rotation by writing $\mathbf{p} = R(t)\mathbf{q}$, where $R(t)$ is a rotation matrix depending on time t, with $R(0) =$ identity. Note that as a point in $\mathbf{R}^3$ the image point is $\mathbf{c}(t) + \mathbf{p}$ and that the distance from this image point to the surface point $\mathbf{r}$ is $\lambda - 1$.

In this paper, we are mostly concerned with 'local' results, that is, we can assume our surface M to be parametrized: $(u, v) \to \mathbf{r}(u, v)$. Notice that equation (1) then becomes one equation in three variables t, u, v, defining a surface in 3-space.

Definition 1 *The surface $\widetilde{M}$ defined by:*

$$(\mathbf{r}(u, v) - \mathbf{c}(t)) \cdot \mathbf{n}(u, v) = 0 \quad (3)$$

is called the spatio-temporal surface. *It is associated to M and the motion* $\mathbf{c}$ *of the camera centre* (*cf.* Faugeras[11]). *There is a natural projection $\pi : \widetilde{M} \to M$ given by $\pi(u, v, t) = \mathbf{r}(u, v)$.*

Thus $\widetilde{M}$ can be regarded as the union of the critical sets on M, spread out in the t-direction. On $\widetilde{M}$ we call the sets $t =$ constant *lifted critical sets*, $\widetilde{\Sigma}_t$. We shall often use the (u, v) parameter space of M in place of M itself. Then π becomes $(u, v, t) \to (u, v)$.

Example 2 The paraboloid

We shall illustrate this and some later concepts by means of a simple example. Consider the surface $M : z = x^2 + y^2$, parametrized by $r(u, v, u^2 + v^2)$, and let $c(t) = (1, t, t^2)$ be the path trace out by the camera centres (*see Figure 2, left*). Equation (3) of $\widetilde{M}$ becomes $f(u, v, t) = 0$, where:

$$f(u, v, t) = (u - 1)^2 + (v - t)^2 - 1 \quad (4)$$

Thus for a fixed t the equation $f = 0$ gives the critical set Σ_t in the u, v parameter space of M. This is clearly a circle, and the circles for increasing t move parallel to the v-axis, forming an *envelope* given by two lines $u = 0$, 2 (*Figure 2, right*). This will be important shortly, for it is clear that the critical sets cannot form part of a coordinate grid along these lines. The surface $\widetilde{M}$ is obtained by spreading the u, v critical sets out in the t direction, and is illustrated in *Figure 3*.

We want to ask when one of the parameters u, v on $\widetilde{M}$ and on M can be replaced by the 'time' parameter t. This is one step towards the 'epipolar parametrization' of M. The result is as follows:

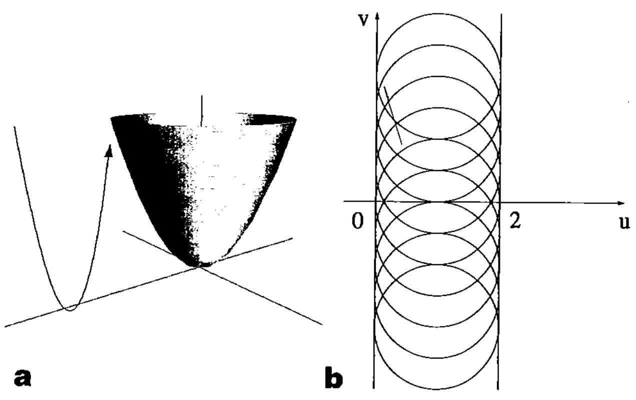

Figure 2 Left: the paraboloid and camera motion in Example 2. Right: the critical sets (circles) and their envelope (straight lines $u = 0, 2$) in the u, v parameter space

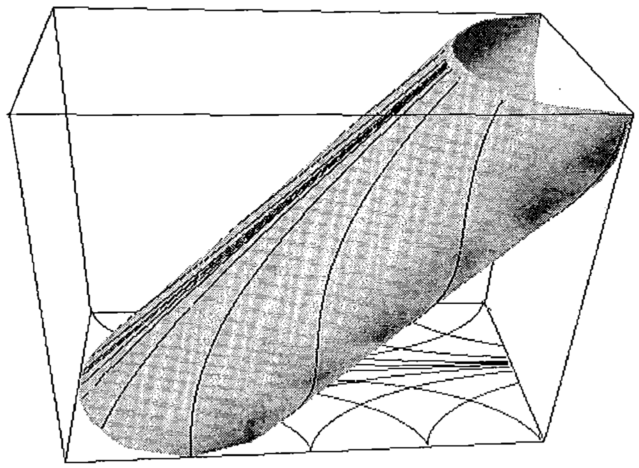

Figure 3 The spatio-temporal surface $\tilde{M}$ (sloping cylinder) in the paraboloid example of *Figure 2*. The base plane is the (u, v) parameter plane of M. Also shown are the epipolar curves on $\tilde{M}$ and in the parameter plane, where they cusp along the frontier (see Example 1)

Proposition 3 (i) *t can be used as one of the local parameters on $\tilde{M}$ unless* $\mathbf{r}$ *is a parabolic point and in addition the viewline* $\mathbf{r} - \mathbf{c}$ *is asymptotic at* $\mathbf{r}$.
(ii) *t can be used as one of the local parameters on M provided it can be used on $\tilde{M}$ and provided also* $\mathbf{c}_t \cdot \mathbf{n} \neq 0$.

Proof See Appendix A.

Note 4
The condition in Proposition 3(i) says precisely that $\mathbf{r}$ is not a 'lips/beaks' point, where the critical sets themselves become singular[12] (pp. 303, 458). There is of course no way in which singular curves can be part of a coordinate grid on a surface. We shall have more to say about lips/beaks singularities below.

The condition $\mathbf{c}_t \cdot \mathbf{n} \neq 0$ in Proposition 3(ii) can be interpreted in many other ways. Note that in Example 2 it holds *away* from the lines $u = 0$, 2 which form the envelope of critical sets. This is no accident. The critical sets are given by $(\mathbf{r} - \mathbf{c}) \cdot \mathbf{n} = 0$, and the envelope is found by adding the condition obtained by differentiating this with respect to t, namely $\mathbf{c}_t \cdot \mathbf{n} = 0$ (cf. Bruce and Giblin[13], p.102). Recall that the *epipolar plane* is spanned by $\mathbf{r} - \mathbf{c}$ and $\mathbf{c}_t$. Since $\mathbf{r} - \mathbf{c}$ is already tangent to M, this plane is the tangent plane to M at $\mathbf{r}$ if and only if $\mathbf{c}_t \cdot \mathbf{n} = 0$.

Note Here we have to exclude points where the viewline $\mathbf{r} - \mathbf{c}$ is *along* $\mathbf{c}_t$, for then the epipolar plane is *undefined.* At such points the motion is instantaneously towards the point $\mathbf{r}$ on M. In the motion literature, $\mathbf{c}$ is called the *focus of expansion* and depth cannot be determined at that point. It turns out that determination of epipolar curves near to these points is an extremely delicate problem, and unless otherwise stated we shall exclude such points from our considerations.

Definition 5 *The* frontier *F of M, relative to the given motion* $\mathbf{c}$, *is the set of points of M for which* $\mathbf{c}_t \cdot \mathbf{n} = 0$. *This can also be described as:*

1. *The envelope of critical sets on M (i.e. the locus of intersections of 'adjacent' critical sets on M);*
2. *The set of points of M where the epipolar plane (assumed defined) is the tangent plane to M: the set of 'epipolar tangency points';*
3. *The set of 'critical values' of the map π in Definition* 1, *i.e. the points of M under the 'fold line' of the projection π. Instead of π here we can equivalently use the linear 'vertical' projection of $\tilde{M}$ to the parameter space of M: $(u, v, t) \rightarrow (u, v)$.*

Thus Proposition (3) says that, except along the frontier of M and at lips/beaks points, the critical sets do form part of a local coordinate grid on M. As remarked earlier, the frontier points do *not* remain stationary as the camera moved, but slip along the frontier. An exception to this rule is motion where $\mathbf{c}$ moves in a plane P. If P is tangent to M, then any point of tangency is a stationary frontier point, but other frontier points may move around. For orthographic projection, motion in any plane P has stationary frontier points where the tangent plane to M is parallel to P. This is used by Kutulakos and Dyer[14] for producing a set of stationary points. Joshi *et al.*[15] used these points to compute structure from unknown motion.

In Example 2, the frontier is given by $u = 0$, 2. The frontier 'lifts' to $\tilde{M}$ as $\tilde{F}$, say, made up of the lines $(0, v, v)$ and $(2, v, v)$. Note that the surface $\tilde{M}$ is 'folded' along $\tilde{F}$ with respect to the linear projection $(u, v, t) \rightarrow (u, v)$, as in Definition 5(3). Note also that the lifted critical sets $\tilde{\Sigma}_t$ *do* form part of a coordinate grid on $\tilde{M}$.

We refer to the whole region of M covered by critical sets as the *visible region* of M. In the example, the visible region is that parametrized by the strip in the (u, v)-plane between the lines $u = 0$, $u = 2$, and in this simple case the frontier is precisely the *boundary of the visible region.*

The paraboloid example does not exhibit all possible features of the projection from $\tilde{M}$ to M. We collect these here for future reference.

Result 6: Local forms of the projection from $\tilde{M}$ to M

For generic M and camera motion, there are four possible patterns for the way in which the projection π of $\tilde{M}$ to M carries lifted critical sets $\tilde{\Sigma}_t$ to critical sets Σ_t. These are shown in *Figure 4*. The cases are:

(i) a non-frontier point, where π is a local diffeomorphism;
(ii) a non-parabolic frontier point which is a 'fold' (see Proposition 7)
(iii) a parabolic frontier point which is a fold. It turns out that the same pattern applies to the case where the motion is instantaneously towards $\mathbf{r}$: $\mathbf{r} - \mathbf{c} \| c_t$. However, the determination of epipolar curves is very delicate and we shall not cover this case;
(iv) a non-parabolic frontier point which is a 'cusp' (Proposition 7).

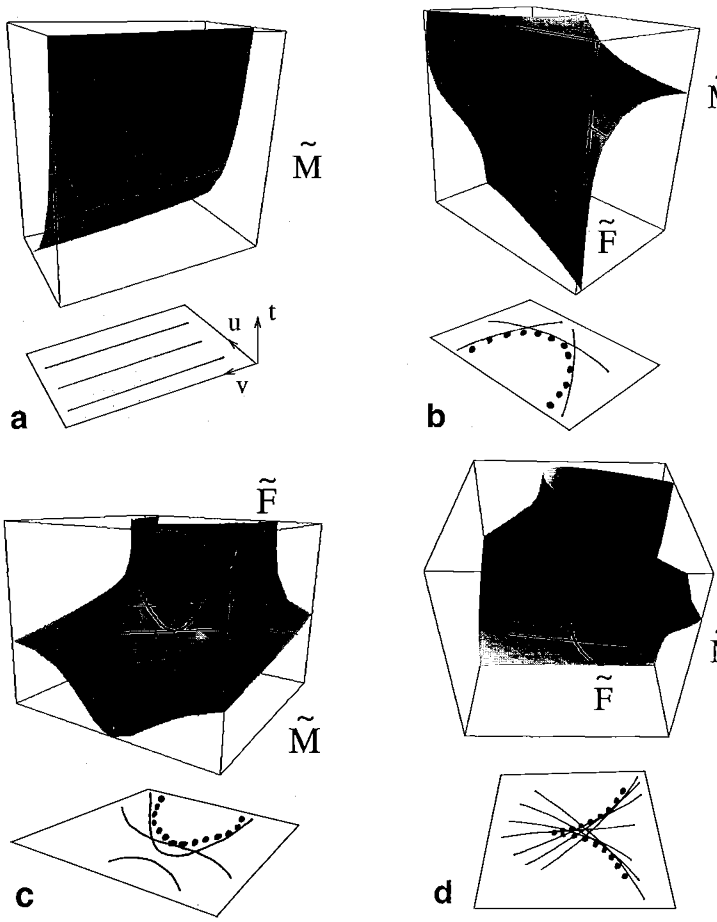

Figure 4 Local diagrams of the map π from $\tilde{M}$ to the (u, v) parameter plane of M. The lifted critical sets in $\tilde{M}$ and the critical sets in the parameter plane are drawn as thin lines. Except in (a), the lifted frontier $\tilde{F}$ in $\tilde{M}$ is a thicker line and the frontier in M is a heavy dotted line. Note that the parameter plane diagrams have been 'exaggerated' to separate frontier from critical sets and so, we hope, achieve greater clarity. The four generic cases are: (a) non frontier point; (b) non-parabolic frontier point; (c) parabolic frontier point; (d) non-parabolic frontier point, where the additional 'cusp' condition $\mathbf{c}_{tt} \cdot \mathbf{n} = 0$ holds

The patterns shown can be deduced from a general result on families of curves in the plane (or in a parametrized surface) by Dufour[16]; it is a straightforward matter to verify that his conditions correspond with the four cases stated above. We record here the simple test for fold and cusp points in our situation. (See, for example, Koenderink[12], pp. 438, 457 and Lu[17] p. 38, for information on fold and cusp points).

Proposition 7 *A frontier point is a fold point of* $\pi : \tilde{M} \to M$ *if and only if* $\mathbf{c}_t \cdot \mathbf{n} = 0$, $\mathbf{c}_{tt} \cdot \mathbf{n} \neq 0$, *and a cusp point if and only if* $\mathbf{c}_t \cdot \mathbf{n} = \mathbf{c}_{tt} \cdot \mathbf{n} = 0, \mathbf{c}_{ttt} \cdot \mathbf{n} \neq 0$. □

EPIPOLAR CURVES

The epipolar curves, together with the critical sets, give us the 'epipolar parametrization' of M used by Blake and Cipolla[2,3]. Let $\mathbf{r}$ be in the visible region of M, lying on a critical set Σ_t. Then at $\mathbf{r}$ the *epipolar field* has a vector along the (tangential) viewline $\mathbf{r} - \mathbf{c}$. For the most part, it will not matter what the length of this vector is: we are interested only in integral curves. Clearly, the epipolar field can be many-valued, since $\mathbf{r}$ may lie on several critical sets (*Figure 1*). It is in fact much better to lift the epipolar field to $\tilde{M}$:

Definition 8 *An* epipolar field *on* $\tilde{M}$ *is a nonzero tangent vector field such that the vector at* (u, v, t) *projects, under the differential of the map* π *(see Definition 1) to a vector parallel to the viewline* $\mathbf{r} - \mathbf{c}$. *An epipolar field on M is the multi-valued tangent vector field obtained by the projection* π *from* $\tilde{M}$. *The* epipolar curves *on* $\tilde{M}$ *are the integral curves of an epipolar field, and those on M are the projections of these curves under* π.

Note We show after Proposition 10 that in some circumstances any epipolar field on $\tilde{M}$ actually has a *zero*, that is, a place where it cannot be continuously defined as a nonzero vector field. Such points are isolated, and present the most complex behaviour of the epipolar curves (case (iii)(d) below).

Example 9 The paraboloid (continued)
We use the notation of Example 2. Given a point $\mathbf{r}(u, v)$, lying on a critical set Σ_t, we want the tangent vector to M which is along the viewline at $\mathbf{r}$, i.e. along the direction $(u, v, u^2 + v^2) - (1, t, t^2)$. The required vector in parameter space is therefore simply along $(u - 1, v - t)$. Of course, we can eliminate t, but at the expense of making the multi-valuedness explicit: using equation (4) we find that the vector at (u, v) in parameter space is $(u - 1, \pm\sqrt{u(2 - u)})$, which happens to be of unit length.

To find the epipolar field on $\tilde{M}$ we need to find a tangent vector to $\tilde{M}$ at (u, v, t) which projects to a vector parallel to $(u - 1, v - t)$ under the projection $(u, v, t) \to (u, v)$. Using the gradient of f from equation (4) as the normal to $\tilde{M}$, we want a vector parallel to $(u - 1, v - t, \xi)$ satisfying:

$$(u-1, v-t, \xi) \cdot (u-1, v-t, -(v-t)) = 0$$

The solution for ξ is $1/(v-t)$, and the vector solution can be written so that the curves are parametrized as $(u(t), v(t), t)$. Such a vector is $((u-1)(v-t), (v-t)^2, 1)$: we can take the epipolar field on $\widetilde{M}$ to be given by this formula. (In Proposition 10, we give a general prescription for finding an epipolar field on $\widetilde{M}$.)

To find the epipolar curves on $\widetilde{M}$ we want the solutions of the differential equation:

$$\frac{dv}{dt} = (v-t)^2$$

Substituting $w = v - t$ turns this into $dw/dt = w^2 - 1$, which gives $w = -\tanh(t+k)$ for any constant k, i.e. $v = t - \tanh(t+k)$. There are two 'exceptional' solutions, namely $v = t \pm 1$, which correspond to '$k = \mp\infty$'. Using equation (4), the corresponding solutions for u are $u = 1 \pm \operatorname{sech}(t+k)$. The exceptional solutions for v both give $u = 1$. So the epipolar curves on M are (for any constant k).

$$(u, v, t) = (1 \pm \operatorname{sech}(t+k), t - \tanh(t+k), t)$$
$$(u, v, t) = (1, t \pm 1, t) \qquad (5)$$

Note that these curves are always nonsingular and are necessarily transverse to the 'lifted critical sets' $\widetilde{\Sigma}_t$, which are given by $t =$ constant. This says that we can always parametrize $\widetilde{M}$ locally with a coordinate grid consisting of the $\widetilde{\Sigma}_t$ and the epipolar curves: 'the epipolar parametrization always works (locally) on $\widetilde{M}$.'

The frontier is given by $\mathbf{c}_t \cdot \mathbf{n} = 0$, where $\mathbf{c}_t = (0, 1, 2t)$ and $\mathbf{n} = (2u, 2v, -1)$. The epipolar field on M is obtained by projection from $\widetilde{M}$ (so of course it becomes zero on the frontier, since $v = t$ there). The epipolar curves on M are obtained by treating the first and second components in (5) as parametrizations with respect to t. For example, consider the curve which, at $t = 0$, passes through $u = v = 0$. This is the curve:

$$u = 1 - \operatorname{sech} t, \; v = t - \tanh t$$

which has initial terms in its MacLaurin expansion:

$$u = \frac{1}{2}t^2 + \ldots, \; v = -\frac{1}{3}t^3 + \ldots.$$

This curve, like all the epipolar curves on M apart from the 'exceptional' curve $u = 1$, has an ordinary cusp where it meets the frontier. (The exceptional curve does not meet the frontier.) The shape of the epipolar curves in $\widetilde{M}$ and in the parameter plane of M is shown in *Figure 3*. The visible region here is that between the lines $u = 0, u = 2$.

It is not difficult to find a general prescription for an epipolar field on $\widetilde{M}$, as follows:

Proposition 10 *An epipolar field on $\widetilde{M}$ has the form:*

$$\left((\mathbf{c}_t \cdot \mathbf{n}) \left(\frac{[\mathbf{r}-\mathbf{c}, \mathbf{r}_v, \mathbf{n}]}{\|\mathbf{n}\|^3} \right), (-\mathbf{c}_t \cdot \mathbf{n}) \left(\frac{[\mathbf{r}-\mathbf{c}, \mathbf{r}_u, \mathbf{n}]}{\|\mathbf{n}\|^3} \right), - \mathrm{II}(\mathbf{r}-\mathbf{c}, \mathbf{r}-\mathbf{c}) \right) \qquad (6)$$

Here, II is the second fundamental form of M (e.g. see Koenderink[12], pp. 226, 232 and O'Neill[18], p. 208), and we can take $\mathbf{n}$ as any nonzero normal vector, not necessarily unit length; for example $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$. For the proof of the proposition, see Appendix B.

Notes on the formula in Proposition 10

1. The only circumstance in which *all three* entries in equation (6) are zero is when $\mathbf{c}_t \cdot \mathbf{n} = 0$ (frontier point, Definition 5) and $\mathrm{II}(\mathbf{r}-\mathbf{c}, \mathbf{r}-\mathbf{c}) = 0$, i.e. the viewline $\mathbf{r} - \mathbf{c}$ is asymptotic at $\mathbf{r}$. This means that the corresponding profile is singular[12] (pp. 422, 437). So, away from singular profiles at frontier points, the epipolar field on $\widetilde{M}$ is smooth and nonzero and the epipolar curves are smooth. We consider singular profiles at frontier points in case (iii)(d) below.
2. The epipolar curve on M will be smooth unless the *first two entries* in (6) are zero, which happens precisely at frontier points. This says that *the epipolar curves on M are smooth except along the frontier of M*, as we observed in example 9. The frontier itself can be singular (have a cusp), and we consider that case in case (iii)(c) below (cf. *Figure 4*).
3. A smooth epipolar curve through $\mathbf{r} \in M$ will be tangent to the corresponding critical set Σ_t precisely when the viewline at $\mathbf{r}$ is tangent to Σ_t. This is the condition for a singular profile. Apart from this, the epipolar curve will cross critical sets transversally, and so *the epipolar curve will be parametrized locally by t.*
4. The formula for the epipolar field on $\widetilde{M}$ looks impressive, but if we regard $\widetilde{M}$ as contained in $M \times \mathbf{R}$ then it really says that the epipolar tangent vector to $\widetilde{M}$ is along $(\mathbf{c}_t \cdot \mathbf{n}(\mathbf{r}-\mathbf{c}), -\mathrm{II}(\mathbf{r}-\mathbf{c}, \mathbf{r}-\mathbf{c}))$.
5. It is a standard fact of surface geometry (ref. 3 eq (6); ref. 18, pp. 200, 208) that $\mathrm{II}(\mathbf{v}, \mathbf{v})$, for a tangent vector $\mathbf{v}$, is just the sectional curvature of M in the direction $\mathbf{v}$, scaled by $\|\mathbf{v}\|^2$. Thus, in our case, the term $\mathrm{II}(\mathbf{r}-\mathbf{c}, \mathbf{r}-\mathbf{c})$ in (6) can be rewritten κ^t/λ^2, where κ^t is the 'transverse curvature', i.e. the sectional curvature of M in the direction of viewing, and λ is the depth as in (2). Both quantities here can be measured from the image[2].
6. Of course, there is a similar formula to (6) in the case of parallel projection with variable viewing direction $\mathbf{w}(t)$. In fact, it is identical to the above formula, replacing $\mathbf{r} - \mathbf{c}$ by $\mathbf{w}$ and $\mathbf{c}_t$ by $\mathbf{w}_t$, except that, for reasons of orientation, the sign in front of II becomes $+$. In the reinterpretation as in the note above, we have simply $\mathrm{II}(\mathbf{w}, \mathbf{w}) = \kappa^t$.

We pause here to mention some geometrical properties of epipolar curves. Suppose that an epipolar curve C is (locally) parametrized by t (see note 3 above) as $\mathbf{r}(t)$, say. Then $\mathbf{r}(t) - \mathbf{c}(t) = \mu(t)\mathbf{r}'(t)$ for some function μ, where $'$ stands for derivative. Differentiating this equation with respect to t shows that r'' is in the plane of r' and c', which is here the epipolar plane. Now the plane of r' and r'' is the *osculating plane* of C (ref. 17, p. 168), so we have shown the first part of the following. The other parts are not difficult to establish; we omit the details:

Proposition 11 *The osculating plane of an epipolar curve is the epipolar plane.*

The epipolar curve has a zero of curvature at **r** *when the corresponding profile is singular. The limit of the osculating planes approaching the point* **r** *is the epipolar plane at* **r**.

The limit of the osculating planes approaching a frontier point **r** *(where the epipolar curve is singular* (see note 2 above), *is the tangent plane to M at* **r**. *In fact this applies to any cusped curve on a surface: the limit of osculating planes at points approaching the cusp is the tangent plane. Note that this agrees with the interpretation of frontier points in Definition* 5(2).

BREAKDOWN OF THE EPIPOLAR PARAMETRIZATION

The (local) epipolar parametrization of M near **r** breaks down when any of the following occur:

- the critical set and epipolar curve on M are smooth and tangent to one another;
- the critical set on M is singular;
- the critical sets form an envelope on M.

In the last case, we know (see note 2 on Proposition 10) that the epipolar curves are singular, which also precludes their use in a parametric grid on M. The epipolar curves do not themselves form an envelope (unless we count this 'singular' envelope along the frontier).

We shall examine those cases which can be expected to happen for a generic surface and generic camera motion.

Away from the frontier, the patterns of critical sets are all well-known[12] (Ch. 8). The patterns of epipolar curves are not hard to determine, since the epipolar field is non-zero (see note 1 on Proposition 10), and we know when it is tangent to the critical set (see note 3 on Proposition 10).

On the frontier, we have more work to do, but we have already exhibited the patterns of critical sets above (see item (6)) and in *Figure 4*. For the epipolar curves, we start with the spatio-temporal surface $\tilde{M}$ and use general theorems on the solution curves of differential equations. The appropriate theorems are given elsewhere[19,20], and in all cases it is a straightforward (though sometimes lengthy) matter to relate the hypotheses of the theorems to the geometry of our present situation. It is usually best in these verifications to set up the surface M in 'Monge form', that is, as the graph of a function:

$$z = h(x, y) = a_0x^2 + a_1xy + a_2y^2 + b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3 + \dots$$

We can take a path of centres of the form:

$$c(t) = (\lambda_0 + c_1(t), c_2(t), c_3(t))$$

where the c_i all vanish at $t = 0$. Thus $\mathbf{O} = (0, 0, 0)$ is on Σ_0, and the viewline here is along the x-axis. The following conditions can be found in Proposition 7 or in Bruce and Giblin[21]; some are also implicit in Koenderink[12] (p. 458):

- a frontier point at **O** corresponds to $c_3'(0) = 0$ (Definition 5);
- a cusp point corresponds to $c_3'(0) = c_3''(0) = 0$ (and $c_3'''(0) \neq 0$);
- a singular profile corresponds to $a_0 = 0$ (viewline is asymptotic (see note 3 to Proposition 10);
- a lips/beaks point corresponds to $a_0 = a_1 = 0$ (viewline is asymptotic and **O** is parabolic – see note 4 above). Also, $3b_2b_0 - b_1^2 \neq 0$: for <0 we get lips and for >0 we get beaks;
- a 'swallowtail' point corresponds to $a_0 = b_0 = 0$ (and $a_1 \neq 0$, coefficient of $x^4 \neq 0$); the viewline has fourfold contact with the surface at **O** and the profile has a swallowtail singularity.

We turn now to a more detailed description of the failures of the epipolar parametrization.

Case (i)(a)

Cusp on profile at **p**, *surface point* **r** *non-parabolic and not on the frontier.*

Thus the viewline $\mathbf{r} - \mathbf{c}$ is asymptotic at **r**. The pattern of critical sets and epipolar curves in M (or equivalently in $\tilde{M}$) is shown in *Figure 5*, where we do not attempt to convey the shape of M but merely the arrangement of curves. Note the 'cusp locus' L, consisting of points of M giving profile cusps, passing through the tangencies of the critical sets and the epipolar curves. The epipolar parametrization breaks down because of the tangency, but a new parametrization can be based on the critical sets and a family of curves containing L. This is used by Cipolla and Giblin[22]. When the surface is opaque, half of each cusp in the image is occluded, and in M the pattern on one side of the line L is removed.

Case (i)(b)

Swallowtail point on profile at **p**, *surface point* **r** *non-parabolic and not on frontier*

Thus the viewline $\mathbf{r} - \mathbf{c}$ is 'flecnodal' at **r**, that is has 4-point contact with M there. The pattern in M is shown in *Figure 6*. Note that here the line of cusps L is actually tangent to a critical set, and the epipolar curve through this tangency inflects the critical set. Between the cusps on the 'swallowtail' figure in the image there is a double point (self-crossing of the profile), and we can trace the locus D of points in M which give rise to these. For an opaque surface the part between one branch of L and the opposite branch of D is occluded.

Case (ii)

Lips/beaks point on profile at **p**, *surface point* **r** *is therefore parabolic, with viewline asymptotic there*

The critical sets here are singular, displaying a 'Morse'

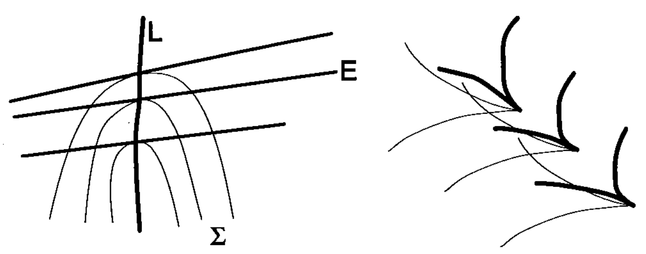

Figure 5 Cusp case (i)(a). Left: critical sets Σ (thin lines), epipolar curves E (thicker lines), and line of cusps L (thick line) in M. (No attempt is made here to indicate the shape of M.) Right: a typical pattern of profiles (thin lines) and epipolar curves (thick lines) in the image sphere (see Proposition 13). Note that one branch of each cusp is occluded in the opaque case

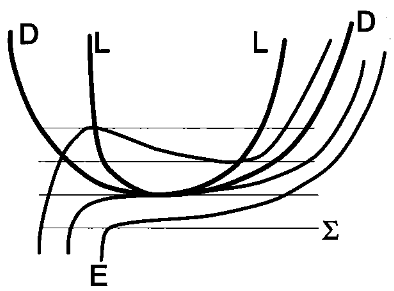

Figure 6 Swallowtail case (i)(b). Critical sets Σ (thin lines), epipolar curves E (thicker lines), line of cusps L and line of double points D (thick lines) in M. Note that the region between one branch of L and the 'opposite' branch of D is occluded for an opaque surface

transition through a crossing or isolated point. The epipolar curves on the other hand remain non-singular and the pattern is shown in *Figure 7*, which also shows the cusp locus L.

Case (iii)(a)

Frontier point **r**, *not parabolic, frontier smooth at* **r**, *profile smooth at* **p** (see Definition 5)
The pattern of critical sets is shown in *Figure 4*, and we show the epipolar curves in *Figure 8*. As in example 9 (*Figure 3*) the epipolar curves cusp along the frontier in M, which means that in $\tilde{M}$ they have 'vertical' tangent lines where they cross the lifted frontier $\tilde{F}$.

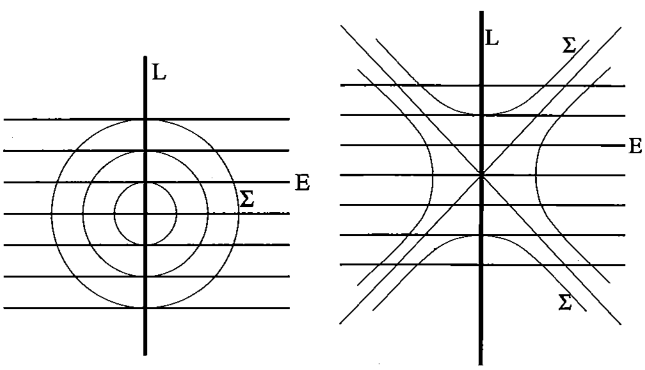

Figure 7 Lips/beaks case (ii). Left: lips, right: beaks. Critical sets Σ (thin lines). (For the beaks case, the two halves of *one* critical set are labelled Σ.) epipolar curves E (thicker lines) and line of cusps L (thick line) in M. For an opaque surface, all to one side of L is occluded

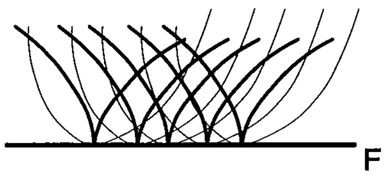

Figure 8 Critical sets (thin lines) and epipolar curves (thicker, cusped lines) along the frontier F at a non-singular point of F and non-parabolic point of M. (No attempt is made to indicate the shape of M; compare *Figure 4 (b)*.) No part is occluded for an opaque surface

Case (iii)(b)

Frontier point **r**, *parabolic, frontier smooth at* **r**, *profile smooth at* **p**
The critical sets have a different pattern here (*Figure 4*), but the epipolar curves have the same pattern as in *Figure 8*. We draw the result schematically in *Figure 9*.

Case (iii)(c)

Frontier point **r**, *non-parabolic, frontier cusped at* **r**, *profile smooth at* **p**
In this case the epipolar curves undergo a 'swallowtail' transition, as shown in *Figure 10*. (The conditions to be checked come from Theorem 2 of Bruce[19], and, setting everything up in Monge form as above, they come to: view direction not asymptotic ($a_0 \neq 0$); camera motion

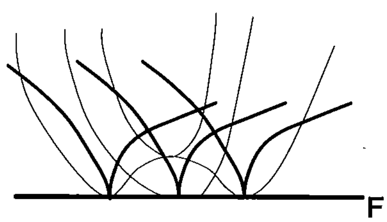

Figure 9 Critical sets (thin lines) and epipolar curves (thicker, cusped lines) at a parabolic point of the frontier F on M. Notice that the pattern of critical sets is different from *Figure 8*; compare *Figure 4 (c)*. No part is occluded for an opaque surface

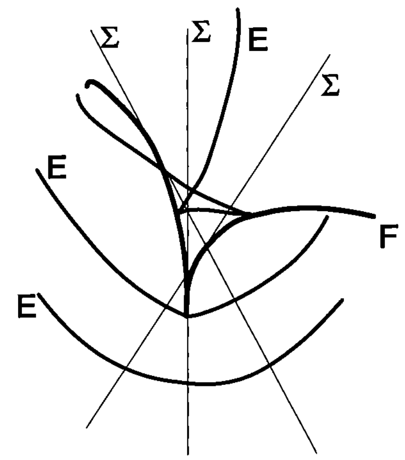

Figure 10 Typical critical sets Σ (thin lines) and epipolar curves E (thicker lines) near a cusp on the frontier F (thick line). The epipolar curves pass through a 'swallowtail transition' while the critical sets all remain smooth. No part is occluded for an opaque surface

not towards **O** ($c_2'(0) \neq 0$); non-parabolic ($a_1^2 \neq 4a_0a_2$); and a genuine cusp point (Proposition 7, $c_3'''(0) \neq 0$).)

Case (iii)(d)

Frontier point **r**, *non-parabolic, frontier smooth at* **r**, *profile singular at* **p**

This is the most bizarre case, because the epipolar field on $\widetilde{M}$ is actually singular (zero) at such a point. To examine the situation, we set up our surface M in Monge form as above. A straightforward, if lengthy, calculation shows that the epipolar field on $\widetilde{M}$ close to $(x, y, t) = (0, 0, 0)$ (using x and y as local coordinates on M) has the following form, up to linear terms in x and t (all derivatives are at 0):

$$\mathrm{d}x/\mathrm{d}t = -\lambda_0 c_2' a_1 x + \lambda_0 c_3'' t$$

$$\mathrm{d}y/\mathrm{d}t = 6\lambda_0^2 b_0 x + 2\lambda_0 c_2' a_1 t$$

This vector field has a *singularity* (a zero) at $x = t = 0$. Hence, there are generically three possibilities for the nature of the integral curves: a node, a focus and a saddle (see, for example, Schwarzenberger[23], Ch. 4, or any book on elementary differential equations). Write A for the quantity:

$$\frac{\lambda_0 c_3'' b_0}{a_1^2 c_2'^2}$$

Then the distinction between the three cases is as follows:

1. **node**, i.e. the matrix of coefficients in the linearized vector field above has real distinct eigenvalues of the same sign, if and only if $-3/8 < A < -1/3$;
2. **saddle**, i.e. the matrix has real distinct eigenvalues of opposite signs, if and only if $A > -1/3$;
3. **focus**, i.e. the matrix has complex conjugate eigenvalues, if and only if $A < -3/8$.

In each case, the behaviour of the epipolar curves is very complex, and we have restricted ourselves to illustrating one case in *Figure 11*, that of a focus. The locus of places where the (spiral) epipolar curve is tangent to the lifted critical sets is the lifted locus of (profile) cusps, $\widetilde{L}$, which passes through the intersection of lifted critical set 2 and the lifted frontier $\widetilde{F}$. In M the epipolar curve cusps infinitely often on the frontier F, and the locus L of (profile) cusps is tangent to F at the point where critical set 2 is tangent to F. In this figure, $\widetilde{L}$ and L are drawn dashed. For an opaque surface, everything to one side of these lines is occluded.

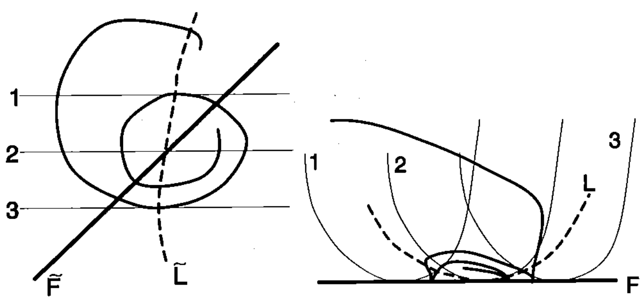

Figure 11 The 'focus' situation for a profile cusp on the frontier. Left: in $\widetilde{M}$, right: in M. The thin numbered lines are critical sets (or lifted sets, in $\widetilde{M}$). The frontier F and lifted frontier $\widetilde{F}$ are drawn as thick straight lines. One (spiral) epipolar curve is drawn in $\widetilde{M}$ and its image is drawn in M. The lines $\widetilde{L}$ and L are lifted locus of (profile) cusps and locus of (profile) cusps, respectively

We have usually excluded the following case, which we now mention briefly.

Case (iii)(e)

Point where the camera motion is instantaneously towards the surface point $\mathbf{r}: \mathbf{r} - \mathbf{c} \| \mathbf{c}_t$. *Automatically a frontier point, frontier smooth,* **r** *non-parabolic, profile smooth at* **p**

As mentioned above, the pattern of critical sets is essentially the same as in Case (iii)(b) (parabolic point on the frontier) above, so we do not repeat them. But the epipolar curves are more subtle and require recent results of Bruce and Tari[24] for analysis. [Added in proof. The epipolar curves also behave exactly as in case (iii)(b).]

EPIPOLAR CURVES IN THE IMAGE SPHERE

For the most part we are concerned here with epipolar curves in the surface M (or in the spatio-temporal surface $\widetilde{M}$), but here we make two remarks on the situation in the image sphere, one concerning the *epipolar correspondence* in the image sphere and the other concerning the case where the profile has a cusp. As in equation (2), we can use unrotated **p** coordinates in the image; we can also use coordinates **q** which are rotating with the moving camera: $\mathbf{p} = R(t)\mathbf{q}$, where R is a rotation matrix. (In Cipolla and Blake[2], **p** appears as **Q** and **q** appears as $\widetilde{\mathbf{Q}}$.)

Let us assume that t and another parameter s are local coordinates on M, where $s =$ constant gives the epipolar curves. Differentiating (2) with respect to t and imposing the epipolar condition $\mathbf{r}_t \| \mathbf{p}$, we find that $\mathbf{p}_t$ is parallel to $\mathbf{c}_t + \mu\mathbf{p}$ for some μ. But $\mathbf{p}_t \cdot \mathbf{p} = 0$ so that $\mu = -\mathbf{c}_t \cdot \mathbf{p}$ and we have the tangent to the epipolar curve in the image sphere in **p** coordinates is

$$\mathbf{p}_t \| (\mathbf{c}_t - (\mathbf{c}_t \cdot \mathbf{p})\mathbf{p})$$

We could call the vector on the right here the 'epipolar field' on the **p** coordinates image sphere.

What about **q** coordinates? For this we shall use the *normal* parametrization, where we retain t as one parameter, but $s =$ constant holds now along the normals to the profiles in the image. Thus $\mathbf{q}(s, t)$ for $t =$ constant is a profile and for $s =$ constant is the orthogonal trajectory of the profiles. Then the epipolar curve in the image will be a curve $s = s(t)$ such that:

$$\mathbf{p}(s(t), t) = R(t)\mathbf{q}(s(t), t)$$

R being the rotation, and:

$$\frac{d}{dt}\mathbf{p}(s(t), t) \quad \text{is parallel to } \mathbf{c}_t - (\mathbf{p}\cdot\mathbf{c}_t)\mathbf{p}$$

(see above). Thus, at $t = 0$, assuming $R(0)$ = identity as usual:

$$\mathbf{q}_s s' + \mathbf{q}_t + \Omega \times \mathbf{q} = \alpha(\mathbf{c}_t - (\mathbf{p}\cdot\mathbf{c}_t)\mathbf{p})$$

for some real function α. Taking the scalar product with $\mathbf{q}_t$ (which is perpendicular to $\mathbf{q}_s$) we get:

$$\alpha = \frac{\mathbf{q}_t \cdot \mathbf{q}_t + [\Omega, \mathbf{q}, \mathbf{q}_t]}{\mathbf{q}_t \cdot \mathbf{c}_t}$$

where it is important to remember that $\mathbf{q}_t$ is measured perpendicular to the profile, so can be measured from the image.

Proposition 12 *A vector tangent to the epipolar curve in the image, in* $\mathbf{q}$ *coordinates and with the normal parametrization, is:*

$$\alpha(\mathbf{c}_t - (\mathbf{c}_t \cdot \mathbf{q})\,\mathbf{q}) - \Omega \times \mathbf{q}$$

for the above α. *(Remember that at* $t = 0, \mathbf{p} = \mathbf{q}$.*)*

Our second observation concerns the form of the epipolar curve, in $\mathbf{p}$ or in $\mathbf{q}$ coordinates, when the profile is singular. We assume that the locus of cusps on M is transverse to the critical sets, as in *Figure 5*, and not tangent as in *Figure 6*. Then we parametrize M locally with t as one parameter and, say s as the other, where $s = 0$ gives the locus of cusps. We are interested in the epipolar curve in the image sphere close to $s = t = 0$. Let the epipolar curve on M through $\mathbf{r}(0,0)$ be $t = T(s)$, say, where $T(0) = T'(0) = 0$ since the epipolar curve is tangent to the critical set. The epipolar curve in the image is $\mathbf{p}(T(s), s)$ with tangent $p_t T' + p_s$. But $p_s(0,0) = 0$ since the profile has a cusp, and $T'(0) = 0$, so the epipolar curve in the image ($\mathbf{p}$ coordinates) is singular too. Note that this only assumes that in M we start with a curve tangent to the critical set; it does not have to be the epipolar curve in fact.

In $\mathbf{q}$ coordinates we have $\mathbf{p}(t,s) = R(t)\mathbf{q}(t,s)$ so that $\mathbf{p}_s = \mathbf{q}_s$. The tangent to the epipolar curve in $\mathbf{q}$ coordinates is $q_t T' + q_s$ and again this is zero at $s = t = 0$.

Proposition 13 *If the profile has a cusp then the epipolar curve in the image, using unrotated* ($\mathbf{p}$) *or rotated* ($\mathbf{q}$) *coordinates, has a cusp at the same point as the profile.*

We have illustrated this in *Figure 5*.

GEOMETRY OF THE VIEWLINES

Consider a regular parametrization of the surface M by $\mathbf{r}(u,t)$ and a camera trajectory $\mathbf{c}(t)$ such that for some fixed u_0 the view lines $\mathbf{c}(t) + \lambda(u_0,t)\,\mathbf{p}(u_0,t)$ are tangent to the surface at $\mathbf{r}(u_0,t)$ are tangent to the surface at $\mathbf{r}(u_0,t)$ (compare (2)). Note that Proposition 3 gives the precise conditions under which this is possible. Thus, for the parameter curve $r(u_0,t)$, parametrized by t, there is a one parameter family of viewlines $l(t)$ such that $l(t)$ is tangent to the surface at $r(u_0,t)$. Intuitively, reconstruction algorithms are based on intersections of viewlines. In practice, these viewlines may not intersect, and the points where they are closest may not even approach a point on the surface. However, there are cases where the closest points do converge to a point on the surface, that is, if $l(t)$ is a tangent line in a family, then the point on $l(t)$ closest to the line $l(t+\varepsilon)$ approaches $\mathbf{r}(u_0,t)$ as $\varepsilon \to 0$. We show that there are two parametrizations which have this geometric property: the epipolar parametrization and the normal parametrization. More formally, this can be stated as follows:

Proposition 14 *The distance from the camera centre to the surface at the point of tangency is to first order given by the distance from the camera centre to the point where this viewline is closest to the viewline for a nearby camera position if and only if the parametrization satisfies either the epipolar or normal constraint.*

One interesting feature of this result is that it gives a *new formula for the depth* (Lemma 15).

Proof Suppose the two viewlines are determined by $(\mathbf{c}(t), \mathbf{r}(u_0,t))$ and $(\mathbf{c}(t+\varepsilon), \mathbf{r}(u_0, t+\varepsilon))$, respectively. These lines in $\mathbf{R}^3$ will not in general intersect, but it is possible to solve for the points where they are closest to each other. Let $\mathbf{a}$ be a point on the first line and $\mathbf{b}$ a point on the second line.

$$\mathbf{a} = \mathbf{c}(t) + \alpha\mathbf{p}(u_0,t)$$
$$\mathbf{b} = \mathbf{c}(t+\varepsilon) + \beta\mathbf{p}(u_0, t+\varepsilon)$$
$$\mathbf{b} - \mathbf{a} = \varepsilon\mathbf{c}_t + (\beta - \alpha)\mathbf{p} + \beta\varepsilon\mathbf{p}_t + \frac{1}{2}\varepsilon^2(\mathbf{c}_{tt} + \beta\mathbf{p}_{tt}) + O(\varepsilon^3)$$

If $\mathbf{a}$ and $\mathbf{b}$ are the closest points, then the line $\mathbf{b} - \mathbf{a}$ is perpendicular to each of these two lines. This can be written as:

$$(\mathbf{b} - \mathbf{a})\cdot\mathbf{p}(u_0,t) = 0$$
$$(\mathbf{b} - \mathbf{a})\cdot\mathbf{p}(u_0,t+\varepsilon) = 0$$

Here, α and β can be thought of as functions of ε and have Taylor series expansions around $\varepsilon = 0$. Solving for α as a Taylor series in ε and taking the limit as ε goes to 0, we can solve for the constant term. Since we are considering a fixed value of u_0 and the expansion is around t, we omit those parameters from the expressions:

$$0 = (\mathbf{b} - \mathbf{a})\cdot\mathbf{p}$$
$$= \varepsilon\mathbf{c}_t\cdot\mathbf{p} + (\beta - \alpha) + \frac{1}{2}\varepsilon^2(\mathbf{c}_{tt}\cdot\mathbf{p} + \beta\mathbf{p}_{tt}\cdot\mathbf{p})$$
$$0 = (\mathbf{b} - \mathbf{a})\cdot\mathbf{p}(t+\varepsilon)$$
$$= (\mathbf{b} - \mathbf{a})\cdot p + \varepsilon^2(\mathbf{c}_t\cdot\mathbf{p}_t + \beta\mathbf{p}_t\cdot\mathbf{p}_t + \frac{1}{2}(\beta - \alpha)\mathbf{p}\cdot\mathbf{p}_{tt})$$
$$+ O(\varepsilon^3)$$

Manipulating these equations to solve for α gives, for $\varepsilon \to 0$:

$$\alpha = \frac{-\mathbf{c}_t\cdot\mathbf{p}_t}{\mathbf{p}_t\cdot\mathbf{p}_t}$$

This is therefore the distance from the camera centre to the point **a** of closest approach of the two viewlines, in the limit. Note that this is defined whenever $\mathbf{p}_t \neq 0$, including most points of the frontier[3], whereas the standard depth formula $\lambda = (-\mathbf{c}_t \cdot \mathbf{n})/(\mathbf{p}_t \cdot \mathbf{n})$ is not defined at the frontier. We now complete the proof of Proposition 14 by showing that this is equal to the distance to the surface if and only if $\mathbf{r}(u, t)$ is an epipolar or normal parametrization, i.e. $\mathbf{r}_t \| \mathbf{p}$ or $\mathbf{p}_t \| \mathbf{n}$.

Lemma 15 *If $\mathbf{r}(u, t)$ is a parametrization of M with t being the parameter of a moving camera centre then for a point not on the frontier:*

$$\frac{-\mathbf{c}_t \cdot \mathbf{p}_t}{\mathbf{p}_t \cdot \mathbf{p}_t} = \frac{-\mathbf{c}_t \cdot \mathbf{n}}{\mathbf{p}_t \cdot \mathbf{n}} \tag{7}$$

if and only if $\mathbf{r}_t \| \mathbf{p}$ or $\mathbf{p}_t \| \mathbf{n}$. (The right side of this is the standard depth formula of Cipolla and Blake[3].)

Proof $\Leftarrow$ Case 1 If $\mathbf{r}_t \| \mathbf{p}$ then differentiating equation (2), it follows that $\mathbf{c}_t$ is in the plane of $\mathbf{p}$ and $\mathbf{p}_t$, so for some ξ, η:

$$\mathbf{c}_t = \xi \mathbf{p} + \eta \mathbf{p}_t$$

Take dot products with the surface normal and the tangent to the trace of the epipolar curve:

$$\mathbf{c}_t \cdot \mathbf{n} = \eta \mathbf{p}_t \cdot \mathbf{n}$$

$$\mathbf{c}_t \cdot \mathbf{p}_t = \eta \mathbf{p}_t \cdot \mathbf{p}_t$$

Eliminating η gives the desired equation.

Case 2 If $p_t \| n$ then substitution gives the desired equation.

$\Rightarrow$ Equation (7) can be rewritten as:

$$(-\mathbf{c}_t \cdot \mathbf{p}_t)(\mathbf{p}_t \cdot \mathbf{n}) - (-\mathbf{c}_t \cdot \mathbf{n})(\mathbf{p}_t \cdot \mathbf{p}_t) = 0$$

This can be written in terms of cross products as:

$$(\mathbf{p}_t \times (\mathbf{c}_t \times \mathbf{p}_t)) \cdot \mathbf{n} = 0$$

This implies that $\mathbf{v} = (\mathbf{p}_t \times (\mathbf{c}_t \times \mathbf{p}_t))$ is in the tangent plane to the surface. Since this is a tangent vector and perpendicular to $\mathbf{p}_t$, either $\mathbf{p}_t \| \mathbf{n}$ and every tangent vector is perpendicular to $\mathbf{p}_t$ or $\mathbf{v}$ is a multiple of $\mathbf{p}$:

- Case 1: $\mathbf{v} \| \mathbf{p}$. By the fact that the cross product of two vectors is always perpendicular to each of the factors, $\mathbf{v}$ is always in the plane spanned by $\mathbf{c}_t$ and $\mathbf{p}_t$. Then it follows that the triple product $[\mathbf{p}, \mathbf{p}_t, \mathbf{c}_t] = 0$. This says that since generically $\mathbf{p}_t \neq 0$, $\mathbf{c}_t$ is in the plane spanned by $\mathbf{p}$ and $\mathbf{p}_t$. Using the equation:

  $$\mathbf{r}_t = \mathbf{c}_t + \lambda_t \mathbf{p} + \lambda \mathbf{p}_t$$

  it follows that $[\mathbf{p}, \mathbf{p}_t, \mathbf{r}_t] = 0$. Thus, $\mathbf{r}_t$ is in the epipolar plane as well, and by definition it is a tangent vector. Except at the frontier, the intersection of the epipolar plane and the tangent plane is just a line in the direction of $\mathbf{p}$. This shows that $\mathbf{r}_t \| \mathbf{p}$.
- Case 2: $\mathbf{p}_t \| \mathbf{n}$. For the spherical image, the normal to the profile is the normal to the surface, so $\mathbf{p}_t$ is normal to the profile. This is the normal correspondence used in Cipolla and Blake[3] for stationary curves on the surface, and results in a parametrization when tracking the profiles of critical sets, except when the profiles are singular. □.

CONCLUSION

The epipolar parametrization of a surface M has been shown elsewhere to be very useful in the reconstruction process. This paper gives some additional geometric properties which suggest that the epipolar curves are easy to compute. We also present the criteria for failure of the epipolar parametrization, namely, at the frontier and at a singularity of the profile, e.g. a cusp point. We have shown that at the frontier we cannot parametrize M using critical sets as parameter curves, but that the epipolar curves can be understood using the 'spatio-temporal surface' $\tilde{M}$, which is (except at a (profile) cusp point on the frontier) parametrized locally by lifted critical sets and lifted epipolar curves. In these cases, we have found the detailed structure of the epipolar curves around the point at which the epipolar parametrization breaks down.

We have described which parts of this structure are occluded when the surface M is opaque. The boundary due to occlusion, which we call the *natural boundary*, has two types of points. These points are cusp points (where the profile ends) and T-junctions (where two points on the critical set project to the same point). As the camera moves these points trace out curves L, T on the surface which are the boundary between the parts of the surface which actually appear on an unoccluded profile and those which do not. In the case of T-junctions, it is the point which is farther from the camera that traces the curve T. *Figures 5, 6, 7, 11* show examples of lines of cusps (and in one case T-junctions). The curves L, T meet at special points to form simple closed curves. For generic surfaces, the special points are the six codimension-one local and multilocal visual events: swallowtail, lips, beaks, triple crossing, cusp crossing and tangent crossing[25]. For example, at a swallowtail a line of cusps meets a T-junction trajectory as shown in *Figure 6*. At a lips or breaks point, two lines of cusps meet as shown in *Figure 7*. At a triple point, two T-junction trajectories meet.

In general, the reconstruction of surfaces from profiles leaves gaps. These gaps are bounded by frontier curves or natural boundaries. Making multiple passes over the surface can reduce these gaps, but may not eliminate them if the object has concavities or the camera trajectory is limited by environmental constraints. Note that the frontier curve can be reconstructed by triangulation. Once one has determined where the frontier and natural boundary curves are, there are three ways to fill in the gaps. One can actively move the camera to a trajectory[26] that will reconstruct some missing parts of the surface (if this is possible). One could also apply different sensors or use information from surface markings and texture (as in multiframe stereo). A third approach is to use other information which is implicitly present in the profiles. The profile of a surface from a given view determines a bounding cone. The intersection of these cones from different viewpoints can be used to construct the bounding volume for an object[27,28]. Merging of the appropriate pieces of these two surfaces along the frontier and the natural boundary can produce a closed surface which uses all of the information available from the profiles. Additional information from surface markings can potentially be combined for visible regions which are not covered by the critical sets.

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APPENDIX A: CONDITION FOR t AS ONE PARAMETER ON M AND $\widetilde{M}$ (PROPOSITION 3)

(i) Noting that $\mathbf{r}_u \cdot \mathbf{n} = \mathbf{r}_v \cdot \mathbf{n} = 0$ since $\mathbf{r}_u, \mathbf{r}_v$ are tangent vectors to M, the Jacobian matrix of the left side of (3) is:

$$((\mathbf{r}-\mathbf{c})\cdot\mathbf{n}_u \qquad (\mathbf{r}-\mathbf{c})\cdot\mathbf{n}_v \qquad -\mathbf{c}_t\cdot\mathbf{n}) \tag{8}$$

Using the implicit function theorem (see, for example, Bruce and Giblin[13] (p 68), we require one of the first two entries to be nonzero, and this is equivalent to saying that the vectors $\mathbf{n}_u, \mathbf{n}_v$, which are automatically tangent vectors to M, are not both perpendicular to $\mathbf{r}-\mathbf{c}$. If this fails, then (i) $\mathbf{n}_u, \mathbf{n}_v$ are parallel, and (ii) their common direction is perpendicular to $\mathbf{r}-\mathbf{c}$. Now $\mathbf{n}_u = -S(\mathbf{r}_u)$, S representing the 'shape operator'[18] (p. 212), so (i) is equivalent to saying that S is degenerate, i.e. that $\mathbf{r}$ is a parabolic point of M where the gaussian curvature vanishes. Then (ii) is equivalent to saying that $\Pi(\mathbf{r}-\mathbf{c},\mathbf{r}_u) = (\mathbf{r}-\mathbf{c})\cdot S(\mathbf{r}_u) = 0$, and similarly for v, where Π represents the second fundamental form of M. But this means that $\Pi(\mathbf{r}-\mathbf{c},\mathbf{v}) = \mathbf{0}$ for every tangent direction $\mathbf{v}$ at $\mathbf{r}$ and hence $\mathbf{r}-\mathbf{c}$ is along the unique asymptotic direction there.

(ii) Suppose that $(\mathbf{r}-\mathbf{c})\cdot\mathbf{n}_v \neq 0$ so that $v = V(u,t)$ say, from (i). We require that $(u,t)\to(u,V(u,t))$ is a regular change of coordinates on M, i.e. that $V_t \neq 0$. Differentiating $(\mathbf{r}(u,V(u,t)) - \mathbf{c}(t))\cdot\mathbf{n}(u,V(u,t)) = 0$ with respect to t rapidly reveals that $V_t = 0$ if and only if $\mathbf{c}_t\cdot\mathbf{n} = 0$. □.

APPENDIX B: EPIPOLAR FIELD ON THE SPATIO-TEMPORAL SURFACE (PROPOSITION 10)

The equation of $\widetilde{M}$ is given in (3) so that a general tangent vector to $\widetilde{M}$ at (u,v,t) is say α,β,τ where:

$$\alpha(\mathbf{r}-\mathbf{c})\cdot\mathbf{n}_u + \beta(\mathbf{r}-\mathbf{c})\cdot\mathbf{n}_v - \tau\mathbf{c}_t\cdot\mathbf{n} = 0 \tag{9}$$

The image of the vector α,β,τ under the projection $\widetilde{M}\to M$ is $\alpha\mathbf{r}_u + \beta\mathbf{r}_v$. We want this to equal $\mathbf{r}-\mathbf{c}$, which determines α and β since $\mathbf{r}_u,\mathbf{r}_v$ are independent. Thus (9) determines τ so long as $\mathbf{c}_t\cdot\mathbf{n}\neq 0$; the contrary case of course occurs precisely at the frontier. The final formula is independent of the frontier restriction since we can clear denominators.

Now $\mathbf{r}-\mathbf{c} = \alpha\mathbf{r}_u + \beta\mathbf{r}_v$ gives:

$$(\mathbf{r}-\mathbf{c})\times\mathbf{r}_u = -\beta\mathbf{r}_u\times\mathbf{r}_v; \quad (\mathbf{r}-\mathbf{c})\times\mathbf{r}_v = \alpha\mathbf{r}_u\times\mathbf{r}_v \tag{10}$$

Hence:

$$[\mathbf{r}-\mathbf{c},\mathbf{r}_u,\mathbf{r}_u\times\mathbf{r}_v] = -\beta\|\mathbf{r}_u\times\mathbf{r}_v\|^2$$
$$[\mathbf{r}-\mathbf{c},\mathbf{r}_v,\mathbf{r}_u\times\mathbf{r}_v] = \alpha\|\mathbf{r}_u\times\mathbf{r}_v\|^2$$

The lifted tangent vector is therefore:

$$(\alpha,\beta,\tau) = (\alpha,\beta,(\alpha(\mathbf{r}-\mathbf{c})\cdot\mathbf{n}_u + \beta(\mathbf{r}-\mathbf{c})\cdot\mathbf{n}_v)/\mathbf{c}_t\cdot\mathbf{n})$$

which is proportional to:

$$(\mathbf{c}_t\cdot\mathbf{n}\alpha, \mathbf{c}_t\cdot\mathbf{n}\beta, \alpha(\mathbf{r}-\mathbf{c})\cdot\mathbf{n}_u + \beta(\mathbf{r}-\mathbf{c})\cdot\mathbf{n}_v)$$

We can now substitute for α,β from (10). Note that $\mathbf{n}$ can here be *any* nonzero normal vector, for example $\mathbf{r}_u\times\mathbf{r}_v$. Now $\mathbf{n}_u\cdot(\mathbf{r}-\mathbf{c}) = -\Pi(\mathbf{r}_u,\mathbf{r}-\mathbf{c})$ provided $\mathbf{n}$ is a *unit* normal (see O'Neill[18], p. 190). Using the linearity of Π, it is a simple matter to reduce the tangent vector to $\widetilde{M}$ to the form given in the statement of the proposition. □.

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