Generic affine differential geometry of curves in $\mathbb{R}^n$.

Declan Davis

The University of Liverpool,
Department of Mathematical Sciences,
Mathematics and Oceanography Building,
Peach Street,
Liverpool L69 7ZL,
United Kingdom.
d.davis@liverpool.ac.uk

April 14, 2005

Abstract

This paper considers curves in $\mathbb{R}^n$. It defines affine arc-length and affine curvatures. The family of affine distance functions is generalised, along with the family of affine height functions. A new basis is constructed that makes the conditions for A_k singularity types easier to calculate, and applications are given to geometrical problems.

1 Introduction

In this paper, we consider the affine differential geometry of space curves, i.e. the geometric properties of smooth curves in $\mathbb{R}^n$, which remain invariant under the actions of $\mathrm{SL}(n, \mathbb{R}) = \{X \in \mathrm{Mat}(n, \mathbb{R}) : \det(X) = 1\}$ and the translation group. Such actions are called *equi-affine* transformations and preserve volume. In particular we generalise some of the results of S. Izumiya and T. Sano, found in [2] and [3], from two and three dimensions to n dimensions.

We generalise the classical ideas of affine curvatures, affine arc-length, families of affine distance functions, and families of affine height functions we also calculate the conditions for A_k singularities of these two families. Our methods are directly applicable to the study of families of functions and their bifurcation sets and our proofs are more direct than those in [3]; they show the underlying, governing dynamic.

In §2 we introduce the affine arc-length parameter for a curve $\gamma : I \to \mathbb{R}^n$. Parametrising a curve by affine arc-length ensures the first n derivatives of γ with respect to affine arc-length always span a volume of $+1$. The condition for such a parametrisation to exist is also found in §2.

In §3 we consider the classical affine curvatures of γ. These arise naturally from the affine arc-length parametrisation. A general formula in terms of n and i is given for calculating the i-th affine curvature of a curve in $\mathbb{R}^n$. Finally a system of affine Serret-Frenet differential equations is given for the classical curvatures.

The family of affine distance functions on a curve parametrised by affine arc-length is defined in §4; we also give a formula for an arbitrarily parametrised curve. Moreover, §5 follows this path exactly, but with the family of affine height functions.

In §6 a new equi-affine frame is found for the curve. This frame has the property that its ordered members span a volume of $+1$; it is defined in terms of the derivatives of γ and the derivatives of the affine curvatures and is denoted as $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\}$. Moreover, a new system of affine Serret-Frenet formulae arise with this frame, giving new affine curvatures (which we call affine torsions and write as σ_i). Again formulae in terms of n and i are given for the i-th affine torsion of a curve in $\mathbb{R}^n$.

In §7 we use the new frame $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\}$ to rewrite the family of affine distance functions and height functions. Conditions for these two families to have A_k singularities are given in terms of the $\mathbf{T}_i$ and σ_j.

Finally, in §§7.1 it is shown that the families are always (p)-versally unfolded for a generic space curve and geometrical applications are given.

Since determinants measure volume, and volume remains unchanged by equi-affine transformations, the determinant is an *affine invariant* and will play a central role in this study. Let $\{v_1, \ldots, v_n\}$ be an ordered set (i.e. a list) of n vectors in $\mathbb{R}^n$. Then let $[v_1, \ldots, v_n]$ denote the determinant of the matrix whose i-th column is the vector v_i. Then $[v_1, \ldots, v_n]$ is equal to the volume spanned by the vectors in $\{v_1, \ldots, v_n\}$.

2 Affine arc-length

Let $I \subseteq \mathbb{R}$ be an open interval, and $\gamma : I \to \mathbb{R}^n$ a smooth space curve. We seek an affine invariant parametrisation for γ of the lowest possible order. As is the convention for $n = 2, 3$ we choose a parametrisation, in terms of the affine arc-length parameter s, such that $[\gamma', \gamma'', \ldots, \gamma^{(n)}] = 1$ for all $s \in I$. Throughout this paper, prime denotes differentiation with respect to the *affine arc-length parameter* s, thus $\gamma' = d\gamma/ds$ etc, whereas a dot is reserved for differentiation with respect to an arbitrary parameter t, thus $\dot{\gamma} = d\gamma/dt$ etc. Using basic properties of determinants, it is easy to show that

$$[\gamma', \gamma'', \ldots, \gamma^{(n)}] = [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}] \left(\frac{dt}{ds}\right)^{n(n+1)/2}, \qquad (1)$$

Assuming that $[\gamma', \gamma'', \ldots, \gamma^{(n)}] = 1$ we obtain

$$s(t) = \int [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^{2/n(n+1)} \; dt \; .$$

Thus for $t_1 \le t \le t_2$, affine arc-length is given by

$$\int_{t_1}^{t_2} [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^{2/n(n+1)} \; dt \; .$$

Remark 1 Let $J \subseteq \mathbb{R}$ and consider a curve $\alpha : J \to \mathbb{R}^n$ parametrised by euclidean arc-length. We define the tangent vector $\mathbf{V}_1$ to be the unit vector in the direction of $\dot{\alpha}$. The second basis vector $\mathbf{V}_2$ is in the subspace $\langle \dot{\alpha}, \ddot{\alpha} \rangle$, is of unit length, is perpendicular to $\mathbf{V}_1$, and together with $\mathbf{V}_1$ spans an area of $+1$. The third basis vector $\mathbf{V}_3$ is in the subspace $\langle \dot{\alpha}, \ddot{\alpha}, \dddot{\alpha} \rangle$, is of unit length, is perpendicular to $\mathbf{V}_1$ and $\mathbf{V}_2$, and together with $\mathbf{V}_1$ and $\mathbf{V}_2$ spans a volume of $+1$. Proceeding in this fashion, the $(k+1)$-st basis vector is in the space $\langle d^i\alpha/dt^i : 1 \le i \le k \rangle$, is of unit length, is perpendicular to $\{\mathbf{V}_i : 1 \le i \le k\}$, and together with $\{\mathbf{V}_i : 1 \le i \le k\}$ spans a volume of $+1$.

Definition 2.1 *Given a smooth curve parameterised by euclidean arc-length, the euclidean curvature is given by* $\kappa = \dot{\mathbf{V}}_1 \cdot \mathbf{V}_2$ *and the higher euclidean torsions are given by* $\tau_i = \dot{\mathbf{V}}_{i+1} \cdot \mathbf{V}_{i+2}$ *for all* $1 \le i \le n-2$.

Remark 2 Letting t be euclidean arc-length and writing κ for the euclidean curvature of γ and $\{\tau_1, \ldots, \tau_{n-2}\}$ for the higher euclidean torsions gives

$$[\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}] = \kappa^{n-1} \prod_{i=1}^{n-2} \tau_i^{n-i-1} \; .$$

Then Equation (1) shows that if $[\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}] = 0$ for some t, then the affine arc-length parametrisation in unobtainable, since $0 \ne 1$. Hence, if any of the euclidean curvatures or euclidean torsions become zero at certain points, the affine arc-length parameter can not be defined at such points. Hence, in all that follows, $I \subseteq \mathbb{R}$ shall be chosen such that the image of γ has everywhere non-zero euclidean curvature and euclidean torsions.

3 Affine curvatures

Here we define the *affine curvatures* of a curve. Let $\gamma : I \to \mathbb{R}^n$ be parametrised by affine arc-length, so that $[\gamma', \gamma'', \ldots, \gamma^{(n)}] = 1$ for all $s \in I$. Then differentiating with respect to s gives $[\gamma', \ldots, \gamma^{(n-1)}, \gamma^{(n+1)}] = 0$. Hence the set of vectors $\{\gamma', \ldots, \gamma^{(n-1)}, \gamma^{(n+1)}\}$ is linearly dependent. Therefore, there must exist functions $\mu_i : I \to \mathbb{R}$ for $1 \le i \le n-1$ such that

$$\gamma^{(n+1)} + \mu_1 \gamma' + \mu_2 \gamma'' + \cdots + \mu_{n-1} \gamma^{(n-1)} = 0 \; . \qquad (2)$$

The functions μ_i are called the *affine curvatures* of γ. Notice that

$$\mu_i = (-1)^{n-i+1}[\gamma', \ldots, \gamma^{(i-1)}, \gamma^{(i+1)}, \ldots, \gamma^{(n+1)}] \ .$$

The μ_i are given by determinants; an equi-affine transformation of $\mathbb{R}^n$ leaves the affine curvatures unchanged. These affine curvatures are truly affine invariants.

These definitions give Serret-Frenet type formulae. Let $\Gamma = (\gamma', \gamma'', \ldots, \gamma^{(n)})^\top$ where $\top$ denotes transpose; then for $M \in \mathrm{Mat}(n, \mathbb{R})$

$$\Gamma' = M\Gamma \ . \tag{3}$$

It follows that if $M = (m_{i,j})$ then

$$m_{i,j} = \begin{cases} 1 & \text{if } j - i = 1 \ , \\ -\mu_j & \text{if } i = n \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{4}$$

Hence $\det(M) = (-1)^n \mu_1$.

Example. Let $n = 3$, so that $\gamma : I \to \mathbb{R}^3$, then

$$\frac{d}{ds}\begin{pmatrix} \gamma' \\ \gamma'' \\ \gamma''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu_1 & -\mu_2 & 0 \end{pmatrix}\begin{pmatrix} \gamma' \\ \gamma'' \\ \gamma''' \end{pmatrix} .$$

4 Affine distance functions

Here we give a general definition of the affine distance function introduced in two and three-dimensions in [3].

Let $\gamma : I \to \mathbb{R}^n$ be parametrised by affine arc-length. Given $\mathbf{x} \in \mathbb{R}^n$ and $s \in I$, we get $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$, an n-parameter family of affine distance functions defined on the curve, where

$$\Delta(\mathbf{x}, s) = [\mathbf{x} - \gamma, \gamma', \ldots, \gamma^{(n-1)}] \ . \tag{5}$$

The zero level-set of $\Delta(\mathbf{x}, s_0)$ is given by $\mathbf{x} \in \mathbb{R}^n$ such that for some $\lambda_i \in \mathbb{R}$

$$\mathbf{x} = \gamma(s_0) + \lambda_1\gamma'(s_0) + \lambda_2\gamma''(s_0) + \cdots + \lambda_{n-1}\gamma^{(n-1)}(s_0) \ .$$

This is the set of points $\mathbf{x} \in \mathbb{R}^n$ of affine distance zero from $\gamma(s_0)$. It is easy to see that the other level-sets are hyperplanes parallel to this one.

Given an open interval $J \subseteq \mathbb{R}$, and an arbitrary parametrisation for the curve $\gamma : J \to \mathbb{R}^n$. The family of affine distance functions $\Delta : \mathbb{R}^n \times J \to \mathbb{R}$ is given by

$$\Delta(\mathbf{x}, t) = [\mathbf{x} - \gamma, \dot{\gamma}, \ldots, \gamma^{(n-1)}][\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^{(1-n)/(1+n)} \ .$$

5 Affine height functions

Let $\gamma : I \to \mathbb{R}^n$ be parametrised by affine arc-length. Let $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| = 1\}$ be the unit hypersphere in $\mathbb{R}^n$. We can define a family of functions on the curve, parametrised by S^{n-1}. This family $H : S^{n-1} \times I \to \mathbb{R}$ is the family of *affine height functions*, where

$$H(\mathbf{x}, s) = [\mathbf{x}, \gamma', \gamma'', \ldots, \gamma^{(n-1)}] \ .$$

Let $J \subseteq \mathbb{R}$ be an open interval, then for an arbitrary parametrisation, the affine height functions are given by $H : S^{n-1} \times I \to \mathbb{R}$ where

$$H(\mathbf{x}, t) = [\mathbf{x}, \dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n-1)}][\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^{(1-n)/(1+n)} \ .$$

6 Equi-affine frames

Let $\mathbf{x} \in \mathbb{R}^n$ and let $\{v_1, \ldots, v_n\}$ be a list of vectors $v_i \in T_{\mathbf{x}}\mathbb{R}^n$. The vectors are said to constitute an *equi-affine frame* if and only if $[v_1, \ldots, v_n] = 1$. It is clear that $\{\gamma', \ldots, \gamma^{(n)}\}$ forms an equi-affine frame with each $\gamma^{(i)} \in T_{\gamma(s)}\mathbb{R}^n$ for all $s \in I$.

The aim here is to define a new equi-affine frame for γ. This is motivated by later applications to singularity theory. Furthermore, the affine Serret-Frenet formulae with respect to this new equi-affine frame will be more analogous to the euclidean Serret-Frenet formulae. For example, if the euclidean torsion τ_{n-2} is zero then the curve can be contained in $\mathbb{R}^{n-1}$. This means the last basis vector, say $\mathbf{V}_n$, is constant. (If $n = 3$ then the binormal vector $\mathbf{B}$ is constant and γ is then a plane curve.) Given the affine Serret-Frenet formulae in Equation (3) and Equation (4), if $\mu_{n-1} = 0$, this in no way means that $\gamma^{(n-1)}$ is constant.

Given any smooth functions $\lambda_{i,j} : I \to \mathbb{R}$, the vectors

$$\gamma^{(i)} + \sum_{j=1}^{i-1} \lambda_{i,j}\gamma^{(j)} \quad \text{for all} \quad 1 \leq i \leq n$$

form an equi-affine frame. The classical case is when $\lambda_{i,j}(s) = 0$ for all $s \in I$ and $(i, j) \in \mathbb{N} \times \mathbb{N}$. Consider the vector given by $i = n$, that is

$$v = \gamma^{(n)} + \lambda_{n,1}\gamma' + \lambda_{n,2}\gamma'' + \cdots + \lambda_{n,n-1}\gamma^{(n-1)} \ .$$

We wish the derivative of v to depend on only one other member of the equi-affine frame. Setting $\lambda_{i,j} \equiv 0$ for all $(i,j) \in \{\mathbb{Z}\times\mathbb{Z} - \mathbb{N}\times\mathbb{N}\}$ gives

$$\begin{aligned} v' &= \sum_{i=1}^{n-1}(\lambda'_{n,i} - \mu_i)\gamma^{(i)} + \lambda_{n,i}\gamma^{(i+1)} \ , \\ &= (\lambda'_{n,1} - \mu_1)\gamma' + \lambda_{n,n-1}\gamma^{(n)} + \sum_{i=2}^{n-1}(\lambda'_{n,i} - \mu_i + \lambda_{n,i-1})\gamma^{(i)} \ . \end{aligned}$$

If v' is to be independent of v it follows that $\lambda_{n,n-1} \equiv 0$. In order to remove dependency on other derivatives set $\lambda_{n,i-1} = \mu_i - \lambda'_{n,i}$ for all $2 \le i \le n-1$. Starting with $i = n-1$ gives $\lambda_{n,n-2} = \mu_{n-1} - \lambda'_{n,n-1} = \mu_{n-1}$. In turn, putting $i = n-2$ gives $\lambda_{n,n-3} = \mu_{n-2} - \mu'_{n-1}$. Putting $i = n-3$ gives $\lambda_{n,n-4} = \mu_{n-3} - \mu'_{n-2} + \mu''_{n-1}$. Continuing this process for $2 \le i \le n-1$ gives

$$\lambda_{n,n-i} = \sum_{j=1}^{i-1}(-1)^{j+1}\mu_{n-i+j}^{(j-1)} \ .$$

Then finally, the vector v' becomes

$$v' = \left(\sum_{i=1}^{n-1}(-1)^i \mu_i^{(i-1)}\right)\gamma' = \sigma_{n-1}\gamma', \text{ say} \ . \tag{6}$$

Thus the derivative of v depends only on one vector and is more analogous to the euclidean Serret-Frenet system.

This has found a new basis vector, namely v. Let us call it $\mathbf{T}_n$ and search for a new basis $\{\mathbf{T}_1 \ldots, \mathbf{T}_n\}$. It is clear that $\mathbf{T}_1 = \gamma'$; this gives the affine tangent vector. Thus we have the identity $\mathbf{T}'_n = -\sigma_{n-1}\mathbf{T}_1$.

We wish to find a new equi-affine frame which satisfies the additional vector differential equations $\mathbf{T}'_1 = \mathbf{T}_2$, $\mathbf{T}'_i = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ for all $2 \le i \le n-1$. These can be written as $\mathbf{T}'_i = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ if we set $\sigma_0 \equiv 0$ and $\mathbf{T}_{n+1} \equiv \mathbf{0}$. From the affine arc-length construction, the functions $\mu_i : I \to \mathbb{R}$ arise naturally. Thus the $\sigma_i : I \to \mathbb{R}$ will be expressed in terms of the μ_i and their derivatives.

Consider the affine Serret-Frenet formulae in matrix form $\Gamma' = M\Gamma$, where Γ and M are defined above in Equation (3) and Equation (4). Each new basis vector $\mathbf{T}_i$ can be expressed in terms of Γ :

$$\mathbf{T}_i = \gamma^{(i)} + \sum_{j=1}^{i-1}\lambda_{i,j}\gamma^{(j)} \quad \text{for all} \quad 1 \le i \le n$$

This can be written in matrix notation as $T = \Lambda\Gamma$ where T is the matrix whose i-th row is the vector $\mathbf{T}_i$. Furthermore we can write $T' = \Sigma T$ where Σ is derived from the identities $\mathbf{T}'_1 = \mathbf{T}_2$, $\mathbf{T}'_i = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ for all $2 \le i \le n-1$, and $\mathbf{T}'_n = -\sigma_{n-1}\mathbf{T}_1$.

Thus we have $\Gamma' = M\Gamma$, $T = \Lambda\Gamma$, and $T' = \Sigma T$. It follows that $\Lambda'\Gamma + \Lambda\Gamma' = \Sigma T$. In turn, this gives $\Lambda'\Gamma + \Lambda M\Gamma = \Sigma T$. This finally yields $\Lambda'\Gamma + \Lambda M\Gamma = \Sigma\Lambda\Gamma$, or simply $\Lambda' + \Lambda M = \Sigma\Lambda$. Here M is known to us, and is given by the identity

$$\gamma^{(n+1)} + \mu_1\gamma' + \cdots + \mu_{n-1}\gamma^{(n-1)} = 0 \ .$$

Writing $\Sigma = (\sigma_{i,j})$ gives $\sigma_{i,j} = 1$ for all $j - i = 1$, $\sigma_{i,1} = -\sigma_{i-1}$ for all $2 \leq i \leq n$, and $\sigma_{i,j} = 0$ otherwise. Writing $\Lambda = (\lambda_{i,j})$ gives $\lambda_{i,j} = 1$ for all $i - j = 0$ and $\lambda_{i,j} = 0$ for all $j - i > 0$, i.e. Λ is a lower triangular matrix with 1 in each position along the leading diagonal.

Let $X = (x_{i,j})$ where $X = \Lambda' + \Lambda M - \Sigma\Lambda$; we wish to make X into the zero matrix. On the leading diagonal of X we have $x_{i,i} = \lambda_{i,i-1} - \lambda_{i+1,i}$. Since $\lambda_{1,0} = 0$ it follows that $\lambda_{i,i-1} = 0$ for all $2 \leq i \leq n$. This implies that Λ has zero along the diagonal $i - j = 1$. Thus each $\mathbf{T}_i$ will not have a component of $\gamma^{(i-1)}$.

Consider $x_{i,j}$ such that $i - j = 1$. It follows that $x_{n,n-1} = \lambda_{n,n-2} - \mu_{n-1}$, $x_{i,i-1} = \lambda_{i,i-2} - \lambda_{i+1,i-1}$ for all $3 \leq i \leq n-1$, and $x_{2,1} = \sigma_1 - \lambda_{3,1}$. Since $x_{i,j} = 0$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$ it follows that

$$\mu_{n-1} = \lambda_{n,n-2} = \lambda_{n-1,n-3} = \ldots = \lambda_{i,i-2} = \ldots = \lambda_{3,1} = \sigma_1 \ .$$

Considering each diagonal in turn, $i-j = 1, 2, 3, \ldots, n-1$ gives the following expressions for the σ_i, we have

$$\begin{aligned}
\sigma_1 &= a_{1,1}\mu_{n-1} \ , \\
\sigma_2 &= a_{2,1}\mu'_{n-1} + a_{2,2}\mu_{n-2} \ , \\
\sigma_3 &= a_{3,1}\mu''_{n-1} + a_{3,2}\mu'_{n-2} + a_{3,3}\mu_{n-3} \ , \\
\sigma_i &= \sum_{j=1}^{i} a_{i,j} \ \mu_{n-j}^{(i-j)} \ ,
\end{aligned}$$

where the $a_{i,j}$ are entries in an $(n-1) \times (n-1)$ lower triangular matrix, we have $a_{i,j} = 1$ for all $i = j$ and $a_{i,j} = 0$ for all $i < j$. When $i > j$ we have

$$a_{i,j} = (-1)^{i+j} \begin{pmatrix} n-j-1 \\ i-j \end{pmatrix} = (-1)^{i+j} \frac{(n-j-1)!}{(i-j)!(n-i-1)!} \ .$$

It follows that the σ_i are then given by

$$\sigma_i = \sum_{j=1}^{i} (-1)^{i+j} \begin{pmatrix} n-j-1 \\ i-j \end{pmatrix} \mu_{n-j}^{(i-j)} \ .$$

Given the existence of Σ and M is known, it is easy to find Λ for all $i - j \geq 1$

$$\lambda_{i,j} = \sum_{k=1}^{i-j-1} (-1)^{i-j-k-1} \begin{pmatrix} n-j-k-1 \\ i-j-k-1 \end{pmatrix} \mu_{n-k}^{(i-j-k-1)} \ .$$

In the present section we have proved the following

Proposition 6.1 *Given a curve $\gamma : I \to \mathbb{R}^n$ parametrised by affine arc-length. An equi-affine basis $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\}$ satisfying the vector differential equations $\mathbf{T}_1' = \mathbf{T}_2$, $\mathbf{T}_i' = \mathbf{T}_{i+1} - \sigma_{i-i}\mathbf{T}_1$ for all $2 \leq i \leq n-2$, and $\mathbf{T}_n = -\sigma_{n-1}\mathbf{T}_1$, can always be found.*

7 Singularities of $\Delta(\mathbf{x}, s)$ and $H(\mathbf{x}, s)$

Given a curve $\gamma : I \to \mathbb{R}^n$, we consider the full bifurcation set of the family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$. Given a fixed $\mathbf{x}_0 \in \mathbb{R}^n$, if there exists $s_0 \in I$ such that $\Delta'(\mathbf{x}_0, s_0) = \Delta''(\mathbf{x}_0, s_0) = 0$ then the family of affine distance functions is said to have a *degenerate singularity* at $\mathbf{x} = \mathbf{x}_0$. Given a fixed $\mathbf{x}_0 \in \mathbb{R}^n$, if there exists $(s_1, s_2) \in I \times I$ such that $\Delta(\mathbf{x}_0, s_1) = \Delta(\mathbf{x}_0, s_2)$ and $\Delta'(\mathbf{x}_0, s_1) = \Delta'(\mathbf{x}_0, s_2) = 0$ then the family of affine distance functions is said to have a *multi-local singularity* at $\mathbf{x} = \mathbf{x}_0$.

The full bifurcation set is then the closure of points $\mathbf{x} \in \mathbb{R}^n$ such that $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ has either a multi-local or degenerate singularity at $\mathbf{x}$. The bifurcation set is thus a subset of the parameter space. Similar ideas apply if we replace $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ by $H : S^{n-1} \times I \to \mathbb{R}$.

We use the standard A_k $(k \geq 2)$ notation for a degenerate singularity and A_1^2, A_1A_2 etc for a multi-local singularity.

Next we consider the condition for $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ to have an A_k singularity.

Theorem 7.1 *Let $\gamma : I \to \mathbb{R}^n$ be a smooth space curve parametrised by affine arc-length. For $0 \leq k \leq n-1$, the family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ has an A_k singularity at $\mathbf{x} \in \mathbb{R}^n$ if and only if, for $\lambda_i \in \mathbb{R}$*

$$\mathbf{x} = \gamma + \lambda_1\mathbf{T}_1 + \cdots + \lambda_{n-k-1}\mathbf{T}_{n-k-1} + \lambda_n\mathbf{T}_n \quad \textit{and} \quad \lambda_{n-k-1} \neq 0 \ .$$

The family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ has an A_n singularity at $\mathbf{x} \in \mathbb{R}^n$ if and only if given $\sigma_{n-1} \neq 0$; $\sigma'_{n-1} \neq 0$, and

$$\mathbf{x} = \gamma + \frac{1}{\sigma_{n-1}}\mathbf{T}_n \ .$$

The family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ has an A_{n+1} singularity at $\mathbf{x} \in \mathbb{R}^n$ if and only if given $\sigma_{n-1} \neq 0$; $\sigma'_{n-1} = 0$, $\sigma''_{n-1} \neq 0$, and

$$\mathbf{x} = \gamma + \frac{1}{\sigma_{n-1}}\mathbf{T}_n \ .$$

Proof Consider the equi-affine basis $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\} \subset T_\gamma\mathbb{R}^n$. We have

$$\Delta(\mathbf{x}, s) = [\mathbf{x} - \gamma, \gamma', \ldots, \gamma^{(n-1)}] = [\mathbf{x} - \gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}] \ .$$

Notice that $\mathbf{T}_1' = \mathbf{T}_2$, $\mathbf{T}_i' = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ for all $2 \le i \le n-1$, and $\mathbf{T}_n' = -\sigma_{n-1}\mathbf{T}_1$. It follows, using also $(\mathbf{x}-\gamma)' = -\mathbf{T}_1$ and $\mathbf{T}_1' = \mathbf{T}_2$, that

$$\begin{aligned}
\Delta' &= \sum_{i=2}^{n-1} [\mathbf{x}-\gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{i-1}, \mathbf{T}_i', \mathbf{T}_{i+1}, \ldots, \mathbf{T}_{n-1}] \ , \\
&= \sum_{i=2}^{n-1} [\mathbf{x}-\gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{i-1}, \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_{n-1}] \ , \\
&= [\mathbf{x}-\gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n] \ .
\end{aligned}$$

Moreover, for all $0 \le m \le n-1$, one can show that

$$\Delta^{(m)} = [\mathbf{x}-\gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{n-m-1}, \mathbf{T}_{n-m+1}, \ldots, \mathbf{T}_n] \ .$$

It follows that, for $\lambda_j \in \mathbb{R}$, $\Delta^{(m)} = 0$ if and only if

$$\mathbf{x}-\gamma = \lambda_1\mathbf{T}_1 + \cdots + \lambda_{n-m-1}\mathbf{T}_{n-m-1} + \lambda_{n-m+1}\mathbf{T}_{n-m+1} + \cdots + \lambda_n\mathbf{T}_n \ .$$

This means that $\mathbf{x} \in \mathbb{R}^n$, for $0 \le k \le n-1$, gives an $A_{\ge k}$ singularity if and only if, for some $\lambda_i \in \mathbb{R}$

$$\mathbf{x}-\gamma = \lambda_1\mathbf{T}_1 + \cdots + \lambda_{n-k-1}\mathbf{T}_{n-k-1} + \lambda_n\mathbf{T}_n \ .$$

The additional condition for exactly A_k is $\lambda_{n-k-1} \neq 0$.

Thus $\Delta' = \ldots = \Delta^{(n-1)} = 0$ if and only if $\mathbf{x}-\gamma = \lambda\mathbf{T}_n$ for some $\lambda \in \mathbb{R}$. This gives the condition for $A_{\ge n-1}$.

Let us now consider higher singularity types. Since $\Delta^{(n-1)} = [\mathbf{x}-\gamma, \mathbf{T}_2, \ldots, \mathbf{T}_n]$, it follows that

$$\begin{aligned}
\Delta^{(n)} &= -1 - \sum_{i=2}^{n} \sigma_{i-1}[\mathbf{x}-\gamma, \mathbf{T}_2, \ldots, \mathbf{T}_{i-1}, \mathbf{T}_1, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_n] \ , \\
&= -1 + \sum_{i=2}^{n} (-1)^{i+1}\sigma_{i-1}\Delta^{(n-i)} \ .
\end{aligned}$$

Hence $\Delta' = \ldots = \Delta^{(n)} = 0$ if and only if $\sigma_{n-1} \neq 0$ and $\mathbf{x}-\gamma = \sigma_{n-1}^{-1}\mathbf{T}_n$. This gives the condition for an $A_{\le n}$ singularity. Next we consider $\Delta^{(n+1)}$ and $\Delta^{(n+2)}$ in turn:

$$\begin{aligned}
\Delta^{(n+1)} &= \sum_{i=2}^{n} (-1)^{i+1}(\sigma_{i-1}'\Delta^{(n-i)} + \sigma_{i-1}\Delta^{(n-i+1)}) \ , \\
\Delta^{(n+2)} &= \sum_{i=2}^{n} (-1)^{i+1}(\sigma_{i-1}''\Delta^{(n-i)} + 2\sigma_{i-1}'\Delta^{(n-i+1)} + \sigma_{i-1}\Delta^{(n-i+2)}) \ .
\end{aligned}$$

Assume that $\sigma_{n-1} \neq 0$ and $\Delta' = \ldots = \Delta^{(n)} = 0$, it follows that $\Delta^{(n+1)} = 0$ if and only if $\sigma'_{n-1}\sigma_{n-1}^{-1} = 0$, i.e. if and only if $\sigma'_{n-1} = 0$.

In order to express $\Delta^{(n+2)}$ in terms of $\Delta^{(k)}$ for $0 \leq k \leq n-1$ it is necessary to consider the case $i = 2$ separately in the formula for $\Delta^{(n+2)}$. Denoting this by α gives

$$\begin{aligned} \alpha &= -(\sigma''_1\Delta^{(n-2)} + 2\sigma'_1\Delta^{(n-1)} + \sigma_1\Delta^{(n)}) \ , \\ &= -(\sigma''_1\Delta^{(n-2)} + 2\sigma'_1\Delta^{(n-1)} + \sigma_1 \left(-1 + \sum_{i=2}^{n} (-1)^{i+1}\sigma_{i-1}\Delta^{(n-i)} \right)) \ . \end{aligned}$$

Thus $\Delta^{(n+2)}$ can be written in terms of $\Delta^{(k)}$ for $0 \leq k \leq n-1$. Assume that $\sigma_{n-1} \neq 0$ and $\Delta' = \ldots = \Delta^{(n+1)} = 0$, it follows that $\Delta^{(n+2)} = 0$ if and only if $\sigma''_{n-1}\sigma_{n-1}^{-1} = 0$, i.e. if and only if $\sigma''_{n-1} = 0$.

Since the condition for type A_k is that $\Delta' = \ldots = \Delta^{(k)} = 0$ and $\Delta^{(k+1)} \neq 0$, the result follows. □

Theorem 7.2 *Let $\gamma : I \to \mathbb{R}^n$ be a smooth space curve parametrised by affine arc-length. Then for $0 \leq k \leq n-1$, the family of affine height functions $H : S^{n-1} \times I \to \mathbb{R}$ has an A_k singularity at $\mathbf{x} \in S^{n-1}$ if and only if, for some $\lambda_i \in \mathbb{R}$*

$$\mathbf{x} = \lambda_1\mathbf{T}_1 + \ldots + \lambda_{n-k-1}\mathbf{T}_{n-k-1} + \lambda_n\mathbf{T}_n \quad \textit{and} \quad \lambda_{n-k-1} \neq 0 \ .$$

The family of affine height functions has an A_n singularity if and only if there exists $\lambda \in \mathbb{R}$, $\lambda \neq 0$ such that

$$\mathbf{x} = \lambda\mathbf{T}_n, \quad \sigma_{n-1} = 0 \quad \textit{and} \quad \sigma'_{n-1} \neq 0 \ .$$

Moreover, the family of affine height functions has an A_{n+1} singularity if and only if there exists $\lambda \in \mathbb{R}$, $\lambda \neq 0$ such that

$$\mathbf{x} = \lambda\mathbf{T}_n, \quad \sigma_{n-1} = \sigma'_{n-1} = 0 \quad \textit{and} \quad \sigma''_{n-1} \neq 0 \ .$$

Proof This is proved similarly to Theorem 7.1. □

7.1 (p)-Versality condition

Here we consider the conditions for the two above families to be a (p)-*versal unfoldings*, i.e. to be versal when considered as potential functions. Due to the uniqueness of bifurcation sets, see [1], if a family of functions is a (p)-versal unfolding then each neighbourhood of its bifurcation set will be locally diffeomorphic to a standard model. Hence the local structure of the bifurcation set is determined up to diffeomorphism. Using the basic ideas of *unfoldings* found in [1] we have the following:

Criterion 7.3 *Let $F : (\mathbb{R}^n \times I, (\mathbf{x}_0, s_0)) \to \mathbb{R}$ be an n-parameter unfolding of $f : (I, s_0) \to \mathbb{R}$ which has type A_k as s_0, and consider*

$$\mathcal{S} = \left\{ j^{k-1} \left(\frac{\partial F}{\partial x_i}(\mathbf{x}_0, s) \right) \Big|_{s=s_0} \ : \ 1 \leq i \leq n \right\}$$

where j^{k-1} denotes the $(k-1)$-jet. Let $\mathbb{R}[s]$ denote the ring of polynomials in s and let $\mathfrak{m}$ denote the maximal ideal consisting of polynomials with zero constant term. Finally let $\langle s^k \rangle$ denote the ideal of polynomial multiples of s^k. Then F is (p)-versal if and only if the elements of $\mathcal{S}$ span the real vector space $\mathfrak{m}/\langle s^k \rangle$.

Criterion 7.3 is equivalent to the following:

Proposition 7.4 *Let $j^{k-1}(\partial F/\partial x_i(\mathbf{x}_0, s_0))(s_0) = \alpha_{1,i}s + \alpha_{2,i}s^2 + \cdots + \alpha_{k-1,i}s^{k-1}$ for $1 \leq i \leq n$. Then F is a (p)-versal unfolding of the singularity of type A_k if and only if the $(k-1) \times n$ matrix of coefficients $(\alpha_{j,i})$ has rank $k-1$.*

Theorem 7.5 *Given a smooth space curve $\gamma : I \to \mathbb{R}^n$ parametrised by affine arc-length. The family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ defined on the curve is a (p)-versal unfolding of the singularity type $A_{\leq n+1}$ if and only if $\sigma_{n-1} \neq 0$, where σ_{n-1} is given in Equation (6). Thus there is no extra condition, the family is implicitly (p)-versal.*

Proof Let $\gamma : I \to \mathbb{R}^n$ be smooth, and let $\gamma(0) = 0$. Consider the frame $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\}$ where $\mathbf{T}_1' = \mathbf{T}_2$, $\mathbf{T}_i' = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ for all $2 \leq i \leq n-1$, and $\mathbf{T}_n' = -\sigma_{n-1}\mathbf{T}_1$. The affine distance function may be rewritten in terms of the $\mathbf{T}_i$, thus

$$\Delta(\mathbf{x}, s) = [\mathbf{x} - \gamma, \gamma', \ldots, \gamma^{(n-1)}] = [\mathbf{x} - \gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}] \ .$$

Let $\Delta_{x_i} = \partial\Delta/\partial x_i$, and consider the vector $\Delta_{\mathbf{x}} = (\Delta_{x_1}, \ldots, \Delta_{x_n})$. Then by Proposition 7.4, to show the family $\Delta(\mathbf{x}, s)$ is (p)-versal, one needs to shows that the first n derivatives of $\mathbf{v}$, with respect to s, are linearly independent.

Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, etc, where $e_i \in T_\gamma\mathbb{R}^n$. Consider $\Delta_{\mathbf{x}}$, we have

$$\Delta_{\mathbf{x}} = ([e_1, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}], \ldots, [e_n, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}]) \ .$$

Notice that each $[e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}]$ is independent of $\mathbf{x}$. In what follows, it is enough to consider $\Delta_{x_i} = [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}]$ alone.

$$\begin{aligned}
\Delta'_{x_i} &= \sum_{j=1}^{n-1} [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{j-1}, \mathbf{T}'_j, \mathbf{T}_{j+1} \ldots, \mathbf{T}_{n-1}] \ , \\
&= \sum_{j=2}^{n-1} [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_{j+1} - \sigma_{j-1}\mathbf{T}_1, \mathbf{T}_{j+1} \ldots, \mathbf{T}_{n-1}] \ , \\
&= [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n] \ .
\end{aligned}$$

Next, consider Δ''_{x_i}, which is found in the same way. Given that $[e_i, \mathbf{T}'_1, \mathbf{T}_2, \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n] = [e_i, \mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{n-2}, \mathbf{T}'_n] = 0$, we have

$$\begin{aligned}
\Delta''_{x_i} &= \sum_{j=2}^{n-2} [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{j-1}, \mathbf{T}'_j, \mathbf{T}_{j+1} \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n] \ , \\
&= \sum_{j=2}^{n-2} [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_{j+1} - \sigma_{j-1}\mathbf{T}_1, \mathbf{T}_{j+1} \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n] \ , \\
&= [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{n-3}, \mathbf{T}_{n-1}, \mathbf{T}_n] \ .
\end{aligned}$$

Continuing in this fashion gives the general answer:

$$\Delta^{(m)}_{x_i} = [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{n-m-1}, \mathbf{T}_{n-m+1}, \ldots, \mathbf{T}_n]$$

for all $1 \leq m \leq n-1$. Thus we need only consider the final case $m = n$. Notice that $\Delta^{(n-1)}_{x_i} = [e_i, \mathbf{T}_2, \ldots, \mathbf{T}_n]$, and so it follows

$$\begin{aligned}
\Delta^{(n)}_{x_i} &= \sum_{j=2}^{n} [e_i, \mathbf{T}_2, \ldots, \mathbf{T}_{j-1}, \mathbf{T}'_j, \mathbf{T}_{j+1}, \ldots, \mathbf{T}_n] \ , \\
&= \sum_{j=2}^{n} [e_i, \mathbf{T}_2, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_{j+1} - \sigma_{j-1}\mathbf{T}_1, \mathbf{T}_{j+1}, \ldots, \mathbf{T}_n] \ , \\
&= -\sum_{j=2}^{n} \sigma_{j-1}[e_i, \mathbf{T}_2, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_1, \mathbf{T}_{j+1}, \ldots, \mathbf{T}_n] \ , \\
&= \sum_{j=2}^{n} (-1)^{j+1}\sigma_{j-1}[e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_{j+1}, \ldots, \mathbf{T}_n] \ , \\
&= \sum_{j=2}^{n} (-1)^{j+1}\sigma_{j-1}\Delta^{(n-j)}_{x_i} \ .
\end{aligned}$$

The aim here is to show that $[\Delta'_{\mathbf{x}}, \ldots, \Delta^{(n)}_{\mathbf{x}}] \neq 0$. Due to the fact that

$$\Delta^{(n)}_{x_i} = \sum_{j=2}^{n} (-1)^{j+1}\sigma_{j-1}\Delta^{(n-j)}_{x_i} \ ,$$

it follows that $\Delta_{\mathbf{x}}^{(n)}$ is a linear combination of $\{\Delta_{\mathbf{x}}, \Delta'_{\mathbf{x}}, \ldots, \Delta_{\mathbf{x}}^{(n-2)}\}$. It follows that $[\Delta'_{\mathbf{x}}, \ldots, \Delta_{\mathbf{x}}^{(n)}] = 0 \iff \sigma_{n-1}[\Delta'_{\mathbf{x}}, \ldots, \Delta_{\mathbf{x}}^{(n-1)}, \Delta_{\mathbf{x}}] = 0$.

The aim now is to show that $[\Delta'_{\mathbf{x}}, \ldots, \Delta_{\mathbf{x}}^{(n-1)}, \Delta_{\mathbf{x}}] \neq 0$. Consider the $n \times n$ matrix $X = (x_{i,j})$ where

$$x_{i,j} = [e_j, \mathbf{T}_1, \ldots, \mathbf{T}_{i-1}, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_n] \ .$$

It follows that $\det(X) = [\Delta_{\mathbf{x}}^{(n-1)}, \Delta_{\mathbf{x}}^{(n-2)}, \ldots, \Delta'_{\mathbf{x}}, \Delta_{\mathbf{x}}] = \pm[\Delta'_{\mathbf{x}}, \ldots, \Delta_{\mathbf{x}}^{(n-1)}, \Delta_{\mathbf{x}}]$. Let T be the matrix whose i-th column is $\mathbf{T}_i$. Furthermore, let $A = (a_{i,j})$ be the adjoint matrix of T. Since

$$a_{i,j} = (-1)^{i+1}[e_j, \mathbf{T}_1, \ldots, \mathbf{T}_{i-1}, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_n]$$

it follows that $a_{i,j} = (-1)^{i+1}x_{i,j}$, which implies $\det(X) = \pm\det(A)$. Next consider the well known identity $T^{-1} = \det(T)^{-1}A$, it follows that $\det(T)^{n-1} = \det(A)$. Thus $\det(X) = \pm\det(T)^{n-1} = \pm 1 \neq 0$. From this and the calculations for type A_k, the result now follows. □

Acknowledgements

The author would like to thank EPSRC for its PhD studentship which has made such research possible. The author would also like to thank his supervisor, and friend, Peter J. Giblin for introducing him to singularities and for his help and support over the last few years. Finally the author would like to thank Shyuichi Izumiya and Victor Flynn for some helpful conversations, the former during the winter singularities school at the CIRM in Marseille, France 2005.

References

[1] J. W. Bruce and P. J. Giblin. "*Curves and singularities.*" Cambridge University Press, Second Edition (1992).

[2] S. Izumiya and T. Sano. "*Generic affine differential geometry of plane curves.*" *Proc. Royal Soc. Edinburgh* 128A (1998) 301-314.

[3] S. Izumiya and T. Sano. "*Generic affine differential geometry of space curves.*" *Proc. Edinburgh Math. Soc.* 41 (1998) 315-324.