THE UNIVERSITY
of LIVERPOOL

Department of Mathematical Sciences
MATH349
Differential Geometry of Curves and Surfaces
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(notes by Bill Bruce and Peter Giblin)
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Introduction

Geometry is an old and venerable subject, going back to ancient Greek times. Indeed to the Greeks any person was not fully educated unless he or she was proficient at geometry. Euclidean geometry was considered one of the pinnacles of man's intellectual achievements, and indeed still is, and its organisation and methodology was a model for all mathematical investigation. The results from the Greeks became an established part of the European heritage via the Moslem civilisation which flourished while Europe was plunged in the so-called dark ages. The rebirth of scientific and mathematical investigation (and much else) in Europe, during the Renaissance, brought geometry to the fore again, and the subject has played a major role ever since. When discussing major scientific revolutions one tends to think of Galileo's early work, Newton's discovery of his laws of motion and gravitational attraction (published in his *Principia* in 1687), Maxwell's theory of electromagnetic forces (1864), Einstein's relativity (1905, 1917), the work concerning quantum theory (from the 1920s monwards).

It is easy to forget two major breakthroughs of a mathematical nature which underpin much of this. The first is the calculus. The fact that this is so commonplace, the fact that many find it dreary should not blind us to its incredible importance. Most of the major laws of physics are expressed in terms of differential equations; calculus is apparently the scientific language of nature. But the calculus is also a powerful tool in the study of things geometrical, and in parallel with its use in the physical sciences it has solved many outstanding problems concerning curves and (later) surfaces in 3-space. Major early contributors included I.Newton (1642–1727), G.Leibniz (1646–1716), C.Huyghens, the Bernoulli family (especially James and John in the early 1700s), L.Euler (1707–83), A.-C. Clairaut (1713–65), at the age of 16, G.Monge (1746–1818) and J.-B.-M.-C. Meusnier (1754–93). The use of the differential calculus to solve geometric problems led to the term *differential geometry*, though this was not actually coined until 1894. Early work was concerned with the properties of curves, often motivated by physical problems—for example, Huyghens in 1673 was interested in pendulums whose time of swing did not depend on the amplitude of the swing, and was led to the ideas of evolute and involute which we shall encounter in Chapter 1. Work on space curves, initiated by Clairaut, followed by 1729, and was taken up by Euler in 1775. Euler's motivation came largely from mechanics. The ideas of curvature and torsion of space curves come from M.-A.Lancret in 1806, and A.Cauchy further developed these ideas in 1826, giving practically a 'modern' treatment.

The other major intellectual advance often overlooked is the change in our concepts of what constitutes space; our understanding of the space in which we live. Great advances were made by J.Bolyai (1802–60) and N.I.Lobachevsky (1793–1856) around 1825 when they proposed (independently) for the first time the existence of non-Euclidean geometry. This is a good example of a problem whose main difficulty lay in a proper formulation. The key advance was made by the C.F. Gauss (1777–1855), who revolutionised the differential geometry of surfaces with the publication of his book Diquisitiones Generales circa Superficies Curvas (General Investigations of Curved Surfaces) in 1827. The theory of surfaces had been developed alongside that of curves in the 18th

century; in particular Euler in 1728 gave equations for the geodesics on surfaces. Euler's major work on surfaces dates from 1760, and his results were obtained later and more elegantly by Meusnier. Much of Gauss's interest in the subject stemmed from his involvement with map-making: how does one represent areas of the earth's surface on the plane, preserving some chosen quantities such as angles, or equality of distance?

Another major step was made by B.Riemann (1826–66) in a seminal essay in 1854. The concepts involved in Riemann's work are rather difficult, but we may give them a brief mention towards the end of the course. It is fair to say that Riemann's view of space proved absolutely essential in Einstein's theory of general relativity.

You should not feel that geometry ended with Riemann. It is generally accepted that more mathematics has been discovered since 1945 than before that date. Geometry has continued to be of fundamental importance, and there are geometers living today whose names will also be remembered by future generations. S.S.Chern, W.Thurston and M.Gromov have proved results which Gauss would have been proud to produce. (Genius is not something that happened only in days gone by!) Indeed geometry's central importance in mathematics has had a decided revival over the last 15 years or so. We shall prove some 20th century results, but must naturally concentrate on the early and fundamental work of earlier generations.

One of many areas where differential geometry has found practical application in recent years is Computer Vision, which is the science of extracting 3-dimensional information about the world from 2-dimensional images. The extraction of differential geometric information from camera images, for the design of industrial robots and other devices, is an ongoing research area.

Outline syllabus

Curves in the plane and in space. Curvature, torsion.
Surface patches in 3-space. Parametrizations of surfaces.
Distance and the first fundamental form on a surface.
Curvature of surfaces and the second fundamental form. Special curves on a surface: principal curves, asymptotic curves, geodesics. Elliptic, hyperbolic and parabolic points. Special kinds of surface.
Gauss's theorem on curvature: the intrinsic nature of Gauss curvature. Area-preserving maps and cartography.
I *hope* to conclude with a detailed study of geodesics on a surface and the Gauss-Bonnet theorem.

Prerequisites

The techniques used in the course are calculus (differential and integral), vectors, matrices and quadratic forms. I shall review any background material which might be unfamiliar.

Organization of the course

I shall provide notes for the course which are reasonably complete, though they will not include all the examples which are done in class. Not all topics in the notes may be covered; the lectures will define the 'examinable material' of the course. From time to time I shall give out extra material, mainly illustrations of curves and surfaces.

I shall set weekly homework assignments, taking problems from the collections at the ends of chapters in the notes—and perhaps an occasional problem from outside.

I shall try to arrange a weekly tutorial where we can discuss the problems and any other difficulties, and perhaps tackle some extra relevant problems.

Assessment

There will be a mid-semester test (announced well in advance) and a final exam. The mid-semester test will count 15% towards the module mark and the final exam will count 75%. The remaining 10% will be awarded for quality of homework answers. Details will be given during the semester.

Books

As stated above, I shall issue notes for this course. But that should not stop you from looking at some of the standard books on the subject. There are many of these, and some are listed below, in alphabetical order of Author.

M.Berger and B.Gostiaux, Differential geometry: manifolds, curves and surfaces, Springer-Verlag Encyclopaedic, very hard to read, has some gems.

M. do Carmo, Differential Geometry of Curves and Surfaces, Prentice Hall Inc., Englewood Cliffs, NJ.
Quite useful, but a little slow, and *very* expensive.

A.Gray, Differential Geometry of Curves and Surfaces, CRC Press
Has an enormous amount of material, so very useful as a resource book. The package Mathematica is incorporated into the material of the text but it can be used without this too. Huge numbers of examples.

J.J.Koenderink, Solid Shape, MIT Press
A unique book, written by someone who has a great visual mastery of the subject (his main interest is in computer and human vision). Almost no proofs, which makes it a wonderful source of ideas for testing out your understanding!

M.M.Lipschutz, Differential Geometry (Schaum's Outline Series), McGraw Hill, New York.
These guides tend to be looked down on by professionals, but they contain a lot of worked examples, which can prove useful. Some of the notation is a bit hard to follow.

J.McCleary, Geometry from a differentiable viewpoint, Cambridge U.P.
Takes a different line from this course, emphasizing non-euclidean geometry and models of the non-euclidean plane. But the sections on curves and surfaces are useful, and there is an excellent treatment of the Gauss-Bonnet theorem. There is also a most interesting section on map projections.

B.O'Neill, Elementary differential geometry, Academic Press
A fine text, marred by the use of differential forms which makes it off-putting at the beginning. But the section on surfaces is very good and has many relevant examples. A new edition was published in 1997.

D. Struik, Lectures on Classical Differential Geometry, Dover Publications, New York.
Quite cheap, but rather old-fashioned: he uses notation which might prove a bit difficult for you to follow. On the other hand there is a lot of good stuff there.

J.A.Thorpe, Elementary topics in differential geometry, Springer-Verlag
Does things in a different order from this course, and takes a somewhat more 'advanced' standpoint (despite the title). But might be worth looking at.

T. Willmore, An Introduction to Differential Geometry, Oxford University Press.
A little too advanced, and not very geometric, but well worth perusing.

Chapter 1

Curves

1.1   Introduction

Our major interest will be with surfaces in 3-space. But it turns out that we can best understand some aspects of the geometry of a surface through the study of curves lying on that surface. For the present we shall study curves in the plane (which is a very flat surface) and curves in higher-dimensional space—usually 3-dimensional space.

Curves arise naturally in all sorts of situations and in many guises. Solutions of Newton's laws of motion give the orbits of the planets as ellipses with the Sun at a focus. A spot of paint on a train wheel describes a cycloid as the wheel rolls. A curve may be traced by a linkage of bars and gearwheels. When the Sun's rays are reflected from the rounded inner surface of a teacup they produce on the surface of the tea a bright 'caustic' curve.

Many naturally occurring curves are traced out with time, that is they come with a 'parametrisation'. We use this idea as our definition.

1.2   Beginning definitions and examples

Definition 1.2.1 A **(parametrised) curve** in the real Euclidean space $\mathbf{R}^n$ (with coordinates $x_1, \ldots, x_n$) is a map

$$\gamma : I \to \mathbf{R}^n, \quad \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$$

where I is an open interval in $\mathbf{R}$. We assume that each of the functions γ_i has derivatives of all orders, for all $t \in I$. Such a function γ_i and curve γ are called *smooth*. The curve γ is called *regular* provided there does not exist $t \in I$ with $\gamma_1' = \gamma_2' = \ldots = \gamma_n' = 0$. The variable t is called the *parameter* : the point $\gamma(t)$ has parameter value t.

Unless otherwise stated all curves will be regular in this course. The vector $(\gamma_1', \gamma_2', \ldots, \gamma_n')$, which is also written $\gamma'(t)$ or $d\gamma/dt$, is called the *velocity vector* at t. For if we (naturally) think of the parameter t as being time and $\gamma(t)$ the position of a particle at time t then $\gamma'(t)$ *is* the particle's velocity. So if a curve is regular the particle tracing out the curve never stops or turns around, since the vector $\gamma'(t)$ is never the zero *vector*. If say $\gamma'(t) = (1, 0, 0, \ldots, 0)$ then this means that instantaneously $\gamma(t)$ is moving parallel to the x_1-axis.

Examples 1.2.2 Curves in $\mathbf{R}^2$ and $\mathbf{R}^3$.

Curves in $\mathbf{R}^2$ and $\mathbf{R}^3$, are referred to as plane and space curves respectively.

(1) $\gamma(t) = (t, t^2)$, $I = \mathbf{R}$; parabola.

(2) $\gamma(t) = (\cos t, \sin t)$, I any open interval containing $[0, 2\pi]$; circle.

(3) $\gamma(t) = (t^3, t^6)$, $I = \mathbf{R}$. The image of γ is still the parabola of (1), but γ is not regular at $t = 0$. What about $\gamma(t) = (t^2, t^4)$?

(4) $\gamma(t) = (A \cos t, B \sin t)$, I any open interval containing $[0, 2\pi]$. The image here (when $AB \neq 0$) is an ellipse in $\mathbf{R}^2$.

(5) $\gamma(t) = (t^2 - 1, t^3 - t)$, $I = \mathbf{R}$. Here the image $\gamma(I)$ is a curve which crosses itself ($t = 1$, $t = -1$ both give $\gamma(t) = (0, 0)$.)

(6) $\gamma(t) = (t^2, t^3)$, $I = \mathbf{R}$. The image here is a cuspidal cubic, and is not regular at $t = 0$.

(7) Quite generally if $f : I \to \mathbf{R}$ is a smooth function then the parametrisation $(graph\ f) : I \to \mathbf{R}^2$ given by $(graph\ f)(t) = (t, f(t))$ is regular for any f, since $(graph f)'(t) = (1, f'(t)) \neq (0, 0)$ for any t.

(8) $\gamma(t) = (t, t^2, t^3)$, $I = \mathbf{R}$, gives a *twisted cubic* in $\mathbf{R}^3$. (This is a bit of old-fashioned terminology; any curve which did not lie in some plane was referred to as 'twisted'.)

(9) $\gamma(t) = (\cos t, \sin t, t)$, $I = \mathbf{R}$, gives a *helix* in $\mathbf{R}^3$. More generally the parametrisation $\gamma(t) = (a \cos t, a \sin t, bt)$, $I = \mathbf{R}$ is also referred to as a helix. We can think of it as the thread on a bolt.

1.3 Tangent Vectors

We shall use vectors a great deal in this course. Given points $\mathbf{q}$, $\mathbf{r}$ in $\mathbf{R}^n$ the segment from $\mathbf{q}$ to $\mathbf{r}$ represents the vector $\mathbf{v} = \mathbf{r} - \mathbf{q}$. So the vector from $\mathbf{O}$ to $\mathbf{p}$ ($\mathbf{O}$ being the origin) represents the vector $\mathbf{p}$, and so does any segment parallel to it. Vectors are indicated by arrowed segments; you should all be familiar with the well-known vector law of addition.

Let $\gamma : I \to \mathbf{R}^n$ be a regular curve. The vector $\mathbf{v} = \gamma(t + h) - \gamma(t)$ corresponds to the chord segment from $\gamma(t)$ to $\gamma(t + h)$. The derivative

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t + h) - \gamma(t)}{h}$$

has as its direction the limit of these chords, i.e. the tangent at $\gamma(t)$.

Before proceeding we recall that the scalar product or dot product of two vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ in $\mathbf{R}^n$ is the real number

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n.$$

Whenever we mention $\mathbf{R}^n$ we shall really be thinking of this space together with this scalar product, in other words we will be dealing with Euclidean space. The scalar product allows us to define angle and distance in $\mathbf{R}^n$.

We note the following facts:

Properties 1.3.1 Scalar (dot) products

(1) $\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \ldots + v_n^2$ and its square root $\sqrt{\mathbf{v} \cdot \mathbf{v}}$ is the length of the vector $\mathbf{v}$, also written $||\mathbf{v}||$.

(2) $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\ ||\mathbf{w}|| \cos \theta$ where θ is the angle ($0 \leq \theta \leq \pi$) between the vectors $\mathbf{v}$ and $\mathbf{w}$. In particular $\mathbf{v} \cdot \mathbf{w} = 0$ means that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular or orthogonal.

(3) If the v_i and w_i are all (smooth) functions of t then

$$(\mathbf{v} \cdot \mathbf{w})' = d(\mathbf{v} \cdot \mathbf{w})/dt = \mathbf{v}' \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}'$$

where $\mathbf{v}' = (v_1', \ldots, v_n')$ etc.

Here are two basic facts we shall need.

(4) If $\mathbf{v} \cdot \mathbf{v} = 1$ for all t (that is $\mathbf{v}$ is a unit vector for all t) then using (3) above $\mathbf{v} \cdot \mathbf{v}' = 0$ for all t. So for each t the vectors $\mathbf{v}$ and $\mathbf{v}'$ are perpendicular.

(5) If $\mathbf{v}$ and $\mathbf{w}$ are functions of t and orthogonal for all t then differentiating the identity $\mathbf{v} \cdot \mathbf{w} = 0$ we find $\mathbf{v}' \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}' = 0$.

Definition 1.3.2 Let $\gamma : I \to \mathbf{R}^n$ be a regular curve. The vector $\mathbf{T}(t) = \gamma'/||\gamma'||$ is called the **unit tangent vector** to γ at t, or at $\gamma(t)$. The length of the velocity vector $||\gamma'(t)||$ is naturally called the **speed** of the curve γ at t, and we say that γ is **unit speed** if $||\gamma'(t)|| = 1$ for all t. The **tangent line** to γ at t is the straight line through $\gamma(t)$, containing the direction $\mathbf{T}(t)$.

Examples 1.3.3 Tangents and speed

(1) For the circle parametrised as $\gamma(t) = (\cos t, \sin t)$ the velocity vector $\gamma'(t)$ is $(-\sin t, \cos t)$, and this is a unit vector, the unit tangent vector to γ at t. Note here that since $\gamma(t)$ is a unit vector for all t the tangent vector at $\gamma(t)$ is perpendicular to $\gamma(t)$ (i.e. the tangent line to a circle is perpendicular to the corresponding diameter).

(2) For the helix $\gamma(t) = (\cos t, \sin t, t)$ the velocity vector is $(-\sin t, \cos t, 1)$, and the unit tangent vector is $(-\sin t, \cos t, 1)/\sqrt{2}$.

(3) For the twisted cubic $\gamma(t) = (t, t^2, t^3)$ the velocity vector is $(1, 2t, 3t^2)$, and the unit tangent vector is $(1, 2t, 3t^2)/\sqrt{(1 + 4t^2 + 9t^4)}$.

(4) Consider a space curve $\gamma(t) = (t, f(t), g(t))$, where f, g are smooth and $f(0) = f'(0) = g(0) = g'(0) = 0$. Thus γ passes through the origin and has tangent $(1, 0, 0)$ there. Note that this γ is automatically regular for all t. If we *project* γ to the y, z-plane we obtain the plane curve $\delta(t) = (f(t), g(t))$. By the assumptions on f and g this curve fails to be regular at $t = 0$. This illustrates the general principle that *projecting a space curve along a tangent line yields a non-regular curve*. In fact experimenting with a piece of bent wire will convince you that what you generally see is a *cusp* when looking along a tangent line to a space curve.

As a particular case of Properties 1.3 above note that the unit tangent vector to a curve γ satisfies $\mathbf{T} \cdot \mathbf{T}' = 0$, so $\mathbf{T}'$ is a (possibly zero) vector perpendicular to $\mathbf{T}$. Since $||\mathbf{T}|| = 1$, $\mathbf{T}'$ measures the rate at which the unit tangent vector is turning; the longer $\mathbf{T}'$ is the faster $\mathbf{T}$ is turning, and roughly speaking, the more curved the curve is.

1.4 Reparametrisation and Arc-length

As usual let $\gamma(t)$ be a regular curve in $\mathbf{R}^n$. The set of points $\mathbf{R}^n$ of the form $\gamma(t)$ is called the *trace* of the curve. Clearly many different curves can have the same trace, for example (t, t^2) and $(3 - 2t, (3 - 2t)^2)$ both trace out the parabola $x_2 = x_1^2$. Think of the trace as being a road, then each car travelling the road gives a parametrisation with respect to the natural parameter of time. The condition for the parametrisation to be regular is simply that the car at no time stops. We would like to choose some standard natural parametrisation of each curve. Clearly one way to proceed is to drive along the road at a constant speed. Of course the choice of speed is rather arbitrary. For curves we can fix even this by asking that in time t we travel distance t. To make this precise we need to define what we mean by distance or arc-length on the curve.

Definition 1.4.1 The **arc-length** of a regular curve $\gamma : I \to \mathbf{R}^n$, measured from $\gamma(t_0)$, where $t_0 \in I$, is

$$l(t) = \int_{t_0}^{t} ||\gamma'(t)||dt \qquad (t \in I).$$

In particular if γ is unit speed (i.e. $||\gamma'(t)|| = 1$ for all t), then $l(t) = t - t_0$. A unit speed curve is often said to be **parametrised by arc-length**. The key equation, valid for all regular curves, is

$$l'(t) = ||\gamma'(t)||.$$

Remark 1.4.2 We often use s to denote the arc-length parameter, that is $s = l(t)$. Then rather than writing $l' = \frac{dl}{dt} = ||\gamma'(t)||$ we write $s' = \frac{ds}{dt} = ||\gamma'(t)||$. As a rule confusing variables with functions does no harm at all, thanks to the robustness of the notation of calculus; one of the few occasions when we need to keep the function l separate from the variable s is the proof of the next result, which says that arclength is always a valid parameter on a regular curve.

Proposition 1.4.3 *Any regular curve $\gamma : I \to \mathbf{R}^n$ can be parametrised with respect to arclength, i.e. has a unit speed parametrisation.*

Proof The map $l : I \to \mathbf{R}$ is smooth and has derivative $l'(t) = ||\gamma'(t)|| > 0$ for all t. It follows that l maps I bijectively to some interval J and the inverse map $l^{-1} : J \to I$ is smooth. We now define $\alpha : J \to \mathbf{R}^n$ by $\alpha(s) = \gamma \circ l^{-1}(s)$, and claim that this is unit speed. First writing h for l^{-1} we note that since $l(h(s)) = s$ we have $l'(h(s))h'(s) = 1$. But $l'(h(s)) = ||\gamma'||$ by definition of l. Since $\alpha'(s) = \gamma'(h(s))h'(s)$ we take lengths to deduce the result.

Examples 1.4.4 Arclengths

It is really useful to know that curves can be parametrized by arclength. It is a different matter to find an explicit formula for arclength in terms of some given parametrization, because integrals are generally very hard to work out explicitly. Here are a few curves whose arclengths we *can* compute. For another example, see Exercise 4 at the end of the Chapter.

(1) $(n = 2) : \gamma(t) = (t - \sin t, 1 - \cos t)$, $0 \le t \le 2\pi$, one arch of a cycloid. Here $(ds/dt)^2$ comes to $2(1 - \cos t) = 4\sin^2 \frac{1}{2}t$. So $s = 4(1 - \cos \frac{1}{2}t)$, taking $s = 0$ when $t = 0$. Note that the length of the arch is exactly 8.

$\gamma(t) = (t, \cosh(t))$: this is easy because $(ds/dt)^2 = 1 + \sinh^2 t = \cosh^2 t$, giving $s = \sinh t$, taking $s = 0$ when $t = 0$.

$\gamma(t) = (R\cos t, R\sin t)$, circle radius R; this is easy (it is constant speed), and $s = Rt$, taking $s = 0$ when $t = 0$.

(2) $(n = 3)$ $\gamma(t) = (1/\sqrt{2})(\cos t, \sin t, t)$, circular helix: here $s = t$.

1.5 Curvature of plane curves

Now that we know that any regular curve has a unit speed parametrisation—even if we can't write down a simple formula for it—we shall often asume that our curves *are* unit speed. When it makes a difference to the formula we shall say so. To make matters (we hope) clearer, for a while unit speed curves will be denoted by α below, and their parameter will be called s. Thus $\alpha'(s) = \mathbf{T}(s)$ for unit speed α, while for a general curve γ we have $\mathbf{T}(t) = \gamma'(t)/||\gamma'(t)||$.

Since $\mathbf{T}(t)$ has constant length 1 its derivative measures the rate at which its *direction* is changing with respect to the parameter t. Of course for different parameterisations we would obtain different

rates. But if we measure the rate of turning of the tangent with respect to arc-length we have a well defined invariant of the curve, that is one which is independent of the parametrisation, and which has some geometric significance. Remember that for the time being we are working with *plane curves.*

Definition 1.5.1 Let $\gamma : I \to \mathbf{R}^2$ be a regular plane curve; then the unit tangent $\mathbf{T}$ is (as above) $\gamma'(t)/||\gamma'(t)||$. The **unit normal** $\mathbf{U}(t)$ is defined as the vector obtained from $\mathbf{T}(t)$ by rotating *anticlockwise* through $\pi/2$. Now $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$, so (as we are in the plane) $\mathbf{T}'$ is parallel to $\mathbf{U}$. Thus there is a real number $\kappa(t)$ such that

$$\mathbf{T}'(t) = \kappa(t)\mathbf{U}(t)||\gamma'(t)||.$$

We call $\kappa(t)$ the **curvature of the plane curve** γ at t. Note that $\kappa(t) = \pm||\mathbf{T}'(t)||/||\gamma'(t)||$.

For a unit speed curve α it's even easier: $\mathbf{T} = \alpha'$, $\mathbf{U}$ is still obtained by rotating $\mathbf{T}$ anticlockwise through $\pi/2$, and $\mathbf{T}' = \kappa\mathbf{U}, \kappa = \pm||\mathbf{T}'||$. The curvature κ is > 0 precisely when $\mathbf{T}'$ is a *positive* multiple of $\mathbf{U}$, that is when the tangent $\mathbf{T}$ is turning *towards* the normal $\mathbf{U}$.

The **centre of curvature** at $\gamma(t)$ is the point $\gamma(t) + (1/\kappa(t))\mathbf{U}(t)$ and the **circle of curvature** is the circle with this point as centre and radius $1/\kappa(t)$. There is no change here if the curve is unit speed.

Note Later on (Chapter 4) we shall have various concepts of curvature for curves lying on *surfaces.* The plane is an example of a surface, and the curvature κ just defined is, from the point of view of Chapter 4, the **geodesic curvature** of γ. So it would also be possible to use this term for κ.

Proposition 1.5.2 *(1) If $\alpha(s) = (X(s), Y(s))$ is a unit speed curve then, using ' for d/ds,*

$$\mathbf{T} = (X', Y'), \ \mathbf{U} = (-Y', X'), \ \kappa = X'Y'' - X''Y'.$$

(2) If $\gamma(t) = (X(t), Y(t))$ is a regular curve then, using ' for d/dt,

$$\mathbf{T} = \frac{(X', Y')}{(X'^2 + Y'^2)^{\frac{1}{2}}}, \ \mathbf{U} = \frac{(-Y', X')}{(X'^2 + Y'^2)^{\frac{1}{2}}}, \ \kappa = \frac{|\gamma' \ \gamma''|}{||\gamma'||^3} = \frac{X'Y'' - X''Y'}{(X'^{\,2} + Y'^{\,2})^{\frac{3}{2}}}.$$

(3) If $\gamma(t) = (t, Y(t))$ for some smooth $Y : I \to \mathbf{R}$ then, using ' for d/dt,

$$\mathbf{T} = \frac{(1, Y')}{(1 + Y'^2)^{\frac{1}{2}}}, \ \mathbf{U} = \frac{(-Y', 1)}{(1 + Y'^2)^{\frac{1}{2}}}, \ \kappa(t) = \frac{Y''}{(1 + Y'^{\,2})^{\frac{3}{2}}}.$$

In particular if the x-axis is the tangent to the curve at $t = 0$, i.e. $Y'(0) = 0$, the curvature $\kappa(0) = Y''(0)$ only depends on the second order terms in the Taylor expansion of Y at $t = 0$.

Note For an arbitrary regular plane curve, it follows from (2) that the curvature is *zero* for $X'Y'' = X''Y'$, which is the same as γ' parallel to γ''. Such points are called **inflexions** of γ. A **vertex** of γ is a point where the curvature is stationary, i.e. where $\kappa' = 0$.

Proof of the Proposition The formulae for $\mathbf{T}$ and $\mathbf{U}$ are immediate from the definition. It is enough to prove the formula for κ in (2). To do this, note that (using s for arclength)

$$\gamma' = \frac{d\gamma}{dt} = \frac{ds}{dt}\frac{d\gamma}{ds} = s'\mathbf{T}.$$

Similarly,

$$\gamma'' = s''\mathbf{T} + \kappa s'^2\mathbf{U}.$$

Forming the 2×2 determinant $|\gamma' \ \gamma''| = X'Y'' - X''Y'$ gives $|\gamma' \ \gamma''| = \kappa s'^3$. The result follows from this and $s' = ||\gamma'|| = (X'^2 + Y'^2)^{\frac{1}{2}}$.

Examples 1.5.3 Curvature of plane curves

(1) For the circle $(r\cos t, r\sin t)$ it is easy to check that the curvature is $1/r$.

(2) For the parabola (t, at^2) the curvature $\kappa(t) = 2a/(1 + 4a^2t^2)^{\frac{3}{2}}$. When $t = 0$ this reduces to $2a$.

(3) For the cubic curve (t, t^3) the curvature is $6t/(1 + 9t^4)^{\frac{1}{2}}$, which is positive for $t > 0$, where the tangent $\mathbf{T}$ is turning *towards* the normal $\mathbf{U}$ and negative for $t < 0$, where $\mathbf{T}$ is turning *away from* $\mathbf{U}$. At $t = 0$, there is an inflexion.

(3) Given a regular plane curve $\gamma : I \to \mathbf{R}^2$, and a point $s_0 \in I$ we can, by a rigid motion of the plane, suppose that $\gamma(s_0) = (0,0)$, and its tangent vector at s_0 is in the direction of the x_1-axis. A little thought shows that locally the image of γ can be written in the form $x_2 = f(x_1)$, with $f(0) = 0$, $f'(0) = 0$. The results above show that the curvature of $\gamma(s_0)$ is $f''(0)$. In particular, if we have a curve $y = ax^2 + bx^3 + \ldots$ where the dots represent terms of degree more than 3, then the curvature at $x = 0$ is $2a$.

Proposition 1.5.4 Serret-Frenet Formulae: plane curves

The Serret-Frenet formulae are the key facts we need in order to establish just about everything that we need to know about curves. Frenet's work dates from 1852 and Serret's from 1851.

Let γ be a regular plane curve. The first formula (see Definition 1.5.1) is

$$\mathbf{T}' = \kappa\mathbf{U}||\gamma'||, \text{ hence } \mathbf{T}' = \kappa\mathbf{U} \text{ for unit speed.} \tag{1.1}$$

Now since $\mathbf{U}$ is a unit vector $\mathbf{U}'$ is orthogonal to $\mathbf{U}$ (Properties 1.3.1,(4)). and so can be written $\mathbf{U}' = \lambda\mathbf{T}$ for some λ. Also differentiating the identity $\mathbf{T} \cdot \mathbf{U} = 0$ we know that $\mathbf{T}' \cdot \mathbf{U} + \mathbf{T} \cdot \mathbf{U}' = 0$, so that $\lambda + \kappa = 0$. Hence we obtain the second formula:

$$\mathbf{U}' = -\kappa\mathbf{T}||\gamma'||, \text{ hence } \mathbf{U}' = -\kappa\mathbf{T} \text{ for unit speed.} \tag{1.2}$$

Definition 1.5.5 Let $\alpha : I \to \mathbf{R}^2$ be a unit speed curve, and let S denote the unit circle in $\mathbf{R}^2$. Then the map $\mathbf{U} : I \to S$ which takes s to $\mathbf{U}(s)$ is called the **Gauss map** of α. (Note that the tangent line to α at s is parallel to the tangent line to S at $\mathbf{U}(s)$.)

Let us choose a fixed unit vector $\mathbf{u}$ in the plane, and, for each s, define the angle ψ by $\mathbf{u} = (\cos\psi)\mathbf{U} + (\sin\psi)\mathbf{T}$. Thus $\psi(s)$ is the anticlockwise angle between $\mathbf{u}$ and the normal $\mathbf{U}(s)$. We have $\mathbf{U}(s) \cdot \mathbf{u} = \cos\psi$ and $\mathbf{T}(s) \cdot \mathbf{u} = \sin\psi$. Differentiating these with respect to s and using (1.1) and (1.2) we obtain

$$\kappa\sin\psi = \psi'\sin\psi, \quad \kappa\cos\psi = \psi'\cos\psi.$$

Since $\sin\psi$ or $\cos\psi$ is non-zero we deduce that

$$\kappa = \psi',$$

which just makes precise the idea that κ is the rate of turning of the normal, or equally of the tangent, since that makes an angle $\psi - \frac{\pi}{2}$ with the fixed direction $\mathbf{u}$.

However note that

$$\psi' = \frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\psi(s + \delta s) - \psi(s)}{\delta s},$$

and since S is a unit circle arc-length on S is the same as angle. So we have in fact

Proposition 1.5.6 *Let I be a small interval containing s_0 in I of length δs, which is mapped by the Gauss map to an arc on S of length $\delta\psi$. Then the absolute value of the curvature of α at s_0, $|\kappa(s_0)|$ is the limit $\lim_{\delta s \to 0} \frac{\delta\psi}{\delta s}$.*

1.6 Curvature of space curves

We now turn to curves in *more* than two dimensions. For a regular curve γ we have the unit tangent $\mathbf{T}(t) = \gamma'(t)/||\gamma'(t)||$, and as before if s is arclength then $ds/dt = ||\gamma'(t)|| = $ speed. Unit speed curves will continue to be denoted by α for the time being.

Definition 1.6.1 (i) Let $\alpha : I \to \mathbf{R}^n$ $(n \geq 3)$ be a unit speed curve. Then the **curvature** $\kappa(s)$ of α at s is given by

$$\kappa(s) = ||\mathbf{T}'(s)|| = ((\alpha_1''(s))^2 + \ldots + (\alpha_n''(s))^2))^{\frac{1}{2}}.$$

So $\kappa(s) \geq 0$ and κ is a smooth function of s away from the points where it vanishes. The positivity of κ for space curves is to be contrasted with the fact that for plane curves we could meaningfully give it a sign (Definition 1.5.1).

   (ii) If $\kappa(s) \neq 0$ then we define the **principal normal** $\mathbf{P}(s)$ of α at s to be the unit vector $\mathbf{T}'(s)/\kappa(s)$. Note that since $\mathbf{T}(s)$ is a unit vector $\mathbf{P}(s)$ is perpendicular to $\mathbf{T}(s)$. When $\kappa(s) = 0$ the vector $\mathbf{P}(s)$ is not defined.

   Note that $\kappa\mathbf{P} = \mathbf{T}' = \alpha''$ for a unit speed curve. Thus the principal normal is in the direction of the *acceleration* vector, just as the tangent is in the direction of the *velocity* vector. It is a good intuition to think of driving along a parametrized space curve at unit speed—always travelling distance d along the curve between parameter values t and $t + d$—in which case the principal normal to the curve is along the direction of your acceleration.

   For an arbitrary (regular) curve $\gamma(t)$ we just define the curvature as $||d\mathbf{T}/ds||$ where s is arclength from some fixed point on the curve. We have

$$\frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \frac{d\mathbf{T}}{dt}$$

and taking lengths we find that, as $\kappa \geq 0$,

$$\kappa(t) = \frac{||d\mathbf{T}/dt||}{ds/dt} = \frac{||d\mathbf{T}/dt||}{||\gamma'||}. \tag{1.3}$$

where $'$ stands for d/dt.

   Hence we have the equations

$$\gamma' = s'\mathbf{T}, \quad \mathbf{T}' = \kappa\mathbf{P}s' = \kappa\mathbf{P}||\gamma'||, \quad \gamma'' = s''\mathbf{T} + \kappa s'^2\mathbf{P}.$$

So in the case of an arbitrary regular curve γ, the principal normal is no longer in the direction of the acceleration γ'', which has a tangential component s''.

Examples 1.6.2 Curvature of space curves

(1) Consider the straight line $\gamma(s) = (s, 0, 0)$. Here $\mathbf{T}(s) = (1, 0, 0)$ and $\mathbf{T}'(s) = 0$, so $\kappa(s) \equiv 0$. Conversely suppose that a curve α has curvature function $\kappa(s) \equiv 0$. Then $\alpha''(s) \equiv 0$ (note the zero here is the zero vector) and we deduce that each component of α must be linear in the arclength function s, i.e. the curve must be a line.

   (2) Consider the unit speed helix $1/\sqrt{2}(\cos s, \sin s, s)$. The unit tangent vector is given by $\mathbf{T}(s) = 1/\sqrt{2}(-\sin s, \cos s, 1)$, and $\mathbf{T}'(s) = 1/\sqrt{2}(-\cos s, -\sin s, 0)$, so the curvature is $1/\sqrt{2}$. We really should expect it to be constant. After all there is no way of distinguishing one point on a helix from any other. (This is precisely why helices are useful for screw threads!)

(3) Finally consider the twisted cubic $\gamma(t) = (t, t^2, t^3)$. So $\gamma'(t) = (1, 2t, 3t^2)$ and $||\gamma'(t)|| = \sqrt{1 + 4t^2 + 9t^4}$. The unit tangent vector at $\gamma(t)$ is $(1, 2t, 3t^2)/\sqrt{1 + 4t^2 + 9t^4}$, and we need its derivative to compute the curvature. The calculation looks fairly unpleasant. Shortly we shall have a better method of doing this calculation—see §1.5.4 and Examples 1.6.6.

We now come to the Serret-Frenet formulae for space curves. Before giving them we introduce the vector cross product and remind you of its basic properties.

Definition 1.6.3 (1) Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors in $\mathbf{R}^3$. Then the **vector cross product** of $\mathbf{a}$ and $\mathbf{b}$ is *perpendicular to* $\mathbf{a}$ *and* $\mathbf{b}$ and is given by

$$\mathbf{a} \times \mathbf{b} = \left(\det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix}, -\det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix}, \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \right) = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the three standard vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ then this can be written in coded form as

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

The cross product is also known as the **wedge product** and written $\mathbf{a} \wedge \mathbf{b}$.

(2) For vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbf{R}^3$ we define the **triple scalar product** written $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ by

$$[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = \det \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Geometrically this is twice the volume of the tetrahedron with vertices at the origin and the points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. It is *zero if and only the the four points* $\mathbf{O}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *lie in a plane* (i.e. if and only if the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are linearly dependent).

Proposition 1.6.4 *The vector product has the following well-known properties.*
  (1) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
  (2) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.
  (3) $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b})$.
  (4) $||\mathbf{a} \times \mathbf{b}||^2 = ||\mathbf{a}||^2 ||\mathbf{b}||^2 - (\mathbf{a} \cdot \mathbf{b})^2$.
  (5) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$.
  (6) $||\mathbf{a} \times \mathbf{b}|| = \sin\theta \ ||\mathbf{a}|| \ ||\mathbf{b}||$ *where* θ *is the angle* $(0 \leq \theta < \pi)$ *between* $\mathbf{a}$ *and* $\mathbf{b}$. *(This is a direct consequence of (4) and the fact that* $\sin^2 + \cos^2 = 1$.)
  (7) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.
  (8) If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *are functions of* t *then*

$$[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]' = [\mathbf{a}' \ \mathbf{b} \ \mathbf{c}] + [\mathbf{a} \ \mathbf{b}' \ \mathbf{c}] + [\mathbf{a} \ \mathbf{b} \ \mathbf{c}'].$$

Now back to the **Serret-Frenet formulae for space curves**. Let $\alpha : I \to \mathbf{R}^3$ be unit speed, so that $\alpha'(s) = \mathbf{T}(s)$ the unit tangent vector. Then we have seen that the curvature

$$\kappa(s) = ||\mathbf{T}'(s)||$$

and provided $\kappa(s) \neq 0$ there is a principal normal vector $\mathbf{P}(s)$, perpendicular to $\mathbf{T}(s)$ with

$$\mathbf{T}'(s) = \kappa(s)\mathbf{P}(s) \text{ [for an arbitrary regular curve } \gamma, \mathbf{T}' = \kappa\mathbf{P}||\gamma'||]. \tag{1.4}$$

We shall continue to assume that $\kappa(s) \neq 0$ in what follows.

Since $\mathbf{T}(s)$ and $\mathbf{P}(s)$ are perpendicular unit vectors there is a unique unit vector $\mathbf{B}(s)$ called the **binormal vector** pependicular to both, and such that $\mathbf{T}(s)$, $\mathbf{P}(s)$, $\mathbf{B}(s)$ is right-handed (i.e. if the components are written as the successive rows of a 3×3 matrix, then the determinant is 1.) Indeed in terms of vector product, $\mathbf{B} = \mathbf{T} \times \mathbf{P}$. The plane through $\alpha(s)$ spanned by $\mathbf{P}(s)$ and $\mathbf{B}(s)$ is called the **normal plane** at s, and the plane through $\alpha(s)$ spanned by $\mathbf{T}(s)$ and $\mathbf{P}(s)$ is called the **osculating plane** there. ('Osculating' means 'kissing' in Greek....the osculating plane and the curve have a 'high contact'.

We know that $\mathbf{P}'$ is perpendicular to $\mathbf{P}$, so $\mathbf{P}' = \lambda\mathbf{T} + \tau\mathbf{B}$ for suitable λ and τ, which depend on s of course. But $\mathbf{T} \cdot \mathbf{P} = 0$ so $\mathbf{T}' \cdot \mathbf{P} + \mathbf{T} \cdot \mathbf{P}' = 0$, which gives $\kappa + \lambda = 0$. Hence

$$\mathbf{P}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s) \quad [\text{for any regular curve } \gamma, \mathbf{P}' = (-\kappa\mathbf{T} + \tau\mathbf{B})||\gamma'||] \qquad (1.5)$$

for some real number $\tau(s)$. This real number is called the **torsion** of the curve. So long as $\kappa(s) \neq 0$, $\tau(s)$ is a smooth function of s.

Next $\mathbf{B}'$ is pependicular to $\mathbf{B}$ and so can be written in the form $\mu\mathbf{T} + \nu\mathbf{P}$ for some μ, $\nu \in \mathbf{R}$. Using $\mathbf{B} \cdot \mathbf{T} = 0$ we have $\mathbf{B}' \cdot \mathbf{T} + \mathbf{B} \cdot \mathbf{T}' = 0$, so $\mu + 0 = 0$. Using $\mathbf{B} \cdot \mathbf{P} = 0$ we have similarly $\nu = -\tau$. Hence

$$\mathbf{B}'(s) = -\tau(s)\mathbf{P}(s) \quad [\text{for any regular curve } \gamma, \mathbf{B}' = -\tau\mathbf{P}||\gamma'||]. \qquad (1.6)$$

The three formulae (1.4), (1.5) and (1.6) are called the Serret-Frenet formulae for space curves. They can be summarized for unit speed curves by the following mnemonic:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{P}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{P} \\ \mathbf{B} \end{pmatrix}.$$

Note the skew-symmetry of the 3×3 matrix. For an arbitrary regular curve γ, multiply the right hand side of the equation by $||\gamma'||$.

In a sense any problem concerning space curves can be solved using them. The curvature and torsion of a curve are its main geometric invariants. The curvature measures the speed at which tangent is turning and the torsion the speed at which the osculating plane is turning (as measured by the rate of turning of the vector $\mathbf{B}$ orthogonal to that plane—see Examples 1.6.6). In fact we shall see that curvature and torsion completely determine a space curve in Proposition 1.6.8.

We already have a nice formula in Proposition 1.5.2 for the curvature of plane curves, and the Proposition below gives similar formulae for κ and τ of a space curve. Mercifully, we do not have to assume unit speed for these formulae.

Proposition 1.6.5 *Let $\gamma : I \to \mathbf{R}^3$ be a regular curve (not necessarily unit speed). Then*

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3}.$$

Thus $\kappa = 0$ if and only if γ' is parallel to γ''. If $\kappa \neq 0$ then

$$\tau = \frac{[\gamma', \gamma'', \gamma''']}{||\gamma' \times \gamma''||^2},$$

and $\gamma' \times \gamma''$ is parallel to the binormal $\mathbf{B}$.

Proof We use t as the parameter for γ and s for arc-length on γ. We proceed as in the plane curve case by calculating derivatives of γ with respect to t. A short calculation, using the Serret-Frenet formulae shows that

$$\begin{aligned}
\gamma' &= s'\mathbf{T} \\
\gamma'' &= s''\mathbf{T} + \kappa s'^2\mathbf{P} \\
\gamma''' &= (s''' - \kappa^2)\mathbf{T} + (\kappa's'^2 + 3\kappa s's'')\mathbf{P} + \kappa\tau s'^3\mathbf{B}.
\end{aligned}$$

From these it follows easily that

$$\gamma' \times \gamma'' = \kappa s'^3\mathbf{B},$$

and

$$[\gamma', \gamma'', \gamma'''] = \kappa^2\tau s'^6.$$

Recalling that $s' = ||\gamma'||$ all the required results now follow.

Here is a good **strategy for finding P**, **B** etc., for a general regular curve:

1. Calculate $\mathbf{T}$ as the unit vector parallel to γ'.

2. Calculate $\gamma' \times \gamma''$. The binormal $\mathbf{B}$ is the unit vector parallel to this. [The only assumption here is that $s' > 0$, i.e. the parameter t and the arclength s increase in the same direction along γ. This is normally taken for granted; if it were not the case, then $\mathbf{B}$ would be *minus* the unit vector parallel to $\gamma' \times \gamma''$.]

3. Calculate $\mathbf{P}$ as $\mathbf{B} \times \mathbf{T}$.

4. Calculate κ and τ from the formulae of Proposition 1.6.5 or from $\mathbf{T}' = \kappa\mathbf{P}||\gamma'||$ and $\mathbf{B}' = -\tau\mathbf{P}||\gamma'||$.

Examples 1.6.6 Curvature and torsion

(1) Consider the curve $\gamma(t) = (t\cos t, t\sin t, t)$. To calculate its curvature and torsion at say $t = 0$ we simply apply the formulae above. It is not hard to see that

$$\gamma'(t) = (-t\sin t + \cos t, t\cos t + \sin t, 1)$$

$$\gamma''(t) = (-t\cos t - 2\sin t, -t\sin t + 2\cos t, 0), \ \gamma'''(t) = (-t\sin t - 3\cos t, -t\cos t - 3\sin t, 0).$$

Evaluating at $t = 0$ and substituting in the above equations we find that $\kappa = 1$ and $\tau = 3/4$.

(2) The example $\gamma(t) = (t, t^2, t^3)$ mentioned above (Examples 1.6.2,(3)) is easy to tackle using the formulae just proved. For example,

$$\kappa(t) = \frac{(36t^4 + 36t^2 + 4)^{\frac{1}{2}}}{(9t^4 + 4t^2 + 1)^{\frac{3}{2}}}.$$

(3) The sine of the (dihedral) angle between the osculating planes of γ at $\gamma(t + h)$ and $\gamma(t)$ is $||\mathbf{B}(t + h) \times \mathbf{B}(t)||$ (see Proposition 1.6.4,(6)). Let us expand $\mathbf{B}(t + h)$ for small h by Taylor's theorem, using unit speed:

$$\mathbf{B}(t + h) = \mathbf{B}(t) + h\mathbf{B}'(t) + \dots.$$

Thus

$$\frac{\mathbf{B}(t + h) \times \mathbf{B}(t)}{h} = \mathbf{B}'(t) \times \mathbf{B}(t) + \dots = \tau\mathbf{P}(t) \times \mathbf{B}(t) + \dots = \tau\mathbf{T}(t) + \dots,$$

where ... here means terms which tend to zero with h. Letting h tend to 0 this means that the sine of the angle between osculating planes—and hence the angle itself—tends to τ in absolute value. So the torsion measures the 'dihedral angle between consecutive osculating planes'. In a similar way the curvature measures the 'angle between consecutive tangents' of γ.

Proposition 1.6.7 *Let $\gamma : I \to \mathbf{R}^3$ be a space curve with non-zero curvature, but whose torsion is identically zero. Then γ lies in a plane. (Note that the converse is clear.)*

Proof Since $\mathbf{B}' = -\tau s' \mathbf{P}$ we deduce that $\mathbf{B}' = 0$ and so $\mathbf{B}$ is constant. Now consider $f(t) = \mathbf{B} \cdot (\gamma(t) - \gamma(t_0))$, where $t_0 \in I$. Differentiating we find that $f'(t) = \mathbf{B} \cdot s' \mathbf{T}(t) = 0$ since $\mathbf{B}$ is the constant binormal. So f is constant. But $f(t_0) = 0$, so the curve γ lies in the plane $\mathbf{B} \cdot (x - \gamma(t2_0)) = 0$ as required.

Finally, as an application of the Serret-Frenet equations, we shall show that the curvature and torsion of a curve determine that curve. More precisely we have

Proposition 1.6.8 *Let γ_1, $\gamma_2 : I \to \mathbf{R}^3$ be unit speed parametrised curves with never-vanishing curvature. Suppose that the curvature and torsion of the γ_i coincide, and, for some $s_0 \in I$, we have $\gamma_1(s_0) = \gamma_2(s_0)$ and the tangents and principal normals coincide there. Then $\gamma_1(s) = \gamma_2(s)$ for all s.*

Remark 1.6.9 Of course any two curves related by a rigid motion have the same torsion and curvature (as functions of arc-length). On the other hand given two curves with this property the other hypotheses of the proposition, namely that the curves have a point in common where the tangent and principal normals coincide, can clearly always be satisfied by moving one of the curves by a rigid motion. So this proposition really does state that any two curves in $\mathbf{R}^3$ with the same κ and τ can be obtained by a rigid motion from each other.

Proof We use the obvious notation, with suffices 1 and 2 to distinguish the two curves. We consider the function

$$
\begin{aligned}
f &= (\mathbf{T}_1 - \mathbf{T}_2) \cdot (\mathbf{T}_1 - \mathbf{T}_2) + (\mathbf{P}_1 - \mathbf{P}_2) \cdot (\mathbf{P}_1 - \mathbf{P}_2) + (\mathbf{B}_1 - \mathbf{B}_2) \cdot (\mathbf{B}_1 - \mathbf{B}_2) \\
&= ||\mathbf{T}_1 - \mathbf{T}_2||^2 + ||\mathbf{P}_1 - \mathbf{P}_2||^2 + ||\mathbf{B}_1 - \mathbf{B}_2||^2.
\end{aligned}
$$

Differentiating we obtain, writing $\kappa_1 = \kappa_2 = \kappa$ and $\tau_1 = \tau_2 = \tau$,

$$
\begin{aligned}
f' &= 2(\mathbf{T}_1 - \mathbf{T}_2) \cdot (\kappa_1 \mathbf{P}_1 - \kappa_2 \mathbf{P}_2) + 2(\mathbf{P}_1 - \mathbf{P}_2) \cdot (-\kappa_1 \mathbf{T}_1 + \tau_1 \mathbf{B}_1 + \kappa_2 \mathbf{T}_2 - \tau_2 \mathbf{B}_2) \\
&\quad + 2(\mathbf{B}_1 - \mathbf{B}_2) \cdot (-\tau_1 \mathbf{P}_1 + \tau_2 \mathbf{P}_2) \\
&= 2\kappa(\mathbf{T}_1 - \mathbf{T}_2) \cdot (N_1 - N_2) - 2\kappa(\mathbf{P}_1 - \mathbf{P}_2) \cdot (\mathbf{T}_1 - \mathbf{T}_2) \\
&\quad + 2\tau(N_1 - N_2) \cdot (\mathbf{B}_1 - \mathbf{B}_2) - 2\tau(\mathbf{B}_1 - \mathbf{B}_2) \cdot (\mathbf{P}_1 - \mathbf{P}_2) \\
&= 0.
\end{aligned}
$$

In other words f is constant. On the other hand by hypothesis $f(s_0) = 0$ (since $\mathbf{T}_1 = \mathbf{T}_2$, $\mathbf{P}_1 = \mathbf{P}_2, \mathbf{B}_1 = \mathbf{B}_2$ there) so f is identically zero. Thus each of the three squares making up f is zero so in particular $\mathbf{T}_1(s) - \mathbf{T}_2(s) = d(\gamma_1(s) - \gamma_2(s))/ds$ is identically zero, So $\gamma_1(s) - \gamma_2(s)$ is constant and since it is zero for $s = s_0$ the result follows.

1.7 Exercises

1. Find a regular parametrised curve $\gamma(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\gamma(t)$ runs clockwise around the circle with $\gamma(0) = (0, 1)$.

2. Check that the curve $\gamma(t) = (t^2 - 1, t^2 + 1, t^3 + t)$ is regular and find its unit tangent vector at $\gamma(t)$.

3. Consider the plane curve $\gamma(t) = (A\cos t + B\sin t, C\cos t + D\sin t)$, where A, B, C, D are constants, not all zero.

   (i) Show that, if $AD \neq BC$, then γ is regular for all values of t.

   (ii) Suppose that $AD = BC$ and (without loss of generality) that $A \neq 0$. Show that γ is regular for all values of t except those with $\tan t = B/A$.

   (iii) Show that, if $AD \neq BC$, then γ is an *ellipse* centred at the origin, that is the equation of γ has the form $px^2 + 2qxy + ry^2 = 1$, where $q^2 < pr$ and $p > 0$. When will the ellipse be a *circle*?

   (iv) What curve is traced out by γ when $AD = BC$ but $A \neq 0$?

4. Show that for the plane curve $\gamma(t) = (x(t), y(t))$, where $x(t) = \cos^3 t$, $y(t) = \sin^3 t$, $0 \leq t \leq \pi/2$, the arclength measured from $t = 0$ is $s = \frac{3}{4}(1 - \cos 2t)$. Sketch the curve over the range $0 \leq t \leq \pi/2$. What do you think happens for larger values of t? For which values of t does the curve fail to be regular? Calculate the curvature of γ.

5. Find the unit tangent, unit normal and curvature of the following plane curves:

   (i) $\gamma(t) = (A\cos t, B\sin t)$, $A > 0$, $B > 0$, I an open interval containing $[0, 2\pi]$.

   (ii) $\gamma(t) = (t^2 - 1, t^3 - t)$, $I = \mathbf{R}$.

6. The cubic curve $y^2 = x - x^3$ consists of an oval, lying in the region $0 \leq x \leq 1$, and another part. Removing the points $(0,0), (1,0)$ the oval splits into two pieces, parametrised by $\gamma(t) = (t, \pm(t - t^3)^{\frac{1}{2}})$. Show that for these curves $(0 < t < 1)$ we have $\kappa(t) = \pm 2(3t^4 - 6t^2 - 1)/(9t^4 - 4t^3 - 6t^2 + 4t + 1)^{\frac{3}{2}}$. Why do the halves have opposite signs for κ? Is κ positive or negative for the top half? (And is that the upper or lower sign?)

7. Let $\alpha : I \to \mathbf{R}^2$ be unit speed, with $\alpha(0) = (0,0)$, $\mathbf{T}(0) = (1,0)$, $\mathbf{U}(0) = (0,1)$. Show that

$$\alpha''(0) = (0, \kappa(0))$$

$$\alpha'''(0) = (-\kappa^2(0), \kappa'(0))$$

$$\alpha^{(4)}(0) = (-3\kappa(0)\kappa'(0), \kappa''(0) - (\kappa(0))^3).$$

8. Let $\alpha : I \to \mathbf{R}^2$ be unit speed, and let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear map with with matrix A relative to the standard basis. Let $\beta = F \circ \alpha$. Show that $\beta'(s) = A\alpha'(s)$ (where we write α' and β' as column vectors), and deduce that β is regular provided that A is non-singular, unit speed provided that A is orthogonal. Deduce that, if A is orthogonal, then the curvatures of α and β differ by the sign of $\det A$. This shows that rotations preserve the curvature, reflections reverse its sign. (Obviously translations preserve curvature: here $\beta(s) = \alpha(s) + v$ for some fixed vector v.)

9. **Parallels** Let γ be a unit speed plane curve, and let d be a fixed real number. The curve δ defined by $\delta(t) = \gamma(t) + d\mathbf{U}(t)$ is called the *parallel* to γ at distance d. Show that δ is a regular curve except for those values of t where $\kappa(t) \neq 0$ and $d = 1/\kappa(t)$; also for these irregular points $\delta(t)$ is the *centre of curvature* of γ at $\gamma(t)$, i.e. the point $\gamma(t) + 1/\kappa(t)\mathbf{U}(t)$.

10. **Evolute** Given a unit speed plane curve γ with $\kappa(t)$ never zero we can consider the locus of centres of curvature of γ, namely the curve

$$\epsilon(t) = \gamma(t) + [1/\kappa(t)]\mathbf{U}(t),$$

which is called the *evolute* of γ.

(i) Show that the evolute is a regular curve except for those values of t for which $\kappa' = 0$. These are the points where the curvature has an extremum, also known as *vertices*. Each one is called a *vertex*.

(ii) With $\kappa' < 0$ again, show that the unit tangent and normal to ϵ satisfy $\mathbf{T}_\epsilon = \mathbf{U}$, $\mathbf{U}_\epsilon = -\mathbf{T}$, and the curvature of ϵ is $\kappa_\epsilon = -(\kappa^3/\kappa')$. Deduce that the evolute has no inflexions.

(iii) Assuming that $\kappa' < 0$ on I, so that the radius of curvature $\rho(t) = 1/\kappa(t)$ is increasing on I, show that the arc-length on ϵ from t_0 to $t_1 > t_0$ is $\rho(t_1) - \rho(t_0)$.

(iv) Deduce from (iii) that

$$||\gamma(t) - \epsilon(t)|| = ||\gamma(t_0) - \epsilon(t_0)|| + (\text{arc-length on } \epsilon \text{ from } t_0 \text{ to } t).$$

Deduce that, if a piece of string is wrapped around the evolute, one end being fastened at $\epsilon(t_1)$ and the other end starting at $\gamma(t_0)$, then as the string is unwrapped the ends are at $\epsilon(t)$ and $\gamma(t)$ for all $t > t_0$.

11. Recall from Exercise 10 that a vertex of a plane curve is a point where the curvature has an extremum ($\kappa' = 0$, where $'$ can be derivative with respect to any regular parameter). Find the vertices of the plane curve $\gamma(t) = (t, t^4)$.

12. For the unit speed helix $\alpha(s) = (1/\sqrt{2})(\cos s, \sin s, s)$, show that $\kappa(s) = 1/\sqrt{2}$ for all s. Find expressions for $\mathbf{T}(s)$, $\mathbf{P}(s)$ and show that $\mathbf{B}(s) = (1/\sqrt{2})(\sin s, -\cos s, 1)$ for all s. Find $\tau(s)$.

13. Find the curvature and torsion at the point t of the helix

$$\gamma(t) = (a\cos t, a\sin t, ct),$$

where $a > 0$. Calculate $\mathbf{P}$ and $\mathbf{B}$ for this curve.

14. Find the curvature and torsion of $\gamma(t) = (t, t^2, t^3)$ at the origin.

15. Show that, for any constants a, b, c, d, e, f, the curve $\gamma : \mathbf{R} \to \mathbf{R}^2$ given by $\gamma(t) = (at + bt^2, ct + dt^2, et + ft^2)$ lies in a plane in $\mathbf{R}^3$, and find the equation of this plane.

16. (January 1999, Qu.1) Let $\gamma : I \to \mathbf{R}^2$ be a regular plane curve with unit tangent $\mathbf{T}$, unit normal $\mathbf{U}$ and curvature κ. Let r be a real number. The *parallel curve* δ to γ at distance r is the curve defined by
$$\delta(t) = \gamma(t) + r\mathbf{U}(t).$$

(i) In this part, you may assume that γ is unit speed. Show that δ is a regular curve except for values of t where $\kappa(t) \neq 0$ and $r = 1/\kappa(t)$.

Assume now that $r\kappa(t) < 1$ for all $t \in I$. Show that the unit tangent $\mathbf{T}_\delta$ and unit normal $\mathbf{U}_\delta$ to δ are the same vectors as $\mathbf{T}$ and $\mathbf{U}$ respectively. Show further that the curvature κ_δ of δ is given by
$$\kappa_\delta = \frac{\kappa}{1 - r\kappa}.$$

Show that the evolute of δ (namely, the curve $\delta + \mathbf{U}_\delta/\kappa_\delta$) coincides with the evolute $\gamma + \mathbf{U}/\kappa$ of γ.

(ii) For the (non unit speed) curve $\gamma(t) = (t, t^2)$ find an explicit parametrization of δ. Also find the curvature κ of γ and show that, for $r > \frac{1}{2}$, the parallel curve δ has exactly two points of non-regularity.

17. (January 1999, Qu.2) Let $\gamma : I \to \mathbf{R}^3$ be a regular space curve.

(i) Supposing γ is unit speed, and γ'' is never zero, define the standard vectors $\mathbf{T}$, $\mathbf{P}$, $\mathbf{B}$ and the curvature κ and torsion τ of γ. Prove that

$$(\gamma' \times \gamma'') \cdot \gamma''' = \kappa^2 \tau.$$

(ii) For a general regular space curve γ with $\kappa \neq 0$, write down formulae for the curvature and torsion of γ. Let u be a real constant and let

$$\gamma(t) = (t, t^3, t^4 + ut^2).$$

Show that, for $u \neq 0$, γ has no points where $\kappa = 0$. Show that for $u > 0$ there are exactly two points where the torsion is zero and for $u < 0$ there are no torsion zero points. Show that, for $u > 0$, binormal vectors at the two torsion zero points are parallel to

$$(-2u^{3/2}, \ -12u^{1/2}, \ \pm 3\sqrt{6}).$$

[Extra, not on exam: what happens when $u \to 0$ through positive values?]

18. (An old exam question) (i) Explain what it means to say that $\gamma : I \to \mathbf{R}^2$ is a unit speed plane curve. Define the unit tangent vector $\mathbf{T}(t)$, the unit normal vector $\mathbf{U}$ and the curvature $\kappa(t)$. Show that $\mathbf{U}' = -\kappa$ (short, as usual, for $\mathbf{U}'(t) = -\kappa(t)\mathbf{T}(t)$).

(ii) Let γ be a regular plane curve. Show that the foot of the perpendicular from the origin to the tangent to the plane curve γ at $\gamma(t)$ is given by $(\gamma(t) \cdot \mathbf{U}(t))\mathbf{U}(t)$. As t varies, this point traces out the *pedal curve* of γ. [The line through the origin parallel to the normal $\mathbf{U}(t)$ meets the tangent to γ in a point $\lambda\mathbf{U}(t)$ where λ is chosen so that this point—the foot of the perpendicular—lies on the tangent to γ.] Let us write

$$\delta(t) = (\gamma(t) \cdot \mathbf{U}(t))\mathbf{U}(t),$$

or just $\delta = (\gamma \cdot \mathbf{U})\mathbf{U}$, and assume from now on that γ is unit speed. Show that

$$\delta' = -\kappa((\gamma \cdot \mathbf{T})\mathbf{U} + (\gamma \cdot \mathbf{U})\mathbf{T}).$$

Show that the second factor $(\gamma \cdot \mathbf{T})\mathbf{U} + (\gamma \cdot \mathbf{U})\mathbf{T}$ is zero if and only if $\gamma(t) = 0$, which says that for this t the curve passes through the origin. [This was probably done as a class example. Write $\gamma = \lambda\mathbf{T} + \mu\mathbf{U}$ and deduce from $(\gamma \cdot \mathbf{T})\mathbf{U} + (\gamma \cdot \mathbf{U})\mathbf{T}) = \mathbf{0}$ that $\lambda = \mu = 0$.] Assuming that $\gamma(t)$ is never zero, deduce that δ is a regular curve except for those points corresponding to inflexions (i.e. where $\kappa = 0$).

For a (non-unit speed) curve in the form $(t, Y(t))$ find an expression for $\delta(t)$. Show that, if $\mathbf{Y}''(t_0) = 0$, then $\delta'(t_0) = \mathbf{0}$.

19. Let γ be a unit speed plane curve with κ never zero. Define the plane curve δ by $\delta(s) = (\gamma(s) \cdot \mathbf{U}(s))\mathbf{U}(s)$. (This is the pedal curve of Exercise 18. It is regular since γ has never zero curvature.) Calculate δ' and δ'' and show that κ_δ, the curvature of δ, is zero if and only if

$$2\kappa||\gamma||^2 = -\gamma \cdot \mathbf{U}.$$

Show that this holds if and only if the origin lies on a circle of radius $1/(4\kappa)$ whose centre is on the normal $\mathbf{U}$ to γ.

20. (An old exam question) Let $\gamma : I \to \mathbf{R}^3$ be a unit speed space curve. Define $\mathbf{T}, \kappa$, and, assuming that $\kappa \neq 0$, define $\mathbf{P}, \mathbf{B}, \tau$. Show that $\mathbf{B}' = -\tau\mathbf{P}$.

    Suppose that $0 \in I$ and $\gamma(0) = (0,0,0), \mathbf{T}(0) = (1,0,0), \kappa(0) \neq 0$ and $\mathbf{P}(0) = (0,1,0)$. Show that $\mathbf{B}(0) = (0,0,1)$ and $\gamma''(0) = (0,\kappa,0), \gamma'''(0) = (-\kappa^2, \kappa', \kappa\tau)$ where here κ, τ and their derivatives are evaluated at $t = 0$. Write down the condition for $\gamma'(0), \gamma''(0)$ and $\gamma'''(0)$ to be linearly independent [i.e., for the triple scalar product or determinant of these 3 vectors to be nonzero].

    Let γ be as above and let $\alpha : I \to \mathbf{R}^2$ be defined by

    $$\alpha(t) = (\gamma(t) \cdot \mathbf{T}(0), \gamma(t) \cdot \mathbf{P}(0)).$$

    Show that α is a regular curve at $t = 0$ [i.e. that $\alpha'(0) \neq \mathbf{0}$] and that its curvature there equals the curvature $\kappa(0)$ of γ at $t = 0$.

21. Let $\gamma(t)$ be a regular parametrised space curve which does not pass through the origin. Show that, if $\gamma(t_0)$ is the point on the curve closest to the origin, then $\gamma(t_0)$ is orthogonal to $\gamma'(t_0)$. [Hint: Let $f(t) = \gamma(t) \cdot \gamma(t) = $ square of the distance of $\gamma(t)$ from the origin. Deduce that $f'(t_0) = 0$.]

22. A parametrised curve $\gamma : I \to \mathbf{R}^3$ has the property that its second derivative $\gamma''(t)$ is identically zero. What can be said about γ?

23. Let $\gamma : I \to \mathbf{R}^3$ be a regular parametrised curve and let $\mathbf{v} \in \mathbf{R}^3$ be a fixed vector. Assume that $\gamma'(t) \cdot \mathbf{v} = 0$ for all $t \in I$, and that $\gamma(0) \cdot \mathbf{v} = 0$ too. Prove that $\gamma(t) \cdot \mathbf{v} = 0$ for all $t \in I$. What does this mean geometrically?

24. Let $\gamma : I \to \mathbf{R}^3$ be a regular parametrised curve. Show that $||\gamma(t)||$ is a non-zero constant if and only if $\gamma(t)$ is orthogonal to $\gamma'(t)$ for all $t \in I$. [Remember that $||\gamma(t)||^2 = \gamma(t) \cdot \gamma(t)$.] What can you say about the curve γ here?

25. Let $\gamma : I \to \mathbf{R}^3$ be a regular parametrised curve and suppose that all of its tangent lines pass through a fixed point (say $(0,0,0)$). Prove that γ is a straight line.

26. The **Darboux vector D** is defined by $\mathbf{D} = \tau\mathbf{T} + \kappa\mathbf{B}$. Show that for a unit speed curve,

    $$\mathbf{T}' = \mathbf{D} \times \mathbf{T}, \quad \mathbf{P}' = \mathbf{D} \times \mathbf{P}, \quad \mathbf{B}' = \mathbf{D} \times \mathbf{B}.$$

    These imply that $\mathbf{T}', \mathbf{P}'$ and $\mathbf{B}'$ are all perpendicular to $\mathbf{D}$. [Physically, this says that $\mathbf{D}$ is in the direction of the 'instantaneous axis of rotation of the triad $\mathbf{T}, \mathbf{P}, \mathbf{B}$'.]

    Show that $\mathbf{D} = \mathbf{P} \times \mathbf{P}'$ (still assuming unit speed). [The *rectifying plane* of a space curve γ at $\gamma(s)$ is the plane through $\gamma(s)$ perpendicular to the principal normal $\mathbf{P}(s)$. The last result says that the Darboux vector $\mathbf{D}(s)$ is in fact the limiting direction of the intersection of the 'rectifying planes' at parameter values s and $s + h$, as $h \to 0$.]

27. Let $\alpha : I \to \mathbf{R}^3$ be unit speed, with $\alpha(0) = (0,0,0)$, $\mathbf{T}(0) = (1,0,0)$, $\mathbf{P}(0) = (0,1,0)$. Show that $\mathbf{B}(0) = (0,0,1)$ and

    $$\alpha''(0) = (0, \kappa, 0)$$
    $$\alpha'''(0) = (-\kappa^2, \kappa', \kappa\tau)$$
    $$\alpha^{(4)}(0) = (-3\kappa\kappa', \kappa'' - \kappa^3 - \kappa\tau^2, 2\kappa'\tau + \kappa\tau')$$

    In these formulae, κ and τ stand for $\kappa(0)$ and $\tau(0)$ respectively.

28. Let $\alpha : I \to \mathbf{R}^3$ be unit speed, and $s_0 \in I$. Define $\gamma : I \to \mathbf{R}^2$ by $\gamma(s) = (\alpha(s) \cdot \mathbf{T}(s_0), \alpha(s) \cdot \mathbf{P}(s_0))$. (This is the projection of α to the osculating plane at $\alpha(s_0)$. Note that γ need not be unit speed.) Show that restricting γ to some suitably small open interval containing s_0, it is a regular curve, and that its curvature at s_0 equals the curvature of α at s_0. Try projecting to the plane spanned by $\mathbf{P}(s_0)$, $\mathbf{B}(s_0)$.

29. Find the equation of the osculating plane at the point t_0 of the curve $\gamma(t) = (t, t^2, t^3)$.

    Prove that, if $\mathbf{a}$ is a point not on the curve there are at most three points on the curve whose osculating plane passes through $\mathbf{a}$, and when there are exactly three points they are coplanar with $\mathbf{a}$.

30. If all of the osculating planes of a curve pass through a fixed point show that the curve lies in a plane. If they are all parallel to a fixed plane show that the curve is planar also.

31. A space curve with κ never zero is such that its tangent $\mathbf{T}$ makes a constant angle α with a fixed direction $\mathbf{a}$. Prove that $\mathbf{P} \cdot \mathbf{a} = 0$ and that $\mathbf{B}$ makes a constant angle, also α, with $\mathbf{a}$. Prove also that $\kappa/\tau = \tan \alpha$.

32. Prove that a curve γ for which the quotient κ/τ is constant has the property that its tangent $\mathbf{T}$ makes a constant angle α with a fixed direction $\mathbf{a}$.

33. Suppose that the curvature and torsion of a curve γ are both non-zero constants. Show that γ is a circular helix i.e. a parametrisation of a curve of the form $(a \cos t, a \sin t, bt)$, for some constants a and b.

34. Consider the unique circle through three nearby points $\gamma(s-h), \gamma(s), \gamma(s+h)$ on a unit speed space curve. (This 'circle' will be a straight line when the three points are collinear.) Here is a formula for the reciprocal $1/R$ of the radius of the circle through three points with position vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in space:
$$\frac{1}{R} = \frac{2\|\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u}\|}{\|\mathbf{u} - \mathbf{v}\| \, \|\mathbf{v} - \mathbf{w}\| \, \|\mathbf{w} - \mathbf{u}\|}.$$
(Maybe you would like to prove this! Of course three points always lie in a plane so you could assume the points lie say in the x, y-plane.) Putting $\mathbf{u} = \gamma(s - h), \mathbf{v} = \gamma(s), \mathbf{w} = \gamma(s + h)$ we expand by Taylor's theorem:

$$\mathbf{u} = \gamma(s) - h\mathbf{T}(s) + \frac{h^2}{2}\kappa(s)\mathbf{P}(s) + \dots, \quad \mathbf{w} = \gamma(s) + h\mathbf{T}(s) + \frac{h^2}{2}\kappa(s)\mathbf{P}(s) + \dots,$$

where each $\dots$ stands for terms of higher order in h than two. Now show that the numerator in the expression for $1/R$ has the form $h^3\kappa(s) + \dots$ and the denominator has the form $2h^3 + \dots$. Deduce that the limit of $1/R$ is indeed $\kappa(s)$.

It would be slightly better to work with say $\gamma(s - k), \gamma(s), \gamma(s + h)$ so that we have three arbitrary nearby points on the curve. This results in

$$\frac{1}{R} = \frac{\kappa hk(h + k) + \text{ terms of degree } \geq 4 \text{ in } h, k}{hk(h + k) + \text{ terms of degree } \geq 4 \text{ in } h, k},$$

so that the same result holds when h and k simultaneously tend to 0.

35. Show that the curvature and torsion of a regular space curve with κ never zero are unaffected by translations and rotations.

36. [This extended example is somewhat 'off the syllabus'.] Closed curves in $\mathbf{R}^3$ can be knotted (we will not say precisely what we mean by that, but you can probably guess). Here is an example which illustrates how a knotted curve can lie on a naturally defined *surface*—of course the idea of surface won't be introduced formally until later but you will undoubtedly know already what one looks like!

Consider a circle, radius $b > 0$, centred at $(a, 0, 0)$, where $a > 0$, and lying in the plane $y = 0$ of x, y, z-space. The circle consists of points $(a + b\cos\phi, 0, b\sin\phi)$ (Why?). Now rotate the circle around the z-axis; this produces the *surface* of points

$$X(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$$

(Why?)

(i) Assume that $a > b$. Why does this imply that the surface does not intersect the z-axis? The surface is called a *torus*, or a *torus of revolution*. Note that $a > b > 0$ implies that $a + b\cos\phi > 0$ for all ϕ.

Assume from now on that $a > b$.

(ii) Why are the circles $X(\theta, \phi)$ for a fixed θ with $0 \le \theta < 2\pi$ all disjoint from one another? Deduce that in fact $X(\theta_1, \phi_1) = X(\theta_2, \phi_2)$, where all angles are in $[0, 2\pi)$, if and only if $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$. That is, no identifications take place under X except when one or other angle is increased by 2π.

(iii) Consider the space curve γ obtained by putting $\theta = 2t, \phi = 3t$ above, that is,

$$\gamma(t) = ((a + b\cos 3t)\cos 2t, (a + b\cos 3t)\sin 2t, b\sin 3t).$$

Show that this curve closes up at $t = 2\pi$, and then repeats. It is marginally harder to show that the points $\gamma(t_1)$ and $\gamma(t_2)$ are never the same, when $0 \le t_1 < t_2 < 2\pi$, but maybe you can see how to do this from (ii) above: you need only check that no two points of the line $\theta = 2t \bmod 2\pi, \phi = 3t \bmod 2\pi$, for $0 \le t < 2\pi$, are the same point of the half-open square $0 \le \theta < 2\pi, 0 \le \phi < 2\pi$. A diagram may suffice for this. The conclusion is that the curve γ, for $0 \le t < 2\pi$, is a 'simple', i.e. non-self-intersecting, space curve lying on the torus.

(iv) Projecting the space curve γ to the (x, y)-plane we get say

$$\delta(t) = ((a + b\cos 3t)\cos 2t, (a + b\cos 3t)\sin 2t).$$

Show that this is a regular curve which self-intersects precisely at points $\gamma(t_1)$ and $\gamma(t_2)$ where (t_1, t_2) are the pairs

$$(\pi/6, 7\pi/6), (\pi/2, 3\pi/2), (5\pi/6, 11\pi/6).$$

(Hint for showing that $\delta'(t)$ is never 0: Work out the square of the length of $\delta'(t)$. It comes to $4(a + b\cos 3t)^2 + 9b^2 \sin^2 3t$.) Show also that the image of δ meets the circles of radius $a - b, a + b$ in the x, y-plane where t is a multiple of $\pi/3$. Perhaps by now you can get a clear idea of what the image of δ looks like, and also which branches of γ go 'over' and which 'under' at the crossings of δ. Perhaps you can even convince yourself that γ is knotted, by taking a closed loop of string with the same over-under arrangement. (This is called a *trefoil knot*. Of course other possibilities are given by $\theta = mt, \phi = nt$ for integers m and n.)

Chapter 2

Surface patches in Euclidean Space

The 2-dimensional analogue of a curve is a surface. However surfaces are much more complicated than curves. This seems natural enough, after all there is a jump from 1 to 2 dimensions! However even allowing for this they are more complicated objects. In some ways this is due to the fact that there is no analogue of the nice parametrisation of curves by arc-length.

Surfaces arise in a great many situations, but chiefly as the "surface" of objects around us. In particular biological objects (for example plants and animals) have geometrically interesting forms. What, for example, do we find attractive in a face? In the past mathematicians have tried to measure the beauty of sculptures using some differential geometry. Moreover manufactured objects can have some interesting geometry associated with them. Objects produced on a lathe correspond to so-called surfaces of revolution, the standard cooling tower is an example of a ruled surface, as are a number of tent-like buildings designed by architects of late. Indeed many of these are examples of developable surfaces (these are surfaces which can be made from sheet metal without any stretching or tearing). Surfaces also arise as soap bubbles spanning wires which can be very beautiful, and are examples of so-called minimal surfaces. We shall spend some time looking at all of these types of surface.

Gauss, who was one of the great figures in mathematics, made an enormously important contribution to differential geometry. His work was at least partly motivated by a surveying project given to him by the Hanoverian and Danish governments. Surveying, of course, is concerned with looking at the surface of the earth. This is related to the subject of cartography, or map making. One problem of great importance to the navigators of the 16th and 17th centuries was how best to represent the surface of the earth (roughly a sphere) on a flat sheet of paper (or vellum or whatever). We shall spend a little time looking at this problem too.

2.1 Basic facts about maps

We shall use parametrizations of surfaces which are maps from some region U in the parameter plane into $\mathbf{R}^3$. In practice we need U to be an *open* subset:

Definition 2.1.1 *A subset U of $\mathbf{R}^2$ is* **open** *if around every point $\mathbf{a} \in U$ there is an open disk contained in U.*

This is not a serious restriction, for example the whole plane is open, and the set $\{(u,v) : u^2 + v^2 < 1\}$ (interior of the unit disk) is open. Basically so long as we define a set by means of $<$ and $>$, and not by means of $=, \leq$ or $\geq$, it will be open. We only need this so that we can differentiate our functions everywhere.

Definition 2.1.2 Let U be an open subset of $\mathbf{R}^2$ and $\mathbf{X} : U \to \mathbf{R}^3$ a map. Write $\mathbf{X} = (X_1, X_2, X_3)$. We say that $\mathbf{X}$ is **smooth** if each of the functions X_1, X_2, X_3 has partial derivatives of all orders which are continuous on U. We shall use coordinates (u, v) as coordinates on U.

Definition 2.1.3 Let $\mathbf{X}$ be as above, and $\mathbf{a} \in U$. The matrix

$$J\mathbf{X}(\mathbf{a}) = \begin{pmatrix} \frac{\partial X_1}{\partial u} & \frac{\partial X_1}{\partial v} \\ \\ \frac{\partial X_2}{\partial u} & \frac{\partial X_2}{\partial v} \\ \\ \frac{\partial X_3}{\partial u} & \frac{\partial X_3}{\partial v} \end{pmatrix}$$

where the partial derivatives are evaluated at $\mathbf{a}$ is called the **jacobian matrix** of $\mathbf{X}$ at $\mathbf{a}$. The linear map $D\mathbf{X}(a) : \mathbf{R}^2 \to \mathbf{R}^3$ with this matrix is called the **derivative** of $\mathbf{X}$ at $\mathbf{a}$. Thus, for any vector $\mathbf{v} \in \mathbf{R}^2$, written as a column, we have $D\mathbf{X}(\mathbf{a})(\mathbf{v}) = J\mathbf{X}(\mathbf{a})\mathbf{v} \in \mathbf{R}^3$, where the vector in $\mathbf{R}^3$ also appears as a column.

The following property says that plenty of smooth functions exist, and reassures us that the order of mixed partial derivatives is unimportant.

Properties 2.1.4 *(1) Polynomials, trigonometric functions, the exponential and log functions are smooth throughout their domains of definition. The sum, product, quotient, and composite of any two smooth functions are smooth wherever they are defined.*
(2) If $f : U \to \mathbf{R}$ is a smooth function, then any partial derivative of f is independent of the order in which the derivatives are taken. In particular $\partial^2 f / \partial u \partial v = \partial^2 f / \partial v \partial u$.

2.2 Local surfaces

We are now in a position to give a precise definition of local surface.

Definition 2.2.1 Let U be an open subset of $\mathbf{R}^2$. A **local surface** or **patch** M is a smooth mapping

$$\mathbf{X} = (X_1, X_2, X_3) : U \to \mathbf{R}^3$$

We may use $\mathbf{X} = (X, Y, Z)$ occasionally. We shall use (x_1, x_2, x_3) or (x, y, z) for coordinates in $\mathbf{R}^3$.

Examples 2.2.2 Patches

(1) Consider $U = \mathbf{R}^2$, and $\mathbf{X}(u, v) = (u, v, au + bv)$, where a and b are real numbers. The image of $\mathbf{X}$ is the plane through the origin orthogonal to the vector $(a, b, -1)$.

(2) Consider $U = \{(u, v) : u^2 + v^2 < 1\}$ and the map $\mathbf{X}$ given by $\mathbf{X}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$. The image of $\mathbf{X}$ is the upper half of the unit sphere in $\mathbf{R}^3$.

(3) Let $\gamma : I \to \mathbf{R}^3$ be a regular curve, and define $\Gamma : I \times \mathbf{R} \to \mathbf{R}^3$ by $\Gamma(s, t) = \gamma(s) + t\gamma'(s)$. Then the image of Γ is the set of all tangent lines to the curve γ. It is called the *tangent developable* of γ.

Let γ be a parametrisation of the unit circle. Then the image of Γ is the set of points on or outside the circle. Note that there are two of these tangent lines through any point of the plane.

On the other hand for the helix $\gamma(t) = (\cos t, \sin t, t)$ Exercise 12 at the end of the chapter shows that at any rate the 'forward' tangents, given by $t > 0$ do not intersect in space.

(4) Examples (1) and (2) are particular cases of the following. Let U be an open subset of $\mathbf{R}^2$ and let $f : U \to \mathbf{R}$ be a smooth function. Then the graph of f is the image of the map $(graph\ f)(u, v) = (u, v, f(u, v))$. Later, we shall be particularly interested in the case when the function f is quadratic, say $f(u, v) = au^2 + 2buv + cv^2$.

(5) Consider $\mathbf{X} : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $\mathbf{X}(u, v) = (v \cos u, v \sin u, v)$. The image of this mapping is the cone given by the set of points (x_1, x_2, x_3) in $\mathbf{R}^3$ such that $x_1^2 + x_2^2 = x_3^2$. Clearly this has a 'bad' point corresponding to the line $v = 0$: this is the cone point on the surface. (It is not difficult to find worse examples. The map $\mathbf{X} : \mathbf{R}^2 \to \mathbf{R}^3$, $\mathbf{X}(u, v) = (u, 0, 0)$ is a big disappointment as a surface; the map $\mathbf{X}(u, v) = (0, 0, 0)$ is even worse.)

Now just as we avoided non-regular points on curves we also wish to avoid bad points like the cone point in the previous example. The idea is really rather simple. We wish to study objects that look locally like a piece of the plane. The condition that our surface should resemble, infinitesimally, the plane is that encompassed in the following definition of a regular surface patch (sometimes called a regular local surface).

Definition 2.2.3 A smooth mapping

$$\mathbf{X} = (X_1, X_2, X_3) : U \to \mathbf{R}^3$$

is called a **regular (surface) patch** provided the jacobian matrix

$$\begin{pmatrix} \dfrac{\partial X_1}{\partial u} & \dfrac{\partial X_1}{\partial v} \\[2mm] \dfrac{\partial X_2}{\partial u} & \dfrac{\partial X_2}{\partial v} \\[2mm] \dfrac{\partial X_3}{\partial u} & \dfrac{\partial X_3}{\partial v} \end{pmatrix}$$

has rank 2 at all points (u, v) of U. If in addition the mapping is *injective*, i.e. the image of $\mathbf{X}$ does not self intersect, then we call $\mathbf{X}$ a **regular injective patch**. To prove injectivity you assume $\mathbf{X}(u_1, v_1) = \mathbf{X}(u_2, v_2)$ and deduce that $u_1 = u_2$ and $v_1 = v_2$.

For the most part we shall be dealing only with such patches and may even refer to them as just 'surfaces'. We denote the image of $\mathbf{X}$ by M, and shall sometimes refer to the surface as M, even though it is really the particular parametrisation given.

Remarks 2.2.4 (1) If we denote $(\partial X_1/\partial u, \partial X_2/\partial u, \partial X_3/\partial u)$ by $\partial \mathbf{X}/\partial u$, or just $\mathbf{X}_u$ for short, and similarly for $\partial \mathbf{X}/\partial v$, we are asking that these two vectors are linearly independent at all points of U. This *is* the analogue of the condition for a curve to be regular. There we only had one variable (say u so that the second column of the matrix is not there). We then asked that the matrix had rank 1, that is, that the matrix was not the zero matrix.

(2) The condition on $\mathbf{X}$ being injective is a bit of a nuisance. You will recall that we did not ask for this when defining a regular curve: we allowed parametrized curves to cross themselves; there was just more than one tangent line at a crossing point. For surfaces, however, it is really better not to allow the possibility of more than one tangent plane at a point.

There is one harmless brand of non-injectivity, exemplified by (5) of Examples 2.2.2 above. Strictly, the map $\mathbf{X}(u, v) = (v \cos u, v \sin u, v)$ is not injective since $\mathbf{X}(u + 2n\pi, v + 2m\pi) = \mathbf{X}(u, v)$ for any integers m, n. But no problems are caused by this. Generally when a formula of $\mathbf{X}$ involves cos and sin of angles the non-injectivity is not a problem.

Example 2.2.5 Graph surface

Let us consider the graph parametrisations of the form $(u, v, f(u, v))$ for some smooth f. Here $\mathbf{X}_u = (1, 0, \partial f / \partial u)$ and $\mathbf{X}_v = (0, 1, \partial f / \partial v)$ and these are clearly linearly independent. So we always have a regular patch; note that injectivity is automatic here.

We now define the tangent plane to a regular injective patch at a point, which in a sense is the best *linear* approximation to the patch at that point.

Let M be a regular injective patch, parametrized by $\mathbf{X}$, and let $\mathbf{p} = \mathbf{X}(u_0, v_0) \in M$. Consider a regular curve $\beta(t) = (u(t), v(t))$ in the parameter plane with $\beta(t_0) = (u_0, v_0)$. The curve $\gamma(t) = \mathbf{X}(\beta(t))$ will then be a regular curve in $\mathbf{R}^3$ which lies on our surface patch M. So we can consider the velocity vector $\gamma'(t_0)$ to γ at $\mathbf{p}$. The vectors $\gamma'(t_0)$ which arise in this way are called **tangent vectors** to M at $\mathbf{p}$. Note that they are not necessarily unit vectors.

Fortunately it is very easy to find all the tangent vectors to M at $\mathbf{p}$, since they are simply linear combinations of two easily calculated vectors, as follows.

Differentiating $\gamma(t) = \mathbf{X}(\beta(t)) = \mathbf{X}(u(t), v(t))$ with respect to t gives

$$\gamma'(t) = \mathbf{X}_u(u(t), v(t))u'(t) + \mathbf{X}_v(u(t), v(t))v'(t) \text{ or } \gamma' = \mathbf{X}_u u' + \mathbf{X}_v v' \text{ for short.} \tag{2.1}$$

Evaluating at $t = t_0$ it is immediate that $\gamma'(t_0)$ is a linear combination of $\mathbf{X}_u$ and $\mathbf{X}_v$, both evaluated at (u_0, v_0).

Conversely let us consider a linear combination $\lambda \mathbf{X}_u + \mu \mathbf{X}_v$. Define $u(t) = u_0 + \lambda t$, $v(t) = v_0 + \mu t$, so that $\beta(t) = (u_0 + \lambda t, v_0 + \mu t)$, which is clearly a regular curve (a straight line) through (u_0, v_0) in the parameter space. Then define $\gamma(t) = \mathbf{X}(\beta(t))$ as before. Thus $u'(t) = \lambda, v'(t) = \mu$ here (for any t). Hence $\gamma'(t_0) = \lambda \mathbf{X}_u + \mu \mathbf{X}_v$, and the given linear combination of $\mathbf{X}_u$ and $\mathbf{X}_v$ has been shown to be a tangent vector to M at $\mathbf{p} = \mathbf{X}(u_0, v_0)$. Summing up:

Proposition 2.2.6 *Let $\mathbf{p} = \mathbf{X}(u_0, v_0)$ be a point on a regular injective patch $\mathbf{X} : U \to \mathbf{R}^3$. Then a vector $\mathbf{v} \in \mathbf{R}^3$ is tangent to M at $\mathbf{p}$ if and only if $\mathbf{v}$ can be written as a linear combination of the vectors $\mathbf{X}_u(u_0, v_0), \mathbf{X}_v(u_0, v_0)$.*

The set of all tangent vectors to M at $\mathbf{p}$ is called the **tangent plane** to M at $\mathbf{p}$:

Tangent plane at $\mathbf{p} = \mathbf{X}(u_0, v_0)$ is the set of vectors $\mathbf{v} = \lambda \mathbf{X}_u + \mu \mathbf{X}_v$,

where $\mathbf{X}_u, \mathbf{X}_v$ are evaluated at (u_0, v_0). We often say that the tangent plane is **spanned** by the vectors $\mathbf{X}_u, \mathbf{X}_v$.

Note This definition only involves vectors and the tangent plane as above might as well be assumed to pass through the origin. To move it to the point $\mathbf{p}$ we simply add $\mathbf{p}$ to all the points $\lambda \mathbf{X}_u + \mu \mathbf{X}_v$. Sometimes this is called the *affine tangent plane* to M at $\mathbf{p}$. I shall just use the same phrase 'tangent plane'; it should always be clear from the context whether it is important to 'move the plane parallel to itself to pass through $\mathbf{p}$' or not. For example if we want to find the *equation* of the tangent plane then obviously we move it to pass through $\mathbf{p}$ first. If we just want to know *directions* in the tangent plane then we might as well leave it passing through the origin.XC

Examples 2.2.7 Surfaces and tangent planes

(1) For the graph of a function $f : U \to \mathbf{R}$, that is the surface $(u, v, f(u, v))$, the tangent plane at the point p corresponding to (u_0, v_0) is spanned by $(1, 0, f_u(u_0, v_0))$ and $(0, 1, f_v(u_0, v_0))$.

In the case when $f(u,v) = \sqrt{1 - u^2 - v^2}$ these are the vectors

$$\left(1, 0, -\frac{u_0}{\sqrt{1 - u_0^2 - v_0^2}}\right), \quad \left(0, 1, -\frac{v_0}{\sqrt{1 - u_0^2 - v_0^2}}\right).$$

Note that these vectors are orthogonal to the vector $(u_0, v_0, \sqrt{1 - u_0^2 - v_0^2})$; it should be geometrically clear why this is so!

(2) Let us consider the tangent developable again—recall Exercise 2.2.2(3). So $\gamma : I \to \mathbf{R}^3$ is a regular curve, which we take to be unit speed, and we define $\mathbf{X} : I \times \mathbf{R} \to \mathbf{R}^3$ by $\mathbf{X}(s, t) = \gamma(s) + t\gamma'(s)$. The image of $\mathbf{X}$ is the set of all tangent lines to the curve γ. When is this a regular patch? To answer this we need to write down the jacobian matrix of Γ, but this is the matrix with columns the vectors $\gamma'(s) + t\gamma''(t)$ and $\gamma'(s)$. These vectors are respectively $\mathbf{T}(s) + t\kappa(s)\mathbf{P}(s)$ and $\mathbf{T}(s)$ and are linearly independent provided $t\kappa(s) \neq 0$. So if the curvature of γ is nowhere zero the restriction of $\mathbf{X}$ to $I \times \mathbf{R}^+$ is a regular injective patch (where $\mathbf{R}^+ = \{t \in \mathbf{R} : t > 0\}$), provided the 'forward' tangent lines, given by $t > 0$, do not intersect one another. (Compare the helix example of Exercise 12.) The tangent vectors to the surface at the point $\mathbf{X}(s_0, t_0)$ are spanned by $\mathbf{T}(s_0)$ and $\mathbf{P}(s_0)$. The same holds if γ is regular but not unit speed.

(3) We next consider a rather interesting class of examples, namely **surfaces of revolution**. For this we consider first a regular parametrised curve $\gamma : I \to \mathbf{R}^3$, whose image lies in a *plane*, and a line in that plane. We use coordinates (x_1, x_2, x_3) in $\mathbf{R}^3$ and take the plane to be given by $x_2 = 0$ and the line to be the x_3-axis. By spinning the curve about this line in 3-space we sweep out a surface called a surface of revolution. So we are dealing with a mapping $\mathbf{X} : I \times \mathbf{R} \to \mathbf{R}^3$ defined by

$$\mathbf{X}(t, \theta) = (\gamma_1(t)\cos\theta, \gamma_1(t)\sin\theta, \gamma_2(t))$$

where $\gamma(t) = (\gamma_1(t), 0, \gamma_2(t))$.

The derivatives are $\mathbf{X}_t = (\gamma_1'(t)\cos\theta, \gamma_1'(t)\sin\theta, \gamma_2'(t))$, and $\mathbf{X}_\theta = (-\gamma_1(t)\sin\theta, \gamma_1(t)\cos\theta, 0)$. One now easily checks that these are linearly independent unless $\gamma_1 = 0$, (using the fact that γ is a regular curve). In other words $\mathbf{X}$ is a regular patch except at points where the axis meets the curve. To make the patch also injective we need to restrict θ to some range like $0 < \theta < 2\pi$ and also to assume that γ does not intersect itself.

As an example we consider the curve $\gamma(t) = (2 + \cos t, 0, \sin t)$, the unit circle centred at $(2, 0, 0)$ in the (x_1, x_3)-plane. When we rotate this about the x_3-axis we obtain a *torus of revolution*.

(4) As a final source of examples we consider **ruled** surfaces. Roughly speaking these are surfaces which are obtained by moving straight lines about. More precisely we shall have as our two ingredients a unit speed space curve $\alpha : I \to \mathbf{R}^3$ and a family of unit vectors $\beta : I \to \mathbf{R}^3$, so that $\|\beta(s)\| = 1$ for all s. The parametrisation is then given by

$$\mathbf{X} : I \times \mathbf{R} \to \mathbf{R}^3, \ \mathbf{X}(s, t) = \alpha(s) + t\beta(s).$$

The derivatives are $\mathbf{X}_s = T(s) + t\beta'(s)$, $\mathbf{X}_t = \beta(s)$ where as usual $T(s)$ is the unit tangent to α at s. For a regular patch we want these vectors to be linearly independent (they will then span the tangent space to the surface at the given point).

The tangent developable (Exercise 2.2.2(3)) is an example of a ruled surface. So are the *cylinders*, which are obtained from this procedure by taking γ to lie in a plane, and β to be a fixed unit vector orthogonal to that plane. So for example we might take $\gamma(s) = (\cos s, \sin s, 0)$ and $\beta = (0, 0, 1)$, and we obtain a classical cylinder. In this case it is easy to see that we have a regular patch, that is that the two derivatives are linearly independent.

As a slightly more interesting example of a ruled surface we choose $\alpha(s) = (\cos s, \sin s, 0)$ to be the unit speed circle, and $\beta(s) = \alpha'(s) + \mathbf{e}_3$ where $\mathbf{e}_3$ is the unit vector $(0, 0, 1)$. In explicit form this is just

$$\mathbf{X}(s, t) = (\cos s - t \sin s, \sin s + t \cos s, t).$$

Note that $X_1^2 + X_2^2 - X_3^2 = 1 - t^2 + t^2 = 1$. This shows that the image of $\mathbf{X}$ is a hyperboloid of revolution. Note also that if we took β to be $-\alpha'(s) + \mathbf{e}_3$ we obtain the same surface. You can check that this means that there are two (straight) lines through each point of this hyperboloid.

Definition 2.2.8 Given a regular injective patch $\mathbf{X} : U \to \mathbf{R}^3$ and a point $(u_0, v_0) \in U$ we define a **normal** at $\mathbf{X}(u_0, v_0)$ to be a vector perpendicular to the tangent plane at $\mathbf{X}(u_0, v_0)$. There are two possible choices of unit normal, indeed these are $\pm(\mathbf{X}_u \times \mathbf{X}_v)/||\mathbf{X}_u \times \mathbf{X}_v||$. With a choice of sign, the normal is denoted by $\mathbf{N}$.

Examples 2.2.9 Normals

(1) Consider the unit (hemi-)sphere: recall that $U = \{(u, v) : u^2 + v^2 < 1\}$ and the map $\mathbf{X}$ is given by $\mathbf{X}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$. The vector product $\mathbf{X}_u \times \mathbf{X}_v$ is given by

$$\begin{aligned}
(1, 0, -(1 - u^2 - v^2)^{-1/2}u) \quad &\times \quad (0, 1, -(1 - u^2 - v^2)^{-1/2}v) \\
&= \quad ((1 - u^2 - v^2)^{-1/2}u, (1 - u^2 - v^2)^{-1/2}v, 1)
\end{aligned}$$

which is a multiple of $(u, v, (1 - u^2 - v^2)^{1/2})$. In other words we recover the geometrically clear fact that one of the unit normals to the unit sphere at a point $\mathbf{p}$ is $\mathbf{p}$ itself.

(2) Consider the quadric surface given by $\mathbf{X}(u, v) = (u, v, uv)$. Note that $\mathbf{X}_u = (1, 0, v)$, $\mathbf{X}_v = (0, 1, u)$ so that a unit normal vector here is $(1 + u^2 + v^2)^{-1/2}(-v, -u, 1)$.

2.3 Exercises

1. Show that
$$\mathbf{X}(u, v) = ((u + v)/2, (u - v)/2, uv)$$
   is a regular injective patch. Check that the image is the hyperbolic paraboloid $x_3 = x_1^2 - x_2^2$, using coordinates (x_1, x_2, x_3) in $\mathbf{R}^3$. Determine the tangent vectors $\mathbf{X}_u, \mathbf{X}_v$, and unit normal $\mathbf{N}$ to this surface patch at the point given by $(u, v) = (1, -1)$.

2. Find a regular patch whose image is the cylinder (in $\mathbf{R}^3$) given by
$$x_1^2/a^2 + x_2^2/b^2 = 1.$$
   Find the tangent plane at a general point $\mathbf{p}$ of the cylinder.

3. Let $U = \{(u, v) : u > 0\}$, and define
$$\mathbf{X}(u, v) = (u \cos v, u \sin v, u + v).$$
   Show that this determines a regular injective patch.

4. Let $\mathbf{X}(u, v) = (u - v, u + v, 2(u^2 + v^2))$, for all $(u, v) \in \mathbf{R}^2$. Show that $\mathbf{X}$ determines a regular injective patch. Can you describe this surface geometrically?

5. Let $\mathbf{X}(u, v) = (\sqrt{1 - v^2} \cos u, \sqrt{1 - v^2} \sin u, v)$, with $-\pi < u < \pi$ and $-1 < v < 1$. Show that $\mathbf{X}$ is a regular injective patch. Describe the image of $\mathbf{X}$ geometrically.

6. Show that a unit normal to a surface of revolution

$$\mathbf{X}(u, v) = (\gamma_1(u) \cos v, \gamma_1(u) \sin v, \gamma_2(u))$$

(where $\gamma_1(u) > 0$ all u) is given by

$$\mathbf{N} = (-\gamma_2' \cos v, -\gamma_2' \sin v, \gamma_1')/||\gamma'||.$$

(The surface is obtained by rotating the curve $\gamma(u) = (\gamma_1(u), 0, \gamma_2(u))$, which lies in the x_1, x_3 plane, about the x_3-axis in (x_1, x_2, x_3)-space.)

7. A *cone* in $\mathbf{R}^3$ is defined by a parametrization $\mathbf{X}(u, v) = \mathbf{p} + v\beta(u)$, where $\mathbf{p}$ is a fixed point and $\beta(u)$ is a unit vector for all u. Show that this is a regular patch if and only if $v \neq 0$ and $\beta'(u)$ is a non-zero vector for all u. If $\mathbf{p} = (0, 0, 1)$ and $\beta(u) = \frac{1}{\sqrt{2}}(\cos u, \sin u, -1)$ then describe the resulting cone.

8. Show that the tangent plane is constant along a ruling of a cone, that is along the lines in the ruled surface.

9. Show that the tangent plane to the tangent developable of a (unit speed) space curve γ, with κ never zero, is the osculating plane at the corresponding point of the curve. (Thus $\mathbf{X}(s, t) = \gamma(s) + t\gamma'(s)$, with $t > 0$ to ensure regularity, and you want to prove that the tangent plane at $\mathbf{X}(s, t)$ is spanned by $\mathbf{T}(s)$ and $\mathbf{P}(s)$.)

10. The Möbius strip, far from being an exotic dance, is a surface which you may have seen before. It can be thought of as a ruled surface where we take

$$\alpha(s) = (\cos s, \sin s, 0) \text{ and } \beta(s) = (\sin(s/2) \cos s, \sin(s/2) \sin s, \cos s/2).$$

Check that this yields a regular patch (parametrised as $\alpha(s) + t\beta(s)$) . Find a normal at $(\cos s, \sin s, 0)$ and compare the expressions you get at $s = 0$, $s = 2\pi$. Can you draw a picture? Take say $-0.5 < t < 0.5$.

11. Let S^2 denote the unit sphere $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$, and $\mathbf{R}^2$ denote the plane in 3-space given by $x_3 = 0$. If $(u, v, 0)$ is a point of $\mathbf{R}^2$ then the line joining $(u, v, 0)$ to $(0, 0, 1)$ meets S^2 in a point other than $(0, 0, 1)$. Denote this point by $\mathbf{X}(u, v)$. Compute the formula for $\mathbf{X}$. (Hint: $\mathbf{X}(u, v) = \lambda(u, v, 0) + (1 - \lambda)(0, 0, 1)$ where $\lambda \neq 0$ is chosen so that $\mathbf{X}(u, v)$ lies on S^2.) Show that the map determines a regular injective patch. (Hint: Prove $\mathbf{X}_u . \mathbf{X}_v = 0$.) How much of the sphere is covered by the parametrization $\mathbf{X}$? The inverse mapping is called **stereographic projection**.

12. Consider the helix $\gamma(s) = (\cos s, \sin s, s)$. Show that the tangent lines to the helix lie entirely in the part of $\mathbf{R}^3$ defined by $x_1^2 + x_2^2 \geq 1$. Suggestion: The general point on a tangent line is

$$(x_1, x_2, x_3) = (\cos s - t \sin s, \sin s + t \cos s, s + t)$$

for $t \in \mathbf{R}$. (The 'forward' part has $t > 0$.) Show $x_1^2 + x_2^2 - 1 = t^2$.

Why can two 'forward' tangents to the helix never meet in space? (The same applies to two 'backward' tangents ($t < 0$) to the helix.) Hint: Project the helix and tangents to the (x_1, x_2) plane. (On the other hand a forward and a backward tangent can intersect; it is actually possible to find these points of intersection. This is an optional extra to the question.)

Deduce that the tangent developable $\mathbf{X}(s, t) = \gamma(s) + t\gamma'(s)$ of the helix (see Example 2.2.2(3)) is a regular injective patch provided we restrict to $t > 0$ or alternatively to $t < 0$.

Chapter 3

Distance and the First Fundamental Form

We shall only be considering regular injective patches $\mathbf{X}$ in this chapter so we'll just use the word **surface** for such an $\mathbf{X}$. Thus the jacobian matrix of $\mathbf{X}$ will be assumed of rank 2 at every point of the domain and the map $\mathbf{X}$ will be assumed injective.

Suppose that we have a regular curve $\gamma : I \to \mathbf{R}^3$ whose image lies in a surface $\mathbf{X} : U \to \mathbf{R}^3$. Thus γ is the image under $\mathbf{X}$ of a some regular plane curve $\beta : I \to U \subset \mathbf{R}^2$. So writing $\beta(t) = (u(t), v(t))$ we have $\gamma(t) = \mathbf{X}(u(t), v(t))$. Now we can work out the length of any part of the curve γ using the fact that the distance from $\gamma(t_0)$ to $\gamma(t_1)$ is $\int_{t_0}^{t_1} ||\gamma'(t)||dt$. The crucial point then is to determine the length of the velocity vector γ'.

Now since $\gamma(t) = \mathbf{X}(u(t), v(t))$ using the chain rule we see that

$$\gamma'(t) = \mathbf{X}_u(u(t), v(t))u'(t) + \mathbf{X}_v(u(t), v(t))v'(t).$$

In other words dropping the t's we see that

$$
\begin{aligned}
||\gamma'|| = ||\mathbf{X}_u u' + \mathbf{X}_v u'|| &= \sqrt{(\mathbf{X}_u u' + \mathbf{X}_v u') \cdot (\mathbf{X}_u u' + \mathbf{X}_v u')} \\
&= \sqrt{(\mathbf{X}_u \cdot \mathbf{X}_u (u')^2 + 2\mathbf{X}_u \cdot \mathbf{X}_v u'v' + \mathbf{X}_v \cdot \mathbf{X}_v (v')^2}.
\end{aligned}
$$

So the three quantities $\mathbf{X}_u \cdot \mathbf{X}_u$, $\mathbf{X}_u \cdot \mathbf{X}_v$, $\mathbf{X}_v \cdot \mathbf{X}_v$ evaluated at say (u_0, v_0) completely determine the length of any tangent vector at $\mathbf{p} = \mathbf{X}(u_0, v_0)$. Moreover these quantities determine the length of any curve on the surface. In other words it is they that determine the notion of distance on the surface.

Definition 3.0.1 The coefficients $E = \mathbf{X}_u \cdot \mathbf{X}_u, F = \mathbf{X}_u \cdot \mathbf{X}_v, G = \mathbf{X}_v \cdot \mathbf{X}_v$ are called the **first fundamental form** coefficients. The first fundamental form I itself associates to any two tangent vectors $\mathbf{a} = a_1 \mathbf{X}_u + a_2 \mathbf{X}_v$ and, $\mathbf{b} = b_1 \mathbf{X}_u + b_2 \mathbf{X}_v$ at $\mathbf{p} = \mathbf{X}(u_0, v_0)$ the number

$$\mathrm{I}(\mathbf{a}, \mathbf{b}) = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Thus $\mathrm{I}(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$ and $\mathrm{I}(\mathbf{a}, \mathbf{a}) = ||\mathbf{a}||^2$.

Examples 3.0.2 First Fundamental Form (1) A co-ordinate system on the plane $P \subset \mathbf{R}^3$ passing through the point $\mathbf{p}$ and containing the orthonormal vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} =$

(b_1, b_2, b_3) is given by $\mathbf{X}(u, v) = \mathbf{p} + u\mathbf{a} + v\mathbf{b}$. We can easily compute the first fundamental form: for $\mathbf{X}_u = \mathbf{a}$, $\mathbf{X}_v = \mathbf{b}$. And since $\mathbf{a}$ and $\mathbf{b}$ are orthonormal we have $E = 1$, $F = 0$, $G = 1$.

(2) Consider the right cylinder $\mathbf{X}(u, v) = (\cos u, \sin u, v)$ where $U = \{(u, v) : 0 < u < 2\pi, \ -\infty < v < +\infty\}$. When we compute the first fundamental form, we notice that

$$\mathbf{X}_u = (-\sin u, \cos u, 0), \ \mathbf{X}_v = (0, 0, 1)$$

and therefore

$$E = \sin^2 u + \cos^2 u = 1, \ F = 0, \ G = 1.$$

So we see that although the cylinder and the plane are distinct surfaces the results are the same. In other words from the point of view of the distance geometry in the surface the two are indistinguishable. When we realise that the cylinder can be obtained from the plane by rolling it up this becomes clear. These are two important examples. For they indicate that there are two types of geometry associated with any surface. There is the geometry of how the surface bends in the ambient space (the so-called *extrinsic* geometry), and the geometry determined by the distance in the surface (the so-called *intrinsic* geometry). We see that the plane and cylinder have the same intrinsic geometry, but quite different extrinsic geometry. For the cylinder curves while the plane is flat. It is an amazing fact that one measure of how curved a surface is depends only on the distance or metrical properties of the surface. This is the so-called Gauss curvature. But we get ahead of our story!

The first fundamental form also allows us to measure angles on the surface.

Definition 3.0.3 Suppose that we are given two regular curves $\gamma_1 : I_1 \to \mathbf{R}^3$, $\gamma_2 : I_2 \to \mathbf{R}^3$ which lie on the surface, and $\gamma_1(t_1) = \gamma_2(t_2)$ for some $t_1 \in I$, $t_2 \in J$. The **angle** between these curves at this common point is defined to be the angle between their tangents there.

Let $\gamma_1'(t_1) = \mathbf{a} = a_1\mathbf{X}_u + a_2\mathbf{X}_v$ and $\gamma_2'(t_2) = \mathbf{b} = b_1\mathbf{X}_u + b_2\mathbf{X}_v$. Then the angle is easily calculated to be θ where

$$\cos\theta = \frac{\mathrm{I}(\mathbf{a}, \mathbf{b})}{\sqrt{\mathrm{I}(\mathbf{a}, \mathbf{a})\mathrm{I}(\mathbf{b}, \mathbf{b})}}.$$

This is entirely determined by I. If the curves γ_i are unit speed, then both terms in the denominator are 1.

Proposition 3.0.4 *(i) Given a surface $\mathbf{X} : U \to \mathbf{R}^3$ the images of the curves $u = $ constant, resp. $v = $ constant, are regular curves, with velocity vectors $\mathbf{X}_v$, resp. $\mathbf{X}_u$, and unit tangent vectors $\mathbf{X}_v/\sqrt{G}$ and $\mathbf{X}_u/\sqrt{E}$.*

(ii) The angle between these two curves at a given point is given by $\cos\theta = F/\sqrt{EG}$. In particular these curves are orthogonal if and only if $F = 0$.

The proof is immediate.

Examples 3.0.5 E, F, G

(1) Consider the cone $\mathbf{X} = (u\cos v, u\sin v, cu)$, $-\pi < v < \pi, u \neq 0$ for some constant c. (Note that the equation of this surface is $(x_1^2 + x_2^2 = x_3^2/c^2)$.) Here

$$\mathbf{X}_u = (\cos v, \sin v, c), \ \mathbf{X}_v = (-u\sin v, u\cos v, 0),$$

so

$$E = \cos^2 v + \sin^2 v + c^2 = 1 + c^2, \ F = 0, \ G = u^2 \sin^2 v + u^2 \cos^2 v = u^2.$$

So the images of the lines $u =$ constant, $v =$ constant are orthogonal.

(2) Consider the sphere, minus the points $(0, 0, \pm 1)$, parametrised as

$$\mathbf{X}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u),$$

where[1] $-\pi/2 < u < \pi/2$. Here

$$\mathbf{X}_u = (-\sin u \cos v, -\sin u \sin v, \cos u), \quad \mathbf{X}_v = (-\cos u \sin v, \cos u \cos v, 0),$$

so $E = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u = 1, \ F = 0, \ G = \cos^2 u \sin^2 v + \cos^2 u \cos^2 v = \cos^2 u$. Here the lines $u =$ constant are the lines of latitude, the lines $v =$ constant the lines of longitude.

(3) Consider the right helicoid parametrised as $\mathbf{X}(u, v) = (u \cos v, u \sin v, cv)$. This is an example of a ruled surface, as can be seen by looking at the images of the line $v =$ constant. Note that $\mathbf{X}_u = (\cos v, \sin v, 0), \mathbf{X}_v = (-u \sin v, u \cos v, c)$. Here $E = 1, \ F = 0, \ G = u^2 + c^2$.

(4) Consider the torus of revolution parametrised as

$$\mathbf{X}(u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u).$$

Here $\mathbf{X}_u = (-\sin u \cos v, -\sin u \sin v, \cos u)$, and $\mathbf{X}_v = (-(2 + \cos u) \sin v, (2 + \cos u) \cos v, 0)$. So we deduce that $E = \mathbf{X}_u \cdot \mathbf{X}_u = 1, \ F = \mathbf{X}_u \cdot \mathbf{X}_v = 0, \ G = \mathbf{X}_v \cdot \mathbf{X}_v = (2 + \cos u)^2$. Again it should be clear why the images of the curves $u =$ constant and $v =$ constant are orthogonal.

(5) We now find the length of the curves $\mathbf{X}(t, c), \ 0 \leq t \leq 2\pi$ on the torus in (4), where c is a constant. The length is given by

$$\int \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} dt.$$

But $u(t) = t, \ v(t) = c$ so $u' = 1, \ v' = 0$ and the integral is $\int_0^{2\pi} dt = 2\pi$. These curves are the *meridians* of the torus.

For the curves $\mathbf{X}(c, t), /0 \leq t \leq 2\pi$ the integral becomes $\int_0^{2\pi} (2 + \cos c)^2 dt = 2\pi(2 + \cos c)^2$, and the values vary between 2π and 6π. These curves are the *parallels* of the torus.

Quite generally given a surface of revolution the curves (indeed circles) obtained by rotating a point of the curve about the axis of revolution are called *parallels*, the curves obtained by rotating the given plane curve through a fixed angle are called *meridians*.

3.1 Exercises

1. Find the first fundamental form for the surface

$$\mathbf{X}(u, v) = (u - v, u + v, 2(u^2 + v^2)).$$

Find the cosine of the angle between the coordinate curves $\mathbf{X}(u, 1)$ and $\mathbf{X}(1, v)$ on this surface at the point $\mathbf{X}(1, 1) = (0, 2, 4)$ where they meet.

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[1] Strictly speaking this is not injective since (u, v) and $(u, v + 2n\pi)$ give the same point for any integer n. However this kind of non-injectivity, arising from the nature of angles as lying on a circle rather than on the real axis, is harmless and will be ignored.

2. Find the first fundamental form for the surface

$$\mathbf{X}(u, v) = (\sqrt{1 - v^2} \cos u, \sqrt{1 - v^2} \sin u, v),$$

with $-\pi < u < \pi$ and $-1 < v < 1$. (Recall Exercise 5 in Chapter 2.)

3. Find the first fundamental form for the cone $\mathbf{X}(u, v) = \mathbf{p} + v\beta(u)$ where $\beta(u)$ is a unit vector for all u. What does this become in the case when $\mathbf{p} = (0, 0, 0)$ and $\beta(u) = (\cos u, \sin u, 1)/\sqrt{2}$? What if $\mathbf{p} = (0, 0, 0)$ and $\beta(u) = (\cos u, \sin u, 0)$? (What is this? It might help to set $v = r$, $u = \theta$.)

4. For which parameter values s, t is the ruled surface $\mathbf{X}(s, t) = \gamma(s) + tB(s)$ generated by the binormals to a unit speed space curve γ regular? Find the first fundamental form for this ruled surface.

5. Find the length of the arc $\mathbf{X}(\exp(v \cot \beta/\sqrt{2}), v)$, $0 \le v \le \pi$ on the cone

$$\mathbf{X}(u, v) = (u \cos v, u \sin v, u),$$

where β is constant.

6. If the first fundamental form of a patch $\mathbf{X} : U \to \mathbf{R}^3$ is of the form $E = 1$, $F = 0$, $G = f(u, v)$ for some smooth f then show that the v parameter curves (i.e., those of the form $\mathbf{X}(u_0, v)$, u_0 constant) cut off equal segments from all u-parameter curves (i.e., those of the form $\mathbf{X}(u, v_0)$, v_0 constant).

7. Show that the u and v parameter curves on the surface $(u, v, f(u, v))$ are orthogonal to each other if and only if $(\partial f/\partial u)(\partial f/\partial v)$ is identically zero. What can you deduce about the surface if say $\partial f/\partial u$ is identically zero?

8. Find the length of the curve $\mathbf{X}(u, \int_u^{\pi/4}(1/\cos t)dt)$, $0 \le u \le \pi/4$ on the sphere

$$\mathbf{X}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u).$$

9. Let I be an open interval and let $\phi : I \to \mathbf{R}$ be a smooth function. Consider the parametrized surface

$$\mathbf{X} : I \times \mathbf{R} \to \mathbf{R}^3, \quad \mathbf{X}(u, v) = (u \cos v, u \sin v, \phi(v)).$$

Show that this is a regular patch provided $u > 0$. We assume from now on that $u > 0$.

Find a unit normal $\mathbf{N}$ to this surface patch and show that $E = 1, F = 0, G = u^2 + (\phi'(v))^2$.

Given a curve $\gamma(v) = \mathbf{X}(u_0, v)$ where u_0 is constant. write down an expression for the length of γ between two points $v = a$ and $v = b$. In the special case $u_0 = 1$, $\phi(v) = \frac{1}{2}v^2$, show that this length is

$$\int_a^b \sqrt{1 + v^2} dv.$$

10. Let $\mathbf{X} : U \to \mathbf{R}^3$ be a surface. Show that u is the arc-length along the curves $\mathbf{X}(u, v_0)$ (v_0 constant) if and only E is identically 1.

11. (January 1999 exam, Qu. 3) Let $\mathbf{X} : U \to \mathbf{R}^3$ be a surface patch. Define the term *regular* and the coefficients E, F, G of the first fundamental form for $\mathbf{X}$.

    Let $\mathbf{X}$ be the surface patch

    $$\mathbf{X}(u, v) = (u \cos v, \ u \sin v, \ u), \quad u > 0.$$

    Show that $\mathbf{X}$ is a regular patch. Draw a sketch of the surface M determined by $\mathbf{X}$. Show that $E = 2, \ F = 0$ and $G = u^2$.

    Let $k > \sqrt{2}$ be constant, and let

    $$\beta(u) = (u, \sqrt{k^2 - 2} \ln u), \quad 1 \le u \le 3,$$

    be a curve in the (u, v) plane. (Here ln is the natural logarithm.) Find the length of the corresponding curve $\gamma(u) = \mathbf{X}(\beta(u))$ on M.

    Calculate a unit tangent to γ and show that this tangent makes a constant angle θ with the direction $(0, 0, 1)$, where $\cos \theta = 1/k$.

Chapter 4

Curvature

As in the previous chapter we shall be dealing almost exclusively with injective regular patches, and shall just refer to such a patch as a 'surface'.

We now wish to measure in some way the manner in which a surface curves or bends at a point. Intuitively a sphere is more curved than a plane. How can we measure this curvature? One way to do this is via sectional curvature. The idea is very simple. Given a point p on the surface M and a tangent vector $\mathbf{a}$ to M at $\mathbf{p}$, we intersect the surface with the plane spanned by the normal at p and a. In this way we obtain a plane curve. The curvature of this curve at the point p is a measure of how fast the surface is bending in the direction a. We shall actually introduce some other ideas first.

In what follows we shall always suppose that our surface M comes a long with a smooth choice of unit normal vector which we shall denote by $\mathbf{N}$.

4.1   Curvatures of a curve on a surface: definitions

Let $\gamma : I \to \mathbf{R}^3$ be a regular space curve whose curvature is never zero, lying in a surface M. Pick a point $\mathbf{p}$ on the image of γ. The tangent $\mathbf{T}$ to the curve γ at $\mathbf{p}$ is, by definition, tangent to M at $\mathbf{p}$. The vector $\mathbf{U} = \mathbf{N} \times \mathbf{T}$, being orthogonal to $\mathbf{N}$, is also tangent to the surface. It is the *normal to γ in the tangent plane to the surface M*. Note that $\mathbf{T}, \mathbf{U}, \mathbf{N}$ form an orthonormal frame. This shares its first vector, $\mathbf{T}$, with the $\mathbf{T}, \mathbf{P}, \mathbf{B}$ frame of Chapter 1. The pair $\mathbf{P}, \mathbf{B}$ can be rotated in the normal plane to γ through an angle, say ϕ, to coincide with $\mathbf{U}, \mathbf{N}$:

$$\mathbf{U} = \cos\phi\mathbf{P} - \sin\phi\mathbf{B}; \qquad \mathbf{N} = \sin\phi\mathbf{P} + \cos\phi\mathbf{B} \tag{4.1}$$

$$\text{or going backwards} \tag{4.2}$$

$$\mathbf{P} = \cos\phi\mathbf{U} + \sin\phi\mathbf{N}; \qquad \mathbf{B} = -\sin\phi\mathbf{U} + \cos\phi\mathbf{N}. \tag{4.3}$$

Hence, from the first Serret-Frenet formula (1.4),

$$\mathbf{T}' = \kappa\mathbf{P}s' = \kappa s'\cos\phi\mathbf{U} + \kappa s'\sin\phi\mathbf{N}.$$

Definition 4.1.1 The scalar $\kappa_n = \mathbf{T}' \cdot \mathbf{N}/s'$ is called the **normal curvature** of the curve γ at $\mathbf{p}$ and the scalar $\kappa_g = \mathbf{T}' \cdot \mathbf{U}/s'$ is called the **geodesic curvature** of γ at $\mathbf{p}$. *This definition is still valid at points where the curvature κ of γ is zero. See the Note below.*

Note Using the above equations, we see that

$$\kappa_n = \mathbf{T}' \cdot \mathbf{N}/s' = \kappa\sin\phi \text{ and } \kappa_g = \mathbf{T}' \cdot \mathbf{U}/s' = \kappa\cos\phi, \text{ so also } \kappa^2 = \kappa_n^2 + \kappa_g^2, \tag{4.4}$$

but in order to define κ_n and κ_g we do not need to set up the $\mathbf{T}, \mathbf{P}, \mathbf{B}$ triad at $\mathbf{p}$. (As above, this can only be done if $\kappa \neq 0$.) When $\kappa = 0$ we have $\mathbf{T}' = 0$ so both κ_n and κ_g are, by the above definition, zero.

Remark 4.1.2 Tangent plane of M and osculating plane of γ. Note that, assuming κ is nonzero, the binormal $\mathbf{B}$ coincides with the surface normal $\mathbf{N}$ if and only if $\sin \phi = 0$, which is equivalent to $\kappa_n = 0$. Thus the osculating plane of γ coincides with the tangent plane to the surface if and only if $\kappa_n = 0$.

Example 4.1.3 Geodesic curvature of a plane curve If M is the *plane* then $\mathbf{N}$ is constant and equal to $\mathbf{B}$, so that $\phi = 0$, $\kappa_n = 0$ and κ_g is the ordinary curvature of γ (Definition 1.5.1).

There is another scalar which fits in naturally here. The derivative of $\mathbf{N}$ will be perpendicular to $\mathbf{N}$ and therefore a linear combination of $\mathbf{T}$ and $\mathbf{U}$. Now using $\mathbf{T} \cdot \mathbf{N} = 0$ we get $\kappa_n = \mathbf{T}' \cdot \mathbf{N}/s' = -\mathbf{T} \cdot \mathbf{N}'/s'$ so this gives us one of the coefficients. The other one is called (minus) the **geodesic torsion** κ_t of γ at $\mathbf{p}$:

$$\mathbf{N}' = -\kappa_n s' \mathbf{T} - \kappa_t s' \mathbf{U}; \quad \kappa_t = -\mathbf{N}' \cdot \mathbf{U}/s', \quad \kappa_n = -\mathbf{N}' \cdot \mathbf{T}/s'.$$

Note. The more usual symbol for the geodesic torsion is τ_g. It isn't really a curvature but the symbol κ_t has the advantage of uniformity and also the matrix below is easy to remember since g, n, t come in alphabetical order!

Thus, for a unit speed curve γ on M, we have the mnemonic

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{U}' \\ \mathbf{N}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \kappa_t \\ -\kappa_n & -\kappa_t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}s' \\ \mathbf{U}s' \\ \mathbf{N}s' \end{pmatrix}, \tag{4.5}$$

where the skew symmetry of the 3×3 matrix follows from differentiating $\mathbf{T} \cdot \mathbf{U} = \mathbf{U} \cdot \mathbf{N} = \mathbf{N} \cdot \mathbf{T} = 0$.

It is not obvious at this point, but in fact $\mathbf{N}'(t_0)$ depends only on the surface M and on the tangent $\mathbf{T}(t_0)$ to the curve γ at $\gamma(t_0)$. Thus κ_n and κ_t depend only on the tangent to γ. This is not so for κ_g. See Proposition 4.3.1.

Before going on to more formal properties of the above scalar functions, let us consider their geometrical import by considering when they are zero.

4.2 Zeros of the three curvatures

Consider first when $\kappa_t = 0$. Using (4.5) this means that $\mathbf{N}'.\mathbf{U} = 0; \mathbf{N}' = -\kappa_n s' \mathbf{T}$. Think of this as saying that, driving along the curve γ, the normal to the surface has no tendency to move to left or right (component of $\mathbf{N}'$ in the $\mathbf{U}$-direction is zero), but moves only in the plane of $\mathbf{N}$ and $\mathbf{T}$. For a car driving along γ, the axles are in direction $\mathbf{U}$ and (supposing it to be vertical!) the radio aerial is in the direction $\mathbf{N}$. Then the aerial is tilting in the plane of $\mathbf{T}$ and $\mathbf{N}$ but not moving side-to-side. When $\kappa_t = 0$ happens at an instant t we say that $\mathbf{T}$ is a **principal direction** on M at the current point $\mathbf{p}$ (see below for formal definitions); when it happens for all t we say that γ is a **principal curve** or **line of curvature** on M.

Example 4.2.1 Curve on a sphere If $\mathbf{X}$ is a parametrization of a sphere[1] of radius r, centre the origin, then $\mathbf{X} \cdot \mathbf{X} = r^2$, and also clearly $\mathbf{N} = \frac{1}{r}\mathbf{X}$. For any regular curve γ on the sphere, we

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[1] more properly of a sphere minus two points, say, using latitude and longitude as in Example 3.0.5,(2). The curve γ will need to avoid these points.

will then have $\mathbf{N}' = \frac{1}{r}\gamma' = \frac{1}{r}s'\mathbf{T}$ so that $\mathbf{N}' \cdot \mathbf{U} = 0$. This means that *every curve on a sphere is a principal curve.* Driving around on a sphere, the radio aerial never has a tendency to move side-to-side, only up and down. Also it is clear that $\kappa_n = -\mathbf{N}' \cdot \mathbf{T}/s' = -\frac{1}{r}$: *the normal curvature of any curve on the sphere equals minus the reciprocal of the radius.* The same result holds, of course, if the sphere has centre any other point $\mathbf{p}$, say. We have $\mathbf{N} = \frac{1}{r}(\mathbf{X} - \mathbf{p})$, etc.

Curves γ on M for which $\kappa_g = 0$ at all points are called **geodesics**. The meaning of $\kappa_g = 0$ is that $\mathbf{T}' \cdot \mathbf{U} = 0$, which says that the tangent to γ has no tendency to move left or right, only up or down in the plane of $\mathbf{T}$ and $\mathbf{N}$. Using the car analogy, the car is 'driven straight': the wheels are not turned left or right. On the plane, this results in travelling in a straight line, since $\kappa_g = \kappa$ for a curve in the plane (Example 4.1.3), and if κ is identically zero then the curve is a straight line (Example 1.6.2,(1)). A point of γ where $\kappa_g = 0$ is called a **geodesic inflexion** of γ. See Proposition 4.3.1 below.

Example 4.2.2 Geodesics on the sphere Take a regular curve γ on a sphere of radius r centre the origin. Thus $s'\mathbf{T} = \gamma'$, $s'\mathbf{T}' + s''\mathbf{T} = \gamma''$, $\mathbf{N} = \frac{1}{r}\gamma$ at all points of the curve. The condition $\mathbf{T} \cdot \mathbf{U} = 0$ becomes $\gamma'' \cdot (\mathbf{N} \times \mathbf{T}) = 0$, that is $[\gamma, \gamma', \gamma''] = 0$, this being the triple scalar product. Assume this holds all along γ. Differentiating, the only possibly non-zero term is $[\gamma, \gamma', \gamma''']$, so this must be identically zero too. Hence $\gamma, \gamma', \gamma''$ and γ''' are all coplanar. But this implies that the torsion of γ is identically zero (1.6.5) and hence that γ is actually a *plane* curve (1.6.7). Therefore γ lies in a plane Π say, and so is a circle on the sphere. Since Π also contains the vectors γ the only possibility is that the plane passes through the centre of the sphere. Hence *geodesics on a sphere are great circles.* Perhaps it is intuitively clear that if you drive straight ahead on the surface of a sphere then you have to go along a great circle.

Remark 4.2.3 Wheels travel equal distances on a geodesic Continuing with the informal analogy of a car driving along curves on a surface, consider a geodesic γ, where $\kappa_g = \mathbf{T}' \cdot \mathbf{U}/s' = 0$, and think of a car with a very short axle driving along γ. For the purpose of discussion assume that the car moves with unit speed (i.e. γ is unit speed, $s' = 1$), though this does not affect the result. At time t, the ends of the axle are at $\gamma(t) \pm \lambda\mathbf{U}(t)$, where λ is the (very small) length of the axle. At time $t + \delta t$ they are at

$$\gamma(t + \delta t) \pm \lambda\mathbf{U}(t + \delta t) = \gamma(t) + \delta t\mathbf{T}(t) + \lambda(\mathbf{U}(t) + \delta t\kappa_t\mathbf{N}(t)),$$

to first order in δt, where we use the fact that $\kappa_g = 0$. See Figure 4.1.

The distances travelled by the two ends of the axle (i.e. by the two wheels) between times t and $t + \delta t$ are, therefore,

$$\delta t\sqrt{1 + \lambda^2\kappa_t^2}.$$

Note that these distances are *equal*. If κ_g has been non-zero the distances would have been unequal (you could check this). So when you drive along a geodesic you keep the wheels turned straight ahead ($\mathbf{T}' \cdot \mathbf{U} = 0$), and the left and right wheels of your car travel the same distance.

Before considering other examples, let us note the following very useful fact which allows us to simplify calculations considerably.

Proposition 4.2.4 *Let γ be any regular curve on the surface M (not necessarily unit speed), with parameter t, and let $\widetilde{\mathbf{T}}, \widetilde{\mathbf{U}}, \widetilde{\mathbf{N}}$ be any non-zero vectors, not necessarily unit, in the directions of*
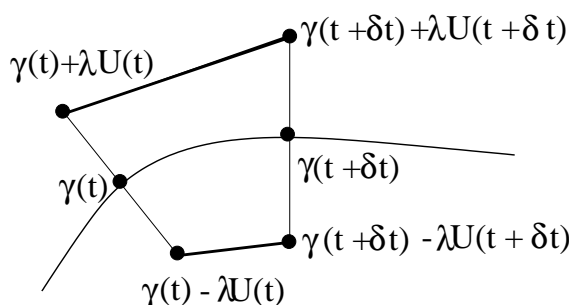
Figure 4.1: Driving along a curve, with the axle along the vector $\mathbf{U}$, the positions of the wheels at the ends of the axles are shown for two time instants. In Remark 4.2.3 we are concerned with the situation where the left and right wheels travel the same distance between times t and $t + \delta t$.

$\mathbf{T}, \mathbf{U}, \mathbf{N}$ *respectively. Then, using* $'$ *for* d/dt,

$$\kappa_n = 0 \Leftrightarrow \widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{N}} = 0 \quad \Leftrightarrow \quad \widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{N}}' = 0;$$
$$\kappa_g = 0 \Leftrightarrow \widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{U}} = 0 \quad \Leftrightarrow \quad \widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{U}}' = 0;$$
$$\kappa_t = 0 \Leftrightarrow \widetilde{\mathbf{U}}' \cdot \widetilde{\mathbf{N}} = 0 \quad \Leftrightarrow \quad \widetilde{\mathbf{U}} \cdot \widetilde{\mathbf{N}}' = 0.$$

The last is the condition for $\widetilde{\mathbf{T}}$ *to be a principal direction.*

Proof   Let us prove the assertion about κ_n. Suppose that $\widetilde{\mathbf{T}} = \lambda \mathbf{T}$, $\widetilde{\mathbf{N}} = \mu \mathbf{N}$ for non-zero scalars λ, μ which just give the lengths of $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{N}}$. Then

$$\widetilde{\mathbf{T}}' = \lambda' \mathbf{T} + \lambda \mathbf{T}' = \lambda' \mathbf{T} + \lambda s'(\kappa_g \mathbf{U} + \kappa_n \mathbf{N}),$$

where s is arclength on γ. Hence $\widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{N}} = \lambda s' \mu \kappa_n$, which is zero if and only if $\kappa_n = 0$. All the other proofs are analogous to this.

The above proof actually establishes slightly more:

Corollary 4.2.5 *Provided* $\widetilde{\mathbf{T}}, \widetilde{\mathbf{N}}$ *are positive multiples of* $\mathbf{T}$, $\mathbf{N}$,

$$\kappa_n = \frac{\widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{N}}}{s' ||\widetilde{\mathbf{T}}|| \, ||\widetilde{\mathbf{N}}||} = -\frac{\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{N}}'}{s' ||\widetilde{\mathbf{T}}|| \, ||\widetilde{\mathbf{N}}||}$$

$$\kappa_g = \frac{\widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{U}}}{s' ||\widetilde{\mathbf{T}}|| \, ||\widetilde{\mathbf{U}}||} = -\frac{\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{U}}'}{s' ||\widetilde{\mathbf{T}}|| \, ||\widetilde{\mathbf{U}}||}$$

$$\kappa_t = \frac{\widetilde{\mathbf{U}}' \cdot \widetilde{\mathbf{N}}}{s' ||\widetilde{\mathbf{U}}|| \, ||\widetilde{\mathbf{N}}||} = -\frac{\widetilde{\mathbf{U}} \cdot \widetilde{\mathbf{N}}'}{s' ||\widetilde{\mathbf{U}}|| \, ||\widetilde{\mathbf{N}}||}$$

If we use (as we normally would) $\widetilde{\mathbf{T}} = \gamma'$, then $s' = ||\widetilde{\mathbf{T}}||$, so that the denominators in some of the above equations can be simplified.

As a example of the above Proposition, let us find all the curves where one of κ_g, κ_n or κ_t is zero on the circular cylinder.

Example 4.2.6 Circular cylinder Take the cylinder with equation $x^2 + y^2 = r^2$ in 3-space, having parametrization

$$\mathbf{X}(u, v) = (r \cos u, r \sin u, v),$$

where $0 < u < 2\pi$ to maintain injectivity of $\mathbf{X}$. For the curve γ, we simply think of u and v as functions of t. The normal to the cylinder is $\mathbf{N} = (\cos u, \sin u, 0)$ (it so happens this is a unit vector), while the (non-unit) tangent and normal to γ are

$$\widetilde{\mathbf{T}} = (-ru' \sin u, ru' \cos u, v'), \quad \widetilde{\mathbf{U}} = \mathbf{N} \times \widetilde{\mathbf{T}} = (v' \sin u, -v' \cos u, ru').$$

The condition for a principal curve, $\mathbf{N}' \cdot \widetilde{\mathbf{U}} = 0$, now easily reduces to $u'v' = 0$, which says that u or v is constant. Thus the principal curves on the cylinder are the straight generators $u = $ constant, and the circular sections $v = $ constant.

The geodesics are given by $\widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{U}} = 0$, which comes to $u'v'' - u''v' = 0$. This says that $\frac{d}{dt} \frac{u'}{v'} = 0$, i.e., that $u' = kv'$ for some constant k. This in turn implies that $u = kv + l$ for constants k, l. So the geodesics are precisely those curves on the cylinder which, when the cylinder is unrolled onto the plane, give straight lines. Note that there are many geodesics joining two given points in this example.

Curves with $\kappa_n = 0$ at all points are called **asymptotic curves**. If $\kappa_n = 0$ at a point of γ we say that the tangent to γ at that point is in an **asymptotic direction**. It is not hard to check that none exists on the sphere or the cylinder—we shall see the reason for this in §4.3 below. It seems an extraordinary condition at first sight: the derivative of $\mathbf{N}$ has no component in the direction of travel: $\mathbf{N}' \cdot \mathbf{T} = 0$ (or $\widetilde{\mathbf{N}}' \cdot \widetilde{\mathbf{T}} = 0$). So the unit surface normal $\mathbf{N}$ is always moving only in the direction $\mathbf{U}$ perpendicular to the direction of travel. Alternatively, the tangent $\mathbf{T}$ moves only left to right, not up and down ($\mathbf{T}' \cdot \mathbf{N} = 0$, or $\widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{N}} = 0$). Certainly this happens if $\mathbf{T}$ is *constant*, which will occur if γ is a straight line. Can a straight line occur on a curved surface? Certainly on a cone or cylinder, but here is a more interesting example. For an example where $\mathbf{T}$ is not constant, see 4.2.8 below.

Example 4.2.7 Saddle surface, or hyperbolic paraboloid Consider the surface given by the equation $z = xy$ in 3-space, parametrized $\mathbf{X}(u, v) = (u, v, uv)$. We have a non-unit normal

$$\widetilde{\mathbf{N}} = \mathbf{X}_u \times \mathbf{X}_v = (1, 0, v) \times (0, 1, u) = (-v, -u, 1).$$

Also a non-unit tangent vector to $\gamma(t) = (u(t), v(t), u(t)v(t))$ is $\widetilde{\mathbf{T}} = (u', v', uv' + u'v)$. So the condition for $\kappa_n = 0$ in Corollary 4.2.5, namely $\widetilde{\mathbf{N}}' \cdot \widetilde{\mathbf{T}} = 0$. becomes simply $u'v' = 0$. So the curves $u = $ constant and $v = $ constant are the asymptotic curves on the saddle surface M. Note that in either case we get a straight line on M. Along any of these straight lines, both κ_n and κ_g are zero, since the unit tangent $\mathbf{T}$ is constant and therefore $\mathbf{T}' = \mathbf{0}$ along a straight line, so that these are automatically geodesics. It is important to notice, however, that the normal $\widetilde{\mathbf{N}} = (-v, -u, 1)$ is not in a constant direction when say u is constant. As you drive along the line $u = $ constant, the radio aerial is turning in space although you are going straight.

Example 4.2.8 The shoe surface Consider the surface M with equation $z = x^2 + y^3$ and parametrization $\mathbf{X}(u, v) = (u, v, u^2 + v^3)$. Taking u and v as functions of t we obtain a curve γ on M. Non-unit tangent $\widetilde{\mathbf{T}}$ and non-unit surface normal $\widetilde{\mathbf{N}}$ are given by

$$\widetilde{\mathbf{T}} = (u', v', 2uu' + 3v^2 v'), \quad \widetilde{\mathbf{N}} = (-2u, -3v^2, 1).$$

Writing down the condition for $\kappa_n = 0$, that is $\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{N}}' = 0$, we get $u'^2 + 3vv'^2 = 0$. To solve this, note that v must be negative, so try $v = -3t^2$; then $u' = \pm 12t^2$ and $u = \pm 4t^3 +$ constant. The alternative sign is not significant since we can replace t by $-t$. The solution curves, in the parameter space (u, v), are given by $(-3t^2, 4t^3 +$ constant$)$, which is a series of cusped curves along the u-axis. Along the corresponding curves on M the curvature κ_n is zero. Unlike the example above, these asymptotic curves really are curved!

Proposition 4.2.9 Coplanar or concurrent normals *Suppose that the normal lines to M at points of γ are all coplanar, or are all concurrent. Then γ is a principal curve.*

Proof Suppose the normal lines at points $\gamma(t)$ all lie in a plane, π say. This automatically implies that the points $\gamma(t)$ lie in π, since $\gamma(t)$ lies on the normal line at this point. (Note that we are not assuming here merely that all the normal *vectors* $\mathbf{N}$ are coplanar, but that the lines themselves are coplanar.) The tangents $\mathbf{T}$ to γ also lie in π and it follows that $\mathbf{U} = \mathbf{N} \times \mathbf{T}$ is normal to π and hence is constant along γ. So $\mathbf{U}' = 0$ and the formulae of Proposition 4.2.4 show that κ_t is zero all along γ so γ is a principal curve. Note that also $\kappa_t = 0$ so γ is a geodesic too.

Suppose on the other hand that the normals all pass through a fixed point, which we can take to be the origin. Thus a non-unit normal $\widetilde{\mathbf{N}}$ is given by

$$\widetilde{\mathbf{N}}(u, v) = \lambda(u, v)\mathbf{X}(u, v),$$

where u, v are functions of the parameter t $(\gamma(t) = \mathbf{X}(u(t), v(t)))$ and λ is never zero. A non-unit tangent to γ is $\widetilde{\mathbf{T}} = \mathbf{X}_u u' + \mathbf{X}_v v'$, while

$$\widetilde{\mathbf{N}}' = (\lambda_u u' + \lambda_v v')\mathbf{X} + \lambda(\mathbf{X}_u u' + \mathbf{X}_v v').$$

It is enough to show that $\widetilde{\mathbf{N}}' \cdot \mathbf{U} = 0$, i.e. $[\widetilde{\mathbf{N}}', \widetilde{\mathbf{T}}, \widetilde{\mathbf{N}}] = 0$. But it is clear from the formulae just given that these three vectors are indeed coplanar.

The following corollary is immediate from the Proposition.

Corollary 4.2.10 *The meridians and parallel circular sections of a surface of revolution are always principal curves. The circular sections of a tube surface are always principal curves.*

4.3 Interpretation of normal and geodesic curvatures

The arguments below illustrate the use of a very powerful technique whereby we take our surface M in a special position relative to (x, y, z) coordinates in space. This involves no loss of generality. Given a point $\mathbf{p}$ of M, we choose the origin at $\mathbf{p}$, and we choose the x, y-plane to be the tangent plane at $\mathbf{p}$. The surface then has a parametrization, valid at any rate for small u, v,

$$\mathbf{X}(u, v) = (u, v, h(u, v)), \tag{4.6}$$

where h is a smooth function of u and v and $h_u(0, 0) = h_v(0, 0) = 0$. When a surface is taken in this form we say that we are using the *Monge form* of the surface at $\mathbf{p}$.

Proposition 4.3.1 *Let γ be a (regular) curve on the surface M, and let $\mathbf{p}$ be a point of M.*
(1) The geodesic curvature κ_g of γ at $\mathbf{p}$ is, up to sign, the (ordinary) curvature of the plane curve obtained by projecting γ orthogonally to the tangent plane to M at $\mathbf{p}$.
(2) The normal curvature κ_n of γ at $\mathbf{p}$ is, up to sign, the (ordinary) curvature of the plane curve obtained by intersecting M with the plane containing the surface normal $\mathbf{N}$ at $\mathbf{p}$ and the tangent $\mathbf{T}$ to γ there.

On account of (2), κ_n is also known as the **sectional curvature** of M in the direction of the tangent to γ at $\mathbf{p}$: it depends only on M and the tangent vector to γ at $\mathbf{p}$, once we are given a choice of unit normal for M (it is always possible to change $\mathbf{N}$ to $-\mathbf{N}$).

Proof We assume that M is in Monge form (4.6). Let t be a regular parameter on γ, and take the tangent to γ at $\mathbf{p} = (0,0,0)$ to be along the x-axis (again this is no loss of generality). The curve γ will then have the form, for small t,

$$\gamma(t) = (t, g(t), h(t, g(t))),$$

for a smooth function g with $g'(0) = 0$. The non-unit tangent $\widetilde{\mathbf{T}}$ to γ is

$$\widetilde{\mathbf{T}} = (1, g', h_u + h_v g'), \ \text{so} \ \widetilde{\mathbf{T}}' = (0, g'', h_{uu} + 2h_{uv}g' + h_{vv}g'^2 + h_v g'').$$

Likewise a non-unit normal to the surface M is

$$\widetilde{\mathbf{N}} = (-h_u, -h_v, 1).$$

Now put $t = 0$. We have, at $t = 0$,

$$\widetilde{\mathbf{N}} = (0, 0, 1),$$
$$\widetilde{\mathbf{T}} = (1, 0, 0), \qquad \widetilde{\mathbf{T}}' = (0, g''(0), h_{uu}(0,0)),$$
$$\widetilde{\mathbf{U}} = (0, 1, 0), \qquad s' = ||\gamma'(0)|| = 1.$$

From these, we work out κ_g and κ_n at the origin from Corollary 4.2.5:

$$\kappa_g = g''(0); \quad \kappa_n = h_{uu}(0,0).$$

But the projection of γ to the tangent x, y-plane is the plane curve $(t, g(t))$ with curvature $g''(0)$ at $t = 0$. The intersection of M with the plane containing the surface normal and tangent to γ, namely the x, z-plane, is the curve parametrized by $(t, 0, h(t, 0))$, which has curvature $h_{uu}(0,0)$ at the origin. This completes the proof.

4.4 Second fundamental form

Here is another formula for the normal curvature, which is significant because it introduces the important concept of 'second fundamental form'.

Theorem 4.4.1 *Let M be a surface with parametrization $\mathbf{X}(u, v)$, as usual, and let $\mathbf{p} = \mathbf{X}(u_0, v_0)$ be a point of M. At $\mathbf{p}$, let $\mathbf{a} = a_1\mathbf{X}_u + a_2\mathbf{X}_v$ be a unit tangent vector. Then the normal curvature of M at $\mathbf{p}$ in the direction $\mathbf{a}$ is*

$$\kappa_n = \mathbf{X}_{uu} \cdot \mathbf{N} a_1^2 + 2\mathbf{X}_{uv} \cdot \mathbf{N} a_1 a_2 + \mathbf{X}_{vv} \cdot \mathbf{N} a_2^2,$$

where the derivatives are evaluated at (u_0, v_0).

Note that the right-hand side really does depend only on the surface, the choice of $\mathbf{N}$ (between two opposite directions) and the direction of the unit tangent vector $\mathbf{a}$.

Proof This is a matter of using Definition 4.1.1, where we use the unit normal $\mathbf{N}$ and unit tangent $\mathbf{T} = \mathbf{a}$. Let $\gamma(t) = (u(t), v(t))$ be a unit speed curve through $\mathbf{p}$ with tangent vector $\mathbf{a}$ there. Then

$$\mathbf{T} = \mathbf{X}_u u' + \mathbf{X}_v v'; \quad \mathbf{T}' = (\mathbf{X}_{uu} u' + \mathbf{X}_{uv} v v')u' + \mathbf{X}_u u'' + (\mathbf{X}_{uv} u' + \mathbf{X}_{vv} v')u' + \mathbf{X}_v u''.$$

Taking the scalar product $\kappa_n = \mathbf{T}' \cdot \mathbf{N}$, all those nasty second derivatives of u and v disappear and we get the stated result.

Definition 4.4.2 We use the notation

$$e = \mathbf{X}_{uu} \cdot \mathbf{N}, \ f = \mathbf{X}_{uv} \cdot \mathbf{N}, \ g = \mathbf{X}_{vv} \cdot \mathbf{N}.$$

Note that e, f and g are defined at each point of M. They are called the **coefficients of the second fundamental form**. The second fundamental form II itself takes any two tangent vectors $\mathbf{a} = a_1\mathbf{X}_u + a_2\mathbf{X}_v$ and $\mathbf{b} = b_1\mathbf{X}_u + b_2\mathbf{X}_v$ at the same point $\mathbf{p}$ of M and associates to them the number

$$\text{II}(\mathbf{a}, \mathbf{b}) = \left(\begin{array}{cc} a_1 & a_2 \end{array} \right) \left(\begin{array}{cc} e & f \\ f & g \end{array} \right) \left(\begin{array}{c} b_1 \\ b_2 \end{array} \right).$$

Notes 1. By differentiating $\mathbf{X}_u \cdot \mathbf{N} = 0$ and $\mathbf{X}_v \cdot \mathbf{N} = 0$ we get the following alternative formulae for e, f and g.

$$e = -\mathbf{X}_u \cdot \mathbf{N}_u, \ f = -\mathbf{X}_u \cdot \mathbf{N}_v = -\mathbf{X}_v \cdot \mathbf{N}_u, \ g = -\mathbf{X}_v \cdot \mathbf{N}_v. \tag{4.7}$$

2. Because the matrix in $\text{II}(\mathbf{a}, \mathbf{b})$ is symmetric, we have $\text{II}(\mathbf{a}, \mathbf{b}) = \text{II}(\mathbf{b}, \mathbf{a})$. The matrix itself will sometimes be denoted by (II), so that $\text{II}(\mathbf{a}, \mathbf{b}) = (a_1, a_2)(\text{II})(b_1, b_2)^{\top}$.

The geometrical interpretation of II is as follows.

Proposition 4.4.3 *Let γ be a curve on M with $\gamma(t_0) = \mathbf{p}$ and $\gamma'(t_0) = \mathbf{a}$. Then, for any tangent vector $\mathbf{b}$ at $\mathbf{p}$, we have*

$$\text{II}(\mathbf{a}, \mathbf{b}) = \text{II}(\mathbf{b}, \mathbf{a}) = -\mathbf{N}'(t_0) \cdot \mathbf{b}.$$

Thus II captures all the derivatives of the unit normal $\mathbf{N}$ in all directions at $\mathbf{p}$.

Proof This is a matter of working out both sides using the definitions. The first equality is true because the matrix (II) is symmetric. For the second, writing $\gamma(t) = \mathbf{X}(u(t), v(t))$ as usual, we have $\mathbf{N}' = u'\mathbf{N}_u + v'\mathbf{N}_v$ and $\mathbf{a} = u'\mathbf{X}_u + v'\mathbf{X}_v$. Also $\mathbf{b} = b_1\mathbf{X}_u + b_2\mathbf{X}_v$ say. Then

$$\begin{aligned} \mathbf{N}' \cdot \mathbf{b} &= (u'\mathbf{N}_u + v'\mathbf{N}_v) \cdot (b_1\mathbf{X}_u + b_2\mathbf{X}_v) \\ &= -(b_1 u'e + (b_1 v' + b_2 u')f + b_2 v'g) \text{ by (4.7)} \\ &= -(b_1, b_2)(\text{II})(u', v')^{\top} \\ &= -\text{II}(\mathbf{b}, \mathbf{a}). \end{aligned}$$

Example 4.4.4 Graph surface

Suppose we are given the graph of a function $h : U \rightarrow \mathbf{R}$ so that our parametrisation is in the form $\mathbf{X}(u, v) = (u, v, h(u, v))$. Then

$$\mathbf{X}_u = (1, 0, h_u), \ \mathbf{X}_v = (0, 1, h_v), \ \mathbf{X}_{uu} = (0, 0, h_{uu}), \ \mathbf{X}_{uv} = (0, 0, h_{uv}), \ \mathbf{X}_{vv} = (0, 0, h_{vv}).$$

We also have $\mathbf{N} = (-h_u, -h_v, 1)/\sqrt{1 + h_u^2 + h_v^2}$. It follows that

$$e = h_{uu}/\sqrt{1 + h_u^2 + h_v^2}, \ f = h_{uv}/\sqrt{1 + h_u^2 + h_v^2}, \ g = h_{vv}/\sqrt{1 + h_u^2 + h_v^2}.$$

In the case when $h(u, v) = uv$, that is for the hyperbolic paraboloid, we have $e = 0$, $f = 1/\sqrt{1 + u^2 + v^2}$, $g = 0$.

4.5   Principal curvatures

For a unit tangent vector $\mathbf{a}$, we have a formula for the normal curvature in Theorem 4.4.1. For a general tangent vector $\mathbf{a} = a_1\mathbf{X}_u + a_2\mathbf{X}_v$ we again appeal to Corollary 4.2.5. Using the new 'e, f, g' notation, and the 'E, F, G' notation from Chapter 3, we can write for the normal (sectional) curvature κ_n in the direction $\mathbf{a}$

$$\kappa_n = \frac{ea_1^2 + 2fa_1a_2 + ga_2^2}{Ea_1^2 + 2Fa_1a_2 + Ga_2^2} = \frac{\mathrm{II}(\mathbf{a}, \mathbf{a})}{\mathrm{I}(\mathbf{a}, \mathbf{a})}. \tag{4.8}$$

The denominator is simply $||\mathbf{a}||^2$.

Consider a curve $\gamma(t) = \mathbf{X}(u(t), v(t))$ on our surface M. We work at a point $\mathbf{p} = \gamma(t_0)$, where γ has a non-unit tangent $\widetilde{\mathbf{T}} = u'\mathbf{X}_u + v'\mathbf{X}_v$. The unit surface normal $\mathbf{N}$ has $\mathbf{N}' = u'\mathbf{N}_u + v'\mathbf{N}_v$. Suppose that $\mathbf{N}' = -\lambda\widetilde{\mathbf{T}}$ for some real number λ. This says precisely that $\widetilde{\mathbf{T}}$ is a principal direction at $\mathbf{p}$. Then the normal curvature is given by (Corollary 4.2.5)

$$\kappa_n = -\frac{\mathbf{N}' \cdot \widetilde{\mathbf{T}}}{||\widetilde{\mathbf{T}}||^2} = \lambda.$$

Also $u'\mathbf{N}_u + v'\mathbf{N}_v = -\lambda(u'\mathbf{X}_u + v'\mathbf{X}_v)$ gives, on taking the dot product with $\mathbf{X}_u$ and $\mathbf{X}_v$, the equation

$$(\mathrm{II})(u', v')^\top = \lambda(\mathrm{I})(u', v')^\top.$$

This says that λ is a 'relative eigenvalue' of the matrices (I) and (II), which is the same as being an eigenvalue of the matrix $(\mathrm{I})^{-1}(\mathrm{II})$. Also $(u', v')^\top$ is the corresponding eigenvector. This corresponds to the tangent vector $\widetilde{\mathbf{T}}$ which is in a principal direction at $\mathbf{p}$.

We make the following definition and immediately deduce the result which follows it.

Definition 4.5.1 The normal (sectional) curvature of M in a principal direction is called the corresponding **principal curvature** of M.

Proposition 4.5.2 *The principal directions at a point $\mathbf{p}$ are the eigenvectors of the matrix $(\mathrm{I})^{-1}(\mathrm{II})$ and the corresponding eigenvalues are the principal curvatures at $\mathbf{p}$.*

So far we do not know that the eigenvalues of $(\mathrm{I})^{-1}(\mathrm{II})$ are real. (Note that this matrix is not generally symmetric.) But it turns out that in fact the principal curvatures are precisely the *maximum and minimum* normal curvatures at $\mathbf{p}$, and these certainly must be real.

Since this is a general 'principle' we pause to state and prove it.

Proposition 4.5.3 Rayleigh's principle *Let A, B be symmetric 2×2 matrices, let $\mathbf{x} = (x_1, x_2)^\top$ and suppose that $\mathbf{x}^\top B\mathbf{x}$ is never zero for $\mathbf{x} \neq \mathbf{0}$ (in particular B is nonsingular). Let*

$$\lambda(x_1, x_2) = \frac{\mathbf{x}^\top A\mathbf{x}}{\mathbf{x}^\top B\mathbf{x}}.$$

Then the gradient $\nabla\lambda = (\partial\lambda/\partial x_1, \partial\lambda/\partial x_2)^\top$ is zero if and only if $A\mathbf{x} = \lambda B\mathbf{x}$.

This says that the turning points of λ are precisely the relative eigenvalues of A and B. Once this is proved we simply have to replace A by (II) and B by (I).
Proof An easy direct verification shows that

$$\nabla(\mathbf{x}^\top A\mathbf{x}) = 2A\mathbf{x}.$$

Since $\lambda(\mathbf{x}^\top B\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$ we have, differentiating with respect to x_1 and x_2 and putting the results together,

$$(\nabla\lambda)(\mathbf{x}^\top B\mathbf{x}) + \lambda(2B\mathbf{x}) = 2A\mathbf{x}.$$

It follows at once that $\nabla\lambda = 0$ if and only if $A\mathbf{x} = \lambda B\mathbf{x}$.

Theorem 4.5.4 *(1) The extreme values of the sectional curvature at $\mathbf{p}$ are the eigenvalues of the matrix $(\mathrm{I})^{-1}(\mathrm{II})$, which are therefore real. They are the principal curvatures and are achieved in the principal directions at the point $\mathbf{p}$ of the surface where we are working. They are denoted by κ_1 and κ_2.*

(2) When the principal curvatures are distinct the corresponding principal directions are orthogonal. The eigenvalues are equal if and only if $e = E\mu, f = F\mu, g = G\mu$ for some real number μ, which is then equal to the common value and to the sectional curvature in any direction at $\mathbf{p}$. These points where the sectional curvatures all coincide are called **umbilics***.*

Proof We only need to prove (2). Write κ_1, κ_2 for the eigenvalues of $(\mathrm{I})^{-1}(\mathrm{II})$ and $\mathbf{v}_1$, $\mathbf{v_2}$ for the corresponding eigenvectors. Then

$$(\mathrm{II})\mathbf{v}_1 = \kappa_1(\mathrm{I})\mathbf{v}_1, \ (\mathrm{II})\mathbf{v}_2 = \kappa_2(\mathrm{I})\mathbf{v}_2.$$

Taking the transpose of the second equation, and using the fact that the matrices are symmetric, we find that $\mathbf{v}_2{}^\top(\mathrm{II}) = \mathbf{v}_2{}^\top\kappa_2(\mathrm{I})$ and so

$$\mathbf{v}_2{}^\top(\mathrm{II})\mathbf{v}_1 = \kappa_2\mathbf{v}_2{}^\top(\mathrm{I})\mathbf{v}_1 = \kappa_1\mathbf{v}_2{}^\top(\mathrm{I})\mathbf{v}_1.$$

Since $\kappa_1 \neq \kappa_2$ we deduce that $\mathbf{v}_2{}^\top(\mathrm{I})\mathbf{v}_1 = 0$, which says that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

The rest of (2) is proved by the following grotty calculation.

First we note that since the first fundamental form is positive definite, that is $Ea_1^2 + 2Fa_1a_2 + Ga_2^2 \geq 0$ for all a_1, a_2. In particular setting $a_1 = 1$, $a_2 = 0$ (resp. $a_1 = 0$, $a_2 = 1$) we see that $E > 0$ and $G > 0$. Also completing the square gives

$$Ea_1^2 + 2Fa_1a_2 + Ga_2^2 = E\left(a_1 + \frac{F}{E}a_2\right)^2 + \left(G - \frac{F^2}{E}\right)a_2^2$$

so we must have $(G - \frac{F^2}{E}) > 0$, in particular $EG - F^2 > 0$.

The equation for λ is

$$(eg - f^2) + \lambda(-Eg - eG + 2fF) + \lambda^2(EG - F^2) = 0.$$

The condition for this to have a repeated root is

$$(-Eg - eG + 2fF)^2 - 4(eg - f^2)(EG - F^2) = 0.$$

Since $E > 0$ and we can rewrite the above expression as

$$4\frac{(EG - F^2)}{E^2}(Ef - Fe)^2 + \left(Eg - Ge - 2\frac{F}{E}(Ef - Fe)\right)^2.$$

Since $EG - F^2 > 0$ this horrible expression is zero if and only if

$$Ef - Fe = Eg - Ge - 2\frac{F}{E}(Ef - Fe) = 0$$

in other words if $Ef - Fe = Eg - Ge = 0$, which is the same as $e = E\mu, f = F\mu, g = G\mu$ for some μ. However the repeated root is

$$\frac{Eg + eG - 2fF}{2(EG - F^2)} = \frac{EG\mu + EG\mu - 2F^2\mu}{2(EG - F^2)} = \mu.$$

Of course, when the maximum and minimum values of the sectional curvature coincide, all of the sectional curvatures must be equal.

Definition 4.5.5 *The product $\kappa_1\kappa_2$ of the principal curvatures at* **p** *is called the* **Gauss Curvature** *at* **p** *and is denoted by K or $K(\mathbf{p})$. The average $(\kappa_1 + \kappa_2)/2$ of the principal curvatures is called the* **mean curvature** *at* **p** *and is denoted by H or $H(\mathbf{p})$.*

If we divide the equation for the principal curvatures

$$(eg - f^2) + \lambda(-Eg - eG + 2fF) + \lambda^2(EG - F^2) = 0$$

by $EG - F^2$ and set this equal to

$$(\lambda - \kappa_1)(\lambda - \kappa_2) = \lambda^2 - (\kappa_1 + \kappa_2)\lambda + \kappa_1\kappa_2 = \lambda^2 - 2H\lambda + K$$

we obtain

$$K = \frac{eg - f^2}{EG - F^2}, \ H = \frac{Eg + eG - 2Ff}{2(EG - F^2)}. \tag{4.9}$$

We note here the following interesting interpretation of K.

Proposition 4.5.6 *Let* **X** *be a parametrization of a surface. The Gauss curvature K satisfies*

$$\mathbf{N}_u \times \mathbf{N}_v = K(\mathbf{X}_u \times \mathbf{X}_v). \tag{4.10}$$

Remark It is fairly clear that the two vectors in (4.10) are parallel. For $\mathbf{N}_u$ and $\mathbf{N}_v$ are both perpendicular to $\mathbf{N}$ and hence their vector product will be parallel to $\mathbf{N}$. Of course $\mathbf{X}_u \times \mathbf{X}_v$ is parallel to $\mathbf{N}$ too.

Proof Since $\mathbf{N}_u, \mathbf{N}_v$ are perpendicular to $\mathbf{N}$ we can write

$$\mathbf{N}_u = a\mathbf{X}_u + b\mathbf{X}_v, \quad \mathbf{N}_v = c\mathbf{X}_u + d\mathbf{X}_v$$

for some scalars a, b, c and d. Hence $\mathbf{N}_u \times \mathbf{N}_v = (ad - bc)\mathbf{X}_u \times \mathbf{X}_v$. On the other hand,

$$\begin{aligned}
-e &= \mathbf{N}_u \cdot \mathbf{X}_u = aE + bF, \\
-f &= \mathbf{N}_u \cdot \mathbf{X}_v = aF + bG, \\
-f &= \mathbf{N}_v \cdot \mathbf{X}_u = cE + dF, \\
-g &= \mathbf{N}_v \cdot \mathbf{X}_v = cF + dG.
\end{aligned}$$

All these equations can be put into matrix form:

$$\begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Taking determinants and using the formula for K in (4.9) we get

$$ad - bc = \frac{eg - f^2}{EG - F^2} = K$$

as required.

4.6 Examples

(1) We shall start with the right circular cylinder, parametrised as $\mathbf{X}(u, v) = (\cos u, \sin u, v)$. Here

$$\mathbf{X}_u = (-\sin u, \cos u, 0), \ \mathbf{X}_v = (0, 0, 1), \ \mathbf{X}_{uu} = (-\cos u, -\sin u, 0), \ \mathbf{X}_{uv} = (0, 0, 0), \ \mathbf{X}_{vv} = (0, 0, 0)$$

so that one finds that $E = 1$, $F = 0$, $G = 1$. If we set $\mathbf{N}(u, v) = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|} = (\cos u, \sin u, 0)$ then we can compute the coefficients in the second fundamental form. For $e = \mathbf{X}_{uu} \cdot n(u, v) = -1$, $f = \mathbf{X}_{uv} \cdot n(u, v) = 0$, $g = \mathbf{X}_{vv} \cdot n(u, v) = 0$. Now it is easy to see that the principal curvatures are 0 and -1, the principal directions $\mathbf{X}_u = (-\sin u, \cos u, 0)$, $\mathbf{X}_v = (0, 0, 1)$, the Gaussian curvature is 0, the mean curvature $-1/2$. Note that if we had chosen the other collection of normals $(-\cos u, -\sin u, 0)$ then the coefficients e, f, g are all multiplied by -1, as are the principal curvatures. So there is always an ambiguity of sign in these quantities. The Gaussian curvature on the other hand is clearly unchanged.

(2) We next consider the sphere which we parametrise as

$$\mathbf{X}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u), \quad -\frac{\pi}{2} < u < \frac{\pi}{2}.$$

We calculate as follows

$$\mathbf{X}_u = (-\sin u \cos v, -\sin u \sin v, \cos u), \ \mathbf{X}_v = (-\cos u \sin v, \cos u \cos v, 0)$$

so

$$\mathbf{X}_{uu} = (-\cos u \cos v, -\cos u \sin v, -\sin u),$$

$$\mathbf{X}_{uv} = (\sin u \sin v, -\sin u \cos v, 0), \ \mathbf{X}_{vv} = (-\cos u \cos v, -\cos u \sin v, 0).$$

We have already seen that $E = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u = 1$, $F = 0$, $G = \cos^2 u \sin^2 v + \cos^2 u \cos^2 v = \cos^2 u$. Now to determine the second fundamental form we need the unit normal vectors, so we need to normalise

$$\mathbf{X}_u(u, v) \times \mathbf{X}_v(u, v) = (-\cos^2 u \cos v, -\cos^2 u \sin v, -\sin u \cos v)$$

which yields $\mathbf{N}(u, v) = (-\cos u \cos v, -\cos u \sin v, -\sin u)$ (we would expect $\pm \mathbf{X}(u, v)$, in fact the minus sign in this case).

Now

$$
\begin{aligned}
e &= \mathbf{N} \cdot \mathbf{X}_{uu} = \cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u = 1, \\
f &= \mathbf{N} \cdot \mathbf{X}_{uv} = -\cos u \sin u \cos v \sin v + \cos u \sin u \cos v \sin v = 0, \\
g &= \mathbf{N} \cdot \mathbf{X}_{vv} = \cos^2 u \cos^2 v + \cos^2 u \sin^2 v = \cos^2 u.
\end{aligned}
$$

We now see that $E = e$, $F = f$, $G = g$ so that every direction is principal, and the sectional curvatures are all 1. Compare Example 4.2.1.

(3) Now for a general class of surfaces, namely the surfaces of revolution. As usual we consider first a regular plane parametrised curve $\gamma : I \to \mathbf{R}^3$, and a line in the plane which does not meet that curve. Using coordinates (x_1, x_2, x_3) in $\mathbf{R}^3$, we take γ to lie in the plane given by $x_2 = 0$ and spin γ about the x_3-axis. The resulting parametrisation of a surface of revolution is of the form

$$\mathbf{X}(u, v) = (\gamma_1(u) \cos v, \gamma_1(u) \sin v, \gamma_2(u))$$

where the plane curve γ is given by $\gamma(u) = (\gamma_1(u), 0, \gamma_2(u))$. The first derivatives are

$$\mathbf{X}_u = (\gamma_1'(u)\cos v, \gamma_1'(u)\sin v, \gamma_2'(u)), \ \ \mathbf{X}_v = (-\gamma_1(u)\sin v, \gamma_1(u)\cos v, 0).$$

The second derivatives are

$$\begin{aligned}
\mathbf{X}_{uu} &= (\gamma_1''(u)\cos v, \gamma_1''(u)\sin v, \gamma_2''(u)), \\
\mathbf{X}_{uv} &= (-\gamma_1'(u)\sin v, \gamma_1'(u)\cos v, 0), \\
\mathbf{X}_{vv} &= (-\gamma_1(u)\cos v, -\gamma_1(u)\sin v, 0).
\end{aligned}$$

Also a suitable collection of unit normals is given by

$$\mathbf{N}(u,v) = (-\gamma_2'(u)\cos v, -\gamma_2'(u)\sin v, \gamma_1')/\sqrt{(\gamma_1'(u))^2 + (\gamma_2'(u))^2}.$$

We can now compute the coefficients occuring in the first and second fundamental forms. So

$$E = (\gamma_1')^2 + (\gamma_2')^2, \ \ F = 0, \ \ G = (\gamma_1)^2$$

while

$$e = (-\gamma_1''\gamma_2' + \gamma_1'\gamma_2'')/\sqrt{(\gamma_1')^2 + (\gamma_2')^2}, \ \ f = 0, \ \ g = \gamma_1\gamma_2'/\sqrt{(\gamma_1')^2 + (\gamma_2')^2}.$$

It is now easy to see that since the matrices associated to each of the fundamental forms are both diagonal the principal curvatures are the quotients e/E and g/G and the corresponding principal directions are $\mathbf{X}_u$ and $\mathbf{X}_v$ respectively. In the first case $e/E = (-\gamma_1''\gamma_2' + \gamma_1'\gamma_2'')/((\gamma_1')^2 + (\gamma_2')^2)^{3/2}$ is just the curvature of the meridian curve (the plane curve we started with).

Consider the torus of revolution obtained by rotating a circle of radius b about a line distance a from its centre, with $a > b > 0$. This is obtained from the above by taking

$$\gamma_1(u) = a + b\cos u, \ \ \gamma_2(u) = b\sin u$$

the coefficients involved are

$$E = b^2\sin^2 u + b^2\cos^2 u = b^2, \ \ F = 0, \ \ G = (a + b\cos u)^2, \ \ e = b, \ \ f = 0, \ \ g = \cos u(a + b\cos u).$$

The principal curvatures are $1/b$ and $\frac{\cos u}{(a+b\cos u)}$. The first is the maximum curvature, which is attained along the meridian, that is the circles $v = $ constant obtained by rotating our original circle of radius b. The minimum curvature varies along a meridian. It takes on its maximum value $1/(a + b)$ on the outside parallel $u = 0$, and is zero on the parallels $u = \pi/2$, $u = -\pi/2$. It has its minimum value $-1/(a - b)$ on the inside parallel $u = \pi$. The Gaussian curvature is

$$K = \kappa_1\kappa_2 = \frac{\cos u}{b(a + b\cos u)}$$

which is > 0 for $-\pi/2 < u < \pi/2$ (the 'outer' part of the torus) and < 0 for $\pi/2 < u < 3\pi/2$ (the 'inner' part).

(4) We finish with a very important set of examples, namely those of the form $\mathbf{X}(u,v) = (u, v, au^2 + bv^2)$ for some constants a and b.

The reason that this is an especially interesting set is the following. Given a surface and a point $\mathbf{p}$ on that surface we can move it, by a rigid motion, so that the point sits at the origin. By a rotation we can then arrange for the tangent plane to the surface to be given by $x_3 = 0$. Now suppose without loss of generality that the parametrisation is in the form $\mathbf{X}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$, with $\mathbf{X}(0,0) = (0,0,0)$. We set $U = x_1(u,v)$, $V = x_2(u,v)$. We claim that these equations locally can be inverted, in other words we

can write $u = y_1(U, V)$, $v = y_2(U, V)$ for some smooth functions y_1, y_2 with $y_1(0, 0) = y_2(0, 0) = 0$. This follows from the inverse function theorem, for the matrix

$$\begin{pmatrix} \partial x_1/\partial u & \partial x_1/\partial v \\ \partial x_2/\partial v & \partial x_2/\partial v \end{pmatrix}$$

has rank 2 at $u = v = 0$. The reason is that the matrix whose rows are $\mathbf{X}_u(0, 0)$ and $\mathbf{X}_v(0, 0)$ is made up of this matrix plus a column of zeros; for these vectors span the tangent space, which is given by $x_3 = 0$.

Now the surface can be written $(U, V, F(U, V))$ where $F(U, V) = x_3(y_1(U, V), y_2(U, V))$. Since the derivatives of x_3 are both zero at $(0, 0)$, the function F starts out with quadratic terms. So near to $(0, 0)$, that is for (u, v) small the surface rather looks like one of the type $(u, v, a_1 u^2 + a_2 uv + a_3 v^2)$. Now by a rotation about the x_3−axis we can kill off the uv-term. In other words if we replace u and v in the third component by $(\cos \theta u + \sin \theta v, -\sin \theta u + \cos \theta v)$ then for some θ we can make the uv term disappear (we need $a_1 \sin 2\theta + a_2 \cos 2\theta - a_3 \sin 2\theta = 0$, i.e. $\tan 2\theta = a_2/(a_1 - a_3)$). We then really do have a surface of the type above.

We consider what happens at the origin. Note that

$$\mathbf{X}_u = (1, 0, 2au), \ \mathbf{X}_v = (0, 1, 2bv), \ \mathbf{X}_{uu} = (0, 0, 2a), \ \mathbf{X}_{uv} = (0, 0, 0), \ \mathbf{X}_{vv} = (0, 0, 2bv)$$

the first two of which are, in turn, $(1, 0, 0)$, $(0, 1, 0)$ when evaluated at $(0, 0, 0)$. For the normals we choose $(-2au, -2bv, 1)/\sqrt{1 + 4a^2u^2 + 4b^2v^2} = (0, 0, 1)$ when evaluated at $u = v = 0$. Now a short calculation shows that

$$E = 1, \ F = 0, \ G = 1, \ e = 2a, \ f = 0, \ g = 2b$$

from whence it follows that the principal curvatures are $2a$, $2b$, the Gauss curvature is $4ab$ and the mean curvature $a + b$. Clearly the principal directions those of the first two coordinate axes.

From this we can see that the second fundamental form at a point essentially determines the surface at that point to second order. Now note that if a and b have the same sign the surface is bowl-shaped, while if they have opposite signs it is saddle-shaped. This distinction is made by the Gaussian curvature. If a or b is zero then we do not have enough information to determine the shape of the surface locally. Note also that when $ab < 0$ there are two straight lines in the surface, namely those given by $ax_1^2 + bx_2^2 = x_3 = 0$. These are in fact in the directions for which the corresponding sectional curvature is zero.

4.7 Elliptic, hyperbolic and parabolic points

Definition 4.7.1 (1) A point **p** of a surface M is **elliptic** (resp. **hyperbolic**) if the Gaussian curvature $K > 0$ (resp. $K < 0$) at **p**. A point where $K = 0$ is called a **parabolic point**.

There is one classical theorem concerning curvature we should prove.

Theorem 4.7.2 *(Euler) The normal curvature at a point of a surface in the direction of a line L is*

$$\kappa_n = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha$$

where κ_1 and κ_2 are the principal curvatures at the point, and α is the angle between the line L and the principal direction corresponding to κ_1.

Proof Let $a_1 \mathbf{X}_u + a_2 \mathbf{X}_v$ be the direction corresponding to κ_1 and $b_1 \mathbf{X}_u + b_2 \mathbf{X}_v$ be the direction corresponding to κ_2, and suppose these are unit vectors. Then $(a_1 \cos \alpha + b_1 \sin \alpha)\mathbf{X}_u + (a_2 \cos \alpha + b_2 \sin \alpha)\mathbf{X}_v$ will be a unit direction corresponding to L. So the value of the curvature is

$$e(a_1 \cos \alpha + b_1 \sin \alpha)^2 + 2f(a_1 \cos \alpha + b_1 \sin \alpha)(a_2 \cos \alpha + b_2 \sin \alpha) + g(a_2 \cos \alpha + b_2 \sin \alpha)^2.$$

Multiplying out we find that this reduces to

$$\{ea_1^2 + 2fa_1a_2 + ga_2^2\}\cos^2\alpha +$$

$$\{eb_1^2 + 2fb_1b_2 + gb_2^2\}\sin^2\alpha +$$

$$+2\{ea_1b_1 + f(a_1b_2 + a_2b_1) + ga_2b_2\}\cos\alpha\sin\alpha.$$

Now we know that

$$ea_1 + fa_2 = \kappa_1(Ea_1 + Fa_2), \ \ fa_1 + ga_2 = \kappa_1(Fa_1 + Ga_2)$$

so that the final term above can be written (ignoring the factor 2)

$$\begin{aligned}
\kappa_1(Ea_1 + Fa_2)b_1 + \kappa_1(Fa_1 + Ga_2)b_2 &= \kappa_1(Ea_1b_1 + F(a_1b_2 + a_2b_1) + Ga_2b_2) \\
&= \kappa(a_1\mathbf{X}_u + a_2\mathbf{X}_v)\cdot(b_1\mathbf{X}_u + b_2\mathbf{X}_v) \\
&= 0,
\end{aligned}$$

since the principal directions are orthogonal.

We have as a corollary

Proposition 4.7.3 *At a hyperbolic point there is one asymptotic direction, at a parabolic point there is one and at an elliptic point there is none. In the hyperbolic case the principal directions bisect the asymptotic directions. If $\kappa_1 = \kappa_2 = 0$ then all directions are asymptotic.*

Proof Clearly $\kappa_1\cos^2\alpha + \kappa_2\sin^2\alpha = 0$ has solutions as an equation in α only if $K = \kappa_1\kappa_2 \leq 0$, and has two real solutions if and only if $K < 0$ and one if and only if $K = 0$. If α corresponds to a solution then so too does $-\alpha$, whence the result.

Finally we establish the following result.

Theorem 4.7.4 *Let $\mathbf{X} : U \to \mathbf{R}^3$ be a parametrisation of a regular injective patch (= 'surface') M, with U connected, and suppose that every point of M is an umbilic. Then M is contained in a sphere, or a plane.*

Proof Since every point of M is an umbilic, every direction on M is principal, and every curve is a principal curve. In particular this is true for the parameter curve $u =$ constant and $v =$ constant. So $\mathbf{N}_u = \lambda\mathbf{X}_u$ and $\mathbf{N}_v = \lambda\mathbf{X}_v$ for some smooth function $\lambda(u,v)$ (the unique sectional curvature at $\mathbf{X}(u,v)$). We now show that λ is constant. For differentiating the first of the above two equations with respect to v and the second with respect to u we obtain $\mathbf{N}_{uv} = \lambda_v\mathbf{X}_u + \lambda\mathbf{X}_{uv} = \lambda_u\mathbf{X}_v + \lambda\mathbf{X}_{uv}$ and so $\lambda_v\mathbf{X}_u = \lambda_u\mathbf{X}_v$. But $\mathbf{X}_u$ and $\mathbf{X}_v$ are linearly independent, so $\lambda_u = \lambda_v = 0$, and hence λ is constant on all of U since U is connected.

It follows that $(\mathbf{N} - \lambda\mathbf{X})_u = (\mathbf{N} - \lambda\mathbf{X})_v = 0$, so for some constant vector $\mathbf{a}$ we have $\mathbf{N} = \lambda\mathbf{X} - \mathbf{a}$. If $\lambda = 0$ then the normal is constant, and equal to $-\mathbf{a}$, so the derivatives of $\mathbf{X}\cdot\mathbf{a} = 0$ are zero, and $\mathbf{X}\cdot\mathbf{a} = c$ for some constant c. So M lies in a plane. If $\lambda \neq 0$ then $||\mathbf{N}||^2 = 1 = ||\lambda\mathbf{X} - \mathbf{a}||^2$, and dividing by λ^2 we see that M lies in the sphere centred at $\mathbf{a}/\lambda$ of radius $1/|\lambda|$.

4.8   Exercises

1. Consider the open subset of the plane $x_3 = 0$ parametrised by

$$\mathbf{X}(r, \theta) = (r \sin \theta, r \cos \theta, 0)$$

   where $r > 0$ and $0 < \theta < 2\pi$. Find the coefficients of the first and second fundamental form E, F, G, e, f, g in terms of r and θ.

2. Find the second fundamental form for the hyperbolic paraboloid

$$\mathbf{X}(u, v) = ((u + v)/2, (u - v)/2, uv).$$

3. Find the second fundamental form for the surface

$$\mathbf{X}(u, v) = (u - v, u + v, 2(u^2 + v^2)).$$

   Find its principal directions and principal curvatures at $(0, 0, 0)$.

4. Find the second fundamental form for the surface

$$\mathbf{X}(u, v) = (\sqrt{1 - u^2} \cos v, \sqrt{1 - u^2} \sin v, u),$$

   with $-1 < u < 1$ and $-\pi < v < \pi$. Find its principal directions and principal curvatures at the point corresponding to the parameter values $(u, v) = (0, 0)$.

5. Find the second fundamental form for the cone $\mathbf{X}(u, v) = \mathbf{p} + v.\beta(u)$ where $v > 0, \beta(u)$ is a unit vector for all u and it is assumed that $\mathbf{X}$ is injective. What is the Gaussian curvature of this surface? Calculate the principal curvatures in the case when $\mathbf{p} = (0, 0, 0)$ and $\beta(u) = (\cos u, \sin u, 1)/\sqrt{2}$.

6. Let U be an open subset of $\mathbf{R}^2$ and $\phi : U \to \mathbf{R}$ a smooth function. Show that the graph of ϕ, namely

$$\mathbf{X} : U \to \mathbf{R}^3, \quad \mathbf{X}(u, v) = (u, v, \phi(u, v)),$$

   is a regular and injective surface patch.

   Write down a unit normal to this surface and compute the first and second fundamental form coefficients in terms of ϕ and its derivatives. Deduce an expression for the Gauss curvature of this surface.

   Consider now the case when $\phi(u, v) = au^2 v + buv^2$ with a, b constants. Show that the Gauss curvature is ≤ 0 on this surface.

7. Find the second fundamental form for the ruled surface $\mathbf{X}(s, t) = \gamma(s) + t\mathbf{B}(s)$ generated by the binormals to a unit speed space curve γ whose curvature κ never vanishes.

8. Let γ be a regular curve on a sphere whose geodesic curvature is constant. Show that the torsion τ of γ is zero at all points. It follows that γ is a *plane* curve (by Proposition 1.6.7), and hence must be a circle. [Hint. Using equation (4.4) and the Example 4.2.1, κ is constant, and non-zero, for γ. Also $\mathbf{P} = (\kappa_g/\kappa)\mathbf{U} + (\kappa_n/\kappa)\mathbf{N}$. Differentiating, you will find that $\mathbf{P}'$ is parallel to $\mathbf{T}$.]

9. For the surface $\mathbf{X} : \mathbf{R}^2 \to \mathbf{R}^3$ given by $\mathbf{X}(u, v) = (u, v, u^2 + v^2)$ find the normal curvature of the curve $\gamma(t) = \mathbf{X}(t^2, t)$ at $t = 1$.

10. Let $\mathbf{X}(u,v) = (u, v, au^2 + bv^2)$ where a, b are constants and $a \neq b, a \neq 0, b \neq 0$. Let $\gamma(t) = \mathbf{X}(t\cos\theta, t\sin\theta)$ where θ is a constant. Find $\gamma' = \widetilde{\mathbf{T}}, \ \widetilde{\mathbf{N}} = \mathbf{X}_u \times \mathbf{X}_v, \ \widetilde{\mathbf{T}}'$ as functions of t, and show that
$$\widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{U}} = 4t(b - a)\cos\theta\sin\theta(a\cos^2\theta + b\sin^2\theta).$$
How many values of θ are there (between 0 and 2π) for which γ is a geodesic?

11. (January 1999 exam)   (i) Given a unit speed curve α on a (regular, injective) surface M parametrized by $\mathbf{X}$, define the three standard vectors $\mathbf{T}, \mathbf{N}, \mathbf{U}$ associated with α at the point $\alpha(s)$. Define the *geodesic curvature* κ_g of α at this point.

    Explain briefly why, for any regular curve γ on M, and any non-zero vectors $\widetilde{\mathbf{T}}, \widetilde{\mathbf{U}}$ in the directions $\mathbf{T}$, $\mathbf{U}$ respectively,
$$\kappa_g = 0 \Leftrightarrow \widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{U}} = 0.$$

    (ii) Now let $\mathbf{X}(s,t) = (x(s), y(s), t)$ be a surface patch, where x and y are functions of s with
$$x'^2 + y'^2 = 1$$
for all s. Show that $\mathbf{X}$ is regular. You may assume that $\mathbf{X}$ is injective.

    Let k be a real constant. Consider the curve
$$\gamma(s) = \mathbf{X}(s, ks) = (x(s), y(s), ks),$$
on the surface M parametrized by $\mathbf{X}$. Find suitable vectors $\widetilde{\mathbf{T}}, \widetilde{\mathbf{U}}$ for γ and show that $\kappa_g = 0$ for all s, that is, γ is a geodesic.

    Sketch the surface and the geodesic γ for the special case $x(s) = \cos s, y(s) = \sin s, k = 1$.

12. Show that the Gaussian curvature of the tangent developable surface of a space curve is identically zero. (Thus the setup here is a regular space curve γ with curvature never zero, and the surface $\mathbf{X}(u,v) = \gamma(u) + v\mathbf{T}(u)$, with $v > 0$ to ensure regularity of $\mathbf{X}$. Show that $f = g = 0$ for this surface.)

13. Consider a surface of revolution of the form
$$\mathbf{X}(u,v) = (h(u)\cos v, h(u)\sin v, u)$$
with $h(u) > 0$ for all u. Show that the Gaussian curvature of this surface is identically zero if and only if it is a right circular cylinder $h(u) = b$, or a cone $h(u) = au + b$ where a and b are constants.

14. (January 1999 exam) Let $\mathbf{X}$ be a parametrization of the surface of revolution obtained by rotating the regular curve $\alpha(u) = (\alpha_1(u), 0, \alpha_2(u))$ about the z-axis, namely
$$\mathbf{X}(u,v) = (\alpha_1(u)\cos v, \ \alpha_1(u)\sin v, \ \alpha_2(u)).$$
We shall assume that $\alpha_1(u)$ is never zero and that $\mathbf{X}$ is injective.

    Find a unit normal $\mathbf{N}$ to this surface and calculate the coefficients of the first and second fundamental forms. Show that the principal directions at every point are given by $\mathbf{X}_u$ and $\mathbf{X}_v$.

Now let $\alpha(u) = (2 + u^2, 0, u)$. Sketch this curve in the (x, z) plane. Show that the resulting surface of revolution has Gauss curvature

$$K = \frac{-2}{(2 + u^2)(4u^2 + 1)^2}.$$

(State without proof any general formula you use for K.)

15. Show that the Gaussian and mean curvatures on the surface

$$\mathbf{X}(u, v) = (u + v, u - v, uv)$$

at $u = v = 1$ are $K = -1/16$ and $H = 1/8\sqrt{2}$.

16. Show that the mean curvature is zero at every point of the surface of revolution

$$\mathbf{X}(u, v) = (\cosh u \cos v, \cosh u \sin v, u).$$

17. In the same way as Proposition 4.5.6, show that for a regular surface $\mathbf{X}$ with mean curvature H,

$$\mathbf{X}_u \times \mathbf{N}_v + \mathbf{N}_u \times \mathbf{X}_v = -2H(\mathbf{X}_u \times \mathbf{X}_v).$$

(Use the formula in equation (4.9) for H.)

18. Let $\mathbf{X}$ be a regular surface and let $\mathbf{X}^r$ be the *parallel* surface to $\mathbf{X}$ at distance r, defined by

$$\mathbf{X}^r(u, v) = \mathbf{X}(u, v) + r\mathbf{N}(u, v).$$

Use the formulae for K and H in Proposition 4.5.6 and the previous question to show that

$$\mathbf{X}_u^r \times \mathbf{X}_v^r = (1 - 2rH + r^2 K)(\mathbf{X}_u \times \mathbf{X}_v).$$

Deduce that $\mathbf{X}^r$ is regular provided r is not equal to a principal radius of curvature at any point of $\mathbf{X}$. (A principal radius is $1/\kappa$ where κ is a principal curvature.)

19. Prove that the sum of the normal curvatures at a point on a surface in any pair of orthogonal directions is constant.

20. Show that the principal curvatures of the surface

$$\mathbf{X}(u, v) = (u \cos v, u \sin v, v)$$

are $\pm 1/(1 + u^2)$.

21. Define the terms *principal curvature* and *principal direction* at a point of a (regular) surface $\mathbf{X}$. (You need not define first and second fundamental forms.) Consider the surface of revolution

$$\mathbf{X}(u, v) = (u \cos v, u \sin v, \phi(u)), \quad u > 0,$$

obtained by rotating the curve $z = \phi(x)$ in the x, z plane about the z axis. Show that $\mathbf{X}$ is regular, and find the principal curvatures and principal directions at each point. [Answers: principal curvatures are

$$\frac{\phi''}{(1 + \phi'^2)^{3/2}}, \quad \frac{\phi'}{u(1 + \phi'^2)^{1/2}}.$$

The principal directions are $\mathbf{X}_u$ and $\mathbf{X}_v$, that is eigenvectors of $(\mathrm{I})^{-1}(\mathrm{II})$ are (1,0) and (0,1).]

22. Consider the parametrised surface (Enneper's surface)

$$\mathbf{X}(u,v) = (u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + vu^2, u^2 - v^2).$$

Show that

(i) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \ F = 0.$$

(ii) The coefficients of the second fundamental form are

$$e = 2, \ g = -2, \ f = 0.$$

(iii) The principal curvatures are

$$\kappa_1 = \frac{2}{(1 + u^2 + v^2)^2}, \ \kappa_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

[When finding $\mathbf{N}$ remember that $||\mathbf{X}_u \times \mathbf{X}_v||^2 = EG - F^2.$]

(iv) The principal curves are the co-ordinate curves $u = $ constant, $v = $ constant.

(v) The asymptotic curves are given by $u + v = $ constant, and $u - v = $ constant. [The usual criterion $\widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{N}} = 0$ gets a bit complicated here. Instead, consider say the curve $\gamma(v) = \mathbf{X}(c - v, v)$, which is $u + v = c$, c being constant. Then $\gamma' = -\mathbf{X}_u + \mathbf{X}_v$. Find the value of κ_n for this curve using equation (4.8), and check that it is zero. Similarly with $\gamma(v) = \mathbf{X}(c+v, v)$.]

23. (January 1999 exam) (i) Let M be a (regular, injective) surface with parametrization $\mathbf{X}$, and let α be a unit speed curve on M. What does it mean to say that the unit tangent $\mathbf{T}$ to α at a point $\mathbf{p}$ is (a) a *principal direction* at $\mathbf{p}$, (b) an *asymptotic direction* at $\mathbf{p}$? What is an *asymptotic curve* on M?

(ii) Let $\mathbf{X}(u,v) = (u, v, u^3 - 3uv^2)$, which is a parametrization of the 'Monkey saddle' M. Find a (non-unit) normal $\widetilde{\mathbf{N}}$ to M.

Consider a curve $\gamma(v) = \mathbf{X}(u(v), v)$ on M, where $u(v)$ is a smooth function of v with $u(0) = 0$ so that γ passes through the origin. Show that

$$\widetilde{\mathbf{T}} = (u', \ 1, \ (3u^2 - 3v^2)u' - 6uv),$$

where the prime means d/dv, is a (non-unit) tangent vector to γ and that γ is an asymptotic curve on M if and only if

$$uu'^2 - 2u'v - u = 0$$

for all v. (State without proof the criterion you use for a curve to be asymptotic.)

By differentiating the last equation with respect to v, or otherwise, show that there are exactly three values for $u'(0)$.

24. Let $\lambda_1, \ldots, \lambda_m$ be the normal curvatures at $\mathbf{p} \in M$ along directions making angles

$$0, \pi/m, 2(\pi/m), \ldots, (m - 1)(\pi/m)$$

with a principal direction. Prove that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_m = mH$$

where H is the mean curvature of M at $\mathbf{p}$.

25. Consider the 'funnel surface' parametrized by $\mathbf{X}(u, v) = (u\cos v, u\sin v, \ln(u))$ where ln is the natural logarithm and $u > 0$. This is the surface of revolution obtained by rotating the graph of the logarithm function, in the x, z-plane, about the z-axis. Find a non-unit normal $\widetilde{\mathbf{N}}$. Let $\gamma(t) = \mathbf{X}(u(t), v(t))$ be a curve on M. Show that $\widetilde{\mathbf{N}}' \cdot \widetilde{\mathbf{T}} = 0$ if and only if $u'/u = \pm v'$. Hence show that the asymptotic curves on M are given by $u = Ae^{\pm v}$ for any constant A. Find the particular asymptotic curves which pass through the point $\mathbf{X}(1, 1)$.

26. For the funnel surface of Exercise 25 find the principal curvatures κ_1, κ_2 and the Gauss curvature K. [Do this by a direct calculation of E, F, G, e, f, g rather than quoting the general result for a surface of revolution.]

27. Show that the co-ordinate curves $u = $ constant and $v = $ constant are principal curves on any (regular) surface if and only if F and f are identically 0.

28. Two tangent vectors $\mathbf{a}$ and $\mathbf{b}$ at a point $\mathbf{p}$ are called **conjugate** if $\mathbf{N}'$ in the direction of $\mathbf{a}$ is perpendicular to $\mathbf{b}$. Show that this is equivalent to $\mathrm{II}(\mathbf{a}, \mathbf{b}) = 0$ and hence by symmetry of II it is also equivalent to the condition that the derivative of $\mathbf{N}$ in the direction of $\mathbf{b}$ is perpendicular to $\mathbf{a}$. Show that $\mathbf{a}$ is an asymptotic direction at $\mathbf{p}$ if and only if $\mathbf{a}$ is conjugate to itself (a 'self-conjugate' direction).

Show that conjugate directions are determined by the following limiting procedure. Take a curve γ on M with $\gamma(t_0) = \mathbf{p}$ and $\gamma'(t_0) = \mathbf{a}$. Consider tangent planes to M at points $\gamma(t_0), \gamma(t_0 + h)$. Show that as $h \to 0$ the line of intersection of these planes tends to the conjugate direction to $\mathbf{a}$.

What can you say about conjugacy at parabolic points (where the determinant of (II) is zero)?

29. Use Euler's Theorem to prove the following. The mean curvature H at a point p of a surface M is given by
$$H = \frac{1}{\pi} \int_0^\pi \kappa_n(\theta) d\theta$$
where $\kappa_n(\theta)$ is the normal curvature at p in the direction making an angle with a fixed direction. (In other words H is the average sectional curvature of M at p.)

30. Assume that the osculating plane of a principal curve C in a surface M makes a constant angle with with the tangent plane, and that C is nowhere tangent to an asymptotic direction. Prove that C is a plane curve. (Hint: the condition about the planes meeting at a constant angle is best interpreted as the normals being inclined at a constant angle.)

31. Prove the following theorem of Beltrami-Enneper. The torsion τ of an asymptotic curve, whose curvature is nowhere zero, is given by
$$\tau^2 = -K$$
where K is the Gaussian curvature of the surface at the given point.

32. Suppose that the surface M_1 intersects the surface M_2 along the regular curve C. Show that the curvature κ of C at $\mathbf{p} \in C$ is given by
$$\kappa^2 \sin^2\theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta$$
where λ_1 and λ_2 are the normal curvatures of C at $\mathbf{p}$ on M_1 and M_2 respectively, and θ is the angle between the normal vectors to M_1 and M_2 at p.

33. Suppose that M_1 and M_2 intersect along a regular curve C and make an angle $\theta(p)$ at $p \in C$. Assume that C is a principal curve of M_1. Prove that $\theta(p)$ is constant if and only if C is a principal curve on M_2.

    A surface is said to be **minimal** if its mean curvature vanishes at every point. The next few problems concern minimal surfaces. According to equation (4.9), this is the same as $Eg + eG - 2Ff = 0$.

34. Let M be the surface of revolution given by

    $$\mathbf{X}(u, v) = ((1/a)\cosh(au + b)\cos v, (1/a)\cosh(au + b)\sin v, u).$$

    Show that M is minimal. M is called a *catenoid*.

35. Let M be the surface $\mathbf{X}(u, v) = (v\cos u, v\sin u, cu)$ where c is a real constant. Show that M is minimal. M is called the *helicoid*.

    The next few problems concern **ruled surfaces.** So let $\alpha : I \to \mathbf{R}^3$ be a space curve, which we take to be unit speed for convenience, and $\beta : I \to \mathbf{R}^3$ a family of unit vectors. The ruled surface is given by $\mathbf{X}(s, t) = \alpha(s) + t\beta(s)$.

36. Prove that a straight line which lies on any surface is necessarily a geodesic and an asymptotic curve. [Hint. What is $\mathbf{T}'$ for a straight line?] Deduce that the curvature of a ruled surface is always ≤ 0.

37. Suppose that $\beta'(s)$ is never zero. Prove that there is a unique function $r(s)$ such that the curve $\gamma(s) = \alpha(s) + r(s)\beta(s)$, which in general is not unit speed, satisfies $\gamma' \cdot \beta' = 0$. This curve γ is called the *line of striction* of the ruled surface.

38. (January 1999 exam, part question) Let $\alpha : I \to \mathbf{R}^3$ be a regular space curve, and let β be a smooth family of unit vectors with β' never zero. Let

    $$\mathbf{X}(s, t) = \alpha(s) + t\beta(s)$$

    be the corresponding ruled surface.

    Show that there is a unique function $r(s)$ such that, with $\gamma(s) = \alpha(s) + r(s)\beta(s)$, we have $\gamma'(s) \cdot \beta'(s) = 0$ for all s. (Thus γ is the *line of striction* on the ruled surface.)

    Now let $\alpha(s) = (\cos s, \sin s, 0)$. Show that, for all s, the straight line through $\alpha(s)$, in the direction of the unit vector
    $$\beta(s) = \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1),$$

    lies on the surface with equation $x^2 + y^2 - z^2 = 1$. Show that, for this ruled surface, α itself is the line of striction.

39. If γ is the line of striction of the ruled surface $\mathbf{X}(s, t) = \alpha(s) + t\beta(s)$ show that $\mathbf{Y}(u, s) = \gamma(s) + u\beta(s)$ parametrises the same ruled surface.

    Show that the parameterisation $\mathbf{Y}$ fails to be regular (i.e. the jacobian matrix has rank < 2) precisely when $u = 0$ and $\gamma'(s)$ is a multiple of $\beta(s)$.

40. Let $\mathbf{X}(s,t) = (t\cos(s/c),\ t\sin(s/c), s)$ where $c > 0$ is constant. Writing $\mathbf{X}$ as $t\beta(s) + \alpha(s)$ find the line of striction on this ruled surface.

A ruled surface is **developable** if along each line $\mathbf{X}(s_0, t)$ of the ruling the tangent planes are all equal. Note that the whole line $s = s_0$ on any ruled surface $\mathbf{X}$ lies in the tangent plane to $\mathbf{X}$ at $\mathbf{X}(s_0, t)$, for any t. So the tangent planes to $\mathbf{X}$ at points $\mathbf{X}(s_0, t)$ all contain the point $\alpha(s_0) = \mathbf{X}(s_0, 0)$. This means that the tangent planes coincide if and only if their normal vectors $\mathbf{N}(s_0, t)$ are all equal.

41. Prove that $\mathbf{X}(s,t) = \alpha(s) + t\beta(s)$ is developable if and only if the triple scalar product $[\alpha', \beta, \beta']$ is identically 0.

42. Prove that each ruling $\mathbf{X}(s_0, t)$ on a developable surface is a principal curve (as well as being a geodesic and an asymptotic curve, which happens for any straight line on a surface).

43. Let $\mathbf{Y}$ be a regular surface and let α be a curve on it, which you can take to be unit speed. Consider the ruled surface $\mathbf{X}(s,t) = \alpha(s) + t\mathbf{N}(s)$, where $\mathbf{N}(s)$ is the unit normal to the surface $\mathbf{Y}$ at the point $\alpha(s)$. This is the ruled surface formed by the normal lines to $\mathbf{Y}$ along the points of the curve α. Show that the ruled surface is developable if and only if the curve α is a principal curve on $\mathbf{Y}$.

44. Let $\mathbf{X}(s,t) = \alpha(s) + t\alpha'(s)$ be the tangent developable of a unit speed space curve α. Show that this surface is developable. (Thus our earlier use of the phrase 'tangent developable' is consistent with the present use of 'developable'!)

45. Show that a surface of revolution generated by a straight line (not crossing the axis of revolution) is developable.

46. Let α be a unit speed space curve with κ never zero. Under what conditions is

$$\mathbf{X}(s,t) = \alpha(s) + t\mathbf{P}(s)$$

developable? ($\mathbf{P}$ is the principal normal to the curve α.)

47. When is $\mathbf{Y}(s,t) = \alpha(s) + t\mathbf{B}(s)$ developable? ($\mathbf{B}$ is the binormal to the curve α.)

48. Show that a developable surface (assumed regular) has Gaussian curvature equal to zero.

49. Let M be a developable surface without umbilics. Let $\gamma(s)$ be a unit speed line of curvature, corresponding to the nonzero principal curvature. Show that γ is orthogonal to each line of the ruling.

50. Let γ be chosen as in the previous problem, and parametrise the developable surface M by

$$\mathbf{Y}(s,t) = \gamma(s) + t\beta(s)$$

with $\beta(s)$ a unit vector. Show that $\beta'(s) = \lambda(s)\gamma'(s)$ for some function $\lambda(s)$.

51. If $\lambda(s)$ is as in the previous problem, show that

(i) $\lambda \equiv 0$ implies that M is a cylinder.

(ii) If λ is constant, but not zero, then all lines of the ruling have a point in common, and in this case that point is the line of striction. (M is a cone, but not necessarily circular.)

(iii) If λ and λ' are both non-zero then M is the tangent developable of its line of striction.

Chapter 5

Gauss Curvature and Geodesic Curvature are 'Intrinsic'

Both Euler and Gauss proved results about curvature; we have already seen Euler's result (Theorem 4.7.2). The theorem of Gauss is much deeper, and much more important, and it is this we will investigate here.

We first recall that the principal curvatures for the right circular cylinder were ± 1 and 0. On the other hand for the plane every point is an umbilic, and the sectional curvature in any direction is 0. Now we know that the plane and cylinder both have first fundamental form coefficients $E = 1, F = 0, G = 1$, so that from the point of view of their intrinsic geometry—measuring distances in the surface itself—the plane and the cylinder are indistinguishable. So we find (not surprisingly) that the principal curvatures of a surface are not intrinsic quantities. Clearly, bending a surface without stretching it does not alter distances but can affect curvatures drastically. Note however that the Gaussian curvature, the product of the principal curvatures, in both cases is zero! Could it be that the Gauss curvature only depends on the intrinsic properties of the surface? At first sight this looks a fairly preposterous suggestion. After all the Gauss curvature is simply the product of the principal curvatures. Nevertheless it turns out to be true! If we think of a surface as a soap film, a 2-dimensional universe in which 2-dimensional beings float, unaware of the enveloping 3-space, then the Gauss curvature is a quantity that the beings can measure. (Just as we might measure the curvature of the space in which we live.) The proof of Gauss' result is rather long, and we first need to discuss the Gauss Weingarten equations. Before that, here is an explicit simple example.

5.1   Example: $E = 1, F = 0, G = 1$

Suppose we *only* know that a surface has first fundamental form coefficients $E = 1, F = 0, G = 1$. Examples are the plane, parametrized $\mathbf{X}(u, v) = (u, v, 0)$, and the unit circular cylinder, parametrized $\mathbf{X}(u, v) = (\cos u, \sin u, v)$. From this alone can we deduce the Gauss curvature?

Here is the pattern, which will be followed in the general case too.

(i) Express $\mathbf{N}_u, \mathbf{N}_v$ in terms of $\mathbf{X}_u, \mathbf{X}_v$.

Now $\mathbf{N}_u$ and $\mathbf{N}_v$ are certainly perpendicular to $\mathbf{N}$, since $\mathbf{N}$ is a unit vector. Since $E = 1, F = 0, G = 1$ we know that $\mathbf{X}_u$ and $\mathbf{X}_v$ are perpendicular unit vectors, so that

$$\mathbf{N}_u = (\mathbf{N}_u \cdot \mathbf{X}_u)\mathbf{X}_u + (\mathbf{N}_u \cdot \mathbf{X}_v)\mathbf{X}_v = -e\mathbf{X}_u - f\mathbf{X}_v,$$

by equation (4.7). Similarly

$$\mathbf{N}_v = -f\mathbf{X}_u - g\mathbf{X}_v.$$

(ii) Express $\mathbf{X}_{uu}, \mathbf{X}_{uv}, \mathbf{X}_{vv}$ in terms of $\mathbf{X}_u, \mathbf{X}_v, \mathbf{N}$.

We know that the component of $\mathbf{X}_{uu}$ in the $\mathbf{N}$ direction is e, by the definition in §4.4. So $\mathbf{X}_{uu} = \lambda\mathbf{X}_u + \mu\mathbf{X}_v + e\mathbf{N}$ for some λ, μ. Using again $E = 1, F = 0, G = 1$ we get

$$\lambda = \mathbf{X}_u \cdot \mathbf{X}_{uu} = \frac{1}{2}(\mathbf{X}_u \cdot \mathbf{X}_u)_u = \frac{1}{2}E_u = 0.$$

Next

$$\mathbf{X}_u \cdot \mathbf{X}_{uv} = \frac{1}{2}(\mathbf{X}_u \cdot \mathbf{X}_u)_v = \frac{1}{2}F_v = 0.$$

Finally

$$0 = F_u = (\mathbf{X}_u \cdot \mathbf{X}_v)_u = \mathbf{X}_{uu} \cdot \mathbf{X}_v + \mathbf{X}_u \cdot \mathbf{X}_{uv} = \mu + 0,$$

using the previous equation. Hence $\mathbf{X}_{uu} = e\mathbf{N}$ and similarly $\mathbf{X}_{uv} = f\mathbf{N}, \quad \mathbf{X}_{vv} = g\mathbf{N}$.

(iii) Express third derivatives two ways and equate coefficients of $\mathbf{X}_u$ or $\mathbf{X}_v$.

We have

$$
\begin{aligned}
\mathbf{X}_{uuv} &= e_v\mathbf{N} + e\mathbf{N}_v = e_v\mathbf{N} + e(-f\mathbf{X}_u - g\mathbf{X}_v) \text{ by (i)}\\
\mathbf{X}_{uvu} &= f_u\mathbf{N} + f\mathbf{N}_u = f_u\mathbf{N} + f(-e\mathbf{X}_u - f\mathbf{X}_v) \text{ by (i)}.
\end{aligned}
$$

But these mixed third derivatives are equal since the function $\mathbf{X}(u, v)$ is assumed smooth; we can therefore compare the coefficients of $\mathbf{X}_v$: $-eg = -f^2$, giving $K = eg - f^2 = 0$. So the Gauss curvature has magically appeared from the calculations, despite the fact the e, f, g themselves cannot be found knowing only E, F, G.

Of course it is not so easy in the general case when E, F, G are functions which vary with u, v. But nevertheless the same pattern holds. First we shall deduce the general equations as in (i), (ii) above for $\mathbf{N}_u, \mathbf{N}_v, \mathbf{X}_{uu}, \mathbf{X}_{uv}, \mathbf{X}_{vv}$.

5.2 Gauss Weingarten Equations

The Gauss Weingarten equations are the analogues for surfaces of the Serret Frenet formulae for curves. Recall that at each point we have three linearly independent vectors $\mathbf{X}_u$, $\mathbf{X}_v$, $\mathbf{N}$. The basic idea is to express the derivatives of these (with respect to u and v) in terms of $\mathbf{X}_u$, $\mathbf{X}_v$, $\mathbf{N}$ themselves. (Just as we expressed the derivatives of $\mathbf{T}$, $\mathbf{P}$, $\mathbf{B}$ in terms of $\mathbf{T}$, $\mathbf{P}$, $\mathbf{B}$ themselves.) We write

$$
\begin{aligned}
\mathbf{X}_{uu} &= \Gamma_{11}^1\mathbf{X}_u + \Gamma_{11}^2\mathbf{X}_v + e\mathbf{N}\\
\mathbf{X}_{uv} &= \Gamma_{12}^1\mathbf{X}_u + \Gamma_{12}^2\mathbf{X}_v + f\mathbf{N}\\
\mathbf{X}_{vv} &= \Gamma_{22}^1\mathbf{X}_u + \Gamma_{22}^2\mathbf{X}_v + g\mathbf{N}\\
\mathbf{N}_u &= \beta_1^1\mathbf{X}_u + \beta_1^2\mathbf{X}_v\\
\mathbf{N}_v &= \beta_2^1\mathbf{X}_u + \beta_2^2\mathbf{X}_v.
\end{aligned}
$$

Note that we already know the coefficients of $\mathbf{N}$ in the first three equations, by definition of e, f, g as in §4.4, and we know that $\mathbf{N}_u$ and $\mathbf{N}_v$, being perpendicular to $\mathbf{N}$, are combinations of $\mathbf{X}_u, \mathbf{X}_v$ alone. The other coefficients β_i^j, Γ_{ij}^k are to be determined. The rather elaborate notation is traditional, and is designed to cover the case of higher dimensions. The coefficients Γ_{ij} are called the *Christoffel symbols*.

Theorem 5.2.1 *The coefficients β_i^j, Γ_{ij}^k, are given by*

$$\beta_1^1 = \frac{Ff - Ge}{EG - F^2}, \ \beta_1^2 = \frac{Fe - Ef}{EG - F^2}, \ \beta_2^1 = \frac{Fg - Gf}{EG - F^2}, \ \beta_2^2 = \frac{Ff - Eg}{EG - F^2}$$

$$\Gamma_{11}^1 = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \ \Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG - F^2)}, \ \Gamma_{22}^1 = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)},$$

$$\Gamma_{11}^2 = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \ \Gamma_{12}^2 = \frac{EG_u - FE_v}{2(EG - F^2)}, \ \Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.$$

The expressions obtained for $\mathbf{X}_{uu}, \mathbf{X}_{uv}, \mathbf{X}_{vv}, \mathbf{N}_u$ and $\mathbf{N}_v$ using these formulae are called the Gauss-Weingarten equations for the surface parametrized by $\mathbf{X}$. Note that according to these equations the Γ_{ij}^k only depend on the first fundamental form.

Proof Since $\mathbf{N}$ is unit length we deduce that $\mathbf{N}_u$ and $\mathbf{N}_v$ are orthogonal to $\mathbf{N}$, so $\mathbf{N} \cdot \mathbf{N}_u = \mathbf{N} \cdot \mathbf{N}_v = 0$, and since $\mathbf{X}_u \cdot \mathbf{N} = \mathbf{X}_v \cdot \mathbf{N} = 0$ we deduce that $\mathbf{N}_u$ and $\mathbf{N}_v$ are combinations of $\mathbf{X}_u, \mathbf{X}_v$ alone.

$$-e = \mathbf{X}_u \cdot \mathbf{N}_u = \beta_1^1 \mathbf{X}_u \cdot \mathbf{X}_u + \beta_1^2 \mathbf{X}_u \cdot \mathbf{X}_v = \beta_1^1 E + \beta_1^2 F$$

$$-f = \mathbf{X}_v \cdot \mathbf{N}_u = \beta_1^1 \mathbf{X}_v \cdot \mathbf{X}_u + \beta_1^2 \mathbf{X}_v \cdot \mathbf{X}_v = \beta_1^1 F + \beta_1^2 G$$

$$-f = \mathbf{X}_u \cdot \mathbf{N}_v = \beta_2^1 \mathbf{X}_u \cdot \mathbf{X}_u + \beta_2^2 \mathbf{X}_u \cdot \mathbf{X}_v = \beta_2^1 E + \beta_2^2 F$$

$$-g = \mathbf{X}_v \cdot \mathbf{N}_v = \beta_2^1 \mathbf{X}_v \cdot \mathbf{X}_u + \beta_2^2 \mathbf{X}_v \cdot \mathbf{X}_v = \beta_2^1 F + \beta_2^2 G$$

and we can now solve for the β's.

Next, observe that

$$\mathbf{X}_u \cdot \mathbf{X}_{uu} = (\mathbf{X}_u \cdot \mathbf{X}_u)_u/2 = E_u/2, \mathbf{X}_u \cdot \mathbf{X}_{uv} = (\mathbf{X}_u \cdot \mathbf{X}_u)_v/2 = E_v/2$$

$$\mathbf{X}_v \cdot \mathbf{X}_{uv} = (\mathbf{X}_v \cdot \mathbf{X}_v)_u/2 = G_u/2, \mathbf{X}_v \cdot \mathbf{X}_{vv} = (\mathbf{X}_v \cdot \mathbf{X}_v)_v/2 = G_v/2$$

while

$$F_u = (\mathbf{X}_u \cdot \mathbf{X}_v)_u = \mathbf{X}_{uu} \cdot \mathbf{X}_v + \mathbf{X}_u \cdot \mathbf{X}_{uv} = \mathbf{X}_{uu} \cdot \mathbf{X}_v + E_v/2$$

$$F_v = (\mathbf{X}_u \cdot \mathbf{X}_v)_v = \mathbf{X}_{uv} \cdot \mathbf{X}_v + \mathbf{X}_u \cdot \mathbf{X}_{vv} = \mathbf{X}_u \cdot \mathbf{X}_{vv} + G_u/2$$

hence

$$\mathbf{X}_v \cdot \mathbf{X}_{uu} = F_u - E_v/2, \ \mathbf{X}_u \cdot \mathbf{X}_{vv} = F_v - G_u/2.$$

Now we deduce

$$E_u/2 = \mathbf{X}_u \cdot \mathbf{X}_{uu} = \Gamma_{11}^1 \mathbf{X}_u \cdot \mathbf{X}_u + \Gamma_{11}^2 \mathbf{X}_u \cdot \mathbf{X}_v = \Gamma_{11}^1 E + \Gamma_{11}^2 F,$$

$$F_u - E_v/2 = \mathbf{X}_v \cdot \mathbf{X}_{uu} = \Gamma_{11}^1 \mathbf{X}_v \cdot \mathbf{X}_u + \Gamma_{11}^2 \mathbf{X}_v \cdot \mathbf{X}_v = \Gamma_{11}^1 F + \Gamma_{11}^2 G.$$

If we solve these equations for Γ_{11}^1, Γ_{11}^2 we get the required expressions. There are two other similar sets of equations

$$E_v/2 = \mathbf{X}_u \cdot \mathbf{X}_{uv} = \Gamma_{12}^1 \mathbf{X}_u \cdot \mathbf{X}_u + \Gamma_{12}^2 \mathbf{X}_u \cdot \mathbf{X}_v = \Gamma_{11}^1 E + \Gamma_{11}^2 F$$

$$G_u/2 = \mathbf{X}_v \cdot \mathbf{X}_{uv} = \Gamma_{12}^1 \mathbf{X}_v \cdot \mathbf{X}_u + \Gamma_{12}^2 \mathbf{X}_v \cdot \mathbf{X}_v = \Gamma_{12}^1 F + \Gamma_{12}^2 G.$$

$$F_v - G_u/2 = \mathbf{X}_u \cdot \mathbf{X}_{vv} = \Gamma_{22}^1 \mathbf{X}_u \cdot \mathbf{X}_u + \Gamma_{22}^2 \mathbf{X}_u \cdot \mathbf{X}_v = \Gamma_{22}^1 E + \Gamma_{22}^2 F$$

$$G_v/2 = \mathbf{X}_v \mathbf{X}_{vv} = \Gamma_{22}^1 \mathbf{X}_v \cdot \mathbf{X}_u + \Gamma_{22}^2 \mathbf{X}_v \cdot \mathbf{X}_v = \Gamma_{22}^1 F + \Gamma_{22}^2 G.$$

These yield similar expressions for the remainder of the Γ_{jk}^i.

5.3 Another example: $E = G = \alpha(v), F = 0$

Let us use the above equations to determine the Gauss curvature in the case when E and G are the same function $\alpha(v)$ of v only, and $F = 0$. We assume that $\alpha(v) > 0$ for all v since this will always be the case when E, F, G come from a genuine surface. There is an interesting special case, the *Poincaré half plane*, which we shall meet again in the next chapter.

Using the Gauss-Weingarten equations we deduce that

$$\mathbf{X}_{uu} = -\frac{\alpha'}{2\alpha}\mathbf{X}_v + e\mathbf{N}, \ \ \mathbf{X}_{uv} = \frac{\alpha'}{2\alpha}\mathbf{X}_u + f\mathbf{N}, \ \ \mathbf{X}_{vv} = \frac{\alpha'}{2\alpha}\mathbf{X}_v + g\mathbf{N},$$

$$\mathbf{N}_u = -\frac{e}{\alpha}\mathbf{X}_u - \frac{f}{\alpha}\mathbf{X}_v, \ \ \mathbf{N}_v = -\frac{f}{\alpha}\mathbf{X}_u - \frac{g}{\alpha}\mathbf{X}_v.$$

Now we proceed to calculate a third derivative in two ways:

$$\mathbf{X}_{uuv} = \frac{\alpha'^2 - \alpha\alpha''}{2\alpha^2}\mathbf{X}_v - \frac{\alpha'}{2\alpha}\left(\frac{\alpha'}{2\alpha}\mathbf{X}_v + g\mathbf{N}\right) + e_v\mathbf{N} + e\left(\frac{-f}{\alpha}\mathbf{X}_u - \frac{g}{\alpha}\mathbf{X}_v\right),$$

where we have used the Gauss-Weingarten equations again to substitute for $\mathbf{X}_{vv}$ and for $\mathbf{N}_v$. Similarly

$$\mathbf{X}_{uvu} = \frac{\alpha'}{2\alpha}\left(-\frac{\alpha'}{2\alpha}\mathbf{X}_v + e\mathbf{N}\right) + f_u\mathbf{N} + f\left(-\frac{e}{\alpha}\mathbf{X}_u - \frac{f}{\alpha}\mathbf{X}_v\right).$$

But $\mathbf{X}_{uuv} = \mathbf{X}_{uvu}$ and comparing coefficients of $\mathbf{X}_v$ we get

$$\frac{\alpha\alpha'' - \alpha'^2}{2\alpha^2} = \frac{eg - f^2}{\alpha}.$$

Finally

$$K = \frac{eg - f^2}{EG - F^2} = \frac{\alpha'^2 - \alpha\alpha''}{2\alpha^3}.$$

Again the Gauss curvature has magically appeared from the calculation.

5.4 The intrinsic nature of the Gauss curvature

The same method as above can be applied in the general case, though the details are naturally very much more complicated. We shall only indicate the argument here.

Theorem 5.4.1 Theorema Egregium *The Gauss curvature depends only on the first fundamental coefficients, i.e it is an intrinsic quantity.*

Proof The Gauss-Weingarten equations show us how to compute second derivatives of $\mathbf{X}$ in terms of the first. We can then write the third derivatives by differentiating both sides, and replacing the second derivatives on the right hand side using the Gauss-Weingarten equations again. Now some of these derivatives can be done in two ways e.g. $(\mathbf{X}_{uu})_v = (\mathbf{X}_{uv})_u$ and $(\mathbf{X}_{uv})_v = (\mathbf{X}_{vv})_u$. So we can compare coefficients of $\mathbf{X}_u, \mathbf{X}_v$ as in the examples above. The resulting expressions can be manipulated to give the following wonderful formula for K.

$$K = \frac{1}{(EG - F^2)^2}\left(\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}\right).$$

A different expression for the same thing is

$$K = \frac{-1}{4W^2} \begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} - \frac{1}{2W} \left(\frac{\partial}{\partial v} \frac{E_v - F_u}{W} - \frac{\partial}{\partial u} \frac{F_v - G_u}{W} \right)$$

where $W = \sqrt{EG - F^2}$.

5.5   Geodesics revisited

A geodesic is a curve on a surface M whose geodesic curvature κ_g vanishes identically. This means that the derivative of the unit tangent to the curve is always in the direction of the surface normal: it has no component perpendicular to the direction of motion. Thus geodesics are curves which 'go straight ahead' on a surface. They are also curves whose projection at each point into the tangent plane at that point has an inflexion. (See Definition 4.1.1, Corollary 4.2.5 and Proposition 4.3.1.)

Proposition 5.5.1 *(1) Let γ be a* unit speed *curve on a surface. Then γ is a geodesic if and only if its acceleration γ'' is always in the direction of the normal $\mathbf{N}$.*
(2) Let γ be a regular *(not necessarily unit speed) curve on a surface which, as a space curve, has non-vanishing curvature. Then γ is a geodesic if and only if the osculating plane of the curve γ at each point contains the surface normal $\mathbf{N}$.*

Note that (1) says that, as far as creatures whose universe is the 2-dimension surface itself are concerned, the acceleration is zero, since they cannot detect vectors out of the surface. Thus motion at constant speed along a geodesic is, so far as these creatures are concerned, uniform motion.

Proof (1) This is merely a restatement of the definition of κ_g. For when γ is unit speed we have $\gamma' = \mathbf{T}$, $\gamma'' = \mathbf{T}' = \kappa_g \mathbf{U} + \kappa_n \mathbf{N}$ and this is in the $\mathbf{N}$ direction if and only if $\kappa_g = 0$.

For (2), as $\kappa \neq 0$ we can use the principal normal $\mathbf{P}$ and the binormal $\mathbf{B}$ of γ. The osculating plane is always perpendicular to the binormal $\mathbf{B}$, so the normal lies in this plane if and only if $\mathbf{B} \cdot \mathbf{N} = 0$. Further, as $\mathbf{P}$, $\mathbf{B}$, $\mathbf{U}$ and $\mathbf{N}$ all lie in the plane perpendicular to $\mathbf{T}$,

$$\mathbf{B} \cdot \mathbf{N} = 0 \Leftrightarrow \mathbf{N} = \pm \mathbf{P} = \pm \mathbf{T}'/(\kappa s') \Rightarrow \mathbf{T}' \cdot \mathbf{U} = 0 \Leftrightarrow \kappa_g = 0.$$

On the other hand, if $\mathbf{T}' \cdot \mathbf{U} = 0$, then $\mathbf{T}'$ is perpendicular to $\mathbf{U}$ and (always) to $\mathbf{T}$, and hence must be parallel to $\mathbf{N}$. This enables all the implications to go both ways.

The expressions given in Chapter 4 for the geodesic curvature are extrinsic in nature, that is they depend on a knowledge of the normal $\mathbf{N}$ to the surface. Nevertheless we do have the following important fact.

Proposition 5.5.2 *(**Minding, 1830***) The geodesic curvature is an intrinsic quantity. In fact if $\gamma(t) = \mathbf{X}(u(t), v(t))$ is a regular curve on M (and $\|\gamma'\| = s'$ where s is arclength and $' = \frac{d}{dt}$ as usual) then*

$$\kappa_g s'^3 = \sqrt{EG - F^2} \begin{vmatrix} u' & \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2 + u'' \\ v' & \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2 + v'' \end{vmatrix}.$$

Proof We have

$$\mathbf{T}s' = \gamma' = \mathbf{X}_u u' + \mathbf{X}_v v'$$

and differentiating again we find that

$$\mathbf{T}s'' + (\kappa_g \mathbf{U} + \kappa_n \mathbf{N})s'^2 = \gamma'' = \mathbf{X}_{uu}u'^2 + 2\mathbf{X}_{uv}u'v' + \mathbf{X}_{vv}v'^2 + \mathbf{X}_u u'' + \mathbf{X}_v v''.$$

The formula on the left comes from (4.5). The formula on the right is obtained by straight differentiation using the chain rule.

We now use Gauss's formulae (Theorem 5.2.1) for the second derivatives of $\mathbf{X}$ to deduce that

$$
\begin{aligned}
\gamma'' & = (\Gamma_{11}^1 \mathbf{X}_u + \Gamma_{11}^2 \mathbf{X}_v + e\mathbf{N})u'^2 + 2(\Gamma_{12}^1 \mathbf{X}_u + \Gamma_{12}^2 \mathbf{X}_v + f\mathbf{N})u'v' \\
& \quad + (\Gamma_{22}^1 \mathbf{X}_u + \Gamma_{22}^2 \mathbf{X}_v + g\mathbf{N})v'^2 + \mathbf{X}_u u'' + \mathbf{X}_v v''.
\end{aligned}
\tag{5.1}
$$

Removing the $\mathbf{N}$ component from γ'' we get

$$
\begin{aligned}
\kappa_g s'^2 \mathbf{U} + s'' \mathbf{T} & = (\Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2 + u'')\mathbf{X}_u + (\Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2 + v'')\mathbf{X}_v \\
& = A\mathbf{X}_u + B\mathbf{X}_v,
\end{aligned}
\tag{5.2}
$$

say.

We then have

$$
\begin{aligned}
\kappa_g s'^2 & = [\kappa_g s'^2 \mathbf{U}, \mathbf{N}, \mathbf{T}] \\
& = [\kappa_g s'^2 \mathbf{U} + s'' \mathbf{T}, \mathbf{N}, \mathbf{T}] \text{ (property of triple scalar product)} \\
& = [A\mathbf{X}_u + B\mathbf{X}_v, (\mathbf{X}_u \times \mathbf{X}_v)/\sqrt{EG - F^2}, (u'\mathbf{X}_u + v'\mathbf{X}_v)/s'],
\end{aligned}
$$

using $||\mathbf{X}_u \times \mathbf{X}_v|| = \sqrt{EG - F^2}$. Now

$$(u'\mathbf{X}_u + v'\mathbf{X}_v) \times (A\mathbf{X}_u + B\mathbf{X}_v) = (u'B - v'A)(\mathbf{X}_u \times \mathbf{X}_v),$$

so evaluating the triple scalar product gives

$$\kappa_g s'^3 = (u'B - v'A)\sqrt{EG - F^2},$$

as required.

In Chapter 4 some simple examples of geodesics were calculated, e.g. on the sphere and the circular cylinder. The general problem of finding geodesics is very hard, but we can characterise them fairly easily in intrinsic terms as follows.

Theorem 5.5.3 *Suppose that the curve $\gamma(t) = \mathbf{X}(u(t), v(t))$, where $t \in I$ and I is an interval, is a unit speed geodesic. Then, for all $t \in I$,*

$$
\begin{aligned}
\Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2 + u'' & = 0, \tag{5.3} \\
\Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2 + v'' & = 0. \tag{5.4}
\end{aligned}
$$

Conversely, any curve satisfying (5.3) and (5.4) is a constant speed geodesic.
There is a unique geodesic emerging from any point $\mathbf{p}$ of a surface in any given direction.

Proof Assuming γ is a unit speed geodesic, we have $\kappa_g = 0$ and $s = t$, $s' = 1$, $s'' = 0$, so that (5.2) implies that $A = B = 0$ as $\mathbf{X}_u$, $\mathbf{X}_v$ are independent vectors.

For the converse, suppose (5.3) and (5.4) hold. Then $A = B = 0$ in (5.2) and, $\mathbf{T}$ and $\mathbf{U}$ being independent vectors, we deduce that $\kappa_g = 0$ and $s'' = 0$. The former shows that γ is a geodesic and the latter that $s' = $ constant, so that γ is a constant speed.

The last part (that is the existence and uniqueness of geodesics) follows from the theory of differential equations.

As you will appreciate this reduces the determination of geodesics to solution of second order differential equations. As you will also appreciate generally speaking the resulting differential equations cannot usually be explicitly solved!

Example 5.5.4 Geodesics on a general cylinder Consider a cylinder $\mathbf{X}(u,v) = \alpha(u) + v\mathbf{e}_3$ for some unit speed plane curve $\alpha : I \to \mathbf{R}^2 \times \{0\}$ lying in the x, y plane, and $\mathbf{e}_3 = (0,0,1)$. A unit speed curve on the surface is of the form $\gamma(s) = \alpha(u(s)) + v(s)\mathbf{e}_3$. It is a geodesic if the acceleration is orthogonal to the surface. Now $\gamma' = \mathbf{T}u' + v'\mathbf{e}_3$ where $\mathbf{T}$ is the unit tangent to α. Thus $\gamma'' = \kappa\mathbf{U}u'^2 + \mathbf{T}u'' + v''\mathbf{e}_3$, where κ and $\mathbf{U}$ refer to the plane curve α. Note that $\mathbf{U}$ is perpendicular to $\mathbf{X}_u$ and $\mathbf{X}_v$ so is normal to the *surface*, while $\mathbf{T}$ and $\mathbf{e}_3$ are tangent to the surface. Thus the condition that γ'' is normal to the surface is that $u'' = v'' = 0$, which says that $u = at + b, v = ct + d$ for constants a, b, c, d. Thus geodesics are the images of straight lines in the parameter space. (We make γ unit speed by $a^2 + c^2 = 1$.)

Of course we expect this result since the cylinder is obtained by bending a vertical sheet of paper so that one edge lies along α, and bending carries geodesics to geodesics.

Note that when α is a closed curve and we allow the surface to consist of the whole cylinder over α (thereby violating the 'injective' assumption of regularity but only in a relatively harmless way) there may be many geodesics joining two points of the cylinder.

We now give a criterion for curves $v = $ constant to be unit speed geodesics.

Proposition 5.5.5 *Let M be a surface parametrised by $\mathbf{X} : U \to \mathbf{R}^3$. Then the curve $\gamma(u) = X(u, c)$, for c constant, is a unit speed geodesic if and only if $E = 1$ and $F_u = 0$ along the curve.*

Proof Suppose that γ is a unit speed geodesic. Since $\gamma' = \mathbf{X}_u$ it follows that $E = 1$. To prove $F_u = 0$, recall that γ is a unit speed geodesic provided the acceleration of the curve is normal to the surface. Now the velocity vector is $\gamma'(u) = \mathbf{X}_u$, so the acceleration is $\mathbf{X}_{uu}$, which is therefore orthogonal to the vector $\mathbf{X}_v$; in other words $\mathbf{X}_{uu} \cdot \mathbf{X}_v = 0$. Differentiating $\mathbf{X}_u \cdot \mathbf{X}_u = 1$ with respect to v we find $\mathbf{X}_u \cdot \mathbf{X}_{uv} = 0$ so $F_u = (\mathbf{X}_u \cdot \mathbf{X}_v)_u = \mathbf{X}_{uu} \cdot \mathbf{X}_v + \mathbf{X}_u \cdot \mathbf{X}_{uv} = 0$.

Conversely, if $E = 1$ then γ is unit speed, and if $F_u = 0$ then $\mathbf{X}_{uu} \cdot \mathbf{X}_v + \mathbf{X}_u \cdot \mathbf{X}_{uv} = \mathbf{0}$, but the second term is zero as above (by differentiating $\mathbf{X}_u \cdot \mathbf{X}_u = 1$ with respect to v), so $\mathbf{X}_{uu}$ is perpendicular to $\mathbf{X}_v$. However, differentiating $\mathbf{X}_u \cdot \mathbf{X}_u = 1$ with respect to u shows that $\mathbf{X}_{uu}$ is perpendicular to $\mathbf{X}_u$. Thus $\mathbf{X}_{uu}$ is perpendicular to both $\mathbf{X}_u$ and $\mathbf{X}_v$ and hence parallel to $\mathbf{N}$. This now shows that γ is a unit speed geodesic.

5.6 Riemannian (abstract) surfaces

Definition 5.6.1 A **Riemannian surface** is defined to be an open subset U of the plane $\mathbf{R}^2$ together with three functions E, F and G with the property that the quadratic form

$$Ea_1^2 + 2Fa_1a_2 + Ga_2^2$$

is positive definite. In other words the quadratic form is ≥ 0 and equals 0 only when $a_1 = a_2 = 0$; this is equivalent to $E > 0, G > 0, EG - F^2 > 0$ for all u, v.

If $\beta : I \to U$, $\beta(t) = (u(t), v(t))$ is a curve then we define the length of the tangent vector $\beta'(t)$ to be $\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}$. and the length of the curve between t_1, $t_2 \in I$ to be

$$\int_{t_1}^{t_2} \sqrt{Eu'^2 + 2Fu'v' + Gv'^2} dt.$$

Quite generally the inner product of two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ is defined to be

$$(a_1, a_2) * (b_1, b_2) = Ea_1b_1 + F(a_1b_2 + a_2b_1) + Ga_2b_2 = (\, a_1 \quad a_2 \,) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We then see that the length of a vector $a = (a_1, a_2)$ is $\sqrt{a * a}$, and denote it by $||a||$. We can now define the angle α between two vectors a and b by $\cos\alpha = a*b/||a||.||b||$. (This quotient does indeed lie between -1 and 1.)

In short we treat U as a surface with E, F, G taking the place of the first fundamental form coefficients. This enables us to define the notion of distance and angle in U. We can then agree that the Gaussian curvature of U is given by the intrinsic version, involving only E, F, G and their derivatives.

Example 5.6.2 Poincaré disk Suppose we consider the open disc of radius 2 centred at the origin in $\mathbf{R}^2$. Of course this is not a very interesting surface. However suppose that we assign a first fundamental form, a notion of distance at each point which does not coincide with the Euclidean one. We can still make sense of distance, angles, and Gaussian curvature. In fact using polar co-ordinates we shall assign to the point (r, θ) the coefficients $E = 1/(1 - r^2/4)^2$, $F = 0$, $G = 1/(1 - r^2/4)^2$. From the formula for the Gauss curvature it is not hard to see that $K = -1$. (Note that the idea of principal curvatures makes no sense here. We really do have to use the fact that Gaussian curvature is intrinsic.) This 'surface' is called the *hyperbolic plane*. It turns out that the geodesics are circles which meet the 'boundary' circle orthogonally. Now suppose we interpret geodesics as the straight lines of this geometry, and by parallel straight lines we mean lines that do not meet. Then we have a geometry which satisfies all of the usual axioms for Euclidean geometry, except that there are many lines parallel to a given line through a given point. On the other hand for the ordinary 2-sphere, where geodesics are great circles, no two 'lines' are parallel. (Actually one has to be a little careful here, since lines meet in two (antipodal) points and not one.)

We shall give another example in more detail. This in fact turns out to be essentially the same as Example 5.6.2 above.

Example 5.6.3 Poincaré upper half plane For our set U we take the upper half plane $\{(u, v) : v > 0\}$ and for the first fundamental form we consider $E = 1/v^2$, $F = 0$, $G = 1/v^2$. Of course this gives a positive definite quadratic form at all points of U. We can compute the Gaussian curvature using the expression

$$K = \frac{1}{(EG - F^2)^2} \left(\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} \right).$$

This gives $K = -1$.

It is interesting to work out the geodesics for this surface. Using the formulae in Chapter 5 for the Christoffel symbols, we get in the present case

$$\begin{vmatrix} u' & u'' - \frac{2}{v}u'v' \\ v' & v'' + \frac{1}{v}u'^2 - \frac{1}{v}v'^2 \end{vmatrix} = 0.$$

Certainly $u = $ constant is a solution of this, so straight lines in the upper half plane parallel to the v-axis are geodesics. To find the other solutions, let $v = v(u)$, so that $'$ means $\frac{d}{du}$, and $u' = 1, u'' = 0$. Then we have $vv' + v'^2 + 1 = 0$. This can be integrated directly:

$$\frac{d}{du}(vv') = -1 \Leftrightarrow \frac{1}{2}\frac{d}{du}(v^2) = vv' = -u + a \Leftrightarrow \frac{1}{2}v^2 = -\frac{1}{2}u^2 + au + b \Leftrightarrow (u - a)^2 + v^2 = 2b + a^2,$$

where a and b are constants. These are circles centred on the u-axis, though only the top half of the circle lies in our domain U.

Hence the geodesics are 'vertical' straight lines and semicircles centred on the u-axis. It is clear that there is a geodesic through every point in every direction. Also there is a unique geodesic joining any two points of U.

A short calculation shows that the length of the geodesic

$$\gamma(t) = (u(t), v(t)) = (u_0, v_1 t + v_0(1 - t))$$

joining two points (u_0, v_0) and (u_0, v_1) with the same u-coordinate is

$$\int_0^1 \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt = |\ln(v_1/v_0)|.$$

Thus, as $v_0 \to 0$, the distance tends to ∞. The u-axis is to be thought of as 'infinitely far away' in this metric.

It is also possible to calculate the length of the (circular arc) geodesic joining two points not in the same vertical line. Suppose that the circle through (u_1, v_1) and (u_2, v_2) has centre at $(c, 0)$ and radius r. The geodesic is then

$$\gamma(t) = (u(t), v(t)) = (c + r\cos t, r\sin t), \quad t_1 \le t \le t_2 \text{ say.}$$

Thus $u'(t) = -r\sin(t), \ v'(t) = r\cos(t)$ and writing down the integral for the length of the geodesic gives

$$\int_{t_1}^{t_2} \operatorname{cosec} t\,dt = \left[\ln\left(\frac{\sin t}{1 + \cos t}\right)\right]_{t_1}^{t_2} = \ln\left(\frac{\sin t_2(1 + \cos t_1)}{\sin t_1(1 + \cos t_2)}\right).$$

But

$$\sin t_1 = \frac{v_1}{r}, \ \cos t_1 = \frac{u_1 - c}{r}, \ \sin t_2 = \frac{v_2}{r}, \ \cos t_2 = \frac{u_2 - c}{r}.$$

The length, which should be positive, then works out as the absolute value of

$$\ln\left(\frac{v_2(r + u_1 - c)}{v_1(r + u_2 - c)}\right).$$

Some exercises are in §5.7 below, and others will be done in class.

What about angles? It turns out that the angle between curves using this new fundamental form is the same as the angle in the Euclidean sense. For if (a_1, a_2) and (b_1, b_2) are two vectors then the angle α between them is given by

$$\cos\alpha = \frac{Ea_1b_1 + F(a_1b_2 + a_2b_1) + Ga_2b_2}{\sqrt{(Ea_1^2 + 2Fa_1a_2 + Ga_2^2)(Eb_1^2 + 2Fb_1b_2 + Gb_2^2)}}.$$

Now $F = 0,\ E = G$ so this reduces to

$$\cos\alpha = \frac{a_1 b_1 + a_2 b_2}{\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}}$$

which is the angle between these vectors for the Euclidean plane (where $E = G = 1,\ F = 0$).

   This is another model for 'non-Euclidean geometry', where by taking 'lines' to mean geodesics, the lines of U obey the axioms of Euclidean geometry apart from the parallel axiom. It is also possible to define circles in this geometry, and many variants of Euclidean theorems can be established.

5.7   Exercises

1. Let $\mathbf{X} : U \to \mathbf{R}^3$ be a parametrization of a regular surface for which the first fundamental form coefficients are $E = \alpha(u,v), F = 0, G = \alpha(u,v)$, where $\alpha(u,v)$ is a smooth function whose values are always > 0. Use the formulae in the text to show that

$$\Gamma_{11}^1 = \alpha_u/2\alpha,\ \Gamma_{12}^1 = \alpha_v/2\alpha,\ \Gamma_{22}^1 = -\alpha_u/2\alpha,$$

$$\Gamma_{11}^2 = -\alpha_v/2\alpha,\ \Gamma_{12}^2 = \alpha_u/2\alpha,\ \Gamma_{22}^2 = \alpha_v/2\alpha.$$

Now write down the Gauss-Weingarten equations for such a surface and using these compute $(\mathbf{X}_{uu})_v$ and $(\mathbf{X}_{uv})_u$. Use the equality of these expressions to equate coefficients of $\mathbf{X}_v$ and deduce that the Gauss curvature of the surface is

$$K = \frac{\alpha_u^2 + \alpha_v^2 - \alpha(\alpha_{uu} + \alpha_{vv})}{2\alpha^3}.$$

Suppose that a surface has $E = G = \mathrm{e}^u,\ F = 0$. Show that the Gauss curvature of this surface is 0.

2. For $\alpha(u,v) = 1/v^2$ in the previous question show that the Gauss curvature is -1 everywhere.

3. Let $\mathbf{X} : U \to \mathbf{R}^3$ be a parametrization of a regular surface for which the first fundamental form coefficients are $E = 1, F = \alpha(u,v), G = 1$, for some function α with $1 - (\alpha(u,v))^2 > 0$ for all $(u,v) \in U$. (Recall that $EG - F^2 > 0$ for any regular surface.) Use the formulae of the text to show that

$$\Gamma_{11}^1 = \frac{-\alpha\alpha_u}{1 - \alpha^2},\ \Gamma_{12}^1 = 0,\ \Gamma_{22}^1 = \frac{\alpha_v}{1 - \alpha^2},$$

$$\Gamma_{11}^2 = \frac{\alpha_u}{1 - \alpha^2},\ \Gamma_{12}^2 = 0,\ \Gamma_{22}^2 = \frac{-\alpha\alpha_v}{1 - \alpha^2}.$$

and give the expressions for the β_i^j.

Write out the Gauss-Weingarten equations for such a surface and use these to compute $(\mathbf{X}_{uu})_v$ and $(\mathbf{X}_{uv})_u$. Using the equality of these expressions equate the coefficients of $\mathbf{X}_v$ to deduce that the Gauss curvature is

$$K = \frac{(1 - \alpha^2)\alpha_{uv} + \alpha\alpha_u\alpha_v}{(1 - \alpha^2)^2}.$$

Deduce that K vanishes if α is a function of u alone.

Let $\gamma : I \to \mathbf{R}^3$ be a unit speed curve and let $\mathbf{a}$ be a (constant) unit vector. Define $\mathbf{X} : I \times \mathbf{R} \to \mathbf{R}^3$ by $\mathbf{X}(u,v) = \gamma(u) + v\mathbf{a}$. Deduce from the above that the Gauss curvature of $\mathbf{X}$ (at regular points) vanishes.

4. Use the formula in the Theorema Egregium to show directly, from E, F, G and their derivatives, that the Gauss curvature of the surface $\mathbf{X}(u, v) = (u, v, uv)$ is

$$K = -\frac{1}{(1 + u^2 + v^2)^2}.$$

Check this formula by calculating e, f, g in the usual way and using $K = (eg - f^2)/(EG - F^2)$.

5. Let M be a surface and Π a plane which meets M in a curve γ. Show that γ traces out a geodesic if Π is a plane of symmetry of M, i.e. reflection in the plane Π leaves M fixed. Use this to find some geodesics on the unit sphere, and the torus of revolution.

6. Suppose that M is a surface of revolution, with γ a geodesic on M. Show that $r \cos \theta = $ constant, where $\theta(s)$ is the angle between $\gamma'(s)$ and the parallel (of radius r) through $\gamma(s)$. [Hints. It is enough to take M generated by a unit speed curve α, and to take γ unit speed. This simplifies the calculation and does not in fact lose generality since curves can always be reparametrized to be unit speed. The geodesic condition becomes γ'' parallel to $\mathbf{N}$, i.e. $\gamma'' \cdot \mathbf{X}_u = \gamma'' \cdot \mathbf{X}_v = 0$.]

7. Let M be the surface parametrised by $\mathbf{X}(u, v) = (u, v, uv)$. Check that the non-unit speed curve $\gamma(t) = (t, -t, -t^2)$ is a geodesic. Can you find any other geodesics on M?

8. Let $\alpha : I \to \mathbf{R}^2$, $\alpha(s) = (x(s), y(s))$ be a unit speed plane curve. Define $\mathbf{X} : I \times \mathbf{R} \to \mathbf{R}^3$ by $\mathbf{X}(s, t) = (x(s), y(s), t)$. This gives the *cylinder* over the curve. Let β be a constant, and let $\gamma(s) = (x(s), y(s), s \tan \beta)$. Prove that γ (which is not generally unit speed) traces out a geodesic.

9. Let $\mathbf{X} : U \to \mathbf{R}^3$ be a parametrisation of a surface, and suppose that we have $F = 0$, while $E = G = a(u) + b(v)$ for some smooth functions a and b. Let $\gamma(s)$ be a unit speed geodesic, and let $\theta(s)$ be the angle between γ and the u-curves (i.e. between $\gamma'(s)$ and $\mathbf{X}_u$). Prove that

$$a \sin^2 \theta - b \cos^2 \theta = \text{ constant.}$$

10. In the upper-half plane model of the hyperbolic plane, find the equation of the geodesic joining $(0, 1)$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, and find the length of the geodesic joining these points.

11. The same as the previous question, for the points $(-1, 1)$ and $(1, 1)$.

12. The same as the previous question, for the points $(0, 2)$ and $(\sqrt{3}, 1)$.

13. The same as the previous question, for the points $(0, 1)$ and $(1, 1)$.