Generic affine differential geometry of curves in $\mathbb{R}^n$.

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Abstract

This paper considers curves in $\mathbb{R}^n$. It defines affine arc-length and affine curvatures. The family of affine distance functions is generalised, along with the family of affine height functions. A new basis is constructed that makes the conditions for $A_k$ singularity types easier to calculate, and applications are given to geometrical problems.

1 Introduction

In this paper, we consider the affine differential geometry of space curves, i.e. the geometric properties of smooth curves in $\mathbb{R}^n$, which remain invariant under the actions of $\text{SL}(n, \mathbb{R}) = \{ X \in \text{Mat}(n, \mathbb{R}) : \det(X) = 1 \}$ and the translation group. Such actions are called equi-affine transformations and preserve volume. In particular, we generalise some of the results of S. Izumiya and T. Sano, found in [2] and [3], from two and three dimensions to $n$ dimensions.

We generalise the classical ideas of affine curvatures, affine arc-length, families of affine distance functions, and families of affine height functions. We also calculate the conditions for $A_k$ singularities of these two families. Our methods are directly applicable to the study of families of functions and their bifurcation sets, and our proofs are more direct than those in [3]; they show the underlying, governing dynamic.

In §2 we introduce the affine arc-length parameter for a curve $\gamma : I \to \mathbb{R}^n$. Parametrising a curve by affine arc-length ensures the first $n$ derivatives of $\gamma$ with respect to affine arc-length always span a volume of $+1$. The condition for such a parametrisation to exist is also found in §2.
In §3 we consider the classical affine curvatures of $\gamma$. These arise naturally from the affine arc-length parametrisation. A general formula in terms of $n$ and $i$ is given for calculating the $i$-th affine curvature of a curve in $\mathbb{R}^n$. Finally a system of affine Serret-Frenet differential equations is given for the classical curvatures.

The family of affine distance functions on a curve parametrised by affine arc-length is defined in §4; we also give a formula for an arbitrarily parametrised curve. Moreover, §5 follows this path exactly, but with the family of affine height functions.

In §6 a new equi-affine frame is found for the curve. This frame has the property that its ordered members span a volume of $+1$; it is defined in terms of the derivatives of $\gamma$ and the derivatives of the affine curvatures and is denoted as $\{T_1, \ldots, T_n\}$. Moreover, a new system of affine Serret-Frenet formulae arise with this frame, giving new affine curvatures (which we call affine torsions and write as $\sigma_i$). Again formulae in terms of $n$ and $i$ are given for the $i$-th affine torsion of a curve in $\mathbb{R}^n$.

In §7 we use the new frame $\{T_1, \ldots, T_n\}$ to rewrite the family of affine distance functions and height functions. Conditions for these two families to have $A_k$ singularities are given in terms of the $T_i$ and $\sigma_j$.

Finally, in §§7.1 it is shown that the families are always $(p)$-versally unfolded for a generic space curve and geometrical applications are given.

Since determinants measure volume, and volume remains unchanged by equi-affine transformations, the determinant is an affine invariant and will play a central role in this study. Let $\{v_1, \ldots, v_n\}$ be an ordered set (i.e. a list) of $n$ vectors in $\mathbb{R}^n$. Then let $[v_1, \ldots, v_n]$ denote the determinant of the matrix whose $i$-th column is the vector $v_i$. Then $[v_1, \ldots, v_n]$ is equal to the volume spanned by the vectors in $\{v_1, \ldots, v_n\}$.

## 2 Affine arc-length

Let $I \subseteq \mathbb{R}$ be an open interval, and $\gamma : I \to \mathbb{R}^n$ a smooth space curve. We seek an affine invariant parametrisation for $\gamma$ of the lowest possible order. As is the convention for $n = 2, 3$ we choose a parametrisation, in terms of the affine arc-length parameter $s$, such that $[\gamma', \gamma'', \ldots, \gamma^{(n)}] = 1$ for all $s \in I$. Throughout this paper, prime denotes differentiation with respect to the affine arc-length parameter $s$, thus $\dot{\gamma} = d\gamma/ds$ etc, whereas a dot is reserved for differentiation with respect to an arbitrary parameter $t$, thus $\ddot{\gamma} = d\dot{\gamma}/dt$ etc. Using basic properties of determinants, it is easy to show that

$$[\gamma', \gamma'', \ldots, \gamma^{(n)}] = [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}] \left(\frac{dt}{ds}\right)^{n(n+1)/2},$$

(1)
Assuming that \( [\gamma', \gamma'', \ldots, \gamma^{(n)}] = 1 \) we obtain
\[
s(t) = \int [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^2/n(n+1) \ dt .
\]
Thus for \( t_1 \leq t \leq t_2 \), affine arc-length is given by
\[
\int_{t_1}^{t_2} [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^2/n(n+1) \ dt .
\]

**Remark 1** Let \( J \subseteq \mathbb{R} \) and consider a curve \( \alpha : J \to \mathbb{R}^n \) parametrised by euclidean arc-length. We define the tangent vector \( V_1 \) to be the unit vector in the direction of \( \dot{\alpha} \). The second basis vector \( V_2 \) is in the subspace \( \langle \dot{\alpha}, \ddot{\alpha} \rangle \), is of unit length, is perpendicular to \( V_1 \), and together with \( V_1 \) spans an area of +1. Proceeding in this fashion, the \((k+1)\)-st basis vector is in the space \( \langle \frac{d\alpha}{dt}, \frac{d^2\alpha}{dt^2}, \ldots, \frac{d^k\alpha}{dt^k} \rangle \), is of unit length, is perpendicular to \( \{V_i : 1 \leq i \leq k\} \), and together with \( \{V_i : 1 \leq i \leq k\} \) spans a volume of +1.

**Definition 2.1** Given a smooth curve parameterised by euclidean arc-length, the euclidean curvature is given by \( \kappa = \dot{V}_1 \cdot V_2 \) and the higher euclidean torsions are given by \( \tau_i = \dot{V}_{i+1} \cdot V_{i+2} \) for all \( 1 \leq i \leq n-2 \).

**Remark 2** Letting \( t \) be euclidean arc-length and writing \( \kappa \) for the euclidean curvature of \( \gamma \) and \( \{\tau_1, \ldots, \tau_{n-2}\} \) for the higher euclidean torsions gives
\[
[\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}] = \kappa^{n-1} \prod_{i=1}^{n-2} \tau_i^{n-i-1} .
\]

Then Equation (1) shows that if \( [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}] = 0 \) for some \( t \), then the affine arc-length parametrisation in unobtainable, since \( 0 \neq 1 \). Hence, if any of the euclidean curvatures or euclidean torsions become zero at certain points, the affine arc-length parameter can not be defined at such points. Hence, in all that follows, \( I \subseteq \mathbb{R} \) shall be chosen such that the image of \( \gamma \) has everywhere non-zero euclidean curvature and euclidean torsions.

### 3 Affine curvatures

Here we define the *affine curvatures* of a curve. Let \( \gamma : I \to \mathbb{R}^n \) be parameterised by affine arc-length, so that \( [\gamma', \gamma'', \ldots, \gamma^{(n)}] = 1 \) for all \( s \in I \). Then differentiating with respect to \( s \) gives \( [\gamma', \ldots, \gamma^{(n-1)}, \gamma^{(n+1)}] = 0 \). Hence the set of vectors \( \{\gamma', \ldots, \gamma^{(n-1)}, \gamma^{(n+1)}\} \) is linearly dependent. Therefore, there must exist functions \( \mu_i : I \to \mathbb{R} \) for \( 1 \leq i \leq n-1 \) such that
\[
\gamma^{(n+1)} + \mu_1 \gamma' + \mu_2 \gamma'' + \cdots + \mu_{n-1} \gamma^{(n-1)} = 0 .
\]
The functions $\mu_i$ are called the affine curvatures of $\gamma$. Notice that

$$\mu_i = (-1)^{n-i+1}[\gamma', \ldots, \gamma^{(i-1)}, \gamma^{(i+1)}, \ldots, \gamma^{(n+1)}].$$

The $\mu_i$ are given by determinants; an equi-affine transformation of $\mathbb{R}^n$ leaves the affine curvatures unchanged. These affine curvatures are truly affine invariants.

These definitions give Serret-Frenet type formulae. Let $\Gamma = (\gamma', \gamma'', \ldots, \gamma^{(n)})^\top$ where $\top$ denotes transpose; then for $M \in \text{Mat}(n, \mathbb{R})$

$$\Gamma' = M\Gamma. \quad (3)$$

It follows that if $M = (m_{i,j})$ then

$$m_{i,j} = \begin{cases} 1 & \text{if } j - i = 1, \\ -\mu_j & \text{if } i = n, \\ 0 & \text{otherwise}. \end{cases} \quad (4)$$

Hence $\det(M) = (-1)^n \mu_1$.

Example. Let $n = 3$, so that $\gamma : I \rightarrow \mathbb{R}^3$, then

$$\frac{d}{ds} \begin{pmatrix} \gamma' \\ \gamma'' \\ \gamma''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu_1 & -\mu_2 & 0 \end{pmatrix} \begin{pmatrix} \gamma' \\ \gamma'' \\ \gamma''' \end{pmatrix}. \quad (3)$$

4 Affine distance functions

Here we give a general definition of the affine distance function introduced in two and three-dimensions in [3].

Let $\gamma : I \rightarrow \mathbb{R}^n$ be parametrised by affine arc-length. Given $x \in \mathbb{R}^n$ and $s \in I$, we get $\Delta : \mathbb{R}^n \times I \rightarrow \mathbb{R}$, an $n$-parameter family of affine distance functions defined on the curve, where

$$\Delta(x, s) = [x - \gamma, \gamma', \ldots, \gamma^{(n-1)}]. \quad (5)$$

The zero level-set of $\Delta(x, s_0)$ is given by $x \in \mathbb{R}^n$ such that for some $\lambda_i \in \mathbb{R}$

$$x = \gamma(s_0) + \lambda_1 \gamma'(s_0) + \lambda_2 \gamma''(s_0) + \cdots + \lambda_{n-1} \gamma^{(n-1)}(s_0).$$

This is the set of points $x \in \mathbb{R}^n$ of affine distance zero from $\gamma(s_0)$. It is easy to see that the other level-sets are hyperplanes parallel to this one.

Given an open interval $J \subseteq \mathbb{R}$, and an arbitrary parametrisation for the curve $\gamma : J \rightarrow \mathbb{R}^n$. The family of affine distance functions $\Delta : \mathbb{R}^n \times J \rightarrow \mathbb{R}$ is given by

$$\Delta(x, t) = [x - \gamma, \gamma', \ldots, \gamma^{(n-1)}] [\gamma, \gamma', \ldots, \gamma^{(n)}]^{(1-n)/(1+n)}.$$
5 Affine height functions

Let \( \gamma : I \to \mathbb{R}^n \) be parametrised by affine arc-length. Let \( S^{n-1} = \{ x \in \mathbb{R}^n : ||x|| = 1 \} \) be the unit hypersphere in \( \mathbb{R}^n \). We can define a family of functions on the curve, parametrised by \( S^{n-1} \). This family \( H : S^{n-1} \times I \to \mathbb{R} \) is the family of affine height functions, where

\[
H(x, s) = [x, \gamma', \gamma'', \ldots, \gamma^{(n-1)}] .
\]

Let \( J \subseteq \mathbb{R} \) be an open interval, then for an arbitrary parametrisation, the affine height functions are given by \( H : S^{n-1} \times I \to \mathbb{R} \) where

\[
H(x, t) = [x, \dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}] [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^{(1-n)/(1+n)} .
\]

6 Equi-affine frames

Let \( x \in \mathbb{R}^n \) and let \( \{v_1, \ldots, v_n\} \) be a list of vectors \( v_i \in T_x \mathbb{R}^n \). The vectors are said to constitute an equi-affine frame if and only if \([v_1, \ldots, v_n] = 1\). It is clear that \( \{\gamma', \ldots, \gamma^{(n)}\} \) forms an equi-affine frame with each \( \gamma^{(i)} \in T_{\gamma(s)} \mathbb{R}^n \) for all \( s \in I \).

The aim here is to define a new equi-affine frame for \( \gamma \). This is motivated by later applications to singularity theory. Furthermore, the affine Serret-Frenet formulae with respect to this new equi-affine frame will be more analogous to the euclidean Serret-Frenet formulae. For example, if the euclidean torsion \( \tau_{n-2} \) is zero then the curve can be contained in \( \mathbb{R}^{n-1} \). This means the last basis vector, say \( V_n \), is constant. (If \( n = 3 \) then the binormal vector \( B \) is constant and \( \gamma \) is then a plane curve.) Given the affine Serret-Frenet formulae in Equation (3) and Equation (4), if \( \mu_{n-1} = 0 \), this in no way means that \( \gamma^{(n-1)} \) is constant.

Given any smooth functions \( \lambda_{i,j} : I \to \mathbb{R} \), the vectors

\[
\gamma^{(i)} + \sum_{j=1}^{i-1} \lambda_{i,j} \gamma^{(j)} \text{ for all } 1 \leq i \leq n
\]

form an equi-affine frame. The classical case is when \( \lambda_{i,j}(s) = 0 \) for all \( s \in I \) and \( (i,j) \in \mathbb{N} \times \mathbb{N} \). Consider the vector given by \( i = n \), that is

\[
v = \gamma^{(n)} + \lambda_{n,1} \gamma' + \lambda_{n,2} \gamma'' + \cdots + \lambda_{n,n-1} \gamma^{(n-1)} .
\]

We wish the derivative of \( v \) to depend on only one other member of the equi-
Then finally, the vector $v'$ becomes

$$v' = \sum_{i=1}^{n-1} (\lambda'_{n,i} - \mu_i) \gamma^{(i)} + \lambda_{n,i} \gamma^{(i+1)} ,$$

$$= (\lambda'_{n,1} - \mu_1) \gamma' + \lambda_{n,n-1} \gamma^{(n)} + \sum_{i=2}^{n-1} (\lambda'_{n,i} - \mu_i + \lambda_{n,i-1}) \gamma^{(i)} .$$

If $v'$ is to be independent of $v$ it follows that $\lambda_{n,n-1} \equiv 0$. In order to remove dependency on other derivatives set $\lambda_{n,i-1} = \mu_i - \lambda'_{n,i}$ for all $2 \leq i \leq n - 1$. Starting with $i = n - 1$ gives $\lambda_{n,n-2} = \mu_{n-1} - \lambda'_{n,n-1} = \mu_{n-1}$. In turn, putting $i = n - 2$ gives $\lambda_{n,n-3} = \mu_{n-2} - \mu'_{n-1}$. Putting $i = n - 3$ gives $\lambda_{n,n-4} = \mu_{n-3} - \mu'_{n-2} + \mu''_{n-1}$. Continuing this process for $2 \leq i \leq n - 1$ gives

$$\lambda_{n,n-i} = \sum_{j=1}^{i-1} (-1)^{j+1} \mu'_{n-i+j} .$$

Then finally, the vector $v'$ becomes

$$v' = \left( \sum_{i=1}^{n-1} (-1)^i \mu_i^{(i-1)} \right) \gamma' = \sigma_{n-1} \gamma' , \text{ say} . \tag{6}$$

Thus the derivative of $v$ depends only on one vector and is more analogous to the euclidean Serret-Frenet system.

This has found a new basis vector, namely $v$. Let us call it $T_n$ and search for a new basis $\{ T_1, \ldots, T_n \}$. It is clear that $T_1 = \gamma'$; this gives the affine tangent vector. Thus we have the identity $T_n = -\sigma_{n-1} T_1$.

We wish to find a new equi-affine frame which satisfies the additional vector differential equations $T_1' = T_2, T_i' = T_{i+1} - \sigma_{i-1} T_1$ for all $2 \leq i \leq n - 1$. These can be written as $T_i' = T_{i+1} - \sigma_{i-1} T_1$ if we set $\sigma_0 \equiv 0$ and $T_{n+1} \equiv 0$. From the affine arc-length construction, the functions $\mu_i : I \to \mathbb{R}$ arise naturally. Thus the $\sigma_i : I \to \mathbb{R}$ will be expressed in terms of the $\mu_i$ and their derivatives.

Consider the affine Serret-Frenet formulae in matrix form $\Gamma' = M \Gamma$, where $\Gamma$ and $M$ are defined above in Equation (3) and Equation (4). Each new basis vector $T_i$ can be expressed in terms of $\Gamma$:

$$T_i = \gamma^{(i)} + \sum_{j=1}^{i-1} \lambda_{i,j} \gamma^{(j)} \text{ for all } 1 \leq i \leq n$$

This can be written in matrix notation as $T = \Lambda \Gamma$ where $T$ is the matrix whose $i$-th row is the vector $T_i$. Furthermore we can write $T' = \Sigma T$ where $\Sigma$ is derived from the identities $T_1' = T_2, T_i' = T_{i+1} - \sigma_{i-1} T_1$ for all $2 \leq i \leq n - 1$, and $T_n' = -\sigma_{n-1} T_1$. 

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Thus we have $\Gamma' = \Lambda T$, $T = \Lambda T$, and $T' = \Sigma T$. It follows that $\Lambda T + \Lambda T' = \Sigma T$. In turn, this gives $\Lambda T + \Lambda M = \Sigma T$. This finally yields $\Lambda T + \Lambda M = \Sigma \Lambda$, or simply $\Lambda' + \Lambda M = \Sigma \Lambda$. Here $M$ is known to us, and is given by the identity

$$\gamma^{(n+1)} + \mu_1 \gamma' + \cdots + \mu_{n-1} \gamma^{(n-1)} = 0.$$ 

Writing $\Sigma = (\sigma_{i,j})$ gives $\sigma_{i,j} = 1$ for all $j - i = 1$, $\sigma_{i,1} = -\sigma_{i-1}$ for all $2 \leq i \leq n$, and $\sigma_{i,j} = 0$ otherwise. Writing $\Lambda = (\lambda_{i,j})$ gives $\lambda_{i,j} = 1$ for all $i - j = 0$ and $\lambda_{i,j} = 0$ for all $j - i > 0$, i.e. $\Lambda$ is a lower triangular matrix with 1 in each position along the leading diagonal.

Let $X = (x_{i,j})$ where $X = \Lambda' + \Lambda M - \Sigma \Lambda$; we wish to make $X$ into the zero matrix. On the leading diagonal of $X$ we have $x_{i,i} = \lambda_{i,i-1} - \lambda_{i+1,i}$. Since $\lambda_{1,0} = 0$ it follows that $\lambda_{i,i-1} = 0$ for all $2 \leq i \leq n$. This implies that $\Lambda$ has zero along the diagonal $i - j = 1$. Thus each $T_i$ will not have a component of $\gamma^{(i-1)}$.

Consider $x_{i,j}$ such that $i - j = 1$. It follows that $x_{n,n-1} = \lambda_{n,n-2} + \mu_{n-1} - \lambda_{n-1,n-3} = \lambda_{n-1,n-3} = \cdots = \lambda_{i,i-2} = \lambda_{i,i-2} = \lambda_{n-1,n-2} = \lambda_{n-1,n-3} = \cdots = \lambda_{3,1} = \sigma_1$.

Considering each diagonal in turn, $i - j = 1, 2, 3, \ldots, n - 1$ gives the following expressions for the $\sigma_i$, we have

$$\sigma_1 = a_{1,1} \mu_{n-1},$$
$$\sigma_2 = a_{2,1} \mu'_{n-1} + a_{2,2} \mu_{n-2},$$
$$\sigma_3 = a_{3,1} \mu''_{n-1} + a_{3,2} \mu'_{n-2} + a_{3,3} \mu_{n-3},$$
$$\sigma_i = \sum_{j=1}^{i} a_{i,j} \mu^{(i-j)}_{n-j},$$

where the $a_{i,j}$ are entries in an $(n - 1) \times (n - 1)$ lower triangular matrix, we have $a_{i,j} = 1$ for all $i = j$ and $a_{i,j} = 0$ for all $i < j$. When $i > j$ we have

$$a_{i,j} = (-1)^{i+j} \binom{n-j-1}{i-j} \frac{(n-j-1)!}{(i-j)!(n-i-1)!}.$$ 

It follows that the $\sigma_i$ are then given by

$$\sigma_i = \sum_{j=1}^{i} (-1)^{i+j} \binom{n-j-1}{i-j} \mu^{(i-j)}_{n-j}.$$ 

Given the existence of $\Sigma$ and $M$ is known, it is easy to find $\Lambda$ for all $i - j \geq 1$

$$\lambda_{i,j} = \sum_{k=1}^{i-j-1} (-1)^{i-j-k-1} \binom{n-j-k-1}{i-j-k-1} \mu^{(i-j-k-1)}_{n-k}.$$ 

In the present section we have proved the following.
Given a curve $\gamma: I \to \mathbb{R}^n$ parametrised by affine arc-length. An equi-affine basis $\{T_1, \ldots, T_n\}$ satisfying the vector differential equations $T_1 = T_2$, $T_i = T_{i+1} - \sigma_i^{-1} T_1$ for all $2 \leq i \leq n-2$, and $T_n = -\sigma_{n-1} T_1$, can always be found.

7 Singularities of $\Delta(x, s)$ and $H(x, s)$

Given a curve $\gamma: I \to \mathbb{R}^n$, we consider the full bifurcation set of the family of affine distance functions $\Delta: \mathbb{R}^n \times I \to \mathbb{R}$. Given a fixed $x_0 \in \mathbb{R}^n$, if there exists $s_0 \in I$ such that $\Delta'(x_0, s_0) = \Delta''(x_0, s_0) = 0$ then the family of affine distance functions is said to have a degenerate singularity at $x = x_0$. Given a fixed $x_0 \in \mathbb{R}^n$, if there exists $(s_1, s_2) \in I \times I$ such that $\Delta(x_0, s_1) = \Delta(x_0, s_2)$ and $\Delta'(x_0, s_1) = \Delta'(x_0, s_2) = 0$ then the family of affine distance functions is said to have a multi-local singularity at $x = x_0$.

The full bifurcation set is then the closure of points $x \in \mathbb{R}^n$ such that $\Delta: \mathbb{R}^n \times I \to \mathbb{R}$ has either a multi-local or degenerate singularity at $x$. The bifurcation set is thus a subset of the parameter space. Similar ideas apply if we replace $\Delta: \mathbb{R}^n \times I \to \mathbb{R}$ by $H: S^{n-1} \times I \to \mathbb{R}$.

We use the standard $A_k$ $(k \geq 2)$ notation for a degenerate singularity and $A^2_k, A_1A_2$ etc for a multi-local singularity.

Next we consider the condition for $\Delta: \mathbb{R}^n \times I \to \mathbb{R}$ to have an $A_k$ singularity.

Theorem 7.1 Let $\gamma: I \to \mathbb{R}^n$ be a smooth space curve parametrised by affine arc-length. For $0 \leq k \leq n-1$, the family of affine distance functions $\Delta: \mathbb{R}^n \times I \to \mathbb{R}$ has an $A_k$ singularity at $x \in \mathbb{R}^n$ if and only if, for $\lambda_1 \in \mathbb{R}$

$$x = \gamma + \lambda_1 T_1 + \cdots + \lambda_{n-k-1} T_{n-k-1} + \lambda_n T_n \quad \text{and} \quad \lambda_{n-k-1} \neq 0.$$  

The family of affine distance functions $\Delta: \mathbb{R}^n \times I \to \mathbb{R}$ has an $A_n$ singularity at $x \in \mathbb{R}^n$ if and only if given $\sigma_n \neq 0; \sigma'_n \neq 0$, and

$$x = \gamma + \frac{1}{\sigma_{n-1}} T_n.$$  

The family of affine distance functions $\Delta: \mathbb{R}^n \times I \to \mathbb{R}$ has an $A_{n+1}$ singularity at $x \in \mathbb{R}^n$ if and only if given $\sigma_n \neq 0; \sigma'_n = 0, \sigma''_n \neq 0$, and

$$x = \gamma + \frac{1}{\sigma_{n-1}} T_n.$$  

Proof Consider the equi-affine basis $\{T_1, \ldots, T_n\} \subset T_x \mathbb{R}^n$. We have

$$\Delta(x, s) = [x - \gamma, \gamma', \ldots, \gamma^{(n-1)}] = [x - \gamma, T_1, \ldots, T_{n-1}].$$

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Notice that $T'_1 = T_2$, $T'_i = T_{i+1} - \sigma_i T_1$ for all $2 \leq i \leq n - 1$, and $T'_n = -\sigma_{n-1} T_1$. It follows, using also $(x - \gamma)' = -T_1$ and $T'_1 = T_2$, that

\[
\Delta' = \sum_{i=2}^{n-1} [x - \gamma, T_1, \ldots, T_{i-1}, T'_i, T_{i+1}, \ldots, T_{n-1}],
\]

\[
= \sum_{i=2}^{n-1} [x - \gamma, T_1, \ldots, T_{i-1}, T_{i+1} - \sigma_i T_1, T_{i+2}, \ldots, T_{n-1}],
\]

\[
= [x - \gamma, T_1, \ldots, T_{n-2}, T_n].
\]

Moreover, for all $0 \leq m \leq n - 1$, one can show that

\[
\Delta^{(m)} = [x - \gamma, T_1, \ldots, T_{n-m-1}, T_{n-m+1}, \ldots, T_n].
\]

It follows that, for $\lambda_j \in \mathbb{R}$, $\Delta^{(m)} = 0$ if and only if

\[
x - \gamma = \lambda_1 T_1 + \cdots + \lambda_{n-m-1} T_{n-m-1} + \lambda_{n-m+1} T_{n-m+1} + \cdots + \lambda_n T_n.
\]

This means that $x \in \mathbb{R}^n$, for $0 \leq k \leq n-1$, gives an $A_{\geq k}$ singularity if and only if, for some $\lambda_k \in \mathbb{R}$

\[
x - \gamma = \lambda_1 T_1 + \cdots + \lambda_{n-k} T_{n-k} + \lambda_n T_n.
\]

The additional condition for exactly $A_k$ is $\lambda_{n-k} \neq 0$.

Thus $\Delta' = \ldots = \Delta^{(n-1)} = 0$ if and only if $x - \gamma = \lambda T_n$ for some $\lambda \in \mathbb{R}$.

This gives the condition for $A_{\geq n-1}$.

Let us now consider higher singularity types. Since $\Delta^{(n-1)} = [x-\gamma, T_2, \ldots, T_n]$, it follows that

\[
\Delta^{(n)} = -1 - \sum_{i=2}^{n} \sigma_{i-1} [x - \gamma, T_2, \ldots, T_{i-1}, T_i, T_{i+1}, \ldots, T_n],
\]

\[
= -1 + \sum_{i=2}^{n} (-1)^{i+1} \sigma_{i-1} \Delta^{(n-i)}.
\]

Hence $\Delta' = \ldots = \Delta^{(n)} = 0$ if and only if $\sigma_{n-1} \neq 0$ and $x - \gamma = \sigma_{n-1}^{-1} T_n$. This gives the condition for an $A_{\leq n}$ singularity. Next we consider $\Delta^{(n+1)}$ and $\Delta^{(n+2)}$ in turn:

\[
\Delta^{(n+1)} = \sum_{i=2}^{n} (-1)^{i+1} (\sigma_{i-1} \Delta^{(n-i)} + \sigma_i \Delta^{(n-i+1)}),
\]

\[
\Delta^{(n+2)} = \sum_{i=2}^{n} (-1)^{i+1} (\sigma_{i-1} \Delta^{(n-i)} + 2\sigma_{i-1} \Delta^{(n-i+1)} + \sigma_i \Delta^{(n-i+2)}).
\]
Assume that $\sigma_{n-1} \neq 0$ and $\Delta' = \ldots = \Delta^{(n)} = 0$, it follows that $\Delta^{(n+1)} = 0$ if and only if $\sigma'_{n-1}\sigma_{n-1} = 0$, i.e. if and only if $\sigma'_{n-1} = 0$.

In order to express $\Delta^{(n+2)}$ in terms of $\Delta^{(k)}$ for $0 \leq k \leq n-1$ it is necessary to consider the case $i = 2$ separately in the formula for $\Delta^{(n+2)}$. Denoting this by $\alpha$ gives

$$\alpha = -(\sigma''_1\Delta^{(n-2)} + 2\sigma'_1\Delta^{(n-1)} + \sigma_1\Delta^{(n)}) + (-1 + \sum_{i=2}^{n} (-1)^{i+1}\sigma_{i-1}\Delta^{(n-i)})$$.

Thus $\Delta^{(n+2)}$ can be written in terms of $\Delta^{(k)}$ for $0 \leq k \leq n-1$. Assume that $\sigma_{n-1} \neq 0$ and $\Delta' = \ldots = \Delta^{(n+1)} = 0$, it follows that $\Delta^{(n+2)} = 0$ if and only if $\sigma''_{n-1}\sigma_{n-1} = 0$, i.e. if and only if $\sigma''_{n-1} = 0$.

Since the condition for type $A_k$ is that $\Delta' = \ldots = \Delta^{(k)} = 0$ and $\Delta^{(k+1)} \neq 0$, the result follows.

**Theorem 7.2** Let $\gamma : I \rightarrow \mathbb{R}^n$ be a smooth space curve parametrised by affine arc-length. Then for $0 \leq k \leq n-1$, the family of affine height functions $H : S^{n-1} \times I \rightarrow \mathbb{R}$ has an $A_k$ singularity at $x \in S^{n-1}$ if and only if, for some $\lambda_i \in \mathbb{R}

$$x = \lambda_1 T_1 + \ldots + \lambda_{n-k-1} T_{n-k-1} + \lambda_n T_n \quad \text{and} \quad \lambda_{n-k-1} \neq 0.$$ The family of affine height functions has an $A_n$ singularity if and only if there exists $\lambda \in \mathbb{R}, \lambda \neq 0$ such that

$$x = \lambda T_n, \quad \sigma_{n-1} = 0 \quad \text{and} \quad \sigma'_{n-1} \neq 0.$$ Moreover, the family of affine height functions has an $A_{n+1}$ singularity if and only if there exists $\lambda \in \mathbb{R}, \lambda \neq 0$ such that

$$x = \lambda T_n, \quad \sigma_{n-1} = \sigma'_{n-1} = 0 \quad \text{and} \quad \sigma''_{n-1} \neq 0.$$ 

**Proof** This is proved similarly to Theorem 7.1.

### 7.1 (p)-Versality condition

Here we consider the conditions for the two above families to be a $(p)$-versal unfoldings, i.e. to be versal when considered as potential functions. Due to the uniqueness of bifurcation sets, see [1], if a family of functions is a $(p)$-versal unfolding then each neighbourhood of its bifurcation set will be locally diffeomorphic to a standard model. Hence the local structure of the bifurcation set is determined up to diffeomorphism. Using the basic ideas of unfoldings found in [1] we have the following:
Let $\Delta (x)$.

**Proposition 7.4** To show the family $\Delta (x)$.

**Theorem 7.5** Given a smooth space curve $\gamma : I \to \mathbb{R}^n$ parametrised by affine arc-length. The family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ defined on the curve is a $(p)$-versal unfolding of the singularity of type $A_k$ if and only if the elements of $S$ span the real vector space $m/\langle s^k \rangle$.

Criterion 7.3 is equivalent to the following:

**Proposition 7.4** Let $j^{k-1}(\partial F/\partial x_i(x_0, s_0))(s_0) = \alpha_1, s + \alpha_2, s^2 + \cdots + \alpha_{k-1}, s^{k-1}$ for $1 \leq i \leq n$. Then $F$ is a $(p)$-versal unfolding of the singularity of type $A_k$ if and only if the $(k-1) \times n$ matrix of coefficients $(\alpha_{j,i})$ has rank $k-1$.

**Theorem 7.5** Given a smooth space curve $\gamma : I \to \mathbb{R}^n$ parametrised by affine arc-length. The family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ defined on the curve is a $(p)$-versal unfolding of the singularity type $A_{n+1}$ if and only if $\alpha_{n-1} \neq 0$, where $\alpha_{n-1}$ is given in Equation (6). Thus there is no extra condition, the family is implicitly $(p)$-versal.

**Proof** Let $\gamma : I \to \mathbb{R}^n$ be smooth, and let $\gamma(0) = 0$. Consider the frame $\{T'_1, \ldots, T'_n\}$ where $T'_1 = T_2, T' = T_{i+1} - \sigma_{i-1} T_1$ for all $2 \leq i \leq n-1$, and $T'_n = -\sigma_{n-1} T_1$. The affine distance function may be rewritten in terms of the $T'_i$, thus

$$\Delta(x, s) = [x - \gamma, \gamma', \ldots, \gamma^{(n-1)}] = [x - \gamma, T_1, \ldots, T_{n-1}] .$$

Let $\Delta_{x_i} = \partial \Delta/\partial x_i$, and consider the vector $\Delta_{x} = (\Delta_{x_1}, \ldots, \Delta_{x_n})$. Then by Proposition 7.4, to show the family $\Delta(x, s)$ is $(p)$-versal, one needs to shows that the first $n$ derivatives of $\gamma$, with respect to $s$, are linearly independent.

Let $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0)$, etc, where $e_i \in T_\gamma \mathbb{R}^n$. Consider $\Delta_x$, we have

$$\Delta_x = ([e_1, T_1, \ldots, T_{n-1}], [e_2, T_1, \ldots, T_{n-1}]) .$$

Notice that each $[e_i, T_1, \ldots, T_{n-1}]$ is independent of $x$. In what follows, it is enough to consider $\Delta_{x_i} = [e_i, T_1, \ldots, T_{n-1}]$ alone.
The aim here is to show that \( \Delta'_{x_1} \) continuing in this fashion gives the general answer:

\[
\Delta'_{x_1} = \sum_{j=1}^{n-1} [e_i, T_1, \ldots, T_{j-1}, T'_j, T_{j+1} \ldots, T_{n-1}] ,
\]

\[
= \sum_{j=2}^{n-1} [e_i, T_1, \ldots, T_{j-1}, T_{j+1} - \sigma_{j-1} T_1, T_{j+1} \ldots, T_{n-1}] ,
\]

\[
= [e_i, T_1, \ldots, T_{n-2}, T_n] .
\]

Next, consider \( \Delta''_{x_1} \), which is found in the same way. Given that \([e_i, T'_1, T_2, \ldots, T_{n-2}, T_n] = [e_i, T_1, T_2, \ldots, T_{n-2}, T_n] = 0\), we have

\[
\Delta''_{x_1} = \sum_{j=2}^{n-2} [e_i, T_1, \ldots, T_{j-1}, T'_j, T_{j+1} \ldots, T_{n-2}, T_n] ,
\]

\[
= \sum_{j=2}^{n-2} [e_i, T_1, \ldots, T_{j-1}, T_{j+1} - \sigma_{j-1} T_1, T_{j+1} \ldots, T_{n-2}, T_n] ,
\]

\[
= [e_i, T_1, \ldots, T_{n-3}, T_{n-1}, T_n] .
\]

Continuing in this fashion gives the general answer:

\[
\Delta^{(m)}_{x_i} = [e_i, T_1, \ldots, T_{n-m-1}, T_{n-m+1}, \ldots, T_n]
\]

for all \( 1 \leq m \leq n - 1 \). Thus we need only consider the final case \( m = n \). Notice that \( \Delta^{(n-1)}_{x_i} = [e_i, T_2, \ldots, T_n] \), and so it follows

\[
\Delta^{(n)}_{x_i} = \sum_{j=2}^{n} [e_i, T_2, \ldots, T_{j-1}, T'_j, T_{j+1} \ldots, T_n] ,
\]

\[
= \sum_{j=2}^{n} [e_i, T_2, \ldots, T_{j-1}, T_{j+1} - \sigma_{j-1} T_1, T_{j+1} \ldots, T_n] ,
\]

\[
= -\sum_{j=2}^{n} \sigma_{j-1} [e_i, T_2, \ldots, T_{j-1}, T_1, T_{j+1}, \ldots, T_n] ,
\]

\[
= \sum_{j=2}^{n} (-1)^{j+1} \sigma_{j-1} [e_i, T_1, \ldots, T_{j-1}, T_{j+1} \ldots, T_n] ,
\]

\[
= \sum_{j=2}^{n} (-1)^{j+1} \sigma_{j-1} \Delta^{(n-j)}_{x_i} .
\]

The aim here is to show that \([\Delta'_x, \ldots, \Delta^{(n)}_x] \neq 0\). Due to the fact that

\[
\Delta^{(n)}_{x_i} = \sum_{j=2}^{n} (-1)^{j+1} \sigma_{j-1} \Delta^{(n-j)}_{x_i} ,
\]

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it follows that $\Delta^{(n)}_x$ is a linear combination of $\{\Delta'_x, \Delta''_x, \ldots, \Delta^{(n-2)}_x\}$. It follows that $[\Delta'_x, \ldots, \Delta^{(n)}_x] = 0 \iff \sigma_{n-1}[\Delta'_x, \ldots, \Delta^{(n-1)}_x, \Delta_x] = 0$.

The aim now is to show that $[\Delta'_x, \ldots, \Delta^{(n-1)}_x, \Delta_x] \neq 0$. Consider the $n \times n$ matrix $X = (x_{i,j})$ where

$$x_{i,j} = [e_j, T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n].$$

It follows that $\det(X) = [\Delta^{(n-1)}_x, \Delta^{(n-2)}_x, \ldots, \Delta'_x, \Delta_x] = \pm[\Delta'_x, \ldots, \Delta^{(n-1)}_x, \Delta_x]$. Let $T$ be the matrix whose $i$-th column is $T_i$. Furthermore, let $A = (a_{i,j})$ be the adjoint matrix of $T$. Since

$$a_{i,j} = (-1)^{i+1}[e_j, T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n]$$

it follows that $a_{i,j} = (-1)^{i+1}x_{i,j}$, which implies $\det(X) = \pm \det(A)$. Next consider the well known identity $T^{-1} = \det(T)^{-1}A$, it follows that $\det(T)^{n-1} = \det(A)$. Thus $\det(X) = \pm \det(T)^{n-1} = \pm 1 \neq 0$. From this and the calculations for type $A_k$, the result now follows.

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**References**

