Affine Differential Geometry & Singularity Theory

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy

Declan Davis

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UNIVERSITY OF LIVERPOOL

Contents

Chapter 1

Introduction

1.1 Mathematical introduction

This thesis studies *affine differential geometry* and *singularity theory.* In 1872, Felix Klein stated his famous "Erlanger programme": geometry is the study of invariants with respect to a given transformation group.

Classical Euclidean differential geometry is the study of differential invariants with respect to the group of rigid motions. Affine differential geometry is the study of differential invariants with respect to the group of affine transformations, i.e. non-singular linear transformations together with translations.

We shall study the group of *affine special linear* transformations: also called the group of *equi-affine* or *unimodular* transformations. This group consists of volume preserving linear transformations together with translations.

This group shall be denoted by $\mathrm{ASL}(n,\mathbb{R}) := \mathrm{SL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$. As a topological space $\mathrm{ASL}(n,\mathbb{R})$ has the local structure of a Cartesian product, i.e. $\mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^n$. However, as a Lie group it has the structure of a semi-direct product.

Throughout the thesis we will use the following notation: let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be n-vectors in $\mathbb{R}^n$, we denote by $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ the oriented volume spanned by the vectors $\mathbf{v}_i$. This is the determinant of the $n \times n$ matrix whose i-th column is the vector $\mathbf{v}_i$.

Singularity theory is a very diverse area of mathematics. It deals with, amongst other things, the classification, up to certain kinds of equivalence relation, of functions and families of functions. The thesis will follow a purely geometrical path, as put forward in the wonderful book "*Curves & Singularities*" (see [4]). However [4] deals with Euclidean differential geometry, and not affine.

1.2 Chapter overview

In chapter 2 we consider plane curves $\gamma : I \to \mathbb{R}^2$. These are the simplest of objects and can be studied with relative ease. We consider an affine arc-length parameter. Given a curve without Euclidean inflexions ($\kappa \neq 0$) we can define a unique parameter for the curve. Using this affine arc-length parameter gives an affine tangent and an affine normal vector. These give a basis for the ambient space. The derivatives of this frame give information about the invariants of the curve. A family of affine distance functions is defined, and the singularities of it are studied. We see that exactly analogous results to the Euclidean theory are discovered. We construct affine Monge-Taylor mappings. These carry with them all of the infinitesimal information about the curve. We use them to prove that generically affine inflexions are isolated and finite in number. The idea of using these Monge-Taylor mappings to prove such results is due to J. W. Bruce and P. J. Giblin in [4].

In chapter 3 we consider space curves $\gamma : I \to \mathbb{R}^3$. The theory proceeds in a similar fashion to that of the plane curve case and Euclidean space curve theory. We define affine arc-length and affine curvatures, we find an affine frame which spans the ambient space. A family of affine distance functions is also defined. Following the work of S. Izumiya and T. Sano in [12] we introduce a slightly modified affine frame which makes the expression of singularities of the distance functions much more natural. Again we consider the affine Monge-Taylor mappings. Conditions for the family of affine distance functions to be versal are computed.

In chapter 4 we consider space curve $\gamma : I \to \mathbb{R}^n$. This chapter has already been published in the Proceeding of the Royal Society of Edinburgh 2006 (see [8]). Again we introduce the usual machinery to study the affine differential geometry of curves in $\mathbb{R}^n$. The unique thing here is that a new basis is computed which makes the A_k conditions for the family of affine distance functions very simple to write down.

Moreover, the conditions for versality are trivial in this basis. We also define, and consider, the family of affine height functions. The new basis is very complicated to write down, but expressions have been given.

In chapter 5 the emphasis shifts to surfaces in three-space. We give a brief introduction to the theory. This begins with a very general approach using connexions and metrics. This approach was put forward by K. Nomizu amongst others (see [13] and [14]). In Riemannian geometry the idea is to introduce a metric, and to study the properties of that metric. In affine differential geometry we introduce volume forms which are defined in a very natural manner. The compatibility of these volume forms with certain induced connexions gives rise to a unique transverse vector field. This transverse vector field, along with the tangent plane, allows us to describe the geometry of our surface. Other standard notions are given in this chapter too, such as the affine shape operator, affine principal direction, affine principal curvatures, asymptotic directions, etc.

In chapter 6 we consider the so-called Pick normal form. In Euclidean geometry we can put a surface into a special form, i.e. $z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2) + \cdots$, where the κ_i are the Euclidean principal curvatures. We can also put a surface into a special form in the affine theory. However, since the special affine group is larger than the group of rigid motions we have more degrees of freedom. We compute explicitly the transformation which takes our surface into Pick normal form.

In chapter 7 we introduce the family of affine distance functions and the family of affine height functions. The former is a three-parameter family of functions of two variables, the later is a two-parameter family. The generic singularities are A_2, A_3, A_4 and D_4. These are the only interesting simple singularities with miniversal deformations (see [1]) of dimension less than or equal to three. We compute the conditions for certain singularity types and the geometric implications. Although the calculations are very explicit, they are also very general, and so the final expressions are very beautiful. Again, the results are almost identical to the Euclidean theory.

In chapter 8 we consider the family of affine surface parallels and the affine focal set. If we consider the set of all points at an affine distance λ from a given surface point then we get a plane. Doing this for every surface point gives a two-parameter family of planes. These generically have an envelope (i.e. a surface tangent to each of them). This envelope is the affine surface parallel of distance λ. We parametrise this surface, and compute conditions for it to be singular. We also find the curvature of the affine surface parallels, i.e. conditions for them to be hyperbolic, parabolic, or elliptic. The affine focal set is the set of points given by the infinitesimal intersection of nearby affine normal lines. If these lines were rays of light then the focal set would be the points where the light was focused. We calculate a parametrisation for this set, and show that it is the bifurcation set of the family of affine distance functions. We show the connexion between this focal surface and the affine principal curvatures and directions. The affine principal directions are the directions in which infinitesimal intersection takes place. The principal curvatures are related to the affine distance of the intersections from the surface.

In chapter 9 we look at special curves which arise from the affine shape operator. We consider the Euclidean parabolic curve (which is actually an affine invariant too), the repeated A-direction curve (where the affine shape operator has repeated eigenvalues), and the affine parabolic curve (where the affine shape operator has at least one zero eigenvalue). We give results for the generic interaction of these three curves on a surface. In classical affine differential geometry, and even in more recent work (see [13] and [14]), only hypersurfaces with non-zero Gaußian curvature are considered. For surface in three-space these are surfaces without Euclidean parabolic points. The unique part of this chapter is that we study the limiting structure of the repeated A-direction curve and the affine parabolic curve as we tend towards the Euclidean parabolic curve, and especially Euclidean cusps of Gauß. Given certain conditions these curves become singular. We use standard techniques to identify these singularities. The techniques include $\mathcal{A}$-equivalence and resolutions via blowing-up.

In chapter 10 we consider affine sectional curvature. This is an idea which is common place in many undergraduate Euclidean differential geometry modules. In the Euclidean theory, you choose a direction in the tangent plane, then intersect the surface with a plane which contains this direction and the Euclidean unit normal vector at that point. This cross-section gives a plane curve, the Euclidean curvature of this plane curve at the surface point is defined to be the Euclidean section curvature of the surface at that point in the chosen direction. The exact analogue fails for the affine case. It is seen to fail because using this idea we lose all analogous properties. We use singularity theory and the family of affine distance functions to reword the standard interpretation. Once this is done, an idea of affine sectional curvature is found which has many analogous properties to Euclidean sectional curvature. We rewrite the affine sectional curvature in terms of metrics (see [13] and [14]).

In chapter 11 we consider in greater detail the affine parabolic curve. We study the generic transitions in a one-parameter family of surfaces. If we have a one-parameter family of surfaces then as the surfaces change so too do the affine parabolic curves. This chapter mirrors the work by J. W. Bruce, P. J. Giblin, and F. Tari in [5] and relies heavily on work by J. W. Bruce in [2] and [3] for its motivation and proofs. All of the generic cases are considered. Each case uses a different method of proof, and this chapter is a nice show piece for singularity theory and its role in geometry.

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Chapter 2

Plane Curves

We consider the differential invariants of the action of the affine special linear group $\mathrm{ASL}(2,\mathbb{R})$ on the plane $\mathbb{R}^2$. The affine special linear group is the special linear group $\mathrm{SL}(2,\mathbb{R})$ combined with the group of translation of the plane. We can recognise this group as the semi-direct product $\mathrm{ASL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$. Special linear transformations are also called *equi-affine*, or *unimodular*.

The study of curves in $\mathbb{R}^2$ is standard (see [11]). In this chapter we recall the standard machinery for dealing with plane curves. The affine Monge-Taylor mappings have not been studied before.

2.1 Affine arc-length

Since equi-affine transformations preserve area it is natural to seek some parametrisation of plane curves which uses area, since it would be affine invariant.

Let $I \subseteq \mathbb{R}$ be an open interval, and let $\gamma : I \to \mathbb{R}^2$ be a smooth plane curve. The simplest way to parametrise such a curve is to use a parameter, called *affine arc-length* and denoted by s, such that $[\gamma'(s), \gamma''(s)] = 1$ for all $s \in I$. Here the prime represents differentiation with respect to the affine arc-length parameter s, and $[\gamma', \gamma'']$ denotes the determinant of the 2×2 matrix whose columns are the vectors γ' and γ'' respectively.

Definition 2.1.1 *Let $\gamma : I \to \mathbb{R}^2$ be a smooth plane curve parametrised using an* *arbitrary parameter. A curve point $\gamma(t_0)$ is called an inflexion if $[\dot{\gamma}(t_0), \ddot{\gamma}(t_0)] = 0$. The curve γ is said to be without inflexions if $[\dot{\gamma}, \ddot{\gamma}] \neq 0$ for all $t \in I$.*

Definition 2.1.2 *Let γ be a smooth plane curve defined using an arbitrary parameter t without inflexions. The affine arc-length parameter is given by*

$$s(t) = \int [\dot{\gamma}(t), \ddot{\gamma}(t)]^{1/3} dt \ .$$

Proposition 2.1.3 *Let γ be a smooth plane curve without inflexions which is parametrised by affine arc-length. Let prime denote differentiation with respect s, so that $\gamma' = d\gamma/ds$ etc. Then $[\gamma'(s), \gamma''(s)] = 1$ for all $s \in I$.*

Proof Using the chain rule, we see that

$$\frac{d\gamma}{ds} = \frac{dt}{ds}\frac{d\gamma}{dt} \qquad \text{and} \qquad \frac{d^2\gamma}{ds^2} = \frac{d^2t}{ds^2}\frac{d\gamma}{dt} + \left(\frac{dt}{ds}\right)^2 \frac{d^2\gamma}{dt^2} \ . \tag{2.1}$$

Putting these two expressions together gives

$$[\gamma', \gamma''] = \left[\frac{dt}{ds}\frac{d\gamma}{dt}, \ \frac{d^2t}{ds^2}\frac{d\gamma}{dt} + \left(\frac{dt}{ds}\right)^2 \frac{d^2\gamma}{dt^2}\right] \ .$$

Using elementary properties of determinants, we see that

$$[\gamma', \gamma''] = \left(\frac{dt}{ds}\right)^3 [\dot{\gamma}, \ddot{\gamma}] \ .$$

From the definition of affine arc-length $ds/dt = [\dot{\gamma}, \ddot{\gamma}]^{1/3}$. The result now follows. □

Remark 2.1.4 Parametrisation by affine arc-length imposes an orientation. Consider the standard Euclidean curvature of a plane curve:

$$\kappa(s) = \frac{[\gamma'(s), \gamma''(s)]}{||\gamma'(s)||^3} \ .$$

Since $[\gamma'(s), \gamma''(s)] = 1$ for all s, it follows that $\kappa(s) = ||\gamma''(s)||^{-3} > 0$ for all s. The Euclidean curvature is always positive.

This is illustrated in the following:

Example. Consider a branch of a hyperbola given by $\gamma(t) = (a\cosh(t), b\sinh(t))$, we have $\dot{\gamma}(t) = (a\sinh(t), b\cosh(t))$ and $\ddot{\gamma}(t) = (a\cosh t, b\sinh(t))$. This means

$$[\dot{\gamma}, \ddot{\gamma}] = \begin{bmatrix} a\sinh(t) & a\cosh(t) \\ b\cosh(t) & b\sinh(t) \end{bmatrix} = -ab\ .$$

Affine arc-length is then given by $s(t) = -(ab)^{1/3}t$. This implies that $t = -(ab)^{-1/3}s$.

2.2 Affine tangents and affine normals

Let $\gamma : I \to \mathbb{R}^2$ be a smooth plane curve without inflexions parametrised by affine arc-length. The vector $\gamma'(s_0)$ defines the *affine tangent* to γ at $\gamma(s_0)$ and the vector $\gamma''(s_0)$ defines the *affine normal* to γ at $\gamma(s_0)$. Equation 2.1 now gives the affine tangent and affine normal vectors for a curve with an arbitrary parametrisation.

Assume the arbitrary parametrisation to be Euclidean arc-length, so that $||\dot{\gamma}|| = 1$ for all t in the domain of definition. The last section can be used to show that

$$ds = [\dot{\gamma}, \ddot{\gamma}]^{1/3}dt = [\mathbf{T}, \kappa\mathbf{N}]^{1/3}dt = \kappa^{1/3}dt\ ,$$

where $\mathbf{T}$ denotes the unit tangent, $\mathbf{N}$ the unit normal and κ the Euclidean curvature. This last expression gives the relationship $\gamma' = \kappa^{-1/3}\mathbf{T}$.

For any regular parametrisation let $k = [\dot{\gamma}, \ddot{\gamma}]$, so that $k^{-1/3} = dt/ds$. It then follows that $\gamma' = k^{-1/3}\dot{\gamma}$ gives an expression for the affine tangent vector in terms of an arbitrary parameter t. Furthermore $\gamma'' = k^{-2/3}\ddot{\gamma} - \frac{1}{3}\dot{k}k^{-5/3}\dot{\gamma}$ gives an expression for the affine normal vector in terms of this arbitrary parameter.

The affine tangent line to γ at $\gamma(t_0)$ is the line parallel to $\gamma'(s(t_0))$ which passes through $\gamma(t_0)$. Writing $\gamma(t) = (X(t), Y(t))$, the affine tangent line to γ at $\gamma(t_0)$ is

$$\{\mathbf{x} \in \mathbb{R}^2 : [\mathbf{x} - \gamma(t_0), \dot{\gamma}(t_0)] = 0\}\ .$$

The affine normal line to γ at $\gamma(t_0)$ is the line parallel to $\gamma''(s(t_0))$ which passes through $\gamma(t_0)$. Using the expression for γ'' we can find the equation of the affine normal line.

Proposition 2.2.1 *Given a plane curve $\gamma = (X, Y)$ parametrised by an arbitrary parameter t. The equation of the affine normal line to γ is:*

$$(y - Y)(\dot{k}\dot{X} - 3k\ddot{X}) + (x - X)(3k\ddot{Y} - \dot{k}\dot{Y}) = 0 \ .$$

Proof Let $\gamma = (X, Y)$ so that

$$\gamma'' = \frac{1}{3}k^{-5/3}(3k\ddot{X} - \dot{k}\dot{X}, 3k\ddot{Y} - \dot{k}\dot{Y}) \ .$$

The line passing through a point of γ which is parallel to γ'' is given by

$$\frac{1}{3}k^{-5/3}\begin{vmatrix} 3k\ddot{X} - \dot{k}\dot{X} & 3k\ddot{Y} - \dot{k}\dot{Y} \\ x - X & y - Y \end{vmatrix} = 0 \ .$$

Multiplying through by $k^{-5/3}/3$ leads to the desired expression:

$$\begin{vmatrix} 3k\ddot{X} - \dot{k}\dot{X} & 3k\ddot{Y} - \dot{k}\dot{Y} \\ x - X & y - Y \end{vmatrix} = 0 \ .$$

□

The affine normal vector can be related to Euclidean geometry. We have had $\gamma'' = k^{-2/3}\ddot{\gamma} - \dot{k}k^{-5/3}\dot{\gamma}/3$. Let γ be parametrised by Euclidean arc-length, so that $k = [\mathbf{T}, \kappa\mathbf{N}] = \kappa$ and $\dot{k} = [\mathbf{T}, \dot{\kappa}\mathbf{N} - \kappa^2\mathbf{T}] = \dot{\kappa}$, then it follows

$$\gamma'' = \kappa^{1/3}\mathbf{N} - \frac{1}{3}\dot{\kappa}\kappa^{-5/3}\mathbf{T} \ .$$

At an Euclidean inflexion $\kappa = 0$, and so in the limit γ'' points in the direction of $\mathbf{T}$, which is the same direction as γ', and it is of infinite length.

2.3 Affine curvature

As with Euclidean curvature, which remains unchanged by isometric transformations, an affine curvature can be defined which is unchanged by equi-affine transformations.

Recall, by the definition of affine arc-length $[\gamma', \gamma''] = 1$. Differentiating with respect to affine arc-length gives $[\gamma', \gamma'''] = 0$. This means the vectors γ' and γ''' are parallel. There must exist $\mu : I \to \mathbb{R}$, such that $\gamma'''(s) + \mu(s)\gamma'(s) = 0$ for all $s \in I$. The real valued function $\mu : I \to \mathbb{R}$ is called the *affine curvature* of γ. Since $\gamma''' + \mu\gamma' = 0$, it follows that $\mu = [\gamma'', \gamma''']$.

Definition 2.3.1 *Let $\gamma : I \to \mathbb{R}^2$ be a smooth plane curve without inflexions parametrised by affine arc-length. The curve γ is said to have an affine inflexion at $\gamma(s_0)$ if $\mu(s_0) = 0$. An affine inflexion is called ordinary if $\mu'(s_0) \neq 0$. An affine inflexion is called higher if $\mu'(s_0) = 0$.*

Definition 2.3.2 *Let $\gamma : I \to \mathbb{R}^2$ be a smooth plane curve without inflexions parametrised by affine arc-length. The curve γ is said to have an affine vertex at $\gamma(s_0)$ if $\mu(s_0) \neq 0$ and $\mu'(s_0) = 0$. An affine vertex is called ordinary if $\mu''(s_0) \neq 0$. An affine vertex is called higher if $\mu''(s_0) = 0$.*

Example. Consider an ellipse parametrised by affine arc-length, where

$$\gamma(s) = \left(a\cos\left(\frac{s}{(ab)^{1/3}}\right), b\sin\left(\frac{s}{(ab)^{1/3}}\right)\right).$$

From the definition of affine curvature, and use of the chain rule

$$[\gamma''(s), \gamma'''(s)] = (ab)^{-2/3}\left(\cos^2\left(\frac{s}{(ab)^{1/3}}\right) + \sin^2\left(\frac{s}{(ab)^{1/3}}\right)\right) = (ab)^{-2/3}.$$

Since $(ab)^{-2/3} > 0$ for all $ab \neq 0$ it follows every ellipse has positive constant affine curvature.

The ellipses are not the only family of plane curves to have constant affine curvature.

Remark 2.3.3 Let G be a Lie transformation group acting on a space M, with the action $\phi : G \times M \to M$. It has been shown that if $x \in M$ belongs to a non-degenerate orbit of the action of G and if $\{g_t : t \in I\}$ is a one-parameter Lie subgroup, then all of the differential invariants of $\phi(g_t, x)$ will be constant. Let $G := \mathrm{SL}(2, \mathbb{R})$ and $M := \mathbb{R}^2$, then given a one-parameter subgroup of G, the action of it upon any point other than the origin will give an ellipse, a hyperbola, or a parabola.

Consider a smooth plane curve in Monge form without an Euclidean inflexion close to the origin, i.e. for $a_i \in \mathbb{R}$, $a_2 \neq 0$, given by

$$\gamma(t) = \left(t, \frac{1}{2}a_2t^2 + \cdots + \frac{1}{k!}a_kt^k + t^{k+1}g(t)\right) ,$$

where $g : I \to \mathbb{R}$ is a smooth function.6 We can calculate the affine curvature of γ at $\gamma(0)$, this is given by

$$\mu(0) = \frac{3a_2a_4 - 5a_3^2}{9a_2^{8/3}} .$$

This means that the affine curvature function is a fourth order affine differential invariant of the plane curve γ.

Proposition 2.3.4 *Let γ be a smooth plane curve without inflexions paramatrised by an arbitrary parameter t. Writing $k = [\dot{\gamma}, \ddot{\gamma}]$ allows the affine curvature to be written as*

$$\mu = \frac{1}{9}(3k\ddot{k} - 5\dot{k}^2 + 9k[\ddot{\gamma}, \dddot{\gamma}])k^{-8/3} . \qquad (2.2)$$

Proof Using the expressions for γ' and γ'' we have

$$\gamma''' = \left(\frac{5}{9}\dot{k}^2k^{-3} - \frac{1}{3}\ddot{k}k^{-2}\right)\dot{\gamma} - \dot{k}k^{-2}\ddot{\gamma} + k^{-1}\dddot{\gamma} .$$

Furthermore, since $k = [\dot{\gamma}, \ddot{\gamma}]$ we have $\dot{k} = [\dot{\gamma}, \dddot{\gamma}]$ etc, we may simplify $\mu = [\gamma'', \gamma''']$. The result then follows from direct computation. □

Corollary 2.3.5 *The affine curvature of a plane curve can be written in terms of the Euclidean curvature in the following way:*

$$\mu = \frac{1}{9}(3\kappa\ddot{\kappa} - 5\dot{\kappa}^2 + 9\kappa^4)\kappa^{-8/3} .$$

Proof Let γ be parametrised by Euclidean arc-length. This means $k = \kappa$, $\dot{k} = \dot{\kappa}$ and $\ddot{k} = \ddot{\kappa}$, also $[\ddot{\gamma}, \dddot{\gamma}] = [\kappa\mathbf{N}, \kappa'\mathbf{N} - \kappa^2\mathbf{T}] = \kappa^3$. Then the result follows by substitution into equation 2.2. □

2.4 Affine distance

It is possible to define affine "distance" using the *affine distance function.* Affine distance uses area to give a value; to find area one needs at least three points or one point and one vector. Thus the definition of the affine distance function involves vectors. This is the reason there is no concept of affine distance between two points. Denoting differentiation with respect to affine arc-length by a prime, we get the following

Definition 2.4.1 *Let $\gamma : I \to \mathbb{R}^2$ be a smooth plane curve without inflexions which is parametrised by affine arc-length. The family of affine distance functions $\Delta : \mathbb{R}^2 \times I \to \mathbb{R}$ is given by*

$$\Delta(\mathbf{x}, s) = [\mathbf{x} - \gamma, \gamma'] \ .$$

This gives a two-parameter family of functions defined over the curve γ. Given a point $\mathbf{x}_0 \in \mathbb{R}^2$ and $s_0 \in I$, we say that the affine distance from $\mathbf{x}_0$ to $\gamma(s_0)$ is $\Delta(\mathbf{x}_0, s_0)$.

Proposition 2.4.2 *Let $\gamma : I \to \mathbb{R}^2$ be parametrised by an arbitrary parameter, say t. Then the family of affine distance functions is given by*

$$\Delta(\mathbf{x}, t) = [\mathbf{x} - \gamma, \dot{\gamma}][\dot{\gamma}, \ddot{\gamma}]^{-1/3} \ .$$

Remark 2.4.3 The family of affine distance functions is clearly invariant under special affine transformations since it is given by a determinant.

Definition 2.4.4 *Let $\gamma : I \to \mathbb{R}^2$ be parametrised by affine arc-length. Given a fixed $\mathbf{x}_0 \in \mathbb{R}^2$, if there exist distinct $s_1, s_2 \in I$ such that $\Delta(\mathbf{x}_0, s_1) = \Delta(\mathbf{x}_0, s_2)$ and $\Delta_s(\mathbf{x}_0, s_1) = \Delta_s(\mathbf{x}_0, s_2) = 0$, then the corresponding function $\Delta(\mathbf{x}_0, s)$ has a multi-local singularity, written in Arnold's A_k notation as $A_{\geq 1}A_{\geq 1}$.*

Definition 2.4.5 *Given a fixed $\mathbf{x}_0 \in \mathbb{R}^2$, if there exists $s_1 \in I$ such that $\Delta_s(\mathbf{x}_0, s_1) = \Delta_{ss}(\mathbf{x}_0, s_1) = 0$, then the corresponding function $\Delta(\mathbf{x}_0, s)$ has a degenerate singularity, written in Arnold's A_k notation as $A_{\geq 2}$.*

It is natural to seek the *full bifurcation set* of this family of functions. The bifurcation set is the closure of points in parameter space which give rise to functions having either *multi-local* or *degenerate* singularities. The bifurcation set is the closure of points in $\mathbb{R}^2$ which satisfy either Definition 2.4.4 or Definition 2.4.5 or both.

2.4.1 Conditions for A_k

Every point $\mathbf{x} \in \mathbb{R}^2$ gives a function with an $A_{\geq 0}$ singularity of the affine distance function. The $A_{\geq 1}$ singularities come from solving $\Delta_s(\mathbf{x}, s_1) = 0$, where

$$\Delta_s(\mathbf{x}, s_1) = [\mathbf{x} - \gamma(s_1), \gamma''(s_1)] \ .$$

Thus $\Delta_s(\mathbf{x}, s_1) = 0$ if, and only if, $\mathbf{x} - \gamma(s_1)$ is parallel to $\gamma''(s_1)$. This means $\mathbf{x}$ is on an affine normal line to γ at $\gamma(s_1)$.

For an $A_{\geq 2}$ at $s = s_1$ one needs $\Delta_s(\mathbf{x}, s_1) = \Delta_{ss}(\mathbf{x}, s_1) = 0$, where

$$\Delta_{ss}(\mathbf{x}, s_1) = [\mathbf{x} - \gamma(s_1), \gamma'''(s_1)] - 1 \ .$$

Given $\Delta_s(\mathbf{x}, s_1) = 0$ it follows $\mathbf{x} - \gamma(s_1) = \lambda\gamma''(s_1)$ for $\lambda \in \mathbb{R}$, and so

$$\Delta_{ss}(\mathbf{x}, s_1) = [\lambda\gamma''(s_1), \gamma'''(s_1)] - 1 = \lambda\mu(s_1) - 1 \ .$$

Hence $\Delta_s(\mathbf{x}, s_1) = \Delta_{ss}(\mathbf{x}, s_1) = 0$ if, and only if, $\mu(s_1) \neq 0$ and $\mathbf{x} = \gamma(s_1) + \mu(s_1)^{-1}\gamma''(s_1)$. The closure of such points is the *affine evolute* and this is part of the bifurcation set of the affine distance function. Similar methods calculate conditions for $A_{\geq 3}$ etc.

Remark 2.4.6 If $\mathbf{x}$ gives a function with an A_3 singularity at $s = s_1$ then there is a conic having six-point contact with γ at $\gamma(s_1)$. The point $\gamma(s_1)$ is called a sextactic point and $\mathbf{x}$ will be an ordinary cusp on the affine evolute. See [15] for more details.

2.5 Affine Monge-Taylor mappings

The aim of the section is to establish something analogous to the Euclidean Monge-Taylor mapping, cf. [4] Chapter 9. The section concludes will a result on the generic behaviour of smooth simple closed plane curves without inflexions.

Definition 2.5.1 *A simple closed plane curve $\gamma : S^1 \to \mathbb{R}^2$ without inflexions is said to be convex.*

Proposition 2.5.2 *Let γ be a smooth plane curve without inflexions which is parametrised by affine arc-length. Given the family of affine distance functions as above, for a generic point $\mathbf{x} \in \mathbb{R}^2$ we have*

$$\mathbf{x} = \gamma + \Delta_s(\mathbf{x}, s)\gamma' - \Delta(\mathbf{x}, s)\gamma'' \ .$$

Proof We have $[\gamma'(s), \gamma''(s)] = 1$ for all $s \in I$. There exist families of functions $u, v : \mathbb{R}^2 \times I \to \mathbb{R}$ such that

$$\mathbf{x} = \gamma(s) + u(\mathbf{x}, s)\gamma'(s) + v(\mathbf{x}, s)\gamma''(s) \ .$$

It follows that

$$\begin{aligned} u(\mathbf{x}, s) &= [\mathbf{x} - \gamma(s), \gamma''(s)] = \Delta_s(\mathbf{x}, s) \ , \\ v(\mathbf{x}, s) &= [\gamma'(s), \mathbf{x} - \gamma(s)] = -\Delta(\mathbf{x}, s) \ . \end{aligned}$$

□

Proposition 2.5.2 shows that $\{\gamma', \gamma''\}$ can be used as a one-parameter family of bases for a one-parameter family of new coordinate systems, whose origins are at some point on the curve γ. Thus fixing $s_0 \in I$, a point $\mathbf{x}$ is given by $(\Delta_s(\mathbf{x}, s_0), -\Delta(\mathbf{x}, s_0))$ in one of these new sets of coordinates.

Restricting $\mathbf{x}$ to the curve, so that $\mathbf{x} = \gamma(r)$ for some $r := s - s_0$, it follows that $u, v : \mathbb{R} \times I \to \mathbb{R}$ where

$$\begin{aligned} u(r, s) &= [\gamma(r) - \gamma(s), \gamma''(s)] = \Delta_s(r, s) \ , \\ v(r, s) &= [\gamma'(s), \gamma(r) - \gamma(s)] = -\Delta(r, s) \ . \end{aligned}$$

In what follows it is important to remember that $r = s - s_0$, and so expressions of the form $\gamma'(r)$ are simply $\gamma'(s - s_0)$. Thus the prime is reserved for differentiation with respect to s. The use of r is designed to simplify notation, and is not an independent variable: it is a translate of s. Fixing $s_0 \in I$ gives a parametrisation for the curve in terms of one of the new coordinate systems, namely $\gamma(r) = (\Delta_s(r, s_0), -\Delta(r, s_0))$. Each choice of coordinate system gives a parametrisation for a curve tangent to one of the basis vectors at the origin, i.e. tangent to $\gamma'(s_0)$ at $\gamma(s_0)$. The curve can be put into Monge form.

For a fixed $s_0 \in I$, the function u can be considered a function of r, where $u(r) = [\gamma(r) - \gamma(s_0), \gamma''(s_0)]$, it follows that $u'(r) = [\gamma'(r), \gamma''(s_0)]$. Thus $u'(s_0) = 1$ and so $u'(s_0) \neq 0$. This allows $u(r)$ to be taken as a local parameter for the curve close to $r = s_0$. This means $v(r) = V(u(r))$ for some function V. Differentiating u

and v leads to the following:

$$
\begin{aligned}
v' &= u'V_u, \\
v'' &= u''V_u + u'^2 V_{uu}, \\
v''' &= u'''V_u + 3u'u''V_{uu} + u'^3 V_{uuu} \ .
\end{aligned}
$$

Expressions can also be calculated for $d^i v/dr^i$ where $1 \leq i \leq k$. Solving each equation, in turn, for $V_{(i)}$ it follows that:

$$
\begin{aligned}
V_u &= v'/u' \ , \\
V_{uu} &= (u'v'' - u''v')/u'^3 \ , \\
V_{uuu} &= (u'^2 v''' - u'v'u''' + 3(v'u''^2 - u'u''v''))/u'^5 \ .
\end{aligned}
$$

Expressions can also be found for higher order derivatives of V. Here, $u^{(i)}(r) = [\gamma^{(i)}(r), \gamma''(s_0)]$ for all $1 \leq i \leq k$ and $v^{(i)}(r) = [\gamma'(s_0), \gamma^{(i)}(r)]$ for all $1 \leq i \leq k$. Thus:

$$
\begin{aligned}
u'(s_0) &= 1 \ , \\
u''(s_0) &= 0 \ , \\
u'''(s_0) &= -\mu(s_0) \ , \\
u^{(4)}(s_0) &= -\mu'(s_0) \ , \\
u^{(5)}(s_0) &= -\mu(s_0)^2 - \mu''(s_0) \ , \\
u^{(6)}(s_0) &= 4\mu(s_0)\mu'(s_0) - \mu'''(s_0) \ .
\end{aligned}
$$

$$
\begin{aligned}
v'(s_0) &= 0 \ , \\
v''(s_0) &= 1 \ , \\
v'''(s_0) &= 0 \ , \\
v^{(4)}(s_0) &= -\mu(s_0) \ , \\
v^{(5)}(s_0) &= -2\mu'(s_0) \ , \\
v^{(6)}(s_0) &= \mu(s_0)^2 - 3\mu''(s_0) \ .
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
V_u(u(s_0)) &= 0 \ , \\
V_{uu}(u(s_0)) &= 1 \ , \\
V_{uuu}(u(s_0)) &= 0 \ , \\
V_{(4)}(u(s_0)) &= 3\mu(s_0) \ , \\
V_{(5)}(u(s_0)) &= 3\mu'(s_0) \ , \\
V_{(6)}(u(s_0)) &= 45\mu(s_0)^2 + 5\mu''(s_0) \ .
\end{aligned}
$$

Next, notice that by Taylor's theorem:

$$V(u(r)) = \sum_{n=0}^{k} \frac{V_{(n)}(u(s_0))}{n!}(u(r) - u(s_0))^n + g(u(r))$$

for some smooth function g with zero k-jet. Given the current coordinate system $V(u(s_0)) = V_u(u(s_0)) = 0$. Thus one may consider the finite approximation:

$$V(u(r)) \sim \sum_{n=2}^{k} \frac{V_{(n)}(u(s_0))}{n!}(u(r) - u(s_0))^n$$

for some $2 \leq k < \infty$. This allows the construction of a mapping into jet space. Expanding V about $u(s_0)$ gives a k-jet, the coefficients of which give a point in

$J^k(\mathbb{R},\mathbb{R})$. Expanding about all other points $u(s)$ gives a curve in $J^k(\mathbb{R},\mathbb{R})$. This mapping is $j^k_{u(s_0)} : C^\infty(\mathbb{R}) \to J^k(\mathbb{R},\mathbb{R})$ where

$$V(u(r)) \mapsto \left(V_u(u(s_0)), \frac{V_{uu}(u(s_0))}{2!}, \ldots, \frac{V_{(k)}(u(s_0))}{k!} \right).$$

Then expanding about all other points on γ gives the curve

$$\eta_\gamma(s) = \left(V_u(u(s)), \frac{V_{uu}(u(s))}{2!}, \ldots, \frac{V_{(k)}(u(s))}{k!} \right).$$

This can then be written

$$\eta_\gamma(u) = \left(V_u, \frac{V_{uu}}{2!}, \ldots, \frac{V_{(k)}}{k!} \right).$$

The curve $\eta_\gamma : I \to J^k(\mathbb{R},\mathbb{R})$ is an equi-affine invariant of the original curve $\gamma : I \to \mathbb{R}^2$. This is clear from the construction. The main use of this affine Monge-Taylor mapping will be in the case of ovals (smooth simple closed convex plane curves), where $I = S^1$ so that $\gamma : S^1 \to \mathbb{R}^2$ and $\eta_\gamma : S^1 \to J^k(\mathbb{R},\mathbb{R})$. It can be used to prove analogous affine results to the Euclidean ones presented in [4].

Writing $a_i(u) = V_{(i)}(u)/i!$, it follows that $a_1(u) = a_3(u) = 0$ and $a_2(u) = 1/2$ for all u. Thus one may consider only the points $(a_4(u), \ldots, a_k(u)) \subset J^k(\mathbb{R},\mathbb{R})$, from now on consider the subset $(a_4, \ldots, a_k) \subset J^k(\mathbb{R},\mathbb{R})$ and write J_k.

Letting $k = 5$, the condition for γ to have a higher affine inflexion is that $\mu(u) = \mu'(u) = 0$ for some u. In an arbitrary coordinate system, a curve $\gamma = (u, V(u))$ has $\mu(0) = 0$ if, and only if, $4a_2(0)a_4(0) - 5a_3(0)^2 = 0$. Also $\mu'(0) = 0$ if, and only if,

$$2a_3(0)^3 - 3a_2(0)a_3(0)a_4(0) + a_5(0)a_2(0)^2 = 0 \ ,$$

In the affine Monge-Taylor coordinates these conditions become $a_4(0) = 0$ and $a_5(0) = 0$ respectively. Thus γ has a higher affine inflexion at some point provided $\eta_\gamma(S^1)$ passes through $(0,0) \in J_5$. If η passes through the point $(0,0)$ then it follows that a small deformation of γ, into $\tilde{\gamma}$, will ensure $\eta_{\tilde{\gamma}}(S^1)$ misses $(0,0)$.

Moreover, letting $k = 6$, the condition for γ to have a higher affine vertex is that $\mu(0) \neq 0$ but $\mu'(0) = \mu''(0) = 0$. We have $\mu''(0) = 0$ if, and only if,

$$10a_2^2a_3a_5 - 24a_2a_3^2a_4 + 11a_3^4 + 6a_2^2a_4^2 - 3a_2^3a_6\big|_{u=0} = 0 \ .$$

In the affine Monge-Taylor system $a_1 = a_3 = 0$ and $a_2 = 1/2$, thus the condition becomes $4a_4(0)^2 - a_6(0) = 0$. The points

$$N = \{(a_4, 0, 4a_4^2) : a_4 \in \mathbb{R} - \{0\}\} \subset J_6$$

give two segments of a parabola. If $\eta_\gamma(S^1)$ passes through N then γ has a higher affine vertex at some point. If $\eta_\gamma(S^1)$ passes through $(0,0,0)$ which is in the closure of N, γ has a very high affine inflexion (corresponding to $\mu = \mu' = \mu'' = 0$). If η is transverse to N then it follows that a small deformation of γ, into $\tilde{\gamma}$, will ensure $\eta_{\tilde{\gamma}}(S^1)$ misses N.

It remains to establish how deformations of γ deform η_γ. Here we use standard *polynomial deformations* which were used in [4]. Let P_k denote the set of maps $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $\pi(x_1, x_2) = (\pi_1(x_1, x_2), \pi_2(x_1, x_2))$ where $\pi_1, \pi_2 \in \mathbb{R}[x_1, x_2]$ with degree $\leq k$, for example

$$\pi_1(x_1, x_2) = \sum_{i=0}^{k} \sum_{j=0}^{i} \alpha_{ij} x_1^{i-j} x_2^j \ .$$

The coefficients of the various monomial give a coordinate system on the space P_k. It follows there are $(k+1)(k+2)/2$ possible choices for π_1 and likewise for π_2, hence $\dim(P_k) = (k+1)(k+2)$.

Assuming $k \geq 1$ it follows that the identity map $id(x_1, x_2) = (x_1, x_2)$ is contained in P_k. Consider an open neighbourhood $B \subset P_k$ with $id \in B$, then for sufficiently small B it follows that $\pi \circ \gamma$ will be smooth for all $\pi \in B$ and will still be an oval.

Theorem 2.5.3 (Bruce and Giblin [4]) *Let M be a submanifold of $J_k = \mathbb{R}^{k-3}$. For some open set $\tilde{B} \subset B$ with $id \in \tilde{B}$, the mapping $\eta : S^1 \times \tilde{B} \to \mathbb{R}^{k-3}$ given by $\eta(u, \pi) = \eta_{\pi \circ \gamma}(u)$ is transverse to M.*

Proof The submanifold M is of no importance here, since η can be shown to be a submersion. Since S^1 is compact and being a submersion is an open condition it is enough to consider a point $(u, id) \in S^1 \times \tilde{B}$. Using an isometry, the curve γ may be put in terms of it affine graph (the so called affine Monge form) so that locally the curve is the graph of a function $x_2 = V(x_1)$ where $V(0) = V_u(0) = 0$. It follows that $\eta(u, id) = j_0^k V$.

Given $F(x_1) \in \mathbb{R}[x_1]$ with $4 \leq \deg F \leq k$, let $\pi_t(x_1, x_2) = (x_1, x_2 + tF(x_1))$ for $t \in \mathbb{R}$. Then for small t, it follows that $\pi_t \in \tilde{B}$. Now consider the curve $\Gamma : \mathbb{R} \to S^1 \times \tilde{B}$ given by $\Gamma(t) = (u, \pi_t(x_1, x_2))$. Then $\pi_t(x_1, V(x_1)) = (x_1, V(x_1) + tF(x_1))$. Since $\deg F \geq 4$ it follows that the affine Monge-Taylor expansion of $(\pi_t \circ \gamma)(S^1)$ at $(\pi_t \circ \gamma)(u) = 0$ is $j_0^k(V + tF)$. The tangent vector in $J^k(\mathbb{R}, \mathbb{R})$ at $\eta(u, id)$ is

$$\lim_{t \to 0} \frac{j_0^k(V + tF) - j_0^k V}{t} = \lim_{t \to 0} \frac{j_0^k V + t j_0^k F - j_0^k V}{t} = j_0^k F = F(u) \ .$$

Since the choice of F was arbitrary and $4 \leq \deg F \leq k$ this clearly shows that η is a submersion at (u, id) and whence the result. □

Lemma 2.5.4 (Thom) *Let $X \subset \mathbb{R}^p$, $Y \subset \mathbb{R}^q$ be smooth manifolds and B an open set in $\mathbb{R}^r$ with $G : X \times U \to \mathbb{R}^q$ a smooth map transverse to Y. Then for almost all $b \in B$ (all b outside a set of measure zero) the maps $G_b : X \to \mathbb{R}^q$ given by $G_b(x) = G(x, b)$ are transverse to Y.*

Applying Lemma 2.5.4 to Theorem 2.5.3 shows that for a dense set of deformations $\pi \in \tilde{U}$ the maps $\eta_{\pi \circ \gamma} : S^1 \to J_k$ will be transverse to M. If $\mathrm{codim}(M) \geq 2$ this implies that $\eta_{\pi \circ \gamma}(S^1)$ misses M.

Definition 2.5.5 *A property P is said to be dense or to hold for a dense set of smooth plane curves if the following holds. For any such $\gamma : I \to \mathbb{R}^2$ there should be some neighbourhood $B \subset \mathbb{R}^n$ with $0 \in B$, and a family of plane curves $\tilde{\gamma} : I \times B \to \mathbb{R}^2$ such that $\tilde{\gamma}(u, 0) = \gamma(u)$ and for some sequence $\{b_n\}$ in B with $\lim_{n \to \infty} b_n = 0$ property P holds for the sequence of curves $\tilde{\gamma}(u, b_n)$.*

Definition 2.5.6 *A property P is said to be open or to hold for an open set of smooth plane curves if given a curve $\gamma : I \to \mathbb{R}^2$ with property P and any family $\tilde{\gamma} : I \times B \to \mathbb{R}^2$ of smooth curves the property P holds for all curves $\tilde{\gamma}(u, b)$ with $b \in B$.*

Definition 2.5.7 *A property P is said to be generic or hold for a generic set of smooth plane curves if it is both open and dense.*

Before stating the main result of this section, we need the following

Proposition 2.5.8 *Let X be compact and $Y \subset \mathbb{R}^m$ be a smooth manifold which is a closed subset of $\mathbb{R}^m$, and consider the set of smooth mappings $f : X \to \mathbb{R}^m$. The property $f \pitchfork Y$ is open.*

Proof See [4] Chapter 8 pp 223. □

Proposition 2.5.9 *If $f : X \to \mathbb{R}^m$ is a smooth map, $Y \subset \mathbb{R}^m$ is a smooth submanifold, f is transverse to Y, and $\dim(X) + \dim(Y) = m$ then the points $f^{-1}(Y)$ are isolated. That is, each $\mathbf{x} \in f^{-1}(Y)$ has a neighbourhood U with $U \cap f^{-1}(Y) = \{\mathbf{x}\}$. Furthermore, if X is compact $f^{-1}(Y)$ is finite.*

Proof Let $g : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map with regular value $z \in \mathbb{R}^n$ such that $g^{-1}(z) = Y$. It follows from the implicit function theorem that Y is a smooth manifold of dimension $m - n$. Since $\dim(X) + \dim(Y) = m$ it now follows that $\dim(X) = n$. If $f \pitchfork Y$ then by the definition of transversality z is a regular value of the map $(g \circ f) : X \to \mathbb{R}^n$. Since $\dim(X) = n$ and $(g \circ f) : X \to \mathbb{R}^n$ has z as a regular value it follows by the inverse function theorem that $(g \circ f)$ is a local diffeomorphism, and so $\dim(X) = 0$. Hence the points $f^{-1}(Y)$ are isolated. The fact that if X is compact $f^{-1}(Y)$ is finite follows from the definition of compactness. □

Corollary 2.5.10 *An open dense set of ovals $\gamma : S^1 \to \mathbb{R}^2$ have only finitely many ordinary affine inflexions and ordinary affine vertices, and no higher affine inflexions or higher affine vertices. Thus these properties are generic.*

Proof Given an oval $\gamma : S^1 \to \mathbb{R}^2$, applying Lemma 2.5.4 to the map $\eta : S^1 \times \tilde{B} \to J_k$ with M the submanifold of higher affine inflexions (resp. affine vertices) proves that a dense set of curves have only ordinary affine inflexions (resp. affine vertices). Furthermore, by Lemma 2.5.4, for a dense set of $\pi \in \tilde{B}$ the map $\eta_\pi : S^1 \to J_k$ is transverse to M. Thus by Proposition 2.5.9 the set $\eta_\pi^{-1}(M)$ of ordinary affine inflexions (resp. affine vertices) is finite for a dense set of $\pi \in \tilde{B}$.

Considering the cases of ordinary affine inflexions and ordinary affine vertices individually we may prove these properties are open. Let M denote either the point $(0,0) \in J_5$ (higher affine inflexions) or the set $a_5 = 4a_4 - a_6 = 0$ in J_6 (higher affine vertices, together with very high affine inflexions, $(0,0,0) \in J_6$). It follows that M is smooth and closed in both cases. Let $\tilde{\gamma} : S^1 \times B \to \mathbb{R}^2$ be a family of curves with $\tilde{\gamma}_0$ having only ordinary affine inflexions and ordinary affine vertices. Let $\eta : S^1 \times B \to J_k$ be the corresponding family of affine Monge-Taylor mappings. Then the compactness of S^1, together with the fact that η_0 is transverse to M, implies by Proposition 2.5.8 that $\eta(S^1 \times \{b\})$ misses M for b in some open neighbourhood of 0. Hence nearby curves $\tilde{\gamma}_u$ in the family also possess no higher affine inflexions and no higher affine vertices.

Finally one may show that having finitely many ordinary affine inflexions and vertices and no higher ones is open. First note that, if γ has an ordinary affine inflexion (resp. affine vertex) at $t \in S^1$, then the image of the map $\eta : S^1 \to J_5$ meets the a_4 (resp. a_5) axis at $\eta(u)$ and is transverse to that axis.

Let $\tilde{\gamma} : S^1 \times B \to J_5$ be a family of curves with $\tilde{\gamma}_0$ having finitely many ordinary affine inflexions and vertices, and no higher ones, so that $\eta_0 : S^1 \to J_5$ is transverse to the a_4-axis and a_5-axis. Since transversality is an open condition when the source is compact and the relevant submanifold closed (see Proposition 2.5.8) it follows that $\eta_b : S^1 \to J_5$ will also be transverse to these axes for all b in some neighbourhood $\tilde{B}$ of $0 \in B$. Hence, if M is either of these axes the set $\eta_b^{-1}(M)$ of ordinary affine inflexions or vertices of $\tilde{\gamma}_b$ is finite and also there are no higher ones (see Proposition 2.5.9). This proves the result.

□

Corollary 2.5.11 *Let $\gamma : I \to \mathbb{R}^2$ be a smooth plane curve without inflexions. Assume that $\gamma(0) = (0,0)$ and $\gamma'(0) \propto (1,0)$. If $\mu(0) \neq 0$ then γ is affinely equivalent to*

$$\gamma(t) = (t, t^2 + a_4 t^4 + a_5 t^5 + \cdots)$$

where $a_i \in \mathbb{R}$ and $a_4 \neq 0$. If $\mu(0) = 0$ but $\mu'(0) \neq 0$ then γ is affinely equivalent to

$$\gamma(t) = (t, t^2 + a_5 t^5 + a_6 t^6 + \cdots)$$

where $a_5 \neq 0$. If $\mu(0) = \mu'(0) = 0$ but $\mu''(0) \neq 0$ then γ is affinely equivalent to

$$\gamma(t) = (t, t^2 + a_6 t^6 + a_7 t^7 + \cdots)$$

where $a_6 \neq 0$. If $\mu^{(i)}(0) = 0$ for all $0 \leq i \leq 2$ then γ is affinely equivalent to

$$\gamma(t) = (t, t^2 + a_7 t^7 + a_8 t^8 + \cdots)$$

where $a_7 \neq 0$.

Proof This comes directly from the computation of the $V_{(i)}(u(s_0))$ on page 25. □

Chapter 3

Space Curves

Here we look at the affine invariant differential geometry of space curves. Let $I \subseteq \mathbb{R}$ be an open interval. Let $\gamma : I \to \mathbb{R}^3$ be a smooth embedding. We can follow the 2-dimensional case to construct a 3-dimensional theory.

The study of curves in $\mathbb{R}^3$ is not as standard as curves in $\mathbb{R}^3$, although they have been considered (see [12]). In this chapter we recall the standard machinery for dealing with plane curves. We give some Euclidean interpretations and compute conditions for A_k singularities of the family of affine distance functions.

3.1 Affine arc-length

We seek a parametrisation for γ, using the parameter s, so that $[\gamma',\ \gamma'',\ \gamma'''] = 1$ for all $s \in I$, where prime denotes differentiation with respect to this special parameter. Consider some arbitrary parameter t, we have

$$\frac{d\gamma}{ds} = \frac{dt}{ds}\frac{d\gamma}{dt}, \quad \frac{d^2\gamma}{ds^2} = \frac{d^2t}{ds^2}\frac{d\gamma}{dt} + \left(\frac{dt}{ds}\right)^2 \frac{d^2\gamma}{dt^2}\ ,$$

$$\frac{d^3\gamma}{ds^3} = \frac{d^3t}{ds^3}\frac{d\gamma}{dt} + 3\frac{dt}{ds}\frac{d^2t}{ds^2}\frac{d^2\gamma}{dt^2} + \left(\frac{dt}{ds}\right)^3 \frac{d^3\gamma}{dt^3}$$

where $d\gamma/dt = \dot{\gamma}$, $d\gamma/ds = \gamma'$ and so on. It then follows that

$$\left(\frac{dt}{ds}\right)^6 [\dot{\gamma},\ \ddot{\gamma},\ \dddot{\gamma}] = [\gamma',\ \gamma'',\ \gamma''']\ .$$

Furthermore, since $[\gamma',\ \gamma'',\ \gamma''']=1$ we conclude that $ds=[\dot\gamma,\ \ddot\gamma,\ \dddot\gamma]^{1/6}dt$ and so

$$s(t)=\int[\dot\gamma,\ \ddot\gamma,\ \dddot\gamma]^{1/6}dt\ .$$

3.1.1 Euclidean interpretations

Let the parameter t be Euclidean arc-length. It follows that $\dot\gamma=\mathbf{T}$, $\ddot\gamma=\kappa\mathbf{N}$ and $\dddot\gamma=\dot\kappa\mathbf{N}+\kappa\tau\mathbf{B}-\kappa^2\mathbf{T}$. Here $\mathbf{B}$ is the unit binormal vector and τ is the Euclidean torsion; $\mathbf{T}$, $\mathbf{N}$ and κ have already been defined. Thus

$$[\dot\gamma,\ \ddot\gamma,\ \dddot\gamma]=[\mathbf{T},\ \kappa\mathbf{N},\ \dot\kappa\mathbf{N}+\kappa\tau\mathbf{B}-\kappa^2\mathbf{T}]=\kappa^2\tau\ .$$

Using Euclidean arc-length also gives us

$$\gamma'=\frac{1}{\kappa^{1/3}\tau^{1/6}}\mathbf{T},\quad \gamma''=\kappa\tau^{-1/3}\mathbf{N}-\frac{2\tau\dot\kappa+\kappa\dot\tau}{6\kappa^{5/3}\tau^{4/3}}\mathbf{T}\ .$$

As with the plane curve case, affine arc-length can only be defined given certain conditions (in the plane curve case it was that $\kappa\neq 0$). Since $(dt/ds)^6[\dot\gamma,\ddot\gamma,\dddot\gamma]=[\gamma',\gamma'',\gamma''']$, and $[\dot\gamma,\ddot\gamma,\dddot\gamma]=\kappa^2\tau$, if $\kappa\tau=0$ it is impossible for $[\gamma',\gamma'',\gamma''']=1$. Thus affine arc-length cannot be defined at points where $\kappa\tau=0$.

3.2 Affine curvature and torsion

If $[\gamma',\gamma'',\gamma''']=1$ for all $s\in I$ then $[\gamma',\gamma'',\gamma^{(4)}]=0$ for all $s\in I$. There exist functions $\mu,\nu:I\to\mathbb{R}$ such that $\gamma^{(4)}+\nu\gamma'+\mu\gamma''=0$ for all $s\in I$. We may give names to these objects. For example γ' is the *affine tangent vector*, γ'' is the *principal affine normal* and γ''' is the *affine binormal*. The function ν is the *affine torsion* of γ and μ is the *affine curvature*.

From this we may construct some Serret-Frenet type formulae, these are

$$\frac{d}{ds}\begin{pmatrix}\gamma'\\ \gamma''\\ \gamma'''\end{pmatrix}=\begin{pmatrix}0&1&0\\ 0&0&1\\ -\nu&-\mu&0\end{pmatrix}\begin{pmatrix}\gamma'\\ \gamma''\\ \gamma'''\end{pmatrix}.$$

3.3 Affine distance

Let $\gamma : I \to \mathbb{R}^3$ be a smooth space curve parametrised by affine arc-length. Then we have the following

Definition 3.3.1 *Given a point $\mathbf{x} \in \mathbb{R}^3$ we define the distance from $\mathbf{x}$ to $\gamma(s_0)$ to be*

$$\Delta(\mathbf{x}, s_0) = [\mathbf{x} - \gamma(s_0), \gamma'(s_0), \gamma''(s_0)] \ .$$

The three-parameter family of functions $\Delta : \mathbb{R}^3 \times I \to \mathbb{R}$ is a family of affine distance functions on γ.

Proposition 3.3.2 *Let $\gamma : I \to \mathbb{R}^3$ be parametrised by an arbitrary parameter, say t. Then the family of affine distance functions is given by*

$$\Delta(\mathbf{x}, t) = [\mathbf{x} - \gamma, \dot{\gamma}, \ddot{\gamma}][\dot{\gamma}, \ddot{\gamma}, \dddot{\gamma}]^{-1/2} \ .$$

3.3.1 Conditions for A_k

Using Arnold's standard A_k notation, we say that $\Delta : \mathbb{R}^3 \times I \to \mathbb{R}$ has an A_k singularity at $\mathbf{x}_0$ if there exists $s_0 \in \mathbb{R}$ such that $\partial^n \Delta / \partial s^n(\mathbf{x}_0, s_0) = 0$ for all $1 \leq n \leq k$ and $\partial^{k+1}\Delta/\partial s^{k+1}(\mathbf{x}_0, s_0) \neq 0$. We can compute the conditions for A_k. Notice that

$$\begin{aligned}
\Delta_s(\mathbf{x}, s) &= [\mathbf{x} - \gamma, \gamma', \gamma'''] \ , \\
\Delta_{ss}(\mathbf{x}, s) &= [\mathbf{x} - \gamma, \gamma'', \gamma'''] + [\mathbf{x} - \gamma, \gamma', -\mu\gamma''] \ , \\
\Delta_{sss}(\mathbf{x}, s) &= [\mathbf{x} - \gamma, \gamma', -\mu'\gamma'' - \mu\gamma'''] - \nu[\mathbf{x} - \gamma, \gamma'', \gamma'] - 1 \ .
\end{aligned}$$

Since $[\gamma', \gamma'', \gamma'''] = 1$, the first three derivatives of γ are always linearly independent, they form a basis for $\mathbb{R}^3$. We can write $\mathbf{x} - \gamma$ as a linear combination of γ', γ'', and γ'''. Then $\Delta_s(\mathbf{x}, s) = 0$ if, and only if, $\mathbf{x} = \gamma + \alpha\gamma' + \beta\gamma'''$ for some $\alpha, \beta \in \mathbb{R}$. In this case the family of affine distance functions has an $A_{\geq 1}$ singularity at $\mathbf{x}$. Also $\Delta_s(\mathbf{x}, s) = \Delta_{ss}(\mathbf{x}, s) = 0$ if, and only if, $\mathbf{x} = \gamma + \alpha\gamma' + \beta\gamma'''$ and $\alpha - \beta\mu = 0$, that is $\mathbf{x} = \gamma + \beta(\mu\gamma' + \gamma''')$. In this case the family of affine distance functions has an $A_{\geq 2}$ singularity at $\mathbf{x}$. Finally $\Delta_s(\mathbf{x}, s) = \Delta_{ss}(\mathbf{x}, s) = \Delta_{sss}(\mathbf{x}, s) = 0$ if, and only if, $\mathbf{x} = \gamma + \beta(\mu\gamma' + \gamma''')$, $\nu - \mu' \neq 0$, and $\beta = 1/(\nu - \mu')$, that is

$$\mathbf{x} = \gamma + \frac{\mu}{\nu - \mu'}\gamma' + \frac{1}{\nu - \mu'}\gamma''' \ .$$

In this case the family of affine functions has an $A_{\geq 3}$ singularity at $\mathbf{x}$.

Notice that the condition $\nu - \mu' \neq 0$ is necessary for an $A_{\geq 3}$. Assume $\mathbf{x}$ is such that Δ has an $A_{\geq 2}$ singularity, i.e. $\mathbf{x}$ is such that $\Delta_s = \Delta_{ss} = 0$. For an $A_{\geq 3}$ we need to solve $\Delta_{sss} = 0$ given $\Delta_s = \Delta_{ss} = 0$, i.e. solve

$$[\mathbf{x} - \gamma, \gamma', -\mu'\gamma'' - \mu\gamma'''] - \nu[\mathbf{x} - \gamma, \gamma'', \gamma'] = 1$$

given $\mathbf{x} = \gamma + \beta(\mu\gamma' + \gamma''')$. In this case the $A_{\geq 3}$ condition becomes $\beta(\nu - \mu') = 1$. This cannot have a solution if $\nu - \mu' = 0$. It follows that there cannot be an $A_{\geq 3}$ singularity when $\nu - \mu' = 0$.

Focal sets

Given a space curve γ, the *focal set* of γ is the closure of points $\mathbf{x} \in \mathbb{R}^3$ such that Δ has an $A_{\geq 2}$ singularity for some $s \in I$. From this we make the following

Definition 3.3.3 *The affine focal set of γ is*

$$\{\mathbf{x} \in \mathbb{R}^3 : \exists\ s \in I \text{ such that } \Delta_s = \Delta_{ss} = 0\}\ .$$

Remark 3.3.4 *The affine focal set of a generic space curve with $\kappa\tau \neq 0$ is a ruled surface.*

Proposition 3.3.5 *Given a space curve γ the affine focal set is parametrised by*

$$\mathbf{x} = \gamma + \beta(\mu\gamma' + \gamma''')\ .$$

3.3.2 Intrinsic affine frames

The set $\{\gamma', \gamma'', \gamma'''\}$ has the property that $[\gamma', \gamma'', \gamma'''] = 1$. Any set of vectors $\{v_1, v_2, v_3\}$ with the property that $[v_1, v_2, v_3] = 1$ is called an *equi-affine frame.* Here we seek to find a new equi-affine frame for a space curve γ which is more analogous to the Euclidean orthonormal frame of $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$.

Notice that for functions $\alpha, \beta : I \to \mathbb{R}$ we have $[\gamma', \gamma'', \gamma''' + \alpha\gamma' + \beta\gamma''] = 1$. Writing $\mathbf{v} := \gamma''' + \alpha\gamma' + \beta\gamma''$, it follows that

$$\begin{aligned} \frac{d\mathbf{v}}{ds} &= \beta\gamma''' + (\beta' + \alpha - \mu)\gamma'' + (\alpha' - \nu)\gamma' \ , \\ \frac{d\mathbf{v}}{ds} &= \beta\mathbf{v} + (\beta' + \alpha - \mu - \beta^2)\gamma'' + (\alpha' - \nu - \alpha\beta)\gamma' \ . \end{aligned}$$

Then $d\mathbf{v}/ds$ depends on $\mathbf{v}$ unless β is identically zero. Hence it is a good idea to take $\beta \equiv 0$ and so $\mathbf{v} := \gamma''' + \alpha\gamma'$ for some $\alpha : I \to \mathbb{R}$.

Consider $\mathbf{v} := \gamma''' + \alpha\gamma'$, then it follows that

$$\mathbf{v}' = (\alpha' - \nu)\gamma' + (\alpha - \mu)\gamma'' \ .$$

For $\mathbf{v}$ to be analogous to the Euclidean binormal, its derivative should be dependent on one vector only. Hence set $\alpha := \mu$ so that $\mathbf{v} = \gamma''' + \mu\gamma'$ and $\mathbf{v}' = (\mu' - \nu)\gamma'$. Thus we define a new vector called the *intrinsic affine binormal*, denoted by $\widetilde{\mathbf{B}} := \mu\gamma' + \gamma'''$. We also let $\widetilde{\mathbf{T}} := \gamma'$ and $\widetilde{\mathbf{N}} := \gamma''$. Clearly $[\widetilde{\mathbf{T}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{B}}] = 1$ for all $s \in I$. The value $\nu - \mu'$ plays an important role. Thus we relabel $\kappa_a := \mu$ and set $\tau_a := \nu - \mu'$, where κ_a is the affine curvature and τ_a is the *intrinsic affine torsion* (from now on called the affine torsion). Then if $\widetilde{\mathbf{T}}$ represents the affine tangent vector and $\widetilde{\mathbf{N}}$ the affine normal we have the following:

$$\frac{d}{ds}\begin{pmatrix} \widetilde{\mathbf{T}} \\ \widetilde{\mathbf{N}} \\ \widetilde{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa_a & 0 & 1 \\ -\tau_a & 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{T}} \\ \widetilde{\mathbf{N}} \\ \widetilde{\mathbf{B}} \end{pmatrix} .$$

Reformulation of A_k conditions

Using the new equi-affine frame $\{\widetilde{\mathbf{T}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{B}}\}$ the conditions for A_k singularities can be reformulated, for $(\alpha, \beta) \in \mathbb{R}^2$, as

$$\begin{aligned} A_{\geq 1} &\iff \mathbf{x} = \gamma + \alpha\widetilde{\mathbf{T}} + \beta\widetilde{\mathbf{B}} \ , \\ A_{\geq 2} &\iff \mathbf{x} = \gamma + \beta\widetilde{\mathbf{B}} \ , \\ A_{\geq 3} &\iff \mathbf{x} = \gamma + \frac{1}{\tau_a}\widetilde{\mathbf{B}} \ . \end{aligned}$$

3.3.3 Local structure of the bifurcation set

Here we classify the singularities of the bifurcation set. Using the basic ideas of *unfoldings* found in [4] we have the following:

Criterion 3.3.6 *Let $F : (\mathbb{R}^n \times \mathbb{R}, (\mathbf{x}_0, s_0)) \to \mathbb{R}$ be an n-parameter unfolding of f, which has type A_k at s_0. Then F is (p)-versal if, and only if, every real polynomial $p(s)$ of degree $\leq k-1$ and without constant term can be written in the form*

$$p(s) = \sum_{i=1}^{n} c_i j^{k-1} \left(\frac{\partial F}{\partial x_i}(\mathbf{x}_0, s_0) \right) \Bigg|_{s=s_0}$$

for real constants c_i, where j^{k-1} denotes the $(k-1)$-jet.

Proposition 3.3.7 *Let $j^{k-1}(\partial F/\partial x_i(\mathbf{x}_0, s_0))(s_0) = \alpha_{1,i}s + \alpha_{2,i}s^2 + \cdots + \alpha_{k-1,i}s^{k-1}$ for $1 \leq i \leq n$. Then F is (p)-versal if, and only if, the $(k-1) \times r$ matrix of coefficients $(\alpha_{j,i})$ has rank $k-1$.*

Proposition 3.3.8 *Let $F : (\mathbb{R}^n \times \mathbb{R}, (\mathbf{x}_0, s_0)) \to \mathbb{R}$ be an n-parameter unfolding of the function $f = F_{\mathbf{x}_0}$. Let $\mathbb{R}[s]$ denote the ring of polynomials in s and let $\mathfrak{m}$ denote the maximal ideal consisting of polynomials with zero constant term. Finally let $\langle s^k \rangle$ denote the ideal of polynomial multiples of s^k. Then F is (p)-versal if, and only if, the $(k-1)$-jets in Proposition 3.3.7 span the real vector space $\mathfrak{m}/\langle s^k \rangle$.*

Remark 3.3.9 *Given a family $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, we wish to consider the local structure of the bifurcation set. Let $\mathbf{x}_0$ be a point of the bifurcation set with corresponding value s_0 (i.e. $\partial F/\partial s = \partial^2 F/\partial s^2 = 0$ at $(\mathbf{x}_0, s_0)$). It is possible to decide if F (p)-versally unfolds f at s_0 by finding $\partial F/\partial x_i$ and using the matrix condition in Proposition 3.3.7. If the condition is satisfied then near to $\mathbf{x}_0$ the bifurcation set is locally diffeomorphic to a standard model, determined by n and k.*

Proposition 3.3.10 *Let $F : (\mathbb{R}^n \times \mathbb{R}, (\mathbf{x}_0, s_0)) \to \mathbb{R}$ be an n-parameter unfolding of $f = F_{\mathbf{x}_0}$ which has an A_k singularity at s_0. If F is a (p)-versal unfolding then*

1. *If $k = 2$, then the bifurcation set is locally diffeomorphic to $\mathbb{R}^{n-1} \times \{0\}$.*

2. *If $k = 3$, then the bifurcation set is locally diffeomorphic to $\mathbb{R}^{n-2} \times C$.*

3. *If $k = 4$, then the bifurcation set is locally diffeomorphic to $\mathbb{R}^{n-3} \times S$.*

Where C is an ordinary cusp and S is a swallow tail:

$$\begin{aligned} C &= \{(u^2, u^3) : u \in \mathbb{R}\} \ , \\ S &= \{(u, 2v^3 + uv, 3v^4 + uv^2) : (u, v) \in \mathbb{R}^2\} \ . \end{aligned}$$

Proposition 3.3.11 *Consider a smooth space curve, with $\kappa\tau \neq 0$, parametrised by affine arc-length. The 3-parameter family of affine distance functions defined on the curve, $\Delta : \mathbb{R}^3 \times I \to \mathbb{R}$, is (p)-versally unfolded provided $\tau_a \neq 0$.*

Proof Consider $\Delta(\mathbf{x}, s) = [\mathbf{x} - \gamma, \gamma', \gamma'']$. Writing $\gamma(s) = (X(s), Y(s), Z(s))$:

$$\begin{aligned} \Delta_x(\mathbf{x}, s) &= \begin{vmatrix} 1 & X' & X'' \\ 0 & Y' & Y'' \\ 0 & Z' & Z'' \end{vmatrix} , \\ \Delta_y(\mathbf{x}, s) &= \begin{vmatrix} 0 & X' & X'' \\ 1 & Y' & Y'' \\ 0 & Z' & Z'' \end{vmatrix} , \\ \Delta_z(\mathbf{x}, s) &= \begin{vmatrix} 0 & X' & X'' \\ 0 & Y' & Y'' \\ 1 & Z' & Z'' \end{vmatrix} . \end{aligned}$$

Writing $\Delta_{\mathbf{x}}(\mathbf{x}, s) = (\Delta_x(\mathbf{x}, s), \Delta_y(\mathbf{x}, s), \Delta_z(\mathbf{x}, s))$ it follows that $\Delta_{\mathbf{x}}(\mathbf{x}, s) = \gamma' \times \gamma''$. Differentiating with respect to affine arc-length yields

$$\begin{aligned} \Delta_{\mathbf{x}s}(\mathbf{x}, s) &= \gamma' \times \gamma''' \ , \\ \Delta_{\mathbf{x}ss}(\mathbf{x}, s) &= \gamma'' \times \gamma''' - \mu(\gamma' \times \gamma'') \ , \\ \Delta_{\mathbf{x}sss}(\mathbf{x}, s) &= (\nu - \mu')(\gamma' \times \gamma'') - \mu(\gamma' \times \gamma''') \ . \end{aligned}$$

Writing these in terms of the basis $\{\widetilde{\mathbf{T}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{B}}\}$ gives

$$\begin{aligned} \Delta_{\mathbf{x}s}(\mathbf{x}, s) &= \widetilde{\mathbf{T}} \times \widetilde{\mathbf{B}} \ , \\ \Delta_{\mathbf{x}ss}(\mathbf{x}, s) &= \widetilde{\mathbf{N}} \times \widetilde{\mathbf{B}} \ , \\ \Delta_{\mathbf{x}sss}(\mathbf{x}, s) &= \tau_a \widetilde{\mathbf{T}} \times \widetilde{\mathbf{N}} - \kappa_a \widetilde{\mathbf{T}} \times \widetilde{\mathbf{B}} \ . \end{aligned}$$

Given vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, an elementary calculation shows that

$$[\mathbf{u} \times \mathbf{v}, \mathbf{u} \times \mathbf{w}, \mathbf{v} \times \mathbf{w}] = [\mathbf{u}, \mathbf{v}, \mathbf{w}]^2 \ .$$

Thus $[\widetilde{\mathbf{T}} \times \widetilde{\mathbf{N}}, \widetilde{\mathbf{T}} \times \widetilde{\mathbf{B}}, \widetilde{\mathbf{N}} \times \widetilde{\mathbf{B}}] = [\widetilde{\mathbf{T}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{B}}]^2 = 1$, which gives

$$[\Delta_{\mathbf{x}sss}(\mathbf{x}, s), \Delta_{\mathbf{x}s}(\mathbf{x}, s), \Delta_{\mathbf{x}ss}(\mathbf{x}, s)] = \tau_a \ ,$$

$\{\Delta_{\mathbf{x}s}(\mathbf{x}, s), \Delta_{\mathbf{x}ss}(\mathbf{x}, s), \Delta_{\mathbf{x}sss}(\mathbf{x}, s)\}$ are linearly independent if, and only if, $\tau_a \neq 0$. It follows that the 3×3 jet matrix

$$\tilde{J} = \begin{pmatrix} \widetilde{\mathbf{T}} \times \widetilde{\mathbf{B}} \\ \frac{1}{2!}(\widetilde{\mathbf{N}} \times \widetilde{\mathbf{B}}) \\ \frac{1}{3!}(\tau_a \widetilde{\mathbf{T}} \times \widetilde{\mathbf{N}} - \kappa_a \widetilde{\mathbf{T}} \times \widetilde{\mathbf{B}}) \end{pmatrix}$$

in Proposition 3.3.7 is non-singular if, and only if, $\tau_a \neq 0$. The result follows. $\square$

Corollary 3.3.12 *Assume that $\tau_a \neq 0$ along γ. At points where $\mathbf{x} = \gamma + \alpha\widetilde{\mathbf{T}} + \beta\widetilde{\mathbf{N}}$ and $\alpha \neq 0$ the affine focal set will be locally a smooth surface. At points where $\mathbf{x} = \gamma + \beta\widetilde{\mathbf{N}}$ and $\beta \neq 1/\tau_a$ the affine focal set will locally be a cuspidal edge. At points where $\mathbf{x} = \gamma + (1/\tau_a)\widetilde{\mathbf{N}}$ the affine focal set will locally be a swallow tail.*

Chapter 4

Curves in $\mathbb{R}^n$

The aim of this section is to generalise some of the affine invariant machinery presented above, for space curves in $\mathbb{R}^n$. These include affine arc-length, affine curvatures, and affine distance. These ideas have been published, the reader is referred to D. Davis [8]. This chapter is by far the author's proudest mathematical achievement.

In this chapter we find a new basis for a space curve which has some very nice properties. This new basis allows us to carry out versal unfolding calculations which would otherwise be very complicated indeed.

4.1 Affine arc-length

Let $I \subseteq \mathbb{R}$ be an open interval, and $\gamma : I \to \mathbb{R}^n$ a smooth space curve. We seek an affine invariant parametrisation for γ of the lowest possible order. As is the convention for $n = 2, 3$ we choose a parametrisation, in terms of the affine arc-length parameter s, such that $[\gamma', \gamma'', \ldots, \gamma^{(n)}] = 1$ for all $s \in I$. Throughout this paper, prime denotes differentiation with respect to the *affine arc-length parameter* s, thus $\gamma' = d\gamma/ds$ etc, whereas a dot is reserved for differentiation with respect to an arbitrary parameter t, thus $\dot{\gamma} = d\gamma/dt$ etc. Using basic properties of determinants, it is easy to show that

$$[\gamma', \gamma'', \ldots, \gamma^{(n)}] = [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}] \left(\frac{dt}{ds}\right)^{n(n+1)/2} . \tag{4.1}$$

Assuming that $[\gamma', \gamma'', \ldots, \gamma^{(n)}] = 1$ we obtain

$$s(t) = \int [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^{2/n(n+1)} \, dt \; .$$

Thus for $t_1 \leq t \leq t_2$, affine arc-length is given by

$$\int_{t_1}^{t_2} [\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^{2/n(n+1)} \, dt \; .$$

Remark 4.1.1 Let $J \subseteq \mathbb{R}$ and consider a curve $\alpha : J \to \mathbb{R}^n$ parametrised by Euclidean arc-length. We define the tangent vector $\mathbf{V}_1$ to be the unit vector in the direction of $\dot{\alpha}$. The second basis vector $\mathbf{V}_2$ is in the subspace $\langle \dot{\alpha}, \ddot{\alpha} \rangle$, is of unit length, is perpendicular to $\mathbf{V}_1$, and together with $\mathbf{V}_1$ spans an area of $+1$. The third basis vector $\mathbf{V}_3$ is in the subspace $\langle \dot{\alpha}, \ddot{\alpha}, \dddot{\alpha} \rangle$, is of unit length, is perpendicular to $\mathbf{V}_1$ and $\mathbf{V}_2$, and together with $\mathbf{V}_1$ and $\mathbf{V}_2$ spans a volume of $+1$. Proceeding in this fashion, the $(k+1)$-st basis vector is in the space $\langle d^i\alpha/dt^i : 1 \leq i \leq k \rangle$, is of unit length, is perpendicular to $\{\mathbf{V}_i : 1 \leq i \leq k\}$, and together with $\{\mathbf{V}_i : 1 \leq i \leq k\}$ spans a volume of $+1$.

Definition 4.1.2 *Given a smooth curve parameterised by Euclidean arc-length, the Euclidean curvature is given by* $\kappa = \dot{\mathbf{V}}_1 \cdot \mathbf{V}_2$ *and the higher Euclidean torsions are given by* $\tau_i = \dot{\mathbf{V}}_{i+1} \cdot \mathbf{V}_{i+2}$ *for all* $1 \leq i \leq n-2$.

Remark 4.1.3 Letting t be Euclidean arc-length and writing κ for the Euclidean curvature of γ and $\{\tau_1, \ldots, \tau_{n-2}\}$ for the higher Euclidean torsions gives

$$[\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}] = \kappa^{n-1} \prod_{i=1}^{n-2} \tau_i^{n-i-1} \; .$$

Then Equation (4.1) shows that if $[\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}] = 0$ for some t, then the affine arc-length parametrisation in unobtainable, since $0 \neq 1$. Hence, if any of the Euclidean curvatures or Euclidean torsions become zero at certain points, the affine arc-length parameter can not be defined at such points. Hence, in all that follows, $I \subseteq \mathbb{R}$ shall be chosen such that the image of γ has everywhere non-zero Euclidean curvature and Euclidean torsions.

4.2 Affine curvatures

Here we define the *affine curvatures* of a curve. Let $\gamma : I \to \mathbb{R}^n$ be parametrised by affine arc-length, so that $[\gamma', \gamma'', \ldots, \gamma^{(n)}] = 1$ for all $s \in I$. Then differentiating with respect to s gives $[\gamma', \ldots, \gamma^{(n-1)}, \gamma^{(n+1)}] = 0$. Hence the set of vectors $\{\gamma', \ldots, \gamma^{(n-1)}, \gamma^{(n+1)}\}$ is linearly dependent. Therefore, there must exist functions $\mu_i : I \to \mathbb{R}$ for $1 \leq i \leq n-1$ such that

$$\gamma^{(n+1)} + \mu_1 \gamma' + \mu_2 \gamma'' + \cdots + \mu_{n-1} \gamma^{(n-1)} = 0 \ . \tag{4.2}$$

The functions μ_i are called the *affine curvatures* of γ. Notice that

$$\mu_i = (-1)^{n-i+1} [\gamma', \ldots, \gamma^{(i-1)}, \gamma^{(i+1)}, \ldots, \gamma^{(n+1)}] \ .$$

The μ_i are given by determinants; an equi-affine transformation of $\mathbb{R}^n$ leaves the affine curvatures unchanged. These affine curvatures are truly affine invariants.

These definitions give Serret-Frenet type formulae. Let $\Gamma = (\gamma', \gamma'', \ldots, \gamma^{(n)})^\top$ where $\top$ denotes transpose; then for $M \in \mathrm{Mat}(n, \mathbb{R})$

$$\Gamma' = M \Gamma \ . \tag{4.3}$$

It follows that if $M = (m_{i,j})$ then

$$m_{i,j} = \begin{cases} 1 & \text{if } j - i = 1 \ , \\ -\mu_j & \text{if } i = n, \ j \neq n \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{4.4}$$

Hence $\det(M) = (-1)^n \mu_1$.

4.3 Affine distance functions

Here we give a general definition of the affine distance function introduced in two and three-dimensions in [12].

Let $\gamma : I \to \mathbb{R}^n$ be parametrised by affine arc-length. Given $\mathbf{x} \in \mathbb{R}^n$ and $s \in I$, we get $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$, an n-parameter family of affine distance functions defined on the curve, where

$$\Delta(\mathbf{x}, s) = [\mathbf{x} - \gamma, \gamma', \ldots, \gamma^{(n-1)}] \ . \tag{4.5}$$

The zero level-set of $\Delta(\mathbf{x}, s_0)$ is given by $\mathbf{x} \in \mathbb{R}^n$ such that for some $\lambda_i \in \mathbb{R}$

$$\mathbf{x} = \gamma(s_0) + \lambda_1\gamma'(s_0) + \lambda_2\gamma''(s_0) + \cdots + \lambda_{n-1}\gamma^{(n-1)}(s_0) \ .$$

This is the set of points $\mathbf{x} \in \mathbb{R}^n$ of affine distance zero from $\gamma(s_0)$. It is easy to see that the other level-sets are hyperplanes parallel to this one.

Given an open interval $J \subseteq \mathbb{R}$, and an arbitrary parametrisation for the curve $\gamma : J \to \mathbb{R}^n$, the family of affine distance functions $\Delta : \mathbb{R}^n \times J \to \mathbb{R}$ is given by

$$\Delta(\mathbf{x}, t) = [\mathbf{x} - \gamma, \dot{\gamma}, \ldots, \gamma^{(n-1)}][\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^{(1-n)/(1+n)} \ .$$

4.4 Affine height functions

Let $\gamma : I \to \mathbb{R}^n$ be parametrised by affine arc-length. Let $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| = 1\}$ be the unit hypersphere in $\mathbb{R}^n$. We can define a family of functions on the curve, parametrised by S^{n-1}. This family $H : S^{n-1} \times I \to \mathbb{R}$ is the family of *affine height functions*, where

$$H(\mathbf{x}, s) = [\mathbf{x}, \gamma', \gamma'', \ldots, \gamma^{(n-1)}] \ .$$

Remark 4.4.1 In the definition of the family of affine height functions we took $\mathbf{x} \in S^{n-1}$; even though hyperspheres do not always go to hyperspheres under affine transformations. This was done for simplicity. We should have $\mathbf{x} \in T_{\gamma(s)}\mathbb{R}^n - \{\mathbf{0}\}$. However it turns out that the singularities of H depend only on $\mathbf{x}$ up to a non-zero multiplicative constant, i.e. $\mathbf{x}$ can be taken as a member of the projectivisation $\mathbb{P}(T_{\gamma(s)}\mathbb{R}^n - \{\mathbf{0}\}) := (T_{\gamma(s)}\mathbb{R}^n - \{\mathbf{0}\})/\sim$ where $\mathbf{x}_1 \sim \mathbf{x}_2$ for all $\mathbf{x}_i \in T_{\gamma(s)}\mathbb{R}^n - \{\mathbf{0}\}$ if, and only if, there exists $\alpha \in \mathbb{R}$, $\alpha \neq 0$, such that $\mathbf{x}_1 = \alpha\mathbf{x}_2$. Since S^{n-1} can be used to parametrised $\mathbb{P}(T_{\gamma(s)}\mathbb{R}^n - \{\mathbf{0}\})$ (in fact it gives a double cover) we shall write S^{n-1} with the understanding that it is an abstract parameter space and should not be considered to be in $\mathbb{R}^n$.

Let $J \subseteq \mathbb{R}$ be an open interval, then for an arbitrary parametrisation, the family of affine height functions is given by $H : S^{n-1} \times I \to \mathbb{R}$ where

$$H(\mathbf{x}, t) = [\mathbf{x}, \dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n-1)}][\dot{\gamma}, \ddot{\gamma}, \ldots, \gamma^{(n)}]^{(1-n)/(1+n)} .$$

4.5 Equi-affine frames

Let $\mathbf{x} \in \mathbb{R}^n$ and let $\{v_1, \ldots, v_n\}$ be a list of vectors $v_i \in T_{\mathbf{x}}\mathbb{R}^n$. The vectors are said to constitute an *equi-affine frame* if $[v_1, \ldots, v_n] = 1$. It is clear that $\{\gamma', \ldots, \gamma^{(n)}\}$ forms an equi-affine frame with each $\gamma^{(i)} \in T_{\gamma(s)}\mathbb{R}^n$ for all $s \in I$.

The aim here is to define a new equi-affine frame for γ. This is motivated by later applications to singularity theory. Furthermore, the affine Serret-Frenet formulae with respect to this new equi-affine frame will be more analogous to the Euclidean Serret-Frenet formulae. For example, if the Euclidean torsion τ_{n-2} is zero then the curve can be contained in $\mathbb{R}^{n-1}$. This means the last basis vector, say $\mathbf{V}_n$, is constant. (If $n = 3$ then the binormal vector $\mathbf{B}$ is constant and γ is then a plane curve.) Given the affine Serret-Frenet formulae in Equation (4.3) and Equation (4.4), if $\mu_{n-1} = 0$, this in no way means that $\gamma^{(n-1)}$ is constant.

Given any smooth functions $\lambda_{i,j} : I \to \mathbb{R}$, the vectors

$$\gamma^{(i)} + \sum_{j=1}^{i-1} \lambda_{i,j}\gamma^{(j)} \quad \text{for all} \quad 1 \leq i \leq n$$

form an equi-affine frame. The classical case is when $\lambda_{i,j}(s) = 0$ for all $s \in I$ and $(i, j) \in [0, n] \times [0, n]$. Consider the vector given by $i = n$, that is

$$v = \gamma^{(n)} + \lambda_{n,1}\gamma' + \lambda_{n,2}\gamma'' + \cdots + \lambda_{n,n-1}\gamma^{(n-1)} .$$

We wish the derivative of v to depend on only one other member of the equi-affine frame. Setting $\lambda_{i,j} \equiv 0$ for all $i \leq 0$ or $j \leq 0$ gives

$$\begin{aligned} v' &= \sum_{i=1}^{n-1}(\lambda'_{n,i} - \mu_i)\gamma^{(i)} + \lambda_{n,i}\gamma^{(i+1)} , \\ &= (\lambda'_{n,1} - \mu_1)\gamma' + \lambda_{n,n-1}\gamma^{(n)} + \sum_{i=2}^{n-1}(\lambda'_{n,i} - \mu_i + \lambda_{n,i-1})\gamma^{(i)} . \end{aligned}$$

If v' is to be independent of v it follows that $\lambda_{n,n-1} \equiv 0$. In order to remove dependency on other derivatives set $\lambda_{n,i-1} = \mu_i - \lambda'_{n,i}$ for all $2 \leq i \leq n-1$. Starting with $i = n-1$ gives $\lambda_{n,n-2} = \mu_{n-1} - \lambda'_{n,n-1} = \mu_{n-1}$. In turn, putting $i = n-2$ gives $\lambda_{n,n-3} = \mu_{n-2} - \mu'_{n-1}$. Putting $i = n-3$ gives $\lambda_{n,n-4} = \mu_{n-3} - \mu'_{n-2} + \mu''_{n-1}$. Continuing this process for $2 \leq i \leq n-1$ gives

$$\lambda_{n,n-i} = \sum_{j=1}^{i-1} (-1)^{j+1} \mu_{n-i+j}^{(j-1)} \ .$$

Then finally, the vector v' becomes

$$v' = \left(\sum_{i=1}^{n-1} (-1)^i \mu_i^{(i-1)} \right) \gamma' = \sigma_{n-1}\gamma', \text{ say } . \tag{4.6}$$

Thus the derivative of v depends only on one vector and is more analogous to the Euclidean Serret-Frenet system.

This has found a new basis vector, namely v. Let us call it $\mathbf{T}_n$ and search for a new basis $\{\mathbf{T}_1 \ldots, \mathbf{T}_n\}$. It is clear that $\mathbf{T}_1 = \gamma'$; this gives the affine tangent vector. Thus we have the identity $\mathbf{T}'_n = -\sigma_{n-1}\mathbf{T}_1$.

We wish to find a new equi-affine frame which satisfies the additional vector differential equations $\mathbf{T}'_1 = \mathbf{T}_2$, $\mathbf{T}'_i = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ for all $2 \leq i \leq n-1$. These can be written as $\mathbf{T}'_i = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ if we set $\sigma_0 \equiv 0$ and $\mathbf{T}_{n+1} \equiv \mathbf{0}$. From the affine arc-length construction, the functions $\mu_i : I \to \mathbb{R}$ arise naturally. Thus the $\sigma_i : I \to \mathbb{R}$ will be expressed in terms of the μ_i and their derivatives.

Consider the affine Serret-Frenet formulae in matrix form $\Gamma' = M\Gamma$, where Γ and M are defined above in Equation (4.3) and Equation (4.4). Each new basis vector $\mathbf{T}_i$ can be expressed in terms of Γ :

$$\mathbf{T}_i = \gamma^{(i)} + \sum_{j=1}^{i-1} \lambda_{i,j}\gamma^{(j)} \quad \text{for all} \quad 1 \leq i \leq n$$

This can be written in matrix notation as $T = \Lambda\Gamma$ where T is the matrix whose i-th row is the vector $\mathbf{T}_i$. Furthermore we can write $T' = \Sigma T$ where Σ is derived from the identities $\mathbf{T}'_1 = \mathbf{T}_2$, $\mathbf{T}'_i = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ for all $2 \leq i \leq n-1$, and $\mathbf{T}'_n = -\sigma_{n-1}\mathbf{T}_1$.

Thus we have $\Gamma' = M\Gamma$, $T = \Lambda\Gamma$, and $T' = \Sigma T$. It follows that $\Lambda'\Gamma + \Lambda\Gamma' = \Sigma T$. In turn, this gives $\Lambda'\Gamma + \Lambda M\Gamma = \Sigma T$. This finally yields $\Lambda'\Gamma + \Lambda M\Gamma = \Sigma\Lambda\Gamma$, or simply $\Lambda' + \Lambda M = \Sigma\Lambda$. Here M is known to us, and is given by the identity

$$\gamma^{(n+1)} + \mu_1\gamma' + \cdots + \mu_{n-1}\gamma^{(n-1)} = 0 \ .$$

Writing $\Sigma = (\sigma_{i,j})$ gives $\sigma_{i,j} = 1$ for all $j - i = 1$, $\sigma_{i,1} = -\sigma_{i-1}$ for all $2 \leq i \leq n$, and $\sigma_{i,j} = 0$ otherwise. Writing $\Lambda = (\lambda_{i,j})$ gives $\lambda_{i,j} = 1$ for all $i - j = 0$ and $\lambda_{i,j} = 0$ for all $j - i > 0$, i.e. Λ is a lower triangular matrix with 1 in each position along the leading diagonal.

Let $X = (x_{i,j})$ where $X = \Lambda' + \Lambda M - \Sigma\Lambda$; we wish to make X into the zero matrix. On the leading diagonal of X we have $x_{i,i} = \lambda_{i,i-1} - \lambda_{i+1,i}$. Since $\lambda_{1,0} = 0$ it follows that $\lambda_{i,i-1} = 0$ for all $2 \leq i \leq n$. This implies that Λ has zero along the diagonal $i - j = 1$. Thus each $\mathbf{T}_i$ will not have a component of $\gamma^{(i-1)}$.

Consider $x_{i,j}$ such that $i - j = 1$. It follows that $x_{n,n-1} = \lambda_{n,n-2} - \mu_{n-1}$, $x_{i,i-1} = \lambda_{i,i-2} - \lambda_{i+1,i-1}$ for all $3 \leq i \leq n-1$, and $x_{2,1} = \sigma_1 - \lambda_{3,1}$. Since $x_{i,j} = 0$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$ it follows that

$$\mu_{n-1} = \lambda_{n,n-2} = \lambda_{n-1,n-3} = \ldots = \lambda_{i,i-2} = \ldots = \lambda_{3,1} = \sigma_1 \ .$$

Considering each diagonal in turn, $i - j = 1, 2, 3, \ldots, n-1$ gives the following expressions for the σ_i, we have

$$\begin{aligned}
\sigma_1 &= a_{1,1}\mu_{n-1} \ , \\
\sigma_2 &= a_{2,1}\mu'_{n-1} + a_{2,2}\mu_{n-2} \ , \\
\sigma_3 &= a_{3,1}\mu''_{n-1} + a_{3,2}\mu'_{n-2} + a_{3,3}\mu_{n-3} \ , \\
\sigma_i &= \sum_{j=1}^{i} a_{i,j} \ \mu_{n-j}^{(i-j)} \ ,
\end{aligned}$$

where the $a_{i,j}$ are entries in an $(n-1) \times (n-1)$ lower triangular matrix, we have $a_{i,j} = 1$ for all $i = j$ and $a_{i,j} = 0$ for all $i < j$. When $i > j$ we have

$$a_{i,j} = (-1)^{i+j} \begin{pmatrix} n-j-1 \\ i-j \end{pmatrix} = (-1)^{i+j} \frac{(n-j-1)!}{(i-j)!(n-i-1)!} \ .$$

It follows that the σ_i are then given by

$$\sigma_i = \sum_{j=1}^{i} (-1)^{i+j} \begin{pmatrix} n-j-1 \\ i-j \end{pmatrix} \mu_{n-j}^{(i-j)} \ .$$

Given the existence of Σ and M is known, it is easy to find Λ for all $i - j \geq 1$

$$\lambda_{i,j} = \sum_{k=1}^{i-j-1} (-1)^{i-j-k-1} \begin{pmatrix} n-j-k-1 \\ i-j-k-1 \end{pmatrix} \mu_{n-k}^{(i-j-k-1)} \ .$$

In the present section we have proved the following

Proposition 4.5.1 *Given a curve $\gamma : I \to \mathbb{R}^n$ parametrised by affine arc-length. An equi-affine basis $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\}$ satisfying the vector differential equations $\mathbf{T}_1' = \mathbf{T}_2$, $\mathbf{T}_i' = \mathbf{T}_{i+1} - \sigma_{i-i}\mathbf{T}_1$ for all $2 \leq i \leq n-2$, and $\mathbf{T}_n = -\sigma_{n-1}\mathbf{T}_1$, can always be found.*

4.6 Singularities of $\Delta(\mathbf{x}, s)$ and $H(\mathbf{x}, s)$

Given a curve $\gamma : I \to \mathbb{R}^n$, we consider the full bifurcation set of the family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$. Given a fixed $\mathbf{x}_0 \in \mathbb{R}^n$, if there exists $s_0 \in I$ such that $\Delta'(\mathbf{x}_0, s_0) = \Delta''(\mathbf{x}_0, s_0) = 0$ then the family of affine distance functions is said to have a *degenerate singularity* at $\mathbf{x} = \mathbf{x}_0$. Given a fixed $\mathbf{x}_0 \in \mathbb{R}^n$, if there exists $(s_1, s_2) \in I \times I$ such that $\Delta(\mathbf{x}_0, s_1) = \Delta(\mathbf{x}_0, s_2)$ and $\Delta'(\mathbf{x}_0, s_1) = \Delta'(\mathbf{x}_0, s_2) = 0$ then the family of affine distance functions is said to have a *multi-local singularity* at $\mathbf{x} = \mathbf{x}_0$.

The full bifurcation set is then the closure of points $\mathbf{x} \in \mathbb{R}^n$ such that $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ has either a multi-local or degenerate singularity at $\mathbf{x}$. The bifurcation set is thus a subset of the parameter space. Similar ideas apply if we replace $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ by $H : S^{n-1} \times I \to \mathbb{R}$.

We use the standard A_k $(k \geq 2)$ notation for a degenerate singularity and A_1^2, A_1A_2 etc for a multi-local singularity.

Next we consider the condition for $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ to have an A_k singularity.

Theorem 4.6.1 *Let $\gamma : I \to \mathbb{R}^n$ be a smooth space curve parametrised by affine arc-length. For $1 \le k \le n-1$, the family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ has an A_k singularity at $\mathbf{x} \in \mathbb{R}^n$ if, and only if,, for $\lambda_i \in \mathbb{R}$*

$$\mathbf{x} = \gamma + \lambda_1 \mathbf{T}_1 + \cdots + \lambda_{n-k-1}\mathbf{T}_{n-k-1} + \lambda_n \mathbf{T}_n \quad \textit{and} \quad \lambda_{n-k-1} \neq 0 \ .$$

The family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ has an A_n singularity at $\mathbf{x} \in \mathbb{R}^n$ if, and only if, $\sigma_{n-1} \neq 0$, $\sigma'_{n-1} \neq 0$, and

$$\mathbf{x} = \gamma + \frac{1}{\sigma_{n-1}} \mathbf{T}_n \ .$$

Lemma 4.6.2 *Consider the equi-affine basis $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\} \subset T_\gamma \mathbb{R}^n$ with the property that $\mathbf{T}'_1 = \mathbf{T}_2$, $\mathbf{T}'_i = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ for all $2 \le i \le n-1$, and $\mathbf{T}'_n = -\sigma_{n-1}\mathbf{T}_1$. For all $0 \le m \le n-1$*

$$\Delta^{(m)} = [\mathbf{x} - \gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{n-m-1}, \mathbf{T}_{n-m+1}, \ldots, \mathbf{T}_n] \ .$$

Proof [Of Theorem 4.6.1] Consider the equi-affine basis $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\} \subset T_\gamma \mathbb{R}^n$:

$$\Delta(\mathbf{x}, s) = [\mathbf{x} - \gamma, \gamma', \ldots, \gamma^{(n-1)}] = [\mathbf{x} - \gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}] \ .$$

Recall that $\mathbf{T}'_1 = \mathbf{T}_2$, $\mathbf{T}'_i = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ for all $2 \le i \le n-1$, and $\mathbf{T}'_n = -\sigma_{n-1}\mathbf{T}_1$. It follows, using also $(\mathbf{x} - \gamma)' = -\mathbf{T}_1$ that

$$\begin{aligned} \Delta' &= \sum_{i=2}^{n-1} [\mathbf{x} - \gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{i-1}, \mathbf{T}'_i, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_{n-1}] \ , \\ &= \sum_{i=2}^{n-1} [\mathbf{x} - \gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{i-1}, \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_{n-1}] \ , \\ &= [\mathbf{x} - \gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n] \ . \end{aligned}$$

From Lemma 4.6.2 we see that for $\lambda_j \in \mathbb{R}$, $\Delta^{(m)} = 0$ if, and only if,

$$\mathbf{x} - \gamma = \lambda_1 \mathbf{T}_1 + \cdots + \lambda_{n-m-1}\mathbf{T}_{n-m-1} + \lambda_{n-m+1}\mathbf{T}_{n-m+1} + \cdots + \lambda_n \mathbf{T}_n \ .$$

This means that $\mathbf{x} \in \mathbb{R}^n$, for $0 \le k \le n-1$, gives an $A_{\ge k}$ singularity if, and only if,, for some $\lambda_i \in \mathbb{R}$

$$\mathbf{x} = \gamma + \lambda_1 \mathbf{T}_1 + \cdots + \lambda_{n-k-1}\mathbf{T}_{n-k-1} + \lambda_n \mathbf{T}_n \ .$$

The additional condition for exactly A_k is $\lambda_{n-k-1} \neq 0$.

Thus $\Delta' = \ldots = \Delta^{(n-1)} = 0$ if, and only if, $\mathbf{x} - \gamma = \lambda \mathbf{T}_n$ for some $\lambda \in \mathbb{R}$. This gives the condition for $A_{\geq n-1}$. We now consider the case A_n. It can easily be shown that

$$\Delta^{(n)} = [-\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n] + \sum_{i=2}^{n} [\mathbf{x} - \gamma, \mathbf{T}_2, \ldots, \mathbf{T}_{i-1}, -\sigma_{i-1}\mathbf{T}_1, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_n] \ .$$

It follows that $\Delta' = \ldots = \Delta^{(n)} = 0$ if, and only if,, for some $\lambda \in \mathbb{R}$

$$[\lambda \mathbf{T}_n, \mathbf{T}_2, \ldots, \mathbf{T}_{n-1}, -\sigma_{n-1}\mathbf{T}_1] = 1 \ .$$

This condition becomes, assuming $\sigma_{n-1} \neq 0$, that $\lambda \sigma_{n-1} = 1$ and thus $\lambda = 1/\sigma_{n-1}$. Finally, we need the condition for $\Delta^{(n+1)} \neq 0$. Given the expression for $\Delta^{(n)}$ and the fact that we will substitute $\mathbf{x} = \gamma + \lambda \mathbf{T}_n$ into the derived expression for $\Delta^{(n+1)}$ we need only consider

$$\begin{aligned} \alpha &= \sum_{i=2}^{n-1} [\mathbf{x} - \gamma, \mathbf{T}_2, \ldots, \mathbf{T}_{i-1}, \mathbf{T}'_i, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_{n-1}, -\sigma_{n-1}\mathbf{T}_1] \ , \\ &= \sum_{i=2}^{n-1} [\mathbf{x} - \gamma, \mathbf{T}_2, \ldots, \mathbf{T}_{i-1}, \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_{n-1}, -\sigma_{n-1}\mathbf{T}_1] \ , \\ &= [\mathbf{x} - \gamma, \mathbf{T}_2, \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n, -\sigma_{n-1}\mathbf{T}_1] \ . \end{aligned}$$

Then $\Delta^{(n+1)} = \alpha + [\mathbf{x} - \gamma, \mathbf{T}_2, \ldots, \mathbf{T}_{n-1}, -\sigma'_{n-1}\mathbf{T}_1 - \sigma_{n-1}\mathbf{T}_2]$

$$= [\mathbf{x} - \gamma, \mathbf{T}_2, \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n, -\sigma_{n-1}\mathbf{T}_1] + [\mathbf{x} - \gamma, \mathbf{T}_2, \ldots, \mathbf{T}_{n-1}, -\sigma'_{n-1}\mathbf{T}_1] \ .$$

Then $\Delta' = \ldots = \Delta^{(n)} = 0$ implies that

$$\begin{aligned} \Delta^{(n+1)} &= [\sigma_{n-1}^{-1}\mathbf{T}_n, \ldots, \mathbf{T}_{n-1}, -\sigma'_{n-1}\mathbf{T}_1] \ , \\ &= [\sigma_{n-1}^{-1}\mathbf{T}_n, \ldots, \mathbf{T}_{n-1}, -\sigma'_{n-1}\mathbf{T}_1] \ , \\ &= \sigma'_{n-1}\sigma_{n-1}^{-1} \ . \end{aligned}$$

Thus $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ has an A_n singularity at $\mathbf{x}$ if, and only if, $\sigma_{n-1} \neq 0$ and

$$\mathbf{x} = \gamma + \frac{1}{\sigma_{n-1}}\mathbf{T}_n \quad \text{and} \quad \sigma'_{n-1} \neq 0 \ .$$

□

Theorem 4.6.3 *Let $\gamma : I \to \mathbb{R}^n$ be a smooth space curve parametrised by affine arc-length. Then for $1 \leq k \leq n-1$, the family of affine height functions $H : S^{n-1} \times I \to \mathbb{R}$ has an A_k singularity at $\mathbf{x} \in S^{n-1}$ if, and only if,, for some $\lambda_i \in \mathbb{R}$*

$$\mathbf{x} = \lambda_1 \mathbf{T}_1 + \ldots + \lambda_{n-k-1} \mathbf{T}_{n-k-1} + \lambda_n \mathbf{T}_n \quad \textit{and} \quad \lambda_{n-k-1} \neq 0 \ .$$

Moreover, the family of affine height functions has an A_n singularity if, and only if, there exists $\lambda \in \mathbb{R}$, $\lambda \neq 0$ such that

$$\mathbf{x} = \lambda \mathbf{T}_n, \quad \sigma_{n-1} = 0 \quad \textit{and} \quad \sigma'_{n-1} \neq 0 \ .$$

Proof This is proved similarly to Theorem 4.6.1. □

4.6.1 The (p)-versality condition

Here we consider the conditions for the two above families to be a (p)-*versal unfoldings*. Due to the uniqueness of bifurcation sets, see [4], if a family of functions is a (p)-versal unfolding then each neighbourhood of its bifurcation set will be locally diffeomorphic to a standard model. Hence the local structure of the bifurcation set is determined up to diffeomorphism. Recall the basic ideas of *unfoldings* found in [4]. The can be found in Criterion 3.3.6 on page 38.

Theorem 4.6.4 *Given a smooth space curve $\gamma : I \to \mathbb{R}^n$ parametrised by affine arc-length. The family of affine distance functions $\Delta : \mathbb{R}^n \times I \to \mathbb{R}$ defined on the curve is (p)-versal if, and only if, $\sigma_{n-1} \neq 0$, where σ_{n-1} is given in Equation (4.6).*

Proof Let $\gamma : I \to \mathbb{R}^n$ be smooth, and let $\gamma(0) = 0$. Consider the frame $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\}$ where $\mathbf{T}'_1 = \mathbf{T}_2$, $\mathbf{T}'_i = \mathbf{T}_{i+1} - \sigma_{i-1}\mathbf{T}_1$ for all $2 \leq i \leq n-1$, and $\mathbf{T}'_n = -\sigma_{n-1}\mathbf{T}_1$. The affine distance function may be rewritten in terms of the $\mathbf{T}_i$, thus

$$\Delta(\mathbf{x}, s) = [\mathbf{x} - \gamma, \gamma', \ldots, \gamma^{(n-1)}] = [\mathbf{x} - \gamma, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}] \ .$$

Let $\Delta_{x_i} = \partial\Delta/\partial x_i$, and consider the vector $\Delta_{\mathbf{x}} = (\Delta_{x_1}, \ldots, \Delta_{x_n})$. Then by Proposition 3.3.7, to show that the family $\Delta(\mathbf{x}, s)$ is (p)-versal, one needs to shows that the first n derivatives of $\Delta_{\mathbf{x}}$, with respect to s, are linearly independent.

Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, etc, where $e_i \in T_\gamma\mathbb{R}^n$. Consider $\Delta_{\mathbf{x}}$, we have

$$\Delta_{\mathbf{x}} = ([e_1, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}], \ldots, [e_n, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}]) \ .$$

Notice that each $[e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}]$ is independent of $\mathbf{x}$. In what follows, it is enough to consider $\Delta_{x_i} = [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}]$ alone.

$$\begin{aligned} \Delta'_{x_i} &= \sum_{j=1}^{n-1} [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{j-1}, \mathbf{T}'_j, \mathbf{T}_{j+1} \ldots, \mathbf{T}_{n-1}] \ , \\ &= \sum_{j=2}^{n-1} [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_{j+1} - \sigma_{j-1}\mathbf{T}_1, \mathbf{T}_{j+1} \ldots, \mathbf{T}_{n-1}] \ , \\ &= [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n] \ . \end{aligned}$$

Next, consider Δ''_{x_i}, which is found in the same way. Given that $[e_i, \mathbf{T}'_1, \mathbf{T}_2, \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n] = [e_i, \mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{n-2}, \mathbf{T}'_n] = 0$, we have

$$\begin{aligned} \Delta''_{x_i} &= \sum_{j=2}^{n-2} [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{j-1}, \mathbf{T}'_j, \mathbf{T}_{j+1} \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n] \ , \\ &= \sum_{j=2}^{n-2} [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_{j+1} - \sigma_{j-1}\mathbf{T}_1, \mathbf{T}_{j+1} \ldots, \mathbf{T}_{n-2}, \mathbf{T}_n] \ , \\ &= [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{n-3}, \mathbf{T}_{n-1}, \mathbf{T}_n] \ . \end{aligned}$$

Continuing in this fashion gives the general answer:

$$\Delta^{(m)}_{x_i} = [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{n-m-1}, \mathbf{T}_{n-m+1}, \ldots, \mathbf{T}_n]$$

for all $1 \leq m \leq n-1$. Thus we need only consider the final case $m = n$. Notice that $\Delta_{x_i}^{(n-1)} = [e_i, \mathbf{T}_2, \ldots, \mathbf{T}_n]$, and so it follows

$$\begin{aligned}
\Delta_{x_i}^{(n)} &= \sum_{j=2}^{n} [e_i, \mathbf{T}_2, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_j', \mathbf{T}_{j+1}, \ldots, \mathbf{T}_n] \ , \\
&= \sum_{j=2}^{n} [e_i, \mathbf{T}_2, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_{j+1} - \sigma_{j-1}\mathbf{T}_1, \mathbf{T}_{j+1}, \ldots, \mathbf{T}_n] \ , \\
&= -\sum_{j=2}^{n} \sigma_{j-1} [e_i, \mathbf{T}_2, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_1, \mathbf{T}_{j+1}, \ldots, \mathbf{T}_n] \ , \\
&= \sum_{j=2}^{n} (-1)^{j+1} \sigma_{j-1} [e_i, \mathbf{T}_1, \ldots, \mathbf{T}_{j-1}, \mathbf{T}_{j+1}, \ldots, \mathbf{T}_n] \ , \\
&= \sum_{j=2}^{n} (-1)^{j+1} \sigma_{j-1} \Delta_{x_i}^{(n-j)} \ .
\end{aligned}$$

The aim here is to show that $[\Delta_{\mathbf{x}}', \ldots, \Delta_{\mathbf{x}}^{(n)}] \neq 0$. Due to the fact that

$$\Delta_{x_i}^{(n)} = \sum_{j=2}^{n} (-1)^{j+1} \sigma_{j-1} \Delta_{x_i}^{(n-j)} \ ,$$

it follows that $\Delta_{\mathbf{x}}^{(n)}$ is a linear combination of $\{\Delta_{\mathbf{x}}, \Delta_{\mathbf{x}}', \ldots, \Delta_{\mathbf{x}}^{(n-2)}\}$. It follows that $[\Delta_{\mathbf{x}}', \ldots, \Delta_{\mathbf{x}}^{(n)}] = 0 \iff \sigma_{n-1}[\Delta_{\mathbf{x}}', \ldots, \Delta_{\mathbf{x}}^{(n-1)}, \Delta_{\mathbf{x}}] = 0$.

The aim now is to show that $[\Delta_{\mathbf{x}}', \ldots, \Delta_{\mathbf{x}}^{(n-1)}, \Delta_{\mathbf{x}}] \neq 0$. Consider the $n \times n$ matrix $X = (x_{i,j})$ where

$$x_{i,j} = [e_j, \mathbf{T}_1, \ldots, \mathbf{T}_{i-1}, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_n] \ .$$

It follows that $\det(X) = [\Delta_{\mathbf{x}}^{(n-1)}, \Delta_{\mathbf{x}}^{(n-2)}, \ldots, \Delta_{\mathbf{x}}', \Delta_{\mathbf{x}}] = \pm[\Delta_{\mathbf{x}}', \ldots, \Delta_{\mathbf{x}}^{(n-1)}, \Delta_{\mathbf{x}}]$. Let T be the matrix whose i-th column is $\mathbf{T}_i$. Furthermore, let $A = (a_{i,j})$ be the adjoint matrix of T. Since

$$a_{i,j} = (-1)^{i+1}[e_j, \mathbf{T}_1, \ldots, \mathbf{T}_{i-1}, \mathbf{T}_{i+1}, \ldots, \mathbf{T}_n]$$

it follows that $a_{i,j} = (-1)^{i+1}x_{i,j}$, which implies $\det(X) = \pm\det(A)$. Next consider the well known identity $T^{-1} = \det(T)^{-1}A$, it follows that $\det(T)^{n-1} = \det(A)$. Thus $\det(X) = \pm\det(T)^{n-1} = \pm 1 \neq 0$. The result now follows. □

Theorem 4.6.5 *Given a smooth space curve $\gamma : I \to \mathbb{R}^n$ parametrised by affine arc-length. The family of affine height functions $H : S^{n-1} \times I \to \mathbb{R}$ defined on the curve is (p)-versal if, and only if, $\sigma_{n-1} \neq 0$, where σ_{n-1} is given in Equation (4.6).*

Proof This follows from the proof of Theorem 4.6.4 since $H_{\mathbf{x}} = \Delta_{\mathbf{x}}$ and

$$H(\mathbf{x}, s) = [\mathbf{x}, \gamma', \ldots, \gamma^{(n-1)}] = [\mathbf{x}, \mathbf{T}_1, \ldots, \mathbf{T}_{n-1}] \ .$$

□

Chapter 5

Basics of Surfaces

5.1 Embeddings and groups

Let $U \subseteq \mathbb{R}^2$ be an open simply connected domain. We consider smooth embeddings $\mathbf{X} : U \to \mathbb{R}^3$, i.e. injective immersions. Since $\mathbf{X}$ is an embedding, the vectors $\mathbf{X}_u$ and $\mathbf{X}_v$ are always linearly independent. The partial derivatives $\mathbf{X}_u$ and $\mathbf{X}_v$ form a basis for the tangent plane to the surface at any given point.

We will study the differential invariants of $\mathbf{X}$ under the action of the special affine group $\mathrm{SA}(3, \mathbb{R}) := \mathrm{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^3$.

5.2 Affine normal for a hypersurface

Here we consider a very general approach to affine differential geometry. This follows from the work of Nomizu in [13] and Nomizu & Sasaki in [14]. Let $\mathfrak{X}(A)$ denote the space of smooth vector fields over some smooth manifold A.

5.2.1 Connexions

Definition 5.2.1 *A connexion on A is a bilinear map $\nabla : \mathfrak{X}(A)^2 \to \mathfrak{X}(A)$, given by $(X, Y) \mapsto \nabla_X Y$, such that for all smooth functions $f : A \to \mathbb{R}$ and all $X, Y \in \mathfrak{X}(A)$ we have $\nabla_{fX} Y = f \nabla_X Y$ and $\nabla_X (fY) = df(X)Y + f \nabla_X Y$.*

Remark 5.2.2 The condition that $\nabla_{fX}Y = f\nabla_X Y$ says that ∇ is $C^\infty(A,\mathbb{R})$-linear in the first variable. The condition that $\nabla_X(fY) = df(X)Y + f\nabla_X Y$ says that ∇ satisfies the Leibnitz rule in the second variable.

Definition 5.2.3 *Given $X, Y \in \mathfrak{X}(A)$, let $[X, Y]$ denote the Lie bracket. A connexion $\nabla : \mathfrak{X}(A)^2 \to \mathfrak{X}(A)$ is called torsion free if for all $X, Y \in \mathfrak{X}(A)$*

$$\nabla_X Y - \nabla_Y X = [X, Y] \ .$$

In what follows, we assume that A is n-dimensional, and that $A \subset \mathbb{R}^{n+1}$. Moreover, we assume that the Gaußian curvature is everywhere non-zero. Let $D : \mathfrak{X}(\mathbb{R}^{n+1})^2 \to \mathfrak{X}(\mathbb{R}^{n+1})$ be the standard covariant derivative on $\mathbb{R}^{n+1}$. This is often called the standard affine connexion on $\mathbb{R}^{n+1}$. We can show that D is a torsion free connexion. This is a standard result: see [13] or [14]

Let ξ be a transverse vector field over A. For $X, Y \in \mathfrak{X}(A)$ we can write

$$D_X Y = \nabla_X Y + h(X, Y)\xi, \tag{5.1}$$

Definition 5.2.4 *The connexion ∇ in Equation (5.1) is called the induced connexion on A with respect to ξ. The bilinear form h is called the affine fundamental form on A with respect to ξ.*

Proposition 5.2.5 *The connexion D is torsion free if, and only if, the connexion ∇ is torsion free and the bilinear for h is symmetric.*

Proof From Equation (5.1) we see that

$$D_X Y - D_Y X = (\nabla_X Y - \nabla_Y X) + (h(X, Y) - h(Y, X))\xi \ .$$

It clearly follows that $D_X Y - D_Y X = [X, Y]$ if, and only if, $\nabla_X Y - \nabla_Y X = [X, Y]$ and $h(X, Y) = h(Y, X)$. □

Remark 5.2.6 Since D is torsion free the induced connexion with respect to ξ is torsion free and the affine fundamental form on A with respect to ξ is symmetric.

In an affine space $\mathbb{R}^k$ neighbouring tangent spaces, and therefore tangent vectors, can be canonically identified by translation, i.e. $T_p\mathbb{R}^k \to T_q\mathbb{R}^k$ via the translation taking p to q. For a manifold this construction does not exist.

A connexion gives a method of identifying (or connecting) neighbouring tangent spaces. This is done by means of parallel transport.

Let $\gamma : [t_1, t_2] \to A$ be a smooth curve segment in A. Consider $X \in \mathfrak{X}(A)$ restricted to $\gamma(I)$. We say that X is the parallel transport, with respect to the connexion ∇, of $X(t_1)$ along $\gamma(I)$ if $X(t)$ is a solution of the system of ordinary differential equation $\nabla_{\dot{\gamma}} X = 0$ for all $t \in I$.

Remark 5.2.7 In ordinary Euclidean space $\mathbb{E}^k$, equipped with the standard covariant derivative as its connexion, parallel transport keeps vectors parallel.

5.2.2 Volume forms

Let $\Omega : \mathfrak{X}(\mathbb{R}^{n+1})^{n+1} \to \mathbb{R}$ be the standard volume element of $\mathbb{R}^{n+1}$. Given an $(n+1)$-tuple of vectors $X_1, \ldots, X_{n+1}$, if we measure $\Omega(X_1, \ldots, X_{n+1})$ before and after the parallel transport with respect to D of the X_i along some curve, then the measurements will be the same. This can be expressed by $D\Omega = 0$ where $D\Omega = 0$ if, and only if, $D_X\Omega = 0$ for all $X \in \mathfrak{X}(\mathbb{R}^{n+1})$. The volume element Ω and the connexion D are said to be compatible.

Definition 5.2.8 *Let $X_i \in \mathfrak{X}(A)$ and ξ be a transverse vector field. The induced primary volume form $\omega : \mathfrak{X}(A)^n \to \mathbb{R}$ on A is given by*

$$\omega(X_1, \ldots, X_n) := \Omega(X_1, \ldots, X_n, \xi) \ .$$

We seek the condition(s) that the induced primary volume form ω and induced connexion ∇ to be compatible, i.e. $\nabla_X\omega = 0$ for all $X \in \mathfrak{X}(A)$.

Definition 5.2.9 *Given a transverse vector field ξ, the affine shape operator S and the transversal connexion form θ are given by the identity*

$$D_X\xi = -SX + \theta(X)\xi \ .$$

Lemma 5.2.10 (Nomizu [13]) *For every $X \in \mathfrak{X}(A)$ we have*

$$\nabla_X \omega = \theta(X)\omega \ .$$

Corollary 5.2.11 *The induced primary volume form ω and induced connexion ∇ are compatible if, and only if, the transversal connexion form is identically zero, i.e.*

$$\nabla\omega = 0 \iff \theta \equiv 0 \ .$$

Using the decomposition $D_X\xi = -SX + \theta(X)\xi$, we see that $\nabla\omega = 0$ if, and only if, $D_X\xi$ is tangent to A for all $X \in \mathfrak{X}(A)$.

We define a second volume element on A :

Definition 5.2.12 *Let $X_i \in \mathfrak{X}(A)$ and ξ be a transverse vector field. Let H be the $n \times n$ matrix, say $H := (h_{i,j})$, where $h_{i,j} := h(X_i, X_j)$. The induced secondary volume form $\nu : \mathfrak{X}(A)^n \to \mathbb{R}$ is given by*

$$\nu(X_1, \ldots, X_n) := |\det(H)|^{1/2} \ .$$

Example. Consider Euclidean 2-space with the usual flat affine connexion D, where h is the scalar product: $h(X,Y) = X \cdot Y$. We wish to calculate $\nu(X,Y)$. We have $h(X,X) = ||X||^2$, $h(X,Y) = h(Y,X) = ||X|| \cdot ||Y|| \cdot \cos\theta$ and $h(Y,Y) = ||Y||^2$, where θ is the angle between X and Y. It follows that

$$H = \begin{pmatrix} ||X||^2 & ||X|| \cdot ||Y|| \cdot \cos\theta \\ ||X|| \cdot ||Y|| \cdot \cos\theta & ||Y||^2 \end{pmatrix}.$$

In turn we see that $\det(H) = ||X||^2||Y||^2(1-\cos^2\theta) = ||X||^2||Y||^2\sin^2\theta$. Finally we see that $\nu(X,Y) = ||X|| \cdot ||Y|| \cdot \sin\theta$. An elementary trigonometric argument shows that the area of the parallelogram with sides X and Y is exactly $\nu(X,Y)$.

Remark 5.2.13 The last example gives motivation for the definition of the induced secondary volume form. To a metric it associates a volume form which in the case of the scalar product is the geometric volume. The induced secondary volume form can be defined for an abstract manifold with metric, not necessarily an embedded manifold. However, once we have an embedding and a transverse vector field we get a metric, and so the induced secondary volume form.

Theorem 5.2.14 (Nomizu [13]) *There is, up to sign, a unique transverse vector field* ξ*, for which the following two conditions are met:*

1. $\nabla_X \omega = 0$ *for all* $X \in \mathfrak{X}(A)$,

2. $\omega(X_1, \ldots, X_n) = \nu(X_1, \ldots, X_n)$ *for all* $(X_1, \ldots, X_n) \in \mathfrak{X}(A)^n$.

Remark 5.2.15 We may decide upon the choice of sign by giving a local orientation to our hypersurface and then using this orientation to co-orient ξ.

Definition 5.2.16 *The unique (after co-orientation) transverse vector field in Theorem 5.2.14 is called the affine normal vector field, and shall be denoted by* **A**. *The affine normal vector field is also known as the Blaschke normal field.*

Remark 5.2.17 In all that follows we consider smooth surfaces embedded in $\mathbb{R}^3$ with non-zero Gaußian curvature.

5.3 Asymptotic directions

Asymptotic curves on the surface are affine invariants. A curve is called asymptotic if at each point its osculating plane when considered as a space curve coincides with the tangent plane to the surface. A direction is called an asymptotic direction if it is the direction of the tangent line to an asymptotic curve through that point. Asymptotic directions are also the direction of lines which have order of contact greater than two with the surface at that point.

Let $I \subseteq \mathbb{R}$ be an open interval, and $\gamma : I \to U$ an embedding. The curve $\mathbf{X} \circ \gamma : I \to \mathbb{R}^3$ is an asymptotic curve if, and only if, $[\mathbf{X}_u, \mathbf{X}_v, (\mathbf{X} \circ \gamma)_{tt}] = 0$ for all $t \in I$. Let $\gamma(t) = (u(t), v(t))$ then using simple properties of determinants

$$[\mathbf{X}_u, \mathbf{X}_v, (\mathbf{X} \circ \gamma)_{tt}] = [\mathbf{X}_u, \mathbf{X}_v, \dot{u}^2 \mathbf{X}_{uu} + 2\dot{u}\dot{v}\mathbf{X}_{uv} + \dot{v}^2 \mathbf{X}_{vv}] \ .$$

Let $L := [\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}]$, $M := [\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}]$, and $N := [\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}]$, then

$$[\mathbf{X}_u, \mathbf{X}_v, (\mathbf{X} \circ \gamma)_{tt}] = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 \ .$$

A curve $\mathbf{X} \circ \gamma$ is therefore an asymptotic curve if, and only if,

$$L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 = 0 \ .$$

A direction $(du : dv)$ is an asymptotic direction if, and only if,

$$L \ du^2 + 2M \ du \ dv + N \ dv^2 = 0 \ .$$

The Gaußian curvature K and the functions L, M, and N are linked, we have

$$\begin{aligned} K < 0 &\iff LN - M^2 < 0 \ , \\ K = 0 &\iff LN - M^2 = 0 \ , \\ K > 0 &\iff LN - M^2 > 0 \ . \end{aligned}$$

Affine fundamental form

Proposition 5.3.1 *Let A be a surface in $\mathbb{R}^3$ paramatrised by $\mathbf{X} : U \to \mathbb{R}^3$, with the affine normal vector field $\mathbf{A}$. The matrix corresponding to the symmetric bilinear form h with respect to the basis $\{\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}\}$ is given by*

$$\frac{1}{|LN - M^2|^{1/4}} \begin{pmatrix} L & M \\ M & N \end{pmatrix} .$$

Proof Due to the properties of any curvature free connexion (see [14]), we need only consider $D_{\mathbf{X}_u}\mathbf{X}_u = \mathbf{X}_{uu}$, $D_{\mathbf{X}_u}\mathbf{X}_v = \mathbf{X}_{uv}$, $D_{\mathbf{X}_v}\mathbf{X}_u = \mathbf{X}_{vu}$, and $D_{\mathbf{X}_v}\mathbf{X}_v = \mathbf{X}_{vv}$. We then solve

$$\begin{aligned} \mathbf{X}_{uu} &= \Gamma^1_{1,1}\mathbf{X}_u + \Gamma^2_{1,1}\mathbf{X}_v + \Gamma^3_{1,1}\mathbf{A} \ , \\ \mathbf{X}_{uv} &= \Gamma^1_{1,2}\mathbf{X}_u + \Gamma^2_{1,2}\mathbf{X}_v + \Gamma^3_{1,2}\mathbf{A} \ , \\ \mathbf{X}_{vu} &= \Gamma^1_{2,1}\mathbf{X}_u + \Gamma^2_{2,1}\mathbf{X}_v + \Gamma^3_{2,1}\mathbf{A} \ , \\ \mathbf{X}_{vv} &= \Gamma^1_{2,2}\mathbf{X}_u + \Gamma^2_{2,2}\mathbf{X}_v + \Gamma^3_{2,2}\mathbf{A} \ . \end{aligned}$$

for smooth functions $\Gamma^k_{i,j}$ where $\Gamma^k_{i,j} = \Gamma^k_{j,i}$. The matrix corresponding to h is then

$(\Gamma^3_{i,j})$. Consider the equation $\mathbf{X}_{uu} = \Gamma^1_{1,1}\mathbf{X}_u + \Gamma^2_{1,1}\mathbf{X}_v + \Gamma^3_{1,1}\mathbf{A}$, we have

$$\begin{aligned}[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}] &= [\mathbf{X}_u, \mathbf{X}_v, \Gamma^1_{1,1}\mathbf{X}_u + \Gamma^2_{1,1}\mathbf{X}_v + \Gamma^3_{1,1}\mathbf{A}] \ , \\ &= [\mathbf{X}_u, \mathbf{X}_v, \Gamma^3_{1,1}\mathbf{A}] \ , \\ &= \Gamma^3_{1,1}[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]\end{aligned}$$

Since $\mathbf{X}$ has no parabolic points we have $[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}] \neq 0$ and it follows that

$$\Gamma^3_{1,1} = \frac{[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}]}{[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]} = \frac{L}{|LN - M^2|^{1/4}} \ .$$

The same methods calculate the $\Gamma^3_{i,j}$. □

Corollary 5.3.2 *At points where $LN - M^2 \neq 0$ the affine fundamental form can be expressed as*

$$\frac{L\ du^2 + 2M\ du\ dv + N\ dv^2}{|LN - M^2|^{1/4}} \ .$$

The denominator makes it invariant under the action of the infinite-dimensional pseudo-group of reparametrisations.

5.4 Affine normal vector

The affine normal vector field $\mathbf{A}$, for a smooth surface in $\mathbb{R}^3$ without parabolic points, has an explicit formulation. If Δ denotes the second differential operator of Beltrami, then $\mathbf{A} = \frac{1}{2}\Delta\mathbf{X}$. Explicitly it has a complicated expression. Assuming that $LN - M^2 \neq 0$ we have

$$\mathbf{A} := \frac{|LN - M^2|}{2\sqrt{LN - M^2}} \left(\frac{\partial}{\partial u} \left(\frac{N\mathbf{X}_u - M\mathbf{X}_v}{\sqrt{LN - M^2}} \right) + \frac{\partial}{\partial v} \left(\frac{L\mathbf{X}_v - M\mathbf{X}_u}{\sqrt{LN - M^2}} \right) \right) .$$

This vector field is invariant under the action of the special affine group and the infinite dimensional pseudo-group of reparametrisations. Two other key facts are that $\mathbb{R}^3 = \langle \mathbf{A} \rangle \oplus \langle \mathbf{X}_u, \mathbf{X}_v \rangle$ (i.e. $[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}] \neq 0$), and $\mathbf{A}_u, \mathbf{A}_v \in \langle \mathbf{X}_u, \mathbf{X}_v \rangle$. Since the transverse connexion form is identically zero, we see that $D_X\mathbf{A} = -SX$ for all

$X \in \mathfrak{X}(A)$. This affine shape operator is of great interest, and will be studied at length in chapter 9 page 101 – 129.

For non-degenerate quadrics the affine normal vector always points towards the centre of the quadric. In this way ellipsoids and hyperploids take the place of spheres, and paraboloids the place of planes, in the Euclidean theory. In the case of a paraboloid the centre is at infinity, and so all of the affine normals are parallel: they point towards a point on the 'plane at infinity'.

Let the affine normal line be the line passing through $\mathbf{X}(u,v)$ and parallel to $\mathbf{A}(u,v)$. There is a natural geometric interpretation of this in the elliptic region (see [7] and [15]). Consider a surface in the neighbourhood of an elliptic point p. Consider the one-parameter family of planes parallel to the tangent plane $T_p\mathbf{X}$. Let P_t be the family of planes, and let $P_0 = T_p\mathbf{X}$. The intersection of P_t, for sufficiently small t, with $\mathbf{X}$ bounds a two-dimensional convex domain in P_t. The locus of centres of mass of these domains is a curve γ in $\mathbb{R}^3$ terminating at p. The limiting tangent line to γ at p is the affine normal line to $\mathbf{X}$ at p.

5.5 Pick normal forms

Consider a non-parabolic surface point p. Take a coordinate system on $\mathbb{R}^3$, based at p, which has $\mathbf{X}_u$, $\mathbf{X}_v$, and $\mathbf{A}$ as its basis. If p is hyperbolic, then via a special affine transformation we can put $\mathbf{X}$ into the form

$$\mathbf{X}(u,v) = \left(u, v, \frac{1}{2}(u^2 - v^2) + \frac{\sigma}{6}(u^3 + 3uv^2) + \sum_{i=0}^{4} a_i u^{4-i} v^i + O(5) \right). \tag{5.2}$$

Where, for $k \in \mathbb{N}$, and the notation $O(k)$ means terms of order k or higher. If p is elliptic, then via a special affine transformation we can put $\mathbf{X}$ into the form

$$\mathbf{X}(u,v) = \left(u, v, \frac{1}{2}(u^2 + v^2) + \frac{\sigma}{6}(u^3 - 3uv^2) + \sum_{i=0}^{4} a_i u^{4-i} v^i + O(5) \right). \tag{5.3}$$

Once a surface point has been chosen, the Pick normal form is unique up to the sign of σ. A rotation of either $\pm\pi/3$ changes the sign. In chapter 6 on page 67 we take a surface in Monge form and, via a series of affine transformations, put the surface into Pick normal form.

Once in Pick normal form, we find that $\mathbf{X}_u(0,0) = (1,0,0)$, $\mathbf{X}_v(0,0) = (0,1,0)$ and $\mathbf{A}(0,0) = (0,0,1)$. These are some of the desired properties which first gave rise to the Pick normal form. See [7] for more details.

5.6 Affine shape operator

Since $\mathbf{A}_u, \mathbf{A}_v \in \langle \mathbf{X}_u, \mathbf{X}_v \rangle$, we may introduce a linear operator $\phi_p : T_p\mathbf{X} \to T_p\mathbf{X}$ given by $\phi_p(\mathbf{v}) := -D_{\mathbf{v}}\mathbf{A}$. If $\mathbf{X}_u$ and $\mathbf{X}_v$ are taken as an ordered basis on $T_p\mathbf{X}$ then there exist functions $a, b, c, d : U \to \mathbb{R}$ such that $\phi(\mathbf{X}_u) = a\mathbf{X}_u + b\mathbf{X}_v$ and $\phi(\mathbf{X}_v) = c\mathbf{X}_u + d\mathbf{X}_v$. Notice that $\phi(\mathbf{X}_u) = -\mathbf{A}_u$ and $\phi(\mathbf{X}_v) = -\mathbf{A}_v$.

Take $\mathbf{X}_u$ and $\mathbf{X}_v$ as an ordered basis for the tangent plane $T_p\mathbf{X}$. We have $\mathbf{A}_u = a\mathbf{X}_u + b\mathbf{X}_v$ and $\mathbf{A}_v = c\mathbf{X}_u + d\mathbf{X}_v$. Since $\mathbf{A}_u = a\mathbf{X}_u + b\mathbf{X}_v$ it follows that $[\mathbf{A}_u, \mathbf{X}_v, \mathbf{A}] = [a\mathbf{X}_u + b\mathbf{X}_v, \mathbf{X}_v, \mathbf{A}]$ and it turn $[\mathbf{A}_u, \mathbf{X}_v, \mathbf{A}] = a[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]$. At non-parabolic points $[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}] \neq 0$ and we can solve for a. Similar methods give expressions for the functions b, c, and d. The expressions are as follows:

$$
\begin{aligned}
a &= \frac{[\mathbf{A}_u, \mathbf{X}_v, \mathbf{A}]}{[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]}, \\
b &= \frac{[\mathbf{X}_u, \mathbf{A}_u, \mathbf{A}]}{[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]}, \\
c &= \frac{[\mathbf{A}_v, \mathbf{X}_v, \mathbf{A}]}{[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]}, \\
d &= \frac{[\mathbf{X}_u, \mathbf{A}_v, \mathbf{A}]}{[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]}.
\end{aligned}
$$

Definition 5.6.1 *Eigenvalues of the affine shape operator are called affine principal curvatures.*

Definition 5.6.2 *The reciprocals of the affine principal curvatures are called the affine radii of curvature.*

Definition 5.6.3 *The eigendirections of the affine shape operator are called affine principal directions.*

Definition 5.6.4 *A surface point is called an affine parabolic point if at least one affine principal curvature is zero at that point.*

Definition 5.6.5 *A point is called a repeated A-direction point if there is a single repeated affine principal direction.*

Definition 5.6.6 *A point is called an affine umbilic if every direction is an affine principal direction. Equivalently, when the affine shape operator is a scalar multiple of the identity.*

Here we give the values of a, b, c, and d at the origin of a surface in Pick normal form. Consider a hyperbolic point, let $\mathbf{X}$ be in the form of Equation (5.2), then

$$\left.\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} 6a_0 - a_2 - \frac{\sigma^2}{2} & \frac{3}{2}(a_1 - a_3) \\ \frac{3}{2}(a_3 - a_1) & 6a_4 - a_2 - \frac{\sigma^2}{2} \end{pmatrix}. \tag{5.4}$$

Consider an elliptic point, let $\mathbf{X}$ be in the form of Equation (5.3), then

$$\left.\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} 6a_0 + a_2 - \frac{\sigma^2}{2} & \frac{3}{2}(a_1 + a_3) \\ \frac{3}{2}(a_1 + a_3) & a_2 + 6a_4 - \frac{\sigma^2}{2} \end{pmatrix}. \tag{5.5}$$

Equation (5.4) and Equation (5.5) give the matrix representation with respect to the basis $\{\mathbf{X}_u, \mathbf{X}_v\}$ of the affine shape operator, at the origin, of a surface in Pick normal form. The actual functions themselves are far too long to be practicably included here, but can be calculated using the Maple computer algebra package.

Definition 5.6.7 *The Dupin indicatrix is given by*

$$L\ du^2 + 2M\ du\ dv + N\ dv^2 = |LN - M^2|^{1/4}\ .$$

Using this we can prove the following

Proposition 5.6.8 *Consider a non-parabolic and non-affine umbilic surface point. When they are real the affine principal directions are conjugate diameters of the Dupin indicatrix.*

Proof Consider a surface in Pick normal form in a neighbourhood of a hyperbolic point (see Equation (5.2) on page 62). In this form, the affine shape operator matrix has the property that $S + S^\top$ is a diagonal matrix (see Equation (5.4) on page 64). All eigendirections must be of the form $(\alpha : \beta)$ and $(\beta : \alpha)$ in this special form. Let the two affine principal directions be $(\cosh\theta : \sinh\theta)$ and $(\sinh\theta : \cosh\theta)$. The Dupin indicatrix has equation $du^2 - dv^2 = 1$. Consider the line with direction $(\cosh\theta : \sinh\theta)$, this meets the indicatrix at the point $(\cosh\theta, \sinh\theta)$. The tangent line to the indicatrix at that point has direction $(\sinh\theta : \cosh\theta)$. Thus the conjugate diameter must have direction $(\sinh\theta : \cosh\theta)$. The result now follows.

Consider a surface in Pick normal form in a neighbourhood of an elliptic point (see Equation (5.3) on page 62). In this form, the affine shape operator matrix is symmetric (see Equation (5.5) on page 64). All eigendirections must be perpendicular in this special form. Let the two affine principal directions be $(\cos\theta : \sin\theta)$ and $(\sin\theta : -\cos\theta)$. The Dupin indicatrix has equation $du^2 + dv^2 = 1$. conjugate diameters of a circle are always perpendicular. The result now follows. □

Corollary 5.6.9 *At non-parabolic points, away from affine umbilics, we have:*

1. *All repeated affine principal directions are asymptotic directions.*

2. *If an asymptotic direction is an affine principal direction then the point must be a repeated A-direction point.*

Proof In the elliptic region this is trivially true. There are no repeated A-direction points which satisfy the hypotheses. Over the hyperbolic region, only the asymptotes of the hyperbolic Dupin indicatrix are conjugate to themselves. The only directions which may be repeated eigendirections of S are the asymptotes. These asymptotes are the asymptotic directions. □

Chapter 6

Pick Normal Form

6.1 Finding the Pick normal form

Consider a surface in a neighbourhood of a non-parabolic point p. We can locally parametrise the surface in the form of Equation (5.2) or Equation (5.3) on page 62.

This Pick normal form is unique (up to the sign of σ) for a surface point under the action of the special affine group $\mathrm{SL}(3,\mathbb{R}) \ltimes \mathbb{R}^3$.

Next we give a method for putting a surface in general position into Pick form.

6.1.1 The elliptic case

Consider a surface in the neighbourhood of an elliptic point. We can assume that the surface is given as a graph $z = f(x,y)$ where

$$f(x,y) = a_0x^2 + a_1xy + a_2y^2 + b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3 + \cdots \ .$$

Since the surface is elliptically curved in a neighbourhood of the origin $4a_0a_2 > a_1^2$. We wish to make a linear transformation of (x,y,z)-space such that the surface can be written as $z = x^2 + y^2 + \cdots$. Consider the following linear transformation

$$\begin{aligned} x &\mapsto x\cos(\theta) - y\sin(\theta) \ , \\ y &\mapsto x\sin(\theta) + y\cos(\theta) \ , \\ z &\mapsto z \ . \end{aligned}$$

Performing this transformation, and evaluating the coefficient of the xy term gives

$$a_1(\cos^2(\theta) - \sin^2(\theta)) + 2(a_2 - a_0)\sin(\theta)\cos(\theta) \ .$$

Thus if the xy term is to be absent from the quadratic terms

$$a_1(\cos^2(\theta) - \sin^2(\theta)) + 2(a_2 - a_0)\sin(\theta)\cos(\theta) = 0 \ .$$

Using well known trigonometric identities this becomes

$$a_1\cos(2\theta) + (a_2 - a_0)\sin(2\theta) = 0 \ .$$

Assuming that $a_1 \neq 0$ and $a_0 \neq a_2$ means that

$$\tan(2\theta) = \frac{a_1}{a_0 - a_2} \ ,$$

if $a_1 \neq 0$ but $a_0 = a_2$ the condition becomes

$$\cos(2\theta) = 0 \ ,$$

finally if $a_1 = 0$ the xy term will already be absent.

Let θ_0 be a solution to one of these three cases, and let us consider the x^2 and y^2 coefficients. Let the post-transform surface be given by $z = A_0x^2 + A_2y^2 + \cdots$. We have

$$\begin{aligned} A_0 &= a_0\cos^2(\theta_0) + a_1\sin(\theta_0)\cos(\theta_0) + a_2\sin^2(\theta_0) \ , \\ A_2 &= a_0\sin^2(\theta_0) - a_1\sin(\theta_0)\cos(\theta_0) + a_2\cos^2(\theta_0) \ . \end{aligned}$$

If $a_1 \neq 0$ and $a_0 \neq a_2$ then we can find θ_0, however no general statements can be made about A_0 and A_2. If $a_1 \neq 0$ and $a_0 = a_2$ then we have $\cos(2\theta_0) = 0$ and hence

$$\theta_0 \in \left\{(2n+1)\frac{\pi}{4} : n \in \mathbb{Z}\right\} \ .$$

It then follows that

$$\begin{aligned} A_0 &= \frac{1}{2}(a_0 \pm a_1 + a_2) \ , \\ A_2 &= \frac{1}{2}(a_0 \mp a_1 + a_2) \ . \end{aligned}$$

In any of the three cases, it is now possible to perform a second transformation of the form $(x, y, z) \mapsto (\alpha x, \beta y, z)$ for $(\alpha, \beta) \in \mathbb{R}^2$ so the surface will be given by

$$z = x^2 + y^2 + b_0 x^3 + b_1 x^2 y + b_2 x y^2 + b_3 y^3 + \cdots \ .$$

Calculating the direction of the affine normal for $x = y = 0$ gives

$$\tilde{\mathbf{A}}(0,0) = (3b_0 + b_2 : b_1 + 3b_3 : -8) \ .$$

For this to be directed along the z-axis $b_1 = -3b_3$ and $b_2 = -3b_0$. We wish to find a transformation which puts the surface into this form.

Consider a linear transformation of (x, y, z)-space that preserves the $z = 0$ plane:

$$\begin{aligned} x &\mapsto px + qy + uz \ , \\ y &\mapsto rx + sy + vz \ , \\ z &\mapsto wz \ . \end{aligned}$$

Writing the image of the surface under this transformation as a graph $z = g(x, y)$:

$$g(x, y) = A_0 x^2 + A_1 xy + A_2 y^2 + B_1 x^3 + B_2 x^2 y + B_3 x y^2 + B_4 y^3 + \cdots \ .$$

Substituting and comparing coefficients gives

$$\begin{aligned} A_0 &= \frac{p^2 + r^2}{w} \ , \\ A_1 &= \frac{2(pq + rs)}{w} \ , \\ A_2 &= \frac{q^2 + s^2}{w} \ . \end{aligned}$$

Setting $s := w\cos(t)$, $p := w\cos(t)$, $q := -w\sin(t)$, and $r := w\sin(t)$ it follows that $A_0 = w$, $A_1 = 0$, and $A_2 = w$. Next we can calculate the B_i coefficients. The conditions that $B_2 = -3B_0$ and $B_1 = -3B_3$ become

$$\begin{aligned} 8u &= -w(b_2 + 3b_0) \ , \\ 8v &= -w(b_1 + 3b_3) \ . \end{aligned}$$

In particular, if the surface $z = f(x, y)$ had its affine normal directed along the z-axis for $x = y = 0$, then we get $u = v = 0$. The condition for $B_1 = 0$, and hence $B_3 = 0$ is that

$$w^2((b_1 - b_3)(4\cos^3(t) - 3\cos(t)) + (b_2 - b_0)(4\sin(t)\cos^2(t) - \sin(t))) = 0 \ .$$

Assuming $w \neq 0$ and $b_0 - b_2 \neq 0$, using well known identities this becomes

$$\tan(3t) = \frac{b_1 - b_3}{b_0 - b_2} \ .$$

This equation imposes a condition on t. There are three choices for t, and these can be shown to correspond to the famous tangents of Darboux. If $w \neq 0$ and $b_0 - b_2 = 0$ then we must solve

$$(b_1 - b_3)(4\cos^3(t) - 3\cos(t)) = 0 \ ,$$

this is just $(b_1 - b_3)\cos(3t) = 0$. Thus either $b_1 = b_3$ or $\cos(3t) = 0$. Letting $0 \leq t_0 < 2\pi$ be a solution to this equation, gives

$$B_0 = \frac{w^2}{4}((b_0 - b_2)\cos(3t_0) + (b_1 - b_3)\sin(3t_0)) \ .$$

Consider once more the linear transformation

$$\begin{aligned} x &\mapsto px + qy + uz \ , \\ y &\mapsto rx + sy + vz \ , \\ z &\mapsto wz \ . \end{aligned}$$

From our investigations this can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} w\cos(t) & -w\sin(t) & -w(b_2 + 3b_0)/8 \\ w\sin(t) & w\cos(t) & -w(b_1 + 3b_3)/8 \\ 0 & 0 & w \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

The determinant of this transformation is w^2. This transformation needs to be combined with

$$\begin{aligned} x &\mapsto x\cos(\theta) - y\sin(\theta) \ , \\ y &\mapsto x\sin(\theta) + y\cos(\theta) \ , \\ z &\mapsto z \ . \end{aligned}$$

This has matrix form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

The composite transformation then becomes

$$\tau : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} w\cos(\theta+t) & -w\sin(\theta+t) & -w(b_2+3b_0)/8 \\ w\sin(\theta+t) & w\cos(\theta+t) & -w(b_1+3b_3)/8 \\ 0 & 0 & w \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where θ and t are solutions to the equations

$$\begin{aligned} a_1\cos(2\theta) + (a_2 - a_0)\sin(2\theta) &= 0 \ , \\ (b_1 - b_3)\cos(3t) + (b_2 - b_0)\sin(3t) &= 0 \ . \end{aligned}$$

The transformation $\tau : \mathbb{R}^3 \to \mathbb{R}^3$ takes a surface in the form

$$z = a_0x^2 + a_1xy + a_2y^2 + b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3 + \cdots$$

and rewrites in the new form

$$z = wx^2 + wy^2 + B_0(x^3 - 3xy^2) + \cdots$$

where $B_0 = (w/2)^2((b_0-b_2)\cos(3t)+(b_1-b_3)\sin(3t))$. The final step is to set $w := 1/2$ and introduce $\sigma := 3((b_0-b_2)\cos(3t)+(b_1-b_3)\sin(3t))/8$, it follows that the surface will be given by

$$z = \frac{1}{2}(x^2+y^2) + \frac{\sigma}{6}(x^3 - 3xy^2) + \cdots \ ,$$

i.e. τ puts the surface into its Pick normal form.

Similar methods using the hyperbolic trigonometric functions will give the transformation which takes a hyperbolic surface patch into Pick normal form.

Chapter 7

Affine Distance and Height Functions

7.1 Singularities of the affine distance functions

Here we consider the family of affine distance functions defined on a surface $\mathbf{X}$.

Definition 7.1.1 *Given $(u, v) \in U$ and $\mathbf{x} \in \mathbb{R}^3$ we define the family of affine distance functions $\Delta : \mathbb{R}^3 \times U \to \mathbb{R}$ to be*

$$\Delta(\mathbf{x}, (u, v)) := \frac{[\mathbf{x} - \mathbf{X}, \mathbf{X}_u, \mathbf{X}_v]}{[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]} \ . \tag{7.1}$$

The family of affine distance functions are also known as the family of support functions relative to $\mathbf{A}$ (see [15] for a review of relative differential geometry). It gives the $\mathbf{A}$ component of the chord joining $\mathbf{x} \in \mathbb{R}^3$ to some surface point.

We may consider the singularities of the family. We always assume to be working away from Euclidean parabolic points.

For simplicity, let us write $F := [\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]$ and consider

$$\Delta \cdot F = [\mathbf{x} - \mathbf{X}, \mathbf{X}_u, \mathbf{X}_v] \ .$$

Using implicit differentiation, we can calculate the conditions on $\mathbf{x}$ such that Δ has an A_1 or A_2 singularity for some $(u, v) \in U$.

Proposition 7.1.2 *The affine distance functions have an $A_{\geq 1}$ singularity if, and only if,* $\mathbf{x} = \mathbf{X} + \lambda\mathbf{A}$ *for some* $\lambda \in \mathbb{R}$.

Proof For an $A_{\geq 1}$ we want an $\mathbf{x} \in \mathbb{R}^3$ such that $\Delta_u = \Delta_v = 0$. Calculating partial derivatives we have

$$\begin{aligned} \Delta_u F + \Delta F_u &= [\mathbf{x} - \mathbf{X}, \mathbf{X}_{uu}, \mathbf{X}_v] + [\mathbf{x} - \mathbf{X}, \mathbf{X}_u, \mathbf{X}_{uv}] \ , \\ \Delta_v F + \Delta F_v &= [\mathbf{x} - \mathbf{X}, \mathbf{X}_{uv}, \mathbf{X}_v] + [\mathbf{x} - \mathbf{X}, \mathbf{X}_u, \mathbf{X}_{vv}] \ . \end{aligned}$$

Away from parabolic points we can write $\mathbf{x} = \mathbf{X} + \lambda_1 \mathbf{X}_u + \lambda_2 \mathbf{X}_v + \lambda_3 \mathbf{A}$ for suitable $\lambda_i \in \mathbb{R}$. Putting this into the above expressions we see that

$$\begin{aligned} \Delta_u F + \lambda_3 F_u &= -\lambda_1 L - \lambda_2 M + \lambda_3 F_u \ , \\ \Delta_v F + \lambda_3 F_v &= -\lambda_1 M - \lambda_2 N + \lambda_3 F_v \ . \end{aligned}$$

Assuming that $F \neq 0$ means we can write

$$\begin{pmatrix} \Delta_u \\ \Delta_v \end{pmatrix} = -\frac{1}{F} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} .$$

Hence $\Delta_u = \Delta_v = 0$ if, and only if, $\lambda_1 = \lambda_2 = 0$, i.e. $\mathbf{x} = \mathbf{X} + \lambda_3 \mathbf{A}$. □

Proposition 7.1.3 *The affine distance functions have an $A_{\geq 2}$ singularity if, and only if,* $\mathbf{x} = \mathbf{X} + \lambda\mathbf{A}$ *where λ is a solution of the quadratic*

$$1 + (a + d)\lambda + (ad - bc)\lambda^2 = 0 \ ,$$

where a, b, c and d are given by $\mathbf{A}_u = a\mathbf{X}_u + b\mathbf{X}_v$ *and* $\mathbf{A}_v = c\mathbf{X}_u + d\mathbf{X}_v$.

Proof Let us assume that the affine distance functions have an $A_{\geq 1}$ singularity, so that $\mathbf{x} = \mathbf{X} + \lambda\mathbf{A}$, and in turn $\Delta_u = \Delta_v = 0$. The condition for an $A_{\geq 2}$ singularity is that the Hessian matrix of Δ is singular. Calculating partial derivatives gives

$$\begin{aligned}
\Delta_{uu}F + 2\Delta_u F_u + \Delta F_{uu} &= L + [\mathbf{x}-\mathbf{X},\mathbf{X}_{uuu},\mathbf{X}_v] + 2[\mathbf{x}-\mathbf{X},\mathbf{X}_{uu},\mathbf{X}_{uv}] + \\
&+ [\mathbf{x}-\mathbf{X},\mathbf{X}_u,\mathbf{X}_{uuv}] \ , \\
\Delta_{uv}F + \Delta_u F_v + \Delta_v F_u + \Delta F_{uv} &= M + [\mathbf{x}-\mathbf{X},\mathbf{X}_{uuv},\mathbf{X}_v] + [\mathbf{x}-\mathbf{X},\mathbf{X}_{uu},\mathbf{X}_{vv}] + \\
&+ [\mathbf{x}-\mathbf{X},\mathbf{X}_u,\mathbf{X}_{uuv}] \ , \\
\Delta_{vu}F + \Delta_v F_u + \Delta_u F_v + \Delta F_{vu} &= M + [\mathbf{x}-\mathbf{X},\mathbf{X}_{uuv},\mathbf{X}_v] + [\mathbf{x}-\mathbf{X},\mathbf{X}_{uu},\mathbf{X}_{vv}] + \\
&+ [\mathbf{x}-\mathbf{X},\mathbf{X}_u,\mathbf{X}_{uuv}] \ , \\
\Delta_{vv}F + 2\Delta_v F_v + \Delta F_{vv} &= N + [\mathbf{x}-\mathbf{X},\mathbf{X}_{uvv},\mathbf{X}_v] + 2[\mathbf{x}-\mathbf{X},\mathbf{X}_{uv},\mathbf{X}_{vv}] + \\
&+ [\mathbf{x}-\mathbf{X},\mathbf{X}_u,\mathbf{X}_{vvv}] \ .
\end{aligned}$$

Now we can set $\mathbf{x} = \mathbf{X} + \lambda\mathbf{A}$ in all of these equations. These then become

$$\begin{aligned}
\Delta_{uu}F + \lambda F_{uu} &= L + \lambda(F_{uu} + aL + bM) \ , \\
\Delta_{uv}F + \lambda F_{uv} &= M + \lambda(F_{uv} + cL + dM) \ , \\
\Delta_{vu}F + \lambda F_{vu} &= M + \lambda(F_{vu} + aM + bN) \ , \\
\Delta_{vv}F + \lambda F_{vv} &= N + \lambda(F_{vv} + cM + dN) \ .
\end{aligned}$$

Using this to construct the Hessian matrix, we have

$$\mathcal{H} = \frac{1}{F}\begin{pmatrix} (1+\lambda a)L + \lambda bM & \lambda cL + (1+\lambda d)M \\ (1+\lambda a)M + \lambda bN & \lambda cM + (1+\lambda d)N \end{pmatrix}.$$

This matrix can be rewritten simply as

$$\mathcal{H} = \frac{1}{F}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} 1+\lambda a & \lambda c \\ \lambda b & 1+\lambda d \end{pmatrix}.$$

Away from parabolic points, $\det(\mathcal{H}) = 0$ if, and only if,

$$1 + (a+d)\lambda + (ad-bc)\lambda^2 = 0 \ .$$

□

Remark 7.1.4 Proposition 7.1.3 shows that the functions with degenerate singularities are given by $\Delta(\mathbf{x}, -) : \{\mathbf{x}\} \times U \to \mathbb{R}$ with $\mathbf{x} = \mathbf{X} + \lambda \mathbf{A}$ where λ is an affine radius of curvature of $\mathbf{X}$.

Proposition 7.1.5 *The function $\Delta(\mathbf{x}, (u, v))$ treated as a function of u and v has no linear terms and no quadratic terms if, and only if, $\mathbf{x}$ gives a function with a degenerate singularity and the base surface point is an affine umbilic.*

Proof We have seen from Proposition 7.1.2 and Proposition 7.1.3 that $\Delta(\mathbf{x}, (u, v))$ will have no linear terms and a degenerate quadratic part if, and only if, $\mathbf{x} = \mathbf{X} + \lambda \mathbf{A}$ where λ is a solution of the quadratic

$$1 + (a + d)\lambda + (ad - bc)\lambda^2 = 0 \ .$$

This is equivalent to $\mathbf{x}$ being an $A_{\geq 2}$ point. The extra condition that $\Delta(\mathbf{x}, (u, v))$ has no quadratic part at all means that the Hessian matrix $\mathcal{H}$ is not only singular, but is the zero matrix. The Hessian matrix is given by

$$\mathcal{H} = \frac{1}{F} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} 1 + \lambda a & \lambda c \\ \lambda b & 1 + \lambda d \end{pmatrix} .$$

Since $LN - M^2 \neq 0$ it follows that $\mathcal{H} = \mathbf{0}$ if, and only if,

$$\begin{pmatrix} 1 + \lambda a & \lambda c \\ \lambda b & 1 + \lambda d \end{pmatrix} = \mathbf{0} \ .$$

By Definition 5.6.6 on page 64 the base point is an affine umbilic.

□

Remark 7.1.6 Generically we have a $D_4^{\pm}$ singularity in Proposition 7.1.5.

7.2 Singularities of the affine height functions

Here we consider the family of affine height functions defined on a surface $\mathbf{X}$.

Definition 7.2.1 *Given* $(u,v) \in U$ *and* $\mathbf{x} \in S^2$ *we define the family of affine height functions* $H : S^2 \times U \to \mathbb{R}$ *to be*

$$H(\mathbf{x},(u,v)) := \frac{[\mathbf{x},\mathbf{X}_u,\mathbf{X}_v]}{[\mathbf{X}_u,\mathbf{X}_v,\mathbf{A}]} \ . \tag{7.2}$$

See Remark 4.4.1 on page 44 about the use of S^2 as an abstract parameter space. Of course we must replace S^{n-1} by S^2, $\mathbb{R}^n$ by $\mathbb{R}^3$, and $\gamma(s)$ by $\mathbf{X}(u,v)$.

The family of affine height functions gives a conormal covector field on $\mathbf{X}$. A conormal at a point $p \in \mathbf{X}$ is a non-zero covector $f \in T_p^*\mathbb{R}^3$ whose kernel is the tangent plane $T_p\mathbf{X}$. A conormal is defined up to a multiplicative constant. We normalise the conormal by the condition that $f(\mathbf{A}_p) = 1$ where $\mathbf{A}_p$ is the affine normal at p. This defines a conormal covector field on $\mathbf{X}$ that can be considered as a covector-valued function on U.

We may now consider the singularities of the family. We always assume to be working away from Euclidean parabolic points.

For simplicity, let us write $F := [\mathbf{X}_u,\mathbf{X}_v,\mathbf{A}]$ and consider

$$HF = [\mathbf{x},\mathbf{X}_u,\mathbf{X}_v] \ .$$

Using implicit differentiation, we can calculate the conditions on $\mathbf{x}$ such that H has an A_1 or A_2 singularity for some $(u,v) \in U$.

Proposition 7.2.2 *The family of affine height functions has an* $A_{\geq 1}$ *singularity if, and only if,* $\mathbf{x}$ *lies on the affine normal line, i.e.* $\mathbf{x} = \lambda_3\mathbf{A}$ *for some* $\lambda_3 \neq 0$.

Proof For an $A_{\geq 1}$ we want an $\mathbf{x} \in S^2$ such that $H_u = H_v = 0$. Calculating the partial derivatives of the function HF we have

$$\begin{aligned} H_uF + HF_u &= [\mathbf{x},\mathbf{X}_{uu},\mathbf{X}_v] + [\mathbf{x},\mathbf{X}_u,\mathbf{X}_{uv}] \ , \\ H_vF + HF_v &= [\mathbf{x},\mathbf{X}_{uv},\mathbf{X}_v] + [\mathbf{x},\mathbf{X}_u,\mathbf{X}_{vv}] \ . \end{aligned}$$

We can write $\mathbf{x} = \lambda_1\mathbf{X}_u + \lambda_2\mathbf{X}_v + \lambda_3\mathbf{A}$ for suitable $\lambda_i \in \mathbb{R}$, not all zero. Putting this into the above expressions we see that

$$\begin{aligned} H_uF + \lambda_3F_u &= -\lambda_1L - \lambda_2M + \lambda_3F_u \ , \\ H_vF + \lambda_3F_v &= -\lambda_1M - \lambda_2N + \lambda_3F_v \ . \end{aligned}$$

Given that $F \neq 0$ by assumption, we may write

$$\begin{pmatrix} H_u \\ H_v \end{pmatrix} = -\frac{1}{F} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

Hence $H_u = H_v = 0$ if, and only if, $\lambda_1 = \lambda_2 = 0$, i.e. $\mathbf{x} = \lambda_3 \mathbf{A}$ for $\lambda_3 \neq 0$. □

Proposition 7.2.3 *The family of affine height functions has an $A_{\geq 2}$ singularity if, and only if, for some $\lambda \in \mathbb{R}$ with $\lambda \neq 0$, we have $\mathbf{x} = \lambda \mathbf{A}$ and $ad - bc = 0$, i.e. the base point is an affine parabolic point.*

Proof Let us assume that the affine distance functions has an $A_{\geq 1}$ singularity, so that $\mathbf{x} = \lambda \mathbf{A}$ for $\lambda \neq 0$. This gives $H_u = H_v = 0$. The condition for an $A_{\geq 2}$ singularity is that the Hessian matrix of H of second order partial derivatives is singular. Calculating partial derivatives gives

$$\begin{aligned}
H_{uu}F + 2H_uF_u + HF_{uu} &= [\mathbf{x}, \mathbf{X}_{uuu}, \mathbf{X}_v] + 2[\mathbf{x}, \mathbf{X}_{uu}, \mathbf{X}_{uv}] + [\mathbf{x}, \mathbf{X}_u, \mathbf{X}_{uuv}] \ , \\
H_{uv}F + H_uF_v + H_vF_u + HF_{uv} &= [\mathbf{x}, \mathbf{X}_{uuv}, \mathbf{X}_v] + [\mathbf{x}, \mathbf{X}_{uu}, \mathbf{X}_{vv}] + [\mathbf{x}\mathbf{X}_u, \mathbf{X}_{uuv}] \ , \\
H_{vu}F + H_vF_u + H_uF_v + HF_{vu} &= [\mathbf{x}, \mathbf{X}_{uuv}, \mathbf{X}_v] + [\mathbf{x}, \mathbf{X}_{uu}, \mathbf{X}_{vv}] + [\mathbf{x}, \mathbf{X}_u, \mathbf{X}_{uuv}] \ , \\
H_{vv}F + 2H_vF_v + HF_{vv} &= [\mathbf{x}, \mathbf{X}_{uvv}, \mathbf{X}_v] + 2[\mathbf{x}, \mathbf{X}_{uv}, \mathbf{X}_{vv}] + [\mathbf{x}, \mathbf{X}_u, \mathbf{X}_{vvv}] \ .
\end{aligned}$$

We may set $\mathbf{x} := \lambda \mathbf{A}$ in all of the above equations. Respectively, they become

$$\begin{aligned}
H_{uu}F + \lambda F_{uu} &= \lambda(F_{uu} + aL + bM) \ , \\
H_{uv}F + \lambda F_{uv} &= \lambda(F_{uv} + cL + dM) \ , \\
H_{vu}F + \lambda F_{vu} &= \lambda(F_{vu} + aM + bN) \ , \\
H_{vv}F + \lambda F_{vv} &= \lambda(F_{vv} + cM + dN) \ .
\end{aligned}$$

Using this to construct the Hessian matrix, we have

$$\mathcal{H} = \frac{\lambda}{F} \begin{pmatrix} aL + bM & cL + dM \\ aM + bN & cM + dN \end{pmatrix}.$$

This matrix can be rewritten simply as

$$\mathcal{H} = \frac{\lambda}{F}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

By the non-parabolic point assumption $\det(\mathcal{H}) = 0$ if, and only if, $ad - bc = 0$. □

Proposition 7.2.4 *The function $H(\mathbf{x}, (u, v))$ treated as a function of u and v has no linear or quadratic part if, and only if, $\mathbf{x}$ is in the direction of the affine normal vector and the base point is both an affine umbilic and an affine parabolic point.*

Proof We have seen from Proposition 7.2.2 and Proposition 7.2.3 that the affine height function has no linear part and a degenerate quadratic part if, and only if, $\mathbf{x} = \lambda\mathbf{A}$ and $\det(\mathcal{H}) = 0$, where $\lambda \neq 0$ and

$$\mathcal{H} = \frac{\lambda}{F}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Clearly, the degenerate quadratic part is identically zero if, and only if, $\mathcal{H}$ is the zero matrix. By assumption $LN - M^2 \neq 0$, and so the first matrix in the product has rank equal to two. It follows that $\mathcal{H}$ is the zero matrix if, and only if, $a = b = c = d = 0$.

□

Remark 7.2.5 Generically we have a $D_4^{\pm}$ singularity in Proposition 7.2.4.

A word of caution is needed. In the Euclidean setting, the Euclidean shape operator has two zero eigenvalues if, and only if, the family of Euclidean height functions has a D_4 singularity or worse, i.e. no linear or quadratic part. This is not the case in the affine setting. We shall see this in the following propositions.

Proposition 7.2.6 *Over the elliptic region and working in the direction $\mathbf{x} = \lambda\mathbf{A}$ for $\lambda \neq 0$, the affine shape operator has two zero eigenvalues if, and only if, the family of affine height functions has a D_4 singularity or worse.*

Proof Working in the direction $\mathbf{x} = \lambda\mathbf{A}$ for $\lambda \neq 0$ means there are no linear terms. The Hessian matrix of the height functions is then

$$\mathcal{H} = \frac{\lambda}{F}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Let us assume that the affine shape operator has two zero eigenvalues. Taking elliptic Pick normal form and evaluating at $u = v = 0$, we have $L = M = F = 1$, $M = 0$, and $c = b$. The characteristic polynomial of the affine shape operator matrix is

$$ad - b^2 - (a+d)\mu + \mu^2 \ .$$

This has $\mu = 0$ as a double solution if, and only if, $a + d = ad - b^2 = 0$, i.e. if, and only if, $d = -a$ and $a^2 + b^2 = 0$, i.e. if, and only if, $a = b = d = 0$. This shows that the affine shape operator has two zero eigenvalues if, and only if, it is the zero operator. The result now follows by Proposition 7.2.4. □

The last Proposition is so because over the elliptic region, the affine principal directions are distinct. Two zero eigenvalues of a 2×2 matrix with a two dimensional eigenspace means it must be the zero matrix. This is just like the Euclidean case where the Euclidean principal directions are always orthogonal.

Proposition 7.2.7 *Over the hyperbolic region and working in the direction* $\mathbf{x} = \lambda\mathbf{A}$ *for* $\lambda \neq 0$, *the affine shape operator can have two zero eigenvalues while the corresponding height function has only an* $A_{\geq 2}$ *singularity.*

Proof Working in the direction $\mathbf{x} = \lambda\mathbf{A}$ for $\lambda \neq 0$ means there are no linear terms. The Hessian matrix of the height function is then

$$\mathcal{H} = \frac{\lambda}{F}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Taking the hyperbolic Pick normal form and evaluating at $u = v = 0$, we see that in the affine shape operator matrix $c = -b$. The characteristic polynomial is then

$$(ad + b^2) - (a+d)\mu + \mu^2 \ .$$

This has $\mu = 0$ as a double solution if, and only if, $a + d = ad + b^2 = 0$, i.e. if, and only if, $d = -a$ and $b^2 - a^2 = 0$, i.e. if, and only if, $b = \pm a$ and $d = -a$. The affine shape operator matrix is then

$$S = \begin{pmatrix} a & \pm a \\ \mp a & -a \end{pmatrix}.$$

If $a \neq 0$ then S has a repeated eigendirection, namely $(1 : \mp 1)$. This means that we have some point which is both an affine parabolic point, and a repeated affine principal direction point.

In the hyperbolic Pick normal form with $u = v = 0$, we also have $L = F = 1$, $M = 0$, and $N = -1$. The Hessian matrix of the affine height functions is then

$$\mathcal{H} = \lambda \begin{pmatrix} a & \pm a \\ \pm a & a \end{pmatrix}.$$

This gives the quadratic terms of the affine height function, they are

$$a\lambda(u^2 \pm 2uv + v^2) = a\lambda(u \pm v)^2 \ .$$

Clearly if $a \neq 0$ the height function has degenerate, yet non-zero, quadratic part. This gives an $A_{\geq 2}$ singularity. □

Proposition 7.2.8 *If the affine height function has an $A_{\geq 2}$ singularity, i.e. treated as a function of u and v with $\mathbf{x} = \lambda \mathbf{A}$ it has no linear terms and a non-zero degenerate quadratic part, the direction of the line given by the zero level of the degenerate quadratic part is a kernel direction of the affine shape operator.*

In order to prove this, we need the following

Lemma 7.2.9 *Let $\mathbf{u} = (u, v)$ be a vector in $\mathbb{R}^2$, and X a symmetric 2×2 real matrix. Consider the quadratic form $\mathbf{u}X\mathbf{u}^\top$. If $\det(X) = 0$ then $\mathbf{u}X\mathbf{u}^\top$ will be a perfect square. Assuming that X has rank one so that $\mathbf{u}X\mathbf{u}^\top = (\alpha u + \beta v)^2$ for some $\alpha, \beta \in \mathbb{R}$, not both zero, then the direction $(\beta : -\alpha)$ is the kernel direction of X.*

Proof [Proposition 7.2.8] Consider the Hessian matrix, as calculated in the proof of Proposition 7.2.3. The family of affine height functions is smooth thus its second order partial derivatives are continuous. This means that $H_{uv} = H_{vu}$ and so the Hessian matrix is symmetric. In fact one may also take the surface in Pick normal form and make the calculations.

Given $\mathbf{x} = \lambda \mathbf{A}$, the quadratic terms are given by the quadratic form whose matrix is the Hessian matrix calculated in the proof of Proposition 7.2.3.

If we further assume that $ad - bc = 0$ for the $A_{\geq 2}$ condition we may apply Lemma 7.2.9. Notice that since we only have an A_k singularity with $k \geq 2$, the rank of the Hessian is one (and not zero).

Lemma 7.2.9 implies that the direction of the line given by the zero level of the degenerate quadratic part is a kernel direction of the Hessian. Since $LN - M^2 \neq 0$, the kernel of the first matrix in the product of the expression of the Hessian has a trivial kernel. Thus the kernel of the Hessian is the kernel of the affine shape operator matrix. □

Along the affine parabolic curve, away from affine umbilics, there is a unique kernel direction of the affine shape operator. In the Euclidean setting, this kernel direction is the asymptotic direction. Proposition 7.2.8 shows that in the affine case, this kernel direction is linked to the affine height function. This leads to the following

Proposition 7.2.10 *Assume that a member of the family of affine height functions has an $A_{\geq 2}$ singularity away from a parabolic point. The family has an $A_{\geq 3}$ singularity if, and only if, the kernel direction of the affine shape operator is tangent to the affine parabolic curve.*

Proof Proposition 7.2.2 and Proposition 7.2.3 show that the family of affine height functions has an $A_{\geq 2}$ singularity in the direction $\mathbf{x} = \lambda \mathbf{A}$ for $\lambda \neq 0$ if and only $ad - bc = 0$. At such points, the kernel direction of the affine shape operator is given by Proposition 7.2.8. We now show that the condition for this direction to be tangent to the affine parabolic curve is that the family of affine height functions has an $A_{\geq 3}$ singularity in the direction $\mathbf{x} = \lambda \mathbf{A}$ for $\lambda \neq 0$.

The condition for an $A_{\geq 2}$ is that $H_u = H_v = H_{uu}H_{vv} - H_{uv}^2 = 0$. Assuming $H_u = H_v = 0$, i.e. we are taking the direction $\mathbf{x} = \lambda\mathbf{A}$ for $\lambda \neq 0$, the equation for the affine parabolic curve is $H_{uu}H_{vv} - H_{uv}^2 = 0$. The tangent direction is

$$(H_{uuv}H_{vv} + H_{uu}H_{vvv} - 2H_{uv}H_{vvv} : 2H_{uv}H_{uuv} - H_{uuu}H_{vv} - H_{uu}H_{uvv}) \ .$$

We want this to be a kernel direction of the Hessian of the height function, and thus a kernel direction of the affine shape operator. Let us assume that $H_{uu} \neq 0$. Then we have $H_{vv} = H_{uv}^2/H_{uu}$. The tangent direction to the affine parabolic curve is then

$$(H_{uv}^2 H_{uuv} + H_{uu}^2 H_{vvv} - 2H_{uu}H_{uv}H_{uvv} : 2H_{uu}H_{uv}H_{uuv} - H_{uv}^2 H_{uuu} - H_{uu}^2 H_{uvv}).$$

The Hessian matrix of the height function is then

$$\mathcal{H} = \begin{pmatrix} H_{uu} & H_{uv} \\ H_{uv} & H_{uv}^2/H_{uu} \end{pmatrix}.$$

Multiplying a tangent vector of the affine parabolic curve by this Hessian matrix gives another vector. This vector is the zero vector if, and only if,

$$3H_{uu}H_{uv}^2H_{uuv} + H_{uu}^3H_{vvv} - 3H_{uu}^2H_{uv}H_{uvv} - H_{uv}^3H_{uuu} = 0 \ .$$

This is then the condition for the kernel direction of the affine shape operator matrix to be tangent to the affine parabolic curve.

The condition for a function of two variables with an $A_{\geq 2}$ singularity to have an $A_{\geq 3}$ singularity is that the square root of the degenerate quadratic part divides the cubic terms. The quadratic terms of the affine height functions are $(uH_{uu} + vH_{uv})^2$ and so we need $u = -H_{uv}/H_{uu}v$ to be a zero of the cubic terms. The cubic terms of the affine height functions are

$$\frac{1}{6}(u^3H_{uuu} + 3u^2vH_{uuv} + 3uv^2H_{uvv} + v^3H_{vvv}) \ .$$

Making the substitution $u = -H_{uv}/H_{uu}v$, the result is zero if, and only if,

$$3H_{uu}H_{uv}^2H_{uuv} + H_{uu}^3H_{vvv} - 3H_{uu}^2H_{uv}H_{uvv} - H_{uv}^3H_{uuu} = 0 \ .$$

If $H_{uu} = 0$ we can assume that $H_{vv} \neq 0$. If $H_{uu} = H_{vv} = 0$, then the quadratic terms form a perfect square if, and only if, $H_{uv} = 0$ also, i.e. a D_4 or worse. However, the proposition deals with an $A_{\geq 2}$. Assuming that $H_{vv} \neq 0$ we continue as above in order to arrive at the same result. $\square$

Definition 7.2.11 (Affine Gauß Map) *The affine Gauß map can be defined as a map $U \to S^2$ where*

$$(u, v) \mapsto \frac{\mathbf{A}}{||\mathbf{A}||} \; .$$

Proposition 7.2.3 shows that the Jacobian matrix of the affine Gauß map and the affine shape operator matrix have the same singular locus. Thus ordinary affine parabolic points (A_2 of the affine height function) give fold points of the affine Gauß map since the kernel field of the Jacobian is transverse to the singular locus. When the affine height functions have an $A_{\geq 3}$ the kernel field of the Jacobian is tangent to the singular locus, and so the affine Gauß map has a cusp point. This gives rise to the following

Definition 7.2.12 (Affine Cusp of Gauß) *If the family of affine height functions has an $A_{\geq 3}$ singularity (or worse) then the corresponding surface point will be called affine cusp of Gauß.*

Remark 7.2.13 Clearly the kernel direction of the affine shape operator at an affine parabolic point is also an affine principal direction. Thus, the affine cusps of Gauß points are points where the affine principal curves are tangent to the affine parabolic curve.

Remark 7.2.14 We see that the bifurcation set of the family of affine height functions is the set of affine normal directions at affine parabolic points. This is exactly analogous to the Euclidean setting.

Chapter 8

Affine Parallel Surfaces and the Affine Focal Set

In this chapter we consider the affine surface parallels for a given surface, we also consider the affine focal set for a given surface. A connexion between the singularities of the former and the points of the latter is established. The reader is referred to §5.6 on page 63 for the definitions associated with the affine shape operator, e.g. affine principal direction etc.

8.1 Affine surface parallels

We can define the family of affine surface parallels in terms of the family of affine distance functions. Recall the definition of the family of affine distance functions (see Equation (7.1) on page 73). Given a surface point $\mathbf{X}(u_0, v_0)$ we can consider the set of points $\mathbf{x} \in \mathbb{R}^3$ of constant affine distance k. This is then

$$\{\mathbf{x} \in \mathbb{R}^3 : \Delta(\mathbf{x}, (u_0, v_0)) = k\} \ .$$

Away from parabolic points we can write $\mathbf{x} = \mathbf{X} + \lambda_1 \mathbf{X}_u + \lambda_2 \mathbf{X}_v + \lambda_3 \mathbf{A}$, where we evaluate $(u, v) = (u_0, v_0)$. It now follows that

$$\mathbf{x} = \mathbf{X} + \lambda_1 \mathbf{X}_u + \lambda_2 \mathbf{X}_v + k\mathbf{A}, \quad \text{where} \quad (u, v) = (u_0, v_0) \ .$$

Thus, for each point on the surface, we have a plane parallel to the tangent plane. A two parameter family of planes in three space generically has an envelope. To see this, consider the two-parameter family of planes given by

$$A(u,v)x + B(u,v)y + C(u,v)z = D(u,v)\ .$$

The envelope points (x, y, z) are solutions to

$$\begin{pmatrix} A & B & C \\ A_u & B_u & C_u \\ A_v & B_v & C_v \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} D \\ D_u \\ D_v \end{pmatrix} .$$

A solution exists if, and only if, the determinant of the matrix on the lefthand side is non-zero. This is, of course, generically true.

The envelope for the family of planes $\Delta = k$ is found by further imposing the conditions that $\Delta_u = \Delta_v = 0$. From Proposition 7.1.2 on page 74 this envelope is given by

$$\mathcal{P}_k := \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{X} + k\mathbf{A}\}\ .$$

For a fixed $k \in \mathbb{R}$ the envelope $\mathcal{P}_k$ may be smooth, or it may be singular. Notice that $\mathcal{P}_0 = \mathbf{X}$ and provided $\mathbf{X}$ is smooth $\mathcal{P}_0$ will be smooth. By continuity, for $\varepsilon \ll 1$, the $\mathcal{P}_k$ will all be smooth for $-\varepsilon < k < \varepsilon$. The singularities on the parallel surfaces $\mathcal{P}_k$ are given by points of regression. Points of regression come from imposing the extra condition that the Hessian matrix of the affine distance function be singular. From Proposition 7.1.3 on page 74 it follows that

$$\mathrm{Sing}(\mathcal{P}_k) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{X} + k\mathbf{A}\} \quad \text{where} \quad 1 + (a+d)k + (ad - bc)k^2 = 0\ .$$

8.1.1 The curvature of affine parallel surfaces

Here we consider the quadratic form $L\ du^2 + 2M\ du\ dv + N\ dv^2$ on the parallel surfaces to a given surface. Let $\mathbf{X} : U \to \mathbb{R}^3$ be the parametrisation of a smooth piece of surface, then the affine parallel surfaces are given by $\mathbf{Y} := \mathbf{X} + k\mathbf{A}$ for some $k \in \mathbb{R}$. We can then treat the functions L, M, and N as a one parameter family of functions $L, M, N : U \times \mathbb{R} \to \mathbb{R}$, where

$$\begin{aligned} L((u,v),k) &= [\mathbf{Y}_u, \mathbf{Y}_v, \mathbf{Y}_{uu}] \ , \\ M((u,v),k) &= [\mathbf{Y}_u, \mathbf{Y}_v, \mathbf{Y}_{uv}] \ , \\ N((u,v),k) &= [\mathbf{Y}_u, \mathbf{Y}_v, \mathbf{Y}_{vv}] \ . \end{aligned}$$

Recall that there are smooth functions $a, b, c, d : U \to \mathbb{R}$ such that

$$\begin{aligned} \mathbf{A}_u &= a\mathbf{X}_u + b\mathbf{X}_v \ , \\ \mathbf{A}_v &= c\mathbf{X}_u + d\mathbf{X}_v \ . \end{aligned}$$

Using this, we can then calculate the following

$$\begin{aligned} \mathbf{Y}_u &= (1+ak)\mathbf{X}_u + bk\mathbf{X}_v \ , \\ \mathbf{Y}_v &= ck\mathbf{X}_u + (1+dk)\mathbf{X}_v \ , \\ \mathbf{Y}_{uu} &= a_u k\mathbf{X}_u + b_u k\mathbf{X}_v + (1+ak)\mathbf{X}_{uu} + bk\mathbf{X}_{uv} \ , \\ \mathbf{Y}_{uv} &= a_v k\mathbf{X}_u + b_v k\mathbf{X}_v + (1+ak)\mathbf{X}_{uv} + bk\mathbf{X}_{vv} \ , \\ \mathbf{Y}_{vu} &= c_u k\mathbf{X}_u + d_u k\mathbf{X}_v + ck\mathbf{X}_{uu} + (1+dk)\mathbf{X}_{uv} \ , \\ \mathbf{Y}_{vv} &= c_v k\mathbf{X}_u + d_v k\mathbf{X}_v + ck\mathbf{X}_{uv} + (1+dk)\mathbf{X}_{vv} \ . \end{aligned}$$

Notice there are two different expressions above, which are actually equal, $\mathbf{Y}_{uv} = \mathbf{Y}_{vu}$. Notice that $\mathbf{Y}_u$ and $\mathbf{Y}_v$ are linearly dependent when $1+(a+d)k+(ad-bc)k^2 = 0$. This means $1/k$ is an eigenvalue of the affine shape operator.

Proposition 8.1.1 *Consider an affine surface parallel* $\mathbf{Y} = \mathbf{X} + k\mathbf{A}$. *Using the above notation for the partial derivatives of* $\mathbf{A}$, *and writing* $L_k := L((u,v),k)$, $M_k := M((u,v),k)$, *and* $N_k := N((u,v),k)$ *it follows that*

$$L_k N_k - M_k^2 = (L_0 N_0 - M_0^2)(1 + (a+d)k + (ad-bc)k^2)^3 \ .$$

Proof

$$\begin{aligned} L_k &= (1+(a+d)k+(ad-bc)k^2)((1+ak)L_0 + bkM_0) \ , \\ M_k &= (1+(a+d)k+(ad-bc)k^2)((1+ak)M_0 + bkN_0) \ , \\ M_k &= (1+(a+d)k+(ad-bc)k^2)(ckL_0 + (1+dk)M_0) \ , \\ N_k &= (1+(a+d)k+(ad-bc)k^2)(ckM_0 + (1+dk)N_0) \ . \end{aligned}$$

Using this, and crucially the two choices for M_k, we see that

$$\begin{pmatrix} L_k \\ M_k \end{pmatrix} = \begin{vmatrix} 1+ak & bk \\ ck & 1+dk \end{vmatrix} \begin{pmatrix} 1+ak & bk \\ ck & 1+dk \end{pmatrix} \begin{pmatrix} L_0 \\ M_0 \end{pmatrix},$$

$$\begin{pmatrix} M_k \\ N_k \end{pmatrix} = \begin{vmatrix} 1+ak & bk \\ ck & 1+dk \end{vmatrix} \begin{pmatrix} 1+ak & bk \\ ck & 1+dk \end{pmatrix} \begin{pmatrix} M_0 \\ N_0 \end{pmatrix}.$$

Setting $\alpha := 1 + (a+d)k + (ad-bc)k^2$ gives

$$\begin{pmatrix} L_k \\ M_k \end{pmatrix} = \alpha \begin{pmatrix} 1+ak & bk \\ ck & 1+dk \end{pmatrix} \begin{pmatrix} L_0 \\ M_0 \end{pmatrix},$$

$$\begin{pmatrix} N_k \\ -M_k \end{pmatrix} = \alpha \begin{pmatrix} 1+dk & -ck \\ -bk & 1+ak \end{pmatrix} \begin{pmatrix} N_0 \\ -M_0 \end{pmatrix}.$$

It is clear that

$$L_k N_k - M_k^2 = \begin{pmatrix} L_k & M_k \end{pmatrix} \begin{pmatrix} N_k \\ -M_k \end{pmatrix}.$$

Finally, we can see that

$$L_k N_k - M_k^2 = \alpha^2 \begin{pmatrix} L_0 & M_0 \end{pmatrix} \begin{pmatrix} 1+ak & bk \\ ck & 1+dk \end{pmatrix}^{\top} \begin{pmatrix} 1+dk & -ck \\ -bk & 1+ak \end{pmatrix} \begin{pmatrix} N_0 \\ -M_0 \end{pmatrix},$$

$$L_k N_k - M_k^2 = \alpha^2 (L_0 N_0 - M_0^2)(1 + (a+d)k + (ad-bc)k^2) \ ,$$

And hence $L_k N_k - M_k^2 = (L_0 N_0 - M_0^2)(1 + (a+d)k + (ad-bc)k^2)^3$. □

Corollary 8.1.2 *Consider a surface in a neighbourhood of a non-parabolic point. The affine surface parallels* $\mathbf{Y} = \mathbf{X} + k\mathbf{A}$ *will have no non-singular parabolic points.*

Proof We have seen that $L_k N_k - M_k^2 = (L_0 N_0 - M_0^2)(1 + (a+d)k + (ad-bc)k^2)$. If $L_0 N_0 - M_0^2 \neq 0$ for all $(u, v) \in U$, then $L_k N_k - M_k^2 = 0$ if, and only if, $1 + (a + d)k + (ad - bc)k^2 = 0$. This corresponds to $\mathbf{Y}_u$ and $\mathbf{Y}_v$ being linearly dependent. □

Remark 8.1.3 Consider a fixed $(u_0, v_0) \in U$. Generically as k varies $\mathbf{Y}(u_0, v_0)$ will be hyperbolic then singular then elliptic or vise versa. Let $f(k) := L_k N_k - M_k^2$, then $f(k) = f'(k) = 0$ if, and only if, $(a-d)^2 + 4bc = 0$ and

$$k = \frac{a+d}{2(bc-ad)} \; .$$

If $\mathbf{X}(u_0, v_0)$ is a repeated A-direction point then $\mathbf{Y}(u_0, v_0)$ will be hyperbolic then singular then hyperbolic or elliptic then singular then elliptic.

8.2 Affine focal surfaces

Here we consider the affine focal set in a number of ways. It is first realised in an analogous way to the Euclidean focal set. We can consider the "infinitesimal intersection" of affine normal lines. This will generate a point set (usually made up of two surfaces) which we call the affine focal set, or affine focal surface. It can also be shown that the affine focal set, as considered as the infinitesimal intersection of affine normal lines, can also be realised as the bifurcation set of the family of affine distance functions, i.e. the set of $\mathbf{x} \in \mathbb{R}^3$ such that $\Delta(\mathbf{x}, -)$ has a degenerate singularity for some $(u, v) \in U$.

8.2.1 Focal surfaces, principal curvatures, and principal directions

Here we consider the affine focal set as the infinitesimal intersection of nearby affine normal lines. Pieces of the affine focal set can be parametrised by $\mathbf{X} + \lambda\mathbf{A}$ where $\lambda : U \to \mathbb{C}$. Here we use an infinitesimal argument to find the function λ and also the directions in which we need to move so that the two nearby affine normal lines remain coplanar, and hence have an intersection point.

We wish to find conditions so that nearby affine normal lines intersect. That means we wish to find conditions such that

$$\mathbf{X}(u,v)+\lambda(u,v)\mathbf{A}(u,v) = \mathbf{X}(u+\delta u, v+\delta v)+\lambda(u+\delta u, v+\delta v)\mathbf{A}(u+\delta u, v+\delta v) \; . \quad (8.1)$$

Proposition 8.2.1 *It is true that*

1. *The directions* $(\delta u : \delta v)$ *such that Equation (8.1) has a solution are affine principal directions.*

2. *The values of* λ *such that Equation (8.1) has a solution are the affine radii of curvature.*

3. *The directions* $(\delta u : \delta v)$ *and the function* λ *satisfy* $\lambda_u \delta u + \lambda_v \delta v = 0$.

Proof

Expanding $\mathbf{X}(u+\delta u, v+\delta v)$, $\lambda(u+\delta u, v+\delta v)$, and $\mathbf{A}(u+\delta u, v+\delta v)$ as power series in δu and δv gives

$$\begin{aligned}
\mathbf{X}(u+\delta u, v+\delta v) &= \mathbf{X}(u,v) + \mathbf{X}_u(u,v)\ \delta u + \mathbf{X}_v(u,v)\ \delta v + \cdots\ , \\
\lambda(u+\delta u, v+\delta v) &= \lambda(u,v) + \lambda_u(u,v)\ \delta u + \lambda_v(u,v)\ \delta v + \cdots\ , \\
\mathbf{A}(u+\delta u, v+\delta v) &= \mathbf{A}(u,v) + \mathbf{A}_u(u,v)\ \delta u + \mathbf{A}_v(u,v)\ \delta v + \cdots\ .
\end{aligned}$$

Substituting these expressions into Equation (8.1) we have

$$\mathbf{X}_u\ \delta u + \mathbf{X}_v\ \delta v + \lambda \mathbf{A}_u\ \delta u + \lambda \mathbf{A}_v\ \delta v + \lambda_u \mathbf{A}\ \delta u + \lambda_v \mathbf{A}\ \delta v + \cdots = \mathbf{0}\ . \tag{8.2}$$

If we let $(u(t), v(t))$ be a smooth curve in the uv-parameter plane, then

$$\begin{aligned}
\delta u &= \dot{u}\ \delta t + \frac{1}{2}\ddot{u}\ \delta t^2 + \cdots\ , \\
\delta v &= \dot{v}\ \delta t + \frac{1}{2}\ddot{v}\ \delta t^2 + \cdots\ .
\end{aligned}$$

Then Equation (8.2) becomes

$$(\dot{u}\mathbf{X}_u + \dot{v}\mathbf{X}_v + \lambda\dot{u}\mathbf{A}_u + \lambda\dot{v}\mathbf{A}_v + \lambda_u\dot{u}\mathbf{A} + \lambda_v\dot{v}\mathbf{A})\ \delta t + \cdots = \mathbf{0}\ ,$$

where the higher order terms are divisible by δt^2. Dividing through by δt and then letting $\delta t \to 0$ we have

$$\dot{u}\mathbf{X}_u + \dot{v}\mathbf{X}_v + \lambda\dot{u}\mathbf{A}_u + \lambda\dot{v}\mathbf{A}_v + \lambda_u\dot{u}\mathbf{A} + \lambda_v\dot{v}\mathbf{A} = \mathbf{0}\ .$$

We can write the equation in two ways, they are

$$(\lambda_u \dot{u} + \lambda_v \dot{v})\mathbf{A} + (\dot{u}\mathbf{X}_u + \dot{v}\mathbf{X}_v) + \lambda(\dot{u}\mathbf{A}_u + \dot{v}\mathbf{A}_v) = \mathbf{0} , \tag{8.3}$$

$$(\lambda_u \dot{u} + \lambda_v \dot{v})\mathbf{A} + (\mathbf{X}_u + \lambda\mathbf{A}_u)\dot{u} + (\mathbf{X}_v + \lambda\mathbf{A}_v)\dot{v} = \mathbf{0} . \tag{8.4}$$

Equation (8.3) says that $\mathbf{A}$, $\dot{u}\mathbf{X}_u + \dot{v}\mathbf{X}_v$, and $\dot{u}\mathbf{A}_u + \dot{v}\mathbf{A}_v$ are linearly dependent, i.e.

$$[\mathbf{A}, \dot{u}\mathbf{X}_u + \dot{v}\mathbf{X}_v, \dot{u}\mathbf{A}_u + \dot{v}\mathbf{A}_v] = 0 .$$

This has eliminated λ and gives an equation in terms of $\dot{u}$ and $\dot{v}$. In fact it will give a quadratic in these, the solutions of which are the directions we need to move in for there to be a solution to equation (8.1). Notice that $\dot{u} = du/dt$ and $\dot{v} = dv/dt$, so

$$[\mathbf{A}, \mathbf{X}_u du + \mathbf{X}_v dv, \mathbf{A}_u du + \mathbf{A}_v dv] = 0 .$$

Using the identities $\mathbf{A}_u = a\mathbf{X}_u + b\mathbf{X}_v$ and $\mathbf{A}_v = c\mathbf{X}_u + d\mathbf{X}_v$ means we can rewrite this last expression as

$$(b\ du^2 + (d - a)\ du\ dv - c\ dv^2)[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}] = 0 .$$

Thus $(du : dv)$ is an eigendirection of the affine shape operator.
Equation (8.4) says that $\mathbf{A}$, $\mathbf{X}_u + \lambda\mathbf{A}_u$, and $\mathbf{X}_v + \lambda\mathbf{A}_v$ are linearly dependent, i.e.

$$[\mathbf{A}, \mathbf{X}_u + \lambda\mathbf{A}_u, \mathbf{X}_v + \lambda\mathbf{A}_v] = 0 .$$

This has eliminated $\dot{u}$ and $\dot{v}$. This will give a quadratic in λ. The solutions give values of λ for which Equation (8.1) will have a solution. Using the identities $\mathbf{A}_u = a\mathbf{X}_u + b\mathbf{X}_v$ and $\mathbf{A}_v = c\mathbf{X}_u + d\mathbf{X}_v$ means we can rewrite this last expression as

$$(1 + (a + d)\lambda + (ad - bc)\lambda^2)[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}] = 0 .$$

Thus $1/\lambda$ is an eigenvalue of the affine shape operator.

Finally, the second and third summands in Equation (8.3) and Equation (8.4) are purely tangential components. It follows that the $\mathbf{A}$ component must be zero for the equations to hold. It follows that $\lambda_u du + \lambda_v dv = 0$.

□

Corollary 8.2.2 *The affine focal set as seen as the infinitesimal intersection of nearby affine normals is the bifurcation set of the family of affine distance functions.*

Proof Proposition 7.1.3 on page 74 gives a means of finding the bifurcation set of the family of affine distance functions in terms of affine principal curvatures. Proposition 8.2.1 shows that this corresponds to the affine focal set. □

Theorem 8.2.3 *At smooth points of the affine focal set ($\mathbf{Y} = \mathbf{X} + \lambda\mathbf{A}$ where $1 + (a+d)\lambda + (ad-bc)\lambda^2 = 0$), the affine normal line at the base point is contained in the tangent plane to the affine focal set.*

Proof At smooth points of the affine focal set the tangent plane is spanned by $\mathbf{Y}_u$ and $\mathbf{Y}_v$. Using the identities $\mathbf{A}_u = a\mathbf{X}_u + b\mathbf{X}_v$ and $\mathbf{A}_v = c\mathbf{X}_u + d\mathbf{X}_v$ we find that

$$\begin{aligned} \mathbf{Y}_u &= (1+\lambda a)\mathbf{X}_u + \lambda b\mathbf{X}_v + \lambda_u\mathbf{A} \ , \\ \mathbf{Y}_v &= \lambda c\mathbf{X}_u + (1+\lambda d)\mathbf{X}_v + \lambda_v\mathbf{A} \ . \end{aligned}$$

Consider the vector $r\mathbf{X}_u + s\mathbf{X}_v + t\mathbf{A}$ for some $r, s, t \in \mathbb{R}$. We seek the condition that $r\mathbf{X}_u + s\mathbf{X}_v + t\mathbf{A}$ is contained in the span of $\mathbf{Y}_u$ and $\mathbf{Y}_v$, namely

$$[r\mathbf{X}_u + s\mathbf{X}_v + t\mathbf{A}, \mathbf{Y}_u, \mathbf{Y}_v] = 0 \ .$$

Since $[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}] \neq 0$ this condition becomes

$$\begin{vmatrix} r & 1+\lambda a & \lambda c \\ s & \lambda b & 1+\lambda d \\ t & \lambda_u & \lambda_v \end{vmatrix} = 0 \ .$$

Since $1 + (a+d)\lambda + (ad-bc)\lambda^2 = 0$ it follows that $r = s = 0$ gives a solution to this equation, i.e. $\langle \mathbf{A} \rangle \subset \langle \mathbf{Y}_u, \mathbf{Y}_v \rangle$. □

Theorem 8.2.4 *At smooth points of the affine focal set the tangent plane to the sheet of the affine focal set corresponding to one affine principal curvature meets the tangent plane to the the surface in the other affine principal direction.*

Proof First let us consider the case where $\mathbf{X}$ has two distinct affine principal directions at the origin, and assume that $\mathbf{X}$ is parametrised in such a way that $\mathbf{X}_u(0,0)$ and $\mathbf{X}_v(0,0)$ are affine principal directions of $\mathbf{X}$ at $\mathbf{X}(0,0)$. Moreover, assume that the focal set is smooth at both corresponding focal points. In such a case we see that $b(0,0) = c(0,0) = 0$. For brevity, let us assume that all of the following expressions are evaluated at $u = v = 0$. We see that

$$\begin{aligned} \mathbf{Y}_u &= (1+\lambda a)\mathbf{X}_u + \lambda_u \mathbf{A} \ , \\ \mathbf{Y}_v &= (1+\lambda d)\mathbf{X}_v + \lambda_v \mathbf{A} \ . \end{aligned}$$

For $u = v = 0$ the affine radii of curvature are simply $\lambda = -1/a, -1/d$. The affine principal direction $\mathbf{X}_u$ corresponds to the affine radius of curvature $\lambda = -1/a$, and the affine principal direction $\mathbf{X}_v$ corresponds to the affine radius of curvature $\lambda = -1/d$.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection along $\mathbf{A}$ so that $\pi : r\mathbf{X}_u + s\mathbf{X}_v + t\mathbf{A} \mapsto r\mathbf{X}_u + s\mathbf{X}_v$. It follows that $\pi(\mathbf{Y}_u) = (1+\lambda a)\mathbf{X}_u$ and $\pi(\mathbf{Y}_v) = (1+\lambda d)\mathbf{X}_v$ where either $\lambda = -1/a$ or $\lambda = -1/d$ (depending on which focal sheet we are considering).

Let us assume, without loss of generality, that $\lambda = -1/a$, so that $\pi(\mathbf{Y}_u) = \mathbf{0}$ and $\pi(\mathbf{Y}_v) \propto \mathbf{X}_v$. Notice that $\pi(\mathbf{Y}_v) = \mathbf{0}$ if, and only if, $a = d$, i.e. if, and only if, the origin is an affine umbilic. This contradicts the assumption that the affine focal set is non-singular. Thus, the tangent plane to the sheet of the affine focal set corresponding to $\lambda = -1/a$ intersects the tangent plane to $\mathbf{X}$ in the direction $\mathbf{X}_v$. However, $\mathbf{X}_v$ is the affine principal direction which corresponds to the affine radius of curvature $\lambda = -1/d$.

Next, let us consider the case where $\mathbf{X}$ has the origin as a repeated A-direction point which is not an affine umbilic. Let us assume that $\mathbf{X}_u(0,0)$ is the single repeated affine principal direction. In such a case we see that $b(0,0) = 0$, $c(0,0) \neq 0$ and $a(0,0) = d(0,0)$. For brevity, let us assume that all of the following expressions are evaluated at $u = v = 0$. We find that

$$\begin{aligned} \mathbf{Y}_u &= (1+\lambda a)\mathbf{X}_u + \lambda_u \mathbf{A} \ , \\ \mathbf{Y}_v &= \lambda c\mathbf{X}_u + (1+\lambda a)\mathbf{X}_v + \lambda_v \mathbf{A} \ . \end{aligned}$$

It follows that $\pi(\mathbf{Y}_u) = (1+\lambda a)\mathbf{X}_u$ and $\pi(\mathbf{Y}_v) = \lambda c\mathbf{X}_u + (1+\lambda a)\mathbf{X}_v$. In this case the repeated affine radius of curvature is $\lambda = -1/a$, and so it follows that $\pi(\mathbf{Y}_u) = \mathbf{0}$ and $\pi(\mathbf{Y}_v) \propto \mathbf{X}_u$. This is, trivially, the other affine principal direction. □

Corollary 8.2.5 *At smooth points of the affine focal set the tangent plane is spanned by the affine normal vector to the surface and the other affine principal direction.*

Theorem 8.2.6 *Consider a neighbourhood of an elliptically curved point. There are two real distinct affine principal curvatures and affine principal directions if, and only if, the Pick normal form coefficients satisfy the equation*

$$(a_1 + a_3)^2 + (a_0 - a_4)^2 \neq 0 \ .$$

Proof Take a surface in the form of Equation (5.3) on page 62. Direct computation using the affine shape operator matrix in Equation (5.5) yields the required result.

□

Theorem 8.2.7 *In the elliptically curved region of a smooth surface the focal points coincide if, and only if, the base point is an affine umbilic.*

Proof This follows from the proof of Theorem 8.2.6. □

Corollary 8.2.8 *Away from affine umbilics there are two distinct real affine focal points for every point in the elliptic region. For a generic surface affine focal points will be isolated and so the focal set will have two sheets except at isolated points.*

Theorem 8.2.9 *Consider a neighbourhood of a hyperbolically curved point. There are two distinct affine principal curvatures and affine principal directions if, and only if, the Pick normal form coefficients satisfy the equation*

$$(2a_0 + a_1 - a_3 - 2a_4)(2a_0 - a_1 + a_3 - 2a_4) \neq 0 \ .$$

Proof Take a surface in the form of Equation (5.2) on page 62. Direct computation using the affine shape operator matrix in Equation (5.4) yields the required result. □

Corollary 8.2.10 *For each point in a hyperbolic region, there are usually two distinct focal points. These need not be real. The affine focal points are real and distinct when the Pick normal form quartic coefficients have the properties that* $2a_0+a_1-a_3-2a_4$ *and* $2a_0-a_1+a_3-2a_4$ *have the same sign and are non-zero. They are complex conjugate focal points when* $2a_0+a_1-a_3-2a_4$ *and* $2a_0-a_1+a_3-2a_4$ *have opposite sign and are non-zero. If either* $2a_0+a_1-a_3-2a_4=0$ *or* $2a_0-a_1+a_3-2a_4=0$ *then there is a single repeated affine focal point.*

The curves along which $2a_0+a_1-a_3-2a_4=0$ or $2a_0-a_1+a_3-2a_4=0$ are the repeated A-direction curves. These two curves correspond to the affine principal directions being along one of the two axes of the Dupin indicatrix. We shall study these in much more detail later on. See chapter 9 on page 101.

8.3 Singular points of the two sets

8.3.1 Singular points of the affine surface parallels

Here we show that the singular points of affine surface parallels sweep out the affine focal set. Let us consider a smooth surface $\mathbf{X}: U \to \mathbb{R}^3$ in a neighbourhood of a non-parabolic point. For a fixed $k \in \mathbb{R}$ we have the parallel surface $\mathbf{Y} = \mathbf{X} + k\mathbf{A}$.

Proposition 8.3.1 *The affine surface parallel of distance* k *is singular if, and only if,* $1/k$ *is an eigenvalue of the affine shape operator.*

Proof Given that $\mathbf{Y} = \mathbf{X} + k\mathbf{A}$ it follows that

$$\begin{aligned}\mathbf{Y}_u &= \mathbf{X}_u + k\mathbf{A}_u \ , \\ \mathbf{Y}_v &= \mathbf{X}_v + k\mathbf{A}_v \ .\end{aligned}$$

Since we are working in non-parabolic regions, we have $\mathbf{A}_u = a\mathbf{X}_u + b\mathbf{X}_v$ and $\mathbf{A}_v = c\mathbf{X}_u + d\mathbf{X}_v$. It follows that

$$\begin{aligned} \mathbf{Y}_u &= (1+ka)\mathbf{X}_u + kb\mathbf{X}_v \ , \\ \mathbf{Y}_v &= kc\mathbf{X}_v + (1+kd)\mathbf{X}_v \ . \end{aligned}$$

The surface $\mathbf{Y}$ is singular if, and only if, $\mathbf{Y}_u$ and $\mathbf{Y}_v$ are linearly dependent. This is the case if, and only if, $1 + (a+d)k + (ad - bc)k^2 = 0$, i.e. if, and only if,

$$\begin{vmatrix} a + \frac{1}{k} & c \\ b & d + \frac{1}{k} \end{vmatrix} = 0 \ .$$

□

8.3.2 Singular points of the affine focal set

Here we consider the affine focal set and its singular points

Proposition 8.3.2 *The affine focal set is singular at a point if, and only if, the derivative of an affine principal curvature in its affine principal direction is zero at the corresponding base point.*

Proof Let the affine focal set be parametrised by $\mathbf{Y} := \mathbf{X} + \lambda\mathbf{A}$ where $1/\lambda$ is an eigenfunction of the affine shape operator, i.e. $1 + (a+d)\lambda + (ad - bc)\lambda^2 = 0$. The affine focal set is singular if, and only if, $\mathbf{Y}_u$ and $\mathbf{Y}_v$ are linearly dependent, where

$$\begin{aligned} \mathbf{Y}_u &= (1+\lambda a)\mathbf{X}_u + \lambda b\mathbf{X}_v + \lambda_u\mathbf{A} \ , \\ \mathbf{Y}_v &= \lambda c\mathbf{X}_u + (1+\lambda d)\mathbf{X}_v + \lambda_v\mathbf{A} \ . \end{aligned}$$

It follows that the affine focal set is singular if, and only if,

$$\text{rank} \begin{pmatrix} 1+\lambda a & \lambda b & \lambda_u \\ \lambda c & 1+\lambda d & \lambda_v \end{pmatrix} < 2.$$

Since $1 + (a+d)\lambda + (ad - bc)\lambda^2 = 0$ by assumption, it follows that that the affine focal set is singular if, and only if,

$$\begin{pmatrix} 1+\lambda a & \lambda c \\ \lambda b & 1+\lambda d \end{pmatrix} \begin{pmatrix} \lambda_v \\ -\lambda_u \end{pmatrix} = \mathbf{0} \; . \tag{8.5}$$

Let $(\alpha : \beta)$ be the corresponding local principal direction field, i.e.

$$\begin{pmatrix} 1+\lambda a & \lambda c \\ \lambda b & 1+\lambda d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mathbf{0} \; .$$

The derivative of λ in the direction $(\alpha : \beta)$ is simply $\alpha\lambda_u + \beta\lambda_v$. It follows that

$$\alpha\lambda_u + \beta\lambda_v = 0 \iff (\alpha : \beta) = (\lambda_v : -\lambda_u) \;\text{ or }\; \lambda_u = \lambda_v = 0 \; .$$

But $(\alpha : \beta) = (\lambda_v : -\lambda_u)$ or $\lambda_u = \lambda_v = 0$ if, and only if, Equation (8.5) is satisfied.

□

Remark 8.3.3 Proposition 8.3.2 gives the same result as in the Euclidean case. Points where a Euclidean principal curvature's derivative is zero in the corresponding Euclidean principal direction are called ridge points. Proposition 8.3.2 gives an affine analogue of this. See [10] for more details.

Proposition 8.3.4 *Let* $\mathbf{Y}$ *and* $(\alpha : \beta)$ *be as above. Consider the differential map* $d\mathbf{Y} : TU \to T\mathbb{R}^3$. *We have that* $\alpha\lambda_u + \beta\lambda_v = 0$ *if, and only if,* $(\alpha : \beta) \in \ker(d\mathbf{Y})$.

Proof With respect to the basis $\{\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}\}$ on $\mathbb{R}^3$, $d\mathbf{Y}$ is given by the matrix

$$J_{\mathbf{Y}} := \begin{pmatrix} 1+\lambda a & \lambda c \\ \lambda b & 1+\lambda d \\ \lambda_u & \lambda_v \end{pmatrix} .$$

It follows that $J_{\mathbf{Y}}(\alpha, \beta)^\top = (0, 0, \alpha\lambda_u + \beta\lambda_v)^\top$, and so

$$(\alpha : \beta) \in \ker(d\mathbf{Y}) \iff \alpha\lambda_u + \beta\lambda_v = 0 \; .$$

□

8.4 Examples

In this section we give explicit parametrisations for surfaces whose family of affine distance functions versally unfold an A_2, A_3, A_4, and a D_4^+ and a D_4^- singularity.

They are the only only simple singularities which have miniversal deformations (see [1]) of dimension less than or equal to three. Since the family of affine distance functions is a three-parameter family we expect it to versally unfold such singularities.

All of the following examples have been calculated using the Maple computer algebra package. The details of the calculations are not given here. The following examples can be seen as existance statements, i.e. each of the singularities A_2, A_3, A_4, D_4^+ and D_4^- can be versally unfolded by the family of affine distnace functions; there are no geometric restrictions.

8.4.1 The A_2 case

In the case of a versally unfolded A_2 singularity, the affine focal surface will be locally smooth. The surface

$$\mathbf{X}(u,v) = \frac{1}{2}(u^2+v^2) - u^2v^2 + \frac{1}{3}v^4 + u^2v^3$$

has the directions $(1:0)$ and $(0:1)$ as affine principal directions when $u=v=0$. The respective affine principal curvatures are -1 and 1. The family of affine distance functions has a versally unfolded A_2 singularity at $\mathbf{x}=(0,0,1)$.

8.4.2 The A_3 case

In the case of a versally unfolded A_3 singularity, the affine focal surface will be locally diffeomorphic to a cusp edge. The surface

$$\mathbf{X}(u,v) = \frac{1}{2}(u^2+v^2) - u^2v^2 + \frac{1}{3}v^4$$

has the directions $(1:0)$ and $(0:1)$ as affine principal directions when $u=v=0$. The respective affine principal curvatures are -1 and 1. The family of affine distance functions has a versally unfolded A_3 singularity at $\mathbf{x}=(0,0,1)$.

8.4.3 The A_4 case

In the case of a versally unfolded A_4 singularity, the affine focal surface will be locally diffeomorphic to a swallow tail. The surface

$$\mathbf{X}(u,v) = \frac{1}{2}(u^2+v^2) - u^2v^2 + \frac{1}{3}v^4 + u^2v^3 + \frac{45}{8}u^2v^4$$

has the directions $(1:0)$ and $(0:1)$ as affine principal directions when $u = v = 0$. The respective affine principal curvatures are -1 and 1. The family of affine distance functions has a versally unfolded A_4 singularity at $\mathbf{x} = (0,0,1)$.

8.4.4 The D_4^+ case

In the case of a versally unfolded D_4^+ singularity, the affine focal surface will be locally diffeomorphic to a purse singularity. The surface

$$\mathbf{X}(u,v) = \frac{1}{2}(u^2+v^2) + u^2v^2 + u^2v^3 + v^5$$

has an affine umbilic at the origin. Every direction is principal at the origin and the repeated affine principal curvature of 1. The family of affine distance functions has a versally unfolded D_4^+ at $\mathbf{x} = (0,0,1)$.

8.4.5 The D_4^- case

In the case of a versally unfolded D_4^- singularity, the affine focal surface will be locally diffeomorphic to a pyramid singularity. The surface

$$\mathbf{X}(u,v) = \frac{1}{2}(u^2+v^2) + u^2v^2 - u^2v^3 + v^5$$

has an affine umbilic at the origin. Every direction is principal at the origin and the repeated affine principal curvature of 1. The family of affine distance functions has a versally unfolded D_4^- at $\mathbf{x} = (0,0,1)$.

Chapter 9

Some Special Curves

Here we consider some invariant curves on a surface. These are the affine parabolic curve, the repeated A-direction curve, and the ordinary Euclidean parabolic curve. We give results on the structure of these curves and on their interactions. We give results on the nature of the affine parabolic curve and the repeated A-direction curve as we approach the Euclidean parabolic curve at both ordinary Euclidean parabolic points and at Euclidean cusps of Gauß. For brevity, we shall call affine parabolic points A-parabolic points. Euclidean parabolic points will be called parabolic points and Euclidean cusps of Gauß will be called cusps of Gauß.

When looking at the limiting behaviour of the sets above, it will be useful to take the surfaces in a form different to Pick normal form. We shall take surfaces

$$\mathbf{X}(u,v) = \left(u, v, u^2 + \sum_{i=0}^{3} b_i u^{3-i} v^i + \sum_{j=0}^{4} c_j u^{4-j} v^j + O(5) \right) . \tag{9.1}$$

The origin is a parabolic point and the unique asymptotic direction at the origin is $u = 0$. The parabolic set is smooth in a neighbourhood of the origin if, and only if, either $b_2 \neq 0$ or $b_3 \neq 0$. The origin is an ordinary parabolic point if, and only if, $b_3 \neq 0$. It is a cusp of Gauß if, and only if, $b_3 = 0$. If $b_3 = 0$ we assume $b_2 \neq 0$. If the origin is a cusp of Gauß it is an ordinary cusp of Gauß if, and only if, $b_2^2 - 4c_4 \neq 0$.

9.1 The repeated A-direction set

The affine shape operator matrix has a repeated eigendirection if, and only if,

$$(a-d)^2 + 4bc = 0 \ .$$

The leading terms in the power series expansion of this function give the tangent cone to the level set.

Proposition 9.1.1 *In the hyperbolic region of a generic surface, the repeated A-direction set is made up of smooth pieces of curve, which can meet to form a transverse crossing.*

Proof Consider a surface given in Pick normal form (see Equation (5.2) on page 62), and consider the power series expansion of $g(u,v) := (a-d)^2 + 4bc$ at $u = v = 0$. The expansion is

$$g(u,v) = (2a_0 + a_1 - a_3 - 2a_4)(2a_0 - a_1 + a_3 - 2a_4) + O(1) \ .$$

The origin is a repeated A-direction point if, and only if, $2a_0 + a_1 - a_3 - 2a_4 = 0$ or $2a_0 - a_1 + a_3 - 2a_4 = 0$. Assume that $2a_0 + a_1 - a_3 - 2a_4 = 0$ and $2a_0 - a_1 + a_3 - 2a_4 \neq 0$. Solving $2a_0 + a_1 - a_3 - 2a_4 = 0$ in terms of a_0 means that $2a_0 - a_1 + a_3 - 2a_4 \neq 0$ if, and only if, $a_1 - a_3 \neq 0$. Expanding to higher powers:

$$g(u,v) = (a_1 - a_3)(\alpha u + \beta v) + O(2) \ ,$$

for some generically non-zero functions, α and β, of the fifth order and lower Pick coefficients at $u = v = 0$. Since $a_1 - a_3 \neq 0$, it follows that $g^{-1}(0)$ is smooth.

Assume that $2a_0 - a_1 + a_3 - 2a_4 = 0$ and $2a_0 + a_1 - a_3 - 2a_4 \neq 0$. Solving $2a_0 - a_1 + a_3 - 2a_4 = 0$ for a_0 means that $2a_0 + a_1 - a_3 - 2a_4 \neq 0$ if, and only if, $a_1 - a_3 \neq 0$. Expanding to higher powers:

$$g(u,v) = (a_1 - a_3)(\gamma u + \delta v) + O(2) \ ,$$

for some generically non-zero functions, γ and δ, of the fifth order and lower Pick coefficients at $u = v = 0$. Since $a_1 - a_3 \neq 0$, it follows that $g^{-1}(0)$ is smooth.

Assume that $2a_0 - a_1 + a_3 - 2a_4 = 0$ and $2a_0 + a_1 - a_3 - 2a_4 = 0$, i.e. $a_0 = a_1$ and $a_1 = a_3$. We find that for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ that

$$g(u,v) = \frac{1}{4}(\alpha u + \beta v)(\gamma u + \delta v) + O(3) \ .$$

Direct computation shows that generically $\alpha\delta - \beta\gamma \neq 0$; so that g is a Morse function. The repeated A-direction set forms a transverse crossing. □

Proposition 9.1.2 *In the elliptic region of a generic surface the repeated A-direction set consists of isolated points, which are the affine umbilic points.*

Proof Consider the power series expansion

$$g(u,v) = 4(a_0 - a_4)^2 + (a_1 + a_3)^2 + O(1) \ .$$

For the origin to be a repeated A-direction point we need $g(0,0) = 0$, i.e. $a_0 = a_4$ and $a_1 = -a_3$. Assuming this to be the case

$$g(u,v) = (\alpha u^2 + 2\beta uv + \gamma v^2) + O(3) \ ,$$

for some generically non-zero functions, α, β, and γ, of the fifth order and lower Pick coefficients at $u = v = 0$. We find that $\alpha\gamma - \beta^2$ is a perfect square and so $\alpha\gamma - \beta^2 \geq 0$. Generically $\alpha\gamma - \beta^2 > 0$. □

9.2 The A-parabolic and repeated A-direction sets

Here we consider only the hyperbolic region of a surface, since otherwise the repeated A-direction set is generically made up of isolated points. Let the surface $\mathbf{X}$ have the form of Equation (5.2) on page 62.

Proposition 9.2.1 *If the A-parabolic set and the repeated A-direction set meet, then one is smooth if, and only if, the other is smooth. If they are smooth, then they will be tangent.*

Proof For the origin to be both an A-parabolic point and a repeated A-direction point we need to solve the equations $ad - bc = 0$ and $(a-d)^2 + 4bc = 0$ when $u = v = 0$. There are two different solutions, they arise from the choice of solutions for $(a-d)^2 + 4bc = 0$ (see the proof of Proposition 9.1.1). The first solution is given by $2a_2 = 6a_0 - \sigma^2 + 6a_4$ and $a_3 = a_1 - 2a_0 + 2a_4$. The second solution is given by $2a_2 = 6a_0 - \sigma^2 + 6a_4$ and $a_3 = 2a_0 + a_1 - 2a_4$.

Imposing these conditions upon the Pick normal form gives parametrisations $\mathbf{X}_i$, for $1 \leq i \leq 2$. Calculating the power series at $u = v = 0$ of the functions $ad - bc$ and $(a-d)^2 + 4bc$ with respect to the $\mathbf{X}_i$ gives

$$\begin{aligned} ad - bc &= p_i u + q_i v + O(2) , \\ (a-d)^2 + 4bc &= r_i u + s_i v + O(2) . \end{aligned}$$

where $\{p_i, q_i, r_i, s_i\}$ are all generically non-zero functions of the coefficients in the Pick normal form at $u = v = 0$ with $p_i s_i - q_i r_i = 0$. The linear parts being generically non-zero and linearly dependent, the result now follows. □

9.3 Limiting behaviour of A.

Proposition 9.3.1 *The limiting direction of the affine normal vector approaching an ordinary parabolic point is the unique asymptotic direction.*

Proof Consider a surface in a neighbourhood of a parabolic point in the form of Equation (9.1) on page 101. It has $(0:1:0)$ as its asymptotic direction at the origin. Assume that $b_3 \neq 0$ so that the origin is not a cusp of Gauß. It is easy to compute the affine normal vector $\mathbf{A}$. Let $\mathbf{A} = (A_1, A_2, A_3)$, and define

$$\widetilde{\mathbf{A}} := \frac{\mathbf{A}}{\sqrt{A_1^2 + A_2^2 + A_3^2}} .$$

Putting $u = r\cos\theta$ and $v = r\sin\theta$ into $\widetilde{\mathbf{A}}$ gives a vector with variables r and θ. Letting $r \to 0$ gives a limit independent of θ. In fact

$$\lim_{r\to 0} \widetilde{\mathbf{A}}(r\cos\theta, r\sin\theta) = \left(0, -\frac{b_3}{|b_3|}, 0\right) .$$

The limiting direction is then $(0:1:0)$. The result now follows. □

Corollary 9.3.2 *The limiting direction of the affine normal vector approaching a cusp of Gauß along the parabolic curve is the unique asymptotic direction.*

Proof Cusps of Gauß are isolated points on the parabolic curve. □

If we approach the cusp of Gauß along in an arbitrary direction the limit is not defined. Consider a surface parametrised in the form of Equation (9.1) on page 101, but with $b_3 = 0$ so that the origin is a cusp of Gauß. Let $(u,v) = (r\cos\theta, r\sin\theta)$ for some $\theta \in [0, 2\pi)$, then as $r \to 0$ the limiting direction of $\mathbf{A}$ is

$$(-b_2^2\cos\theta : (2b_1b_2 - 3c_3)\cos\theta + (3b_2^2 - 12c_4)\sin\theta : 0)\ .$$

This is a map $S^1 \to \mathbb{R}^2$ which is the restriction of a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ to the unit circle S^1. The linear map is given by

$$(u,v) \mapsto (-b_2^2 u, (2b_1b_2 - 3c_3)u + (3b_2^2 - 12c_4)v)\ .$$

This has matrix representation of

$$\mathcal{M} := \begin{pmatrix} -b_2^2 & 0 \\ 2b_1b_2 - 3c_3 & 3b_2^2 - 12c_4 \end{pmatrix}.$$

Notice that $\det(\mathcal{M}) = 3b_2^2(4c_4 - b_2^2)$, i.e. $\det(\mathcal{M}) = 0$ if, and only if, the parabolic curve is singular or the cusp of Gauß is degenerate. The eigendirections of $\mathcal{M}$ are $(0:1)$ and $(4(3c_4 - b_2^2) : 2b_1b_2 - 3c_3)$. We shall return to this in § 9.5.

9.4 The parabolic and A-parabolic sets

Here we consider surfaces with a parabolic point at the origin. Consider a surface in the form of Equation (9.1) on page 101. Assume that either $b_2 \neq 0$ or $b_3 \neq 0$ so that the origin is a parabolic point on a smooth parabolic curve.

Let us define $p : U \to \mathbb{R}$, where $p(u,v) := ad - bc$, so that the A-parabolic set is the real part of the complex closure of $\mathbf{X}(\{(u,v) \in U : p(u,v) = 0\})$. The function $p : U \to \mathbb{R}$ is not defined over the parabolic set. Calculating p, we find that p can be factorised, say $p = p_1 p_2$. The function p_1 is not defined over the parabolic set, where as the function p_2 is. In fact

$$p_1 = \frac{\sqrt{|LN - M^2|}}{(LN - M^2)^5} \ .$$

There is no such expression for p_2; it is a very complicated function.

The approach is to consider only the function $p_2 : U \to \mathbb{R}$ and its zero-level set. If a point belongs to the parabolic set and the A-parabolic set, we must show that this point is isolated in the intersection of the parabolic set and the A-parabolic set.

Given a surface in the form of Equation (9.1), we can parametrise the parabolic curve by γ by writing u as a function of v. In fact

$$\gamma(v) = -\frac{3b_3}{b_2}v + O(2) \ .$$

First, assume that $b_3 \neq 0$, i.e. the origin is not a cusp of Gauß. We then see that

$$(p_2 \circ \gamma)(v) = b_3(3b_1b_3 - b_2^2) + O(1) \ .$$

We want $(p_2 \circ \gamma)(0) = 0$, but $(p_2 \circ \gamma)(\varepsilon) \neq 0$ for all $0 < |\varepsilon| \ll 1$. Solving $3b_1b_3 - b_2^2 = 0$ for b_1 gives $(p_2 \circ \gamma)(v) = \alpha v + O(2)$ where α is a generically non-zero function of the third order and lower Monge form coefficients. By Hadamard's Lemma (see [4]) we can write $(p_2 \circ \gamma)(v) = vf(v)$ for some smooth function f with $f(0) \neq 0$. The zeros of $(p_2 \circ \gamma)(\varepsilon)$ for $0 < |\varepsilon| \ll 1$ are given by $f(\varepsilon) = 0$. Since f is smooth and $f(0) \neq 0$, we see that f is non-zero in a sufficiently small neighbourhood of zero.

Let us assume that $b_3 = 0$, i.e. the origin is a cusp of Gauß. Here we have

$$\gamma(v) = \frac{b_2^2 - 6c_4}{b_2}v^2 + O(3) \ .$$

Making the substitution as above, up to a non-zero constant we have

$$(p_2 \circ \gamma)(v) = b_2^2(b_2^2 - 4c_4)^2 v^2 + O(3) \ .$$

Provided the origin is not a degenerate cusp of Gauß we have $(p_2 \circ \gamma)(v) = v^2 g(v)$ for a smooth function g with $g(0) \neq 0$. We may now follow the previous argument.

Proposition 9.4.1 *Consider a point which is both a parabolic point and an A-parabolic point. We have*

1. *If the parabolic point is an ordinary parabolic point or a non-degenerate cusp of Gauß then the A-parabolic set will be non-singular.*
2. *If the parabolic point is an ordinary parabolic point then the parabolic set and the A-parabolic set will meet transversely.*
3. *If the parabolic point is a non-degenerate cusp of Gauß then the parabolic set and the A-parabolic set will be tangent.*

Proof Consider a surface in the form of Equation (9.1). Expanding as a power series about $u = v = 0$ gives

$$p_2(u, v) = b_3^2(3b_1b_3 - b_2^2) + O(1) .$$

The origin is an A-parabolic point if, and only if, either $b_3 = 0$ or $3b_1b_3 - b_2^2 = 0$. Let us assume that $b_3 \neq 0$ and $3b_1b_3 - b_2^2 = 0$, then

$$p_2(u, v) = \lambda u + O(2) ,$$

where λ is a generically non-zero function of the third order and lower Monge form coefficients. The zero-level set is non-singular. The tangent line to the A-parabolic set is the $u = 0$, but since $b_3 \neq 0$, this is not the same tangent direction as that of the parabolic set.

Let us assume that $b_3 = 0$. Since $b_3 = 0$ the tangent line to the parabolic curve at the origin is given by $u = 0$. We find that

$$p_2(u, v) = b_2^3(b_2^2 - 4c_4)u + O(2) .$$

The zero-level set is smooth, and has tangent line $u = 0$ if, and only if, $b_2^3(b_2^2 - 4c_4) \neq 0$, i.e. if, and only if, the parabolic set is non-singular and the cusp of Gauß is not a degenerate cusp of Gauß.

If $b_3 = 3b_1b_3 - b_2^2 = 0$ then $b_3 = b_2 = 0$ and this contradicts the assumption that the parabolic set is a smooth curve close to the origin. □

9.5 The parabolic and repeated A-direction sets

Let us define $q : U \to \mathbb{R}$, where $q(u,v) := (a-d)^2 + 4bc$, so that the repeated A-direction set is the real part of the complex closure of $\mathbf{X}(\{(u,v) \in U : q(u,v) = 0\})$. The function $q : U \to \mathbb{R}$ is not defined over the parabolic set. Calculating q, we find that q can be factorised, say $q = q_1q_2$. The function q_1 is not defined over the parabolic set, where as the function q_2 is. In fact

$$q_1 = \frac{\sqrt{|LN - M^2|}}{(LN - M^2)^6} \ .$$

There is no such expression for q_2; it is a very complicated function. Note the power of six in the denominator, instead of the fifth in p_1.

Consider a surface in the form of Equation (9.1) on page 101. In what follows we consider only the function $q_2 : U \to \mathbb{R}$ and its zero-level set. We show the validity of this approach after the following

Proposition 9.5.1 *The repeated A-direction curve and the parabolic curve meet only at cusps of Gauß.*

Proof Consider the power series expansion of q_2 about $u = v = 0$, which has the form

$$q_2(u,v) = b_3^4 + O(1) \ .$$

It follows that $q_2(0,0) = 0$ if, and only if, $b_3 = 0$. □

Remark 9.5.2 For some cusps of Gauß the only repeated A-direction point in a neighbourhood is the cusp of Gauß itself. This is because the repeated A-direction set is complex.

We can parametrise the parabolic curve by writing u as a function of v. The origin is a repeated A-direction point if, and only if, it is a cusp of Gauß. Let us assume that $b_3 = 0$, then as in §9.4 we have

$$\gamma(v) = \frac{b_2^2 - 6c_4}{b_2} v^2 + O(3) \ .$$

Up to a non-zero constant, we have

$$(q_2 \circ \gamma)(v) = (b_2^2 - 4c_4)^4 v^4 + O(5) \ .$$

If the origin is a non-degenerate cusp of Gauß then by Hadamard's Lemma (see [4]) we can write $(q_2 \circ \gamma)(v) = v^4 f(v)$ for some smooth function f with $f(0) \neq 0$. The origin is an isolated point in the intersection of the parabolic set and the repeated A-direction set.

Remark 9.5.3 When $b_3 = 0$ the linear terms of q_2 also vanish, meaning $q_2 \in O(2)$.

Proposition 9.5.4 *Let the origin be a cusp of Gauß (i.e. $b_3 = 0$). If $16c_4 - 3b_2^2 \neq 0$ then the repeated A-direction set has the Euclidean asymptotic direction with multiplicity two as its tangent cone. If $16c_4 - 3b_2^2 = 0$ then the repeated A-direction set has the Euclidean asymptotic direction with multiplicity one and a line transverse to that with multiplicity two as its tangent cone.*

Proof Given that $b_3 = 0$ we have

$$q_2(u, v) = b_2^2(3b_2^2 - 16c_4)^2 u^2 + O(3) \ .$$

By assumption $b_2 \neq 0$. Thus $\{u = 0\}$ is the tangent cone with multiplicity two if, and only if, $3b_2^2 - 16c_4 \neq 0$. Assume that $3b_2^2 - 16c_4 = 0$, then

$$q_2(u, v) = b_2^3 u(4(2b_1b_2 - 3c_3)u + 7b_2^2 v)^2 + O(4) \ .$$

Since $b_2 \neq 0$, the line $u = 0$ can never be given by $4(2b_1b_2 - 3c_3)u + 7b_2^2 v = 0$. □

Remark 9.5.5 If $16c_4 - 3b_2^2 = 0$ then the eigendirections of $\mathcal{M}$ become $(0 : 1)$ and $(-7b_2^2 : 4(2b_1b_2 - 3c_3))$. These are exactly the directions given by

$$b_2^3 u(4(2b_1b_2 - 3c_3)u + 7b_2^2 v)^2 = 0 \ ,$$

i.e. the directions in the tangent cone of $q_2 = 0$ at the origin.

The next result show what to expect when $16c_4 - 3b_2^2 \neq 0$.

Remark 9.5.6 Cusps of Gauß where $3b_2^2 - 16c_4 = 0$ are non-generic for a given surface. They will arise in generic one-parameter families of surfaces.

Before we continue we give an example of completing the square. This method will be used in a following proof.

Example. Consider the function germ $f(u, v) = u^2 + 2u^2v + 2uv^2$. We can complete the square on the u^2, uv, and v^2 terms to give $(u + uv + v^2)^2 - (uv + v^2)^2$. We relabel $u + uv + v^2 := \tilde{u}$. We can solve the equality $\tilde{u} = u + uv + v^2$ for u as a formal power series in $\tilde{u}$ and v. See [17] for more details. In this example

$$u = \tilde{u} - \tilde{u}v - \tilde{u}^2 + v^3 + O(4) \ .$$

Substituting this formal power series into f and rewriting $\tilde{u}$ as u we have

$$u^2 - 2uv^3 - u^2v^2 - v^4 + O(5) \ .$$

Completing the square on the u^2, uv^3, and u^2v^2 terms gives

$$\left(u - \frac{1}{2}uv^2 - v^3\right)^2 - v^4 + O(5) \ .$$

We relabel $u - (uv^2)/2 - v^3 := \tilde{u}$. We solve $2\tilde{u} - 2u + uv^2 + v^3 = 0$ for u as a formal power series in $\tilde{u}$ and v. In this example we have

$$u = \tilde{u} + \frac{1}{2}\tilde{u}v^2 + v^3 + O(4) \ .$$

Substituting this formal power series into f and rewriting $\tilde{u}$ as u we have

$$u^2 - v^5 + O(6) \ .$$

Substituting v for $-v$ gives $u^2+v^5+O(6)$. Therefore $u^2+2u^2v+2uv^2$ has type A_4, and hence is a degenerate cusp of Gauß. It is easy to check that the three substitutions were diffeomorphic changes of variable.

Proposition 9.5.7 *Let the origin be an ordinary cusp of Gauß with* $16c_4 - 3b_2^2 \neq 0$.

1. *If* $b_2^2 - 3c_4 < 0$ *then repeated A-direction set will be a single point, namely the cusp of Gauß.*

2. *If* $b_2^2 - 3c_4 > 0$ *then near the cusp of Gauß the repeated A-direction set will be locally diffeomorphic to a tacnode, i.e. the* $\{(x, y) \in \mathbb{R}^2 : x^2 - y^4 = 0\}$. *The singular point is at the cusp of Gauß.*

Proof The condition for an ordinary cusp of Gauß is that $b_2^2 - 4c_4 \neq 0$. Let us assume that this holds.

The proposition says that the function germ $q_2 : U, 0 \to \mathbb{R}, 0$ has an $A_3^{\pm}$ singularity. We now employ the method of the example on page 110.

We find that $q_2(u, v) = b_2^2(3b_2^2 - 16c_4)^2u^2 + O(3)$. Moreover, there is no v^3 term. By completing the square with the u^2, u^3, u^2v, and uv^2 terms and relabelling as $\tilde{u}$ the 4-jet becomes $\alpha\tilde{u}^2 + O(4)$ for $\alpha \neq 0$. Then rewrite the variables as u and v once more. The new v^4 term is absent if, and only if,

$$b_2^4\,(b_2^2 - 3c_4)(b_2^2 - 4c_4)^2 = 0\ .$$

By assumption $b_2(b_2^2 - 4c_4) \neq 0$. If $b_2^2 - 3c_4 \neq 0$ then there is a v^4 coefficient. Completing the square with the u^2, u^4, u^3v, u^2v^2, and uv^3 terms and relabelling as $\tilde{u}$ gives a 4-jet of $\lambda\tilde{u}^2 + \mu v^4$ for $\lambda > 0$ and $\mu \neq 0$. In fact

$$\mu = -b_2^4\,(b_2^2 - 3c_4)(b_2^2 - 4c_4)^2\ ,$$

so the sign of $3c_4 - b_2^2$ dictates the choice of $A_3^{\pm}$. rewriting the variables as u and v and making a final scaling brings the 4-jet into the form $u^2 \pm v^4$.

It is easy to check that all of the mappings induced by the completing the square and relabelling are diffeomorphisms. □

Remark 9.5.8 If $16c_4 - 3b_2^2 \neq 0$ but $b_2^2 - 3c_4 = 0$ then q_2 will not be $\mathcal{A}$-equivalent to $u^2 \pm v^4$. It will have a more degenerate singularity type. Since $16c_4 - 3b_2^2 \neq 0$, there will be a u^2 terms, so q_2 will always have an A_k.

Next we consider the case when $16c_4 - 3b_2^2 = 0$. The tangent cone here is the Euclidean asymptotic direction with multiplicity one and a line transverse to this with multiplicity two. The classification of singularities of this degeneracy is much more complicated. Here we use the techniques of blow-ups to classify the normal forms. See [17] for more information about blow-ups.

9.5.1 Blow-up calculations

We assume that $b_3 = 0$, $16c_4 - 3b_2^2 = 0$, and $b_2 \neq 0$. The power series expansion is a non-zero constant multiple of

$$q_2(u,v) = b_2^3\, u\,((8b_1b_2 - 12c_3)u + 7b_2^2v)^2 + O(4)\ .$$

We wish to perform a linear transformation to simplify this. Let $u \mapsto \tilde{u}$ and

$$v \mapsto \frac{1}{7b_2^2}((12c_3 - 8b_1b_2)\tilde{u} - \tilde{v})\ .$$

Under this transformation, we find that

$$q_2(\tilde{u},\tilde{v}) = b_2^3\tilde{u}\tilde{v}^2 + O(4)\ .$$

In order to use the resolution of singularities and the blowing-up techniques, we need to examine more than just the cubic term, we need to examine all of the monomials which lie on the convex hull of the Newton Polygon. Consider

$$A_2uv^2 + B_0u^4 + B_1u^3v + B_2u^2v^2 + B_3uv^3 + B_4v^4 + O(5)\ .$$

We have seen that q_2 can be brought into this form by a linear change of coordinates.

Proposition 9.5.9 *If $B_0 \neq 0$ then the repeated A-direction set has branches locally diffeomorphic to a straight line and an ordinary cusp, the singular point of the ordinary cusp being at the cusp of Gauß, and the limiting tangent line to the cusp at the singular point transverse to the smooth branch.*

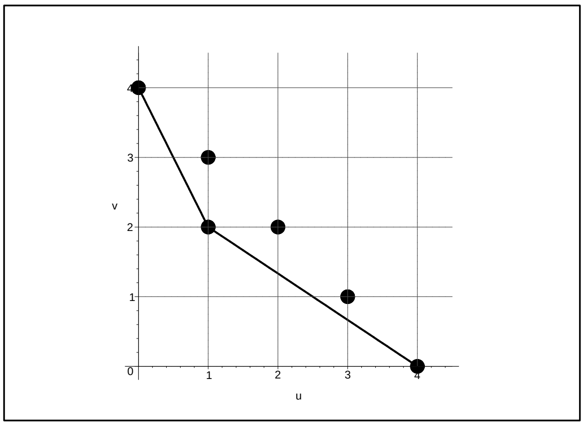

Figure 9.1: Newton Polygon for $B_0 \neq 0$.

Proof Calculating the Newton Polygon, we find that only the monomials uv^2, u^4, and v^4 lie on the convex hull, all other monomials lie above it. This can be seen in Figure 9.1 which shows the Newton Polygon and its convex hull. In Figure 9.1 only the third and fourth order monomials have been included. Any higher degree monomials will lie above the ones shown, and so will play no part. Let us write

$$F := A_2uv^2 + B_0u^4 + B_4v^4 + \cdots .$$

The terms in the tail are multiples of monomials of the form u^mv^n where $m+n \geq 4$ and $(m,n) \neq (0,4)$. These are the terms above the Newton Polygon. We apply the blowing-up method to this function. Let us remark that A_2 is a non-zero constant multiple of b_2^3 and so is always non-zero by assumption. Also, B_4 is a non-zero constant rational number. Thus, we may conclude that $A_2B_4 \neq 0$.

We now use the standard blowing-up method. For the first blow-up we make the substitution $(u,v) \rightsquigarrow (u_1v_1, v_1)$. This gives the total transform

$$F_1 = A_2u_1v_1^3 + B_0u_1^4v_1^4 + B_4v_1^4 + \cdots .$$

The terms in the tail are multiples of the monomial of the form $u_1^m v_1^{m+n}$ where $m+n \geq 4$ and $(m,n) \neq (0,4)$. Clearly, v_1^3 divides F_1 and so

$$F_1 = v_1^3(A_2u_1 + B_0u_1^4v_1 + B_4v_1 + \cdots) ,$$

where the terms in the tail are multiples of monomials of the form $u_1^m v_1^{m+n-3}$. The exceptional divisor E_0 is the line $v_1 = 0$ counted thrice, this shows that $\{F = 0\}$ has intersection number three when $u = v = 0$. The proper transform $C^{(1)}$ is

$$\{A_2 u_1 + B_0 u_1^4 v_1 + B_4 v_1 + \cdots = 0\} \ .$$

Notice that $C^{(1)}$ is a smooth curve close to $u_1 = v_1 = 0$. Provided $A_2 \neq 0$, which it is, the proper transform and the exceptional divisor meet transversally at a single point, and so form a normal crossing divisor.

By the Implicit Function Theorem we can write v_1 as a function of u_1 close to $u_1 = v_1 = 0$. By Hadamard's Lemma (see [4]), there exists a smooth $h \in O(5)$ such that

$$v_1(u_1) = -\frac{A_2}{B_4} u_1 + \frac{A_2 B_0}{B_4^2} u_1^4 + h(u_1) \ .$$

Using the blow-down map $(u_1, v_1) \mapsto (u_1 v_1, v_1)$ we find that

$$(u_1, v_1(u_1)) \mapsto \left(u_1, -\frac{A_2}{B_4} u_1^2 + \cdots \right) .$$

This is clearly a smooth branch of $\{F = 0\}$.

We now consider the alternative blow-up map. Let us make the substitution $(u, v) \rightsquigarrow (u_1, u_1 v_1)$. The total transform of F is then

$$F_1 = u_1^3 (A_2 v_1^2 + B_0 u_1 + B_4 u_1 v_1^4 + \cdots) \ ,$$

where the terms in the tail are multiples of monomials of the form $u_1^{m+n-3} v_1^n$. The monomials above the convex hull of the Newton Polygon are of the form $u^m v^n$ where $m + n \geq 4$ and $(m, n) \neq (0, 4)$. For such monomials $m + n - 3 \geq 0$. The exceptional divisor E_0 is the line $u_1 = 0$ counted thrice. And $C^{(1)}$ is

$$\{A_2 v_1^2 + B_0 u_1 + B_4 u_1 v_1^4 + \cdots = 0\} \ .$$

Although the proper transform $C^{(1)}$ is a smooth curve close to $u_1 = v_1 = 0$, it is tangent to the exceptional divisor E_0; they do not form a normal crossing divisor. Their positions can be seen in the schematic diagram in Figure 9.2. It is therefore necessary to blow-up at least once more.

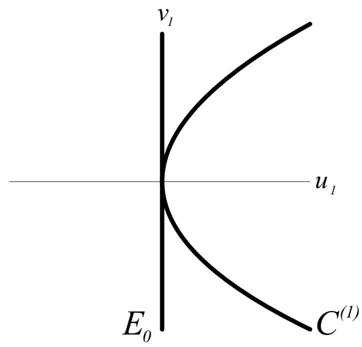

Figure 9.2: The picture after the first blow-up

For the second blow-up, make the substitution $(u_1, v_1) \rightsquigarrow (u_2v_2, v_2)$. The total transform of the equation of $C^{(1)}$ is

$$F_2 = v_2(A_2v_2 + B_0u_2 + B_4u_2v_2^4 + \cdots) ,$$

the terms in the tail are multiples of monomials of the form $u_2^{m+n-3}v_2^{m+2n-4}$. The terms above the Newton Polygon have $m + 2n - 4 \geq 0$. The exceptional divisor E_1 is the line $v_2 = 0$. The proper transform $C^{(2)}$ is

$$\{A_2v_2 + B_0u_2 + B_4u_2v_2^4 + \cdots = 0\} .$$

The image of E_0 is the line $u_2 = 0$. Again $C^{(2)}$ is a smooth curve close to $u_2 = v_2 = 0$. However, $C^{(2)}$, E_1, and the image of E_0 do not form a normal crossing divisor because all three meet at one point. Their positions can be seen in the schematic diagram in Figure 9.3. It is necessary to blow-up at least once more.

Before we do that, we need to check that no branches have been sent to infinity, i.e. to the origin in the other chart. We make the substitution $(u_1, v_1) \rightsquigarrow (u_2, u_2v_2)$. This gives a total transform

$$\tilde{F}_2 = u_2(B_0 + A_2u_2v_2^2 + B_4u_2^4v_2^4 + \cdots) .$$

If $B_0 \neq 0$ the proper transform does not pass through the origin in this chart.

For the third blow-up, make the substitution $(u_2, v_2) \rightsquigarrow (u_3v_3, v_3)$. The total transform of the equation of $C^{(2)}$ is then

$$F_3 = v_3(A_2 + B_0u_3 + B_4u_3v_3^3 + \cdots) ,$$

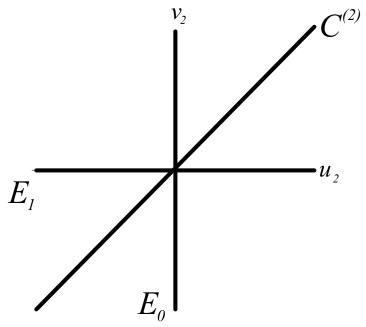

Figure 9.3: The picture after the second blow-up

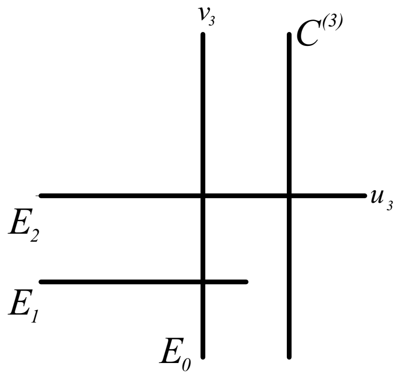

Figure 9.4: The picture after the third blow-up

where the terms in the tail are multiples of monomials of the form $u_3^{m+n-3} v_3^{2m+3n-8}$. Again we find that for terms above the Newton Polygon $2m + 3n - 8 \geq 0$. The exceptional divisor E_2 is the line $v_3 = 0$. The proper transform $C^{(3)}$ is

$$\{A_2 + B_0 u_3 + B_4 u_3 v_3^3 + \cdots = 0\} \ .$$

The image of E_1 does not appear in this chart. The image of E_0 is the line $u_3 = 0$. The proper transform $C^{(3)}$ is a smooth curve, and together with E_2 and the images of E_1 and E_0, we have a normal crossing divisor. We can see their positions in the schematic diagram in Figure 9.4. It is necessary to blow-up at least once more.

Checking the other chart we make the substitution $(u_2, v_2) \rightsquigarrow (u_3, u_3v_3)$. This gives the total transform

$$\tilde{F}_3 = u_3(B_0 + A_2v_3 + B_4u_3^4v_3^4 + \cdots) \ .$$

Again, since $B_0 \neq 0$ by assumption, the proper transform does not pass through the origin in this chart. We have not lost any branches by making the original substitution.

Assume that $B_0 \neq 0$; by the Inverse Function Theorem we can write u_3 as a function of v_3. By Hadamard's Lemma (see [4]), for a smooth function $h \in O(5)$, we can write

$$u_3(v_3) = -\frac{A_2}{B_0} + \frac{A_2B_4}{B_0^2}v_3^4 + h(v_3) \ .$$

We need to find the image of $C^{(3)}$ under the blow-down map. In this case, the blow-down map is the composite of the three individual blow-down maps. This yields $(u_3, v_3) \mapsto (u_3v_3^2, u_3v_3^3)$. Clearly the image of $C^{(3)}$ is

$$(u_3(v_3), v_3) \mapsto \left(-\frac{A_2}{B_0}v_3^2 + \cdots, -\frac{A_2}{B_0}v_3^3 + \cdots\right) .$$

By the k-determinacy of plane curve germs, we find that this is locally diffeomorphic to the normal form (t^2, t^3) and is therefore an ordinary cusp. □

Remark 9.5.10 Consider the normal form of a $D_k^{\pm}$ (see [1]), this is given by $(u, v) \mapsto v(u^2 \pm v^{k-2})$ where the normal forms for odd k give equivalent functions. A D_5 has normal form $v(u^2 \pm v^3)$; this is a line and an ordinary cusp meeting at the origin. The limiting tangent to the cusp at its singular point being transverse to the line. This is exactly the case in Proposition 9.5.9 on page 112.

We now consider the case when $B_0 = 0$. Provided that $C_0 \neq 0$, we find that the first candidate for the Newton Polygon contains the monomials uv^2, u^3v, v^4, and u^5. This can be seen in Figure 9.5. Only the third, fourth, and fifth order monomials are shown. However, direct computation shows that $B_1^2 - 4A_2C_0 = 0$; the polynomial

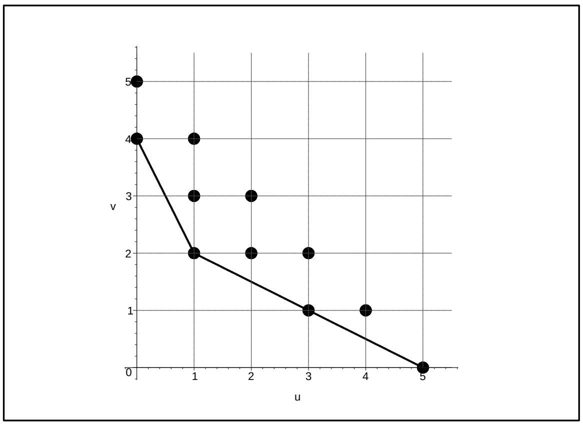

Figure 9.5: Newton Polygon for $B_0 = 0$ and $C_0 \neq 0$.

$A_2uv^2 + B_1u^3v + C_0u^5$ has a repeated root. Given that $A_2 \neq 0$ by assumption (recall that this means that the parabolic set is non-singular) and that $B_1^2 - 4A_2C_0 = 0$, we have

$$A_2uv^2 + B_1u^3v + C_0u^5 = A_2u\left(v + \frac{B_1}{2A_2}u^2\right)^2.$$

We need to make a change of variable, let $u \mapsto \tilde{u}$ and

$$v \mapsto \tilde{v} - \frac{B_1}{2A_2}\tilde{u}^2 \ .$$

This is a diffeomorphic change of variable. If

$$F(u,v) = A_2uv^2 + B_1u^3v + B_2u^2v^2 + B_3uv^3 + B_4v^4 + C_0u^5 \cdots \ ,$$

then making the substitution we find that

$$F(\tilde{u},\tilde{v}) = A_2\tilde{u}\tilde{v}^2 + B_4\tilde{v}^4 + \left(\frac{3B_1^2B_3}{4A_2^2} - \frac{B_1C_2}{A_2} + D_1\right)\tilde{u}^6 + \cdots \ ,$$

where the quintic terms in $F(u,v)$ were $\sum_{i=0}^{5} D_iu^{5-i}v^i$. The terms in the tail of $F(\tilde{u},\tilde{v})$ are multiples of monomials of the form $\tilde{u}^m\tilde{v}^n$ where $2m + 5n > 12$ and $(m,n) \neq (0,3)$ or $(0,4)$, and so lie above the Newton Polygon. Generically the coefficient of $\tilde{u}^6$ is non-zero. To simplify the notation replace $\tilde{u}$ by u and $\tilde{v}$ by v. The monomials on the new Newton Polygon are uv^2, v^4, and u^6. The diagram can

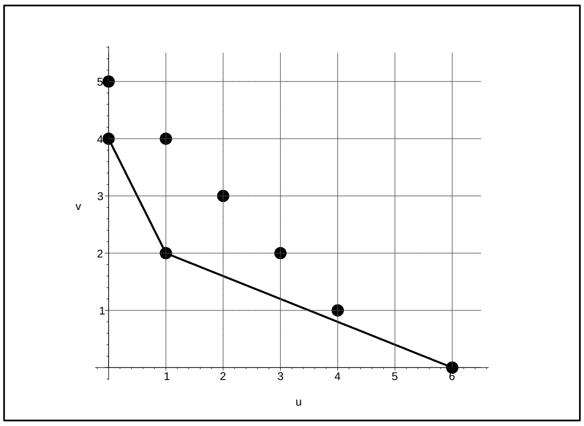

Figure 9.6: The new, post-transform Newton Polygon

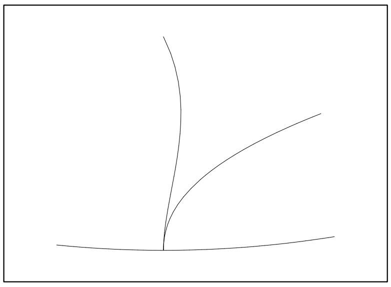

Figure 9.7: The local picture for $B_0 = 0$ and $\Omega \neq 0$.

be found in Figure 9.6. Only the monomials of order less than or equal to five are shown. In particular, notice that there is no u^5 term. Let Ω denote the new u^6 coefficient.

Proposition 9.5.11 *If $\Omega \neq 0$ then the repeated A-direction set will be locally diffeomorphic to a straight line and a rhamphoid cusp, with the singular point at the cusp of Gauß.*

A picture of this can be found in Figure 9.7.

Proof Consider

$$F := A_2uv^2 + B_4v^4 + \Omega u^6 + \cdots ,$$

where the terms are multiples of monomials of the form $u^m v^n$ where $2m + 5n > 12$ and $(m, n) \neq (0, 3)$ or $(0, 4)$. For the first blow-up, make the substitution $(u, v) \rightsquigarrow$

(u_1v_1, v_1). This gives the total transform of F, namely

$$F_1 = v_1^3(A_2u_1 + B_4v_1 + \Omega u_1^6v_1^3 + \cdots) ,$$

where the terms in the tail are monomials of the form $u^m v^{m+n-3}$. For all of the terms above the Newton Polygon, we have $m + n - 3 > 0$. The exceptional divisor E_0 is the line $v_1 = 0$ counted thrice. The proper transform $C^{(1)}$ is

$$\{A_2u_1 + B_4v_1 + \Omega u_1^6v_1^3 + \cdots = 0\} .$$

The proper transform is smooth close to the origin since $A_2B_4 \neq 0$. The exceptional divisor and the proper transform form a normal crossing divisor. By the Implicit Function Theorem, we are able to parametrise $C^{(1)}$ close to the origin by writing u_1 as a function of v_1. Looking for a formal power series solution:

$$u_1(v_1) = -\frac{B_4}{A_2}v_1 - \frac{B_4^6\Omega}{A_2^7}v_1^9 + \cdots .$$

The blow-down map is $(u_1, v_1) \mapsto (u_1v_1, v_1)$, and so

$$(u_1(v_1), v_1) \mapsto \left(-\frac{B_4}{A_2}v_1^2 + \cdots , -\frac{B_4}{A_2}v_1 + \cdots\right) .$$

This is a smooth branch tangent to the $u = 0$ axis.

For the other blow-up, make the substitution $(u, v) \rightsquigarrow (u_1, u_1v_1)$. The total transform of F is given by

$$F_1 = u_1^3(A_2v_1^2 + B_4u_1v_1^4 + \Omega u_1^3 + \cdots) ,$$

where the terms in the tail are multiples of monomials of the form $u^{m+n-3}v^n$. For all of the terms above the Newton Polygon $m + n - 3 > 0$. The exceptional divisor E_0 is the line $u_1 = 0$ counted thrice. The proper transform $C^{(1)}$ is

$$\{A_2v_1^2 + B_4u_1v_1^4 + \Omega u_1^3 + \cdots = 0\} .$$

The proper transform is singular, it is generically a cusp, with its singular point at the origin. The schematic diagram can be seen in Figure 9.8. It is necessary to blow-up at least once more.

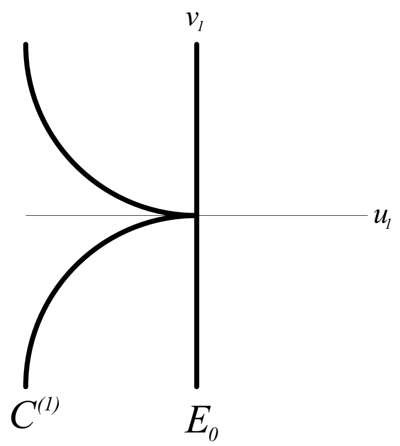

Figure 9.8: The Picture After The First Blow-Up

For the second blow-up, make the substitution $(u_1, v_1) \rightsquigarrow (u_2, u_2 v_2)$. The total transform of F_1 is

$$F_2 = u_2^2(A_2 v_2^2 + B_4 u_2^2 v_2^2 + \Omega u_2 + \cdots) \ ,$$

where the terms in the tail are multiples of monomials of the form $u^{m+2n-5}v^n$. For all of the terms above the Newton Polygon $m + 2n - 5 > 0$. The exceptional divisor E_1 is the line $u_2 = 0$ counted twice. The image of E_0 does not appear in this chart. The proper transform $C^{(2)}$ is

$$\{A_2 v_2^2 + B_4 u_2^2 v_2^2 + \Omega u_2 + \cdots = 0\} \ .$$

The proper transform $C^{(2)}$ is a smooth curve close to the origin, however it is also tangent to the exceptional divisor E_1 : they do not form a normal crossing divisor. The schematic diagram can be seen in Figure 9.9. It is necessary to blow-up at least once more.

The only point in this chart which is missing is the origin in the other chart. We find that the total transform with respect to the substitution $(u_1, v_1) \rightsquigarrow (u_2 v_2, v_2)$ is

$$\tilde{F}_2 = v_2^2(A_2 + B_4 u_2 v_2^3 + \Omega u_2^3 v_2 + \cdots) \ .$$

The proper transform does not pass through the origin, and so nothing had been missed in the original chart.

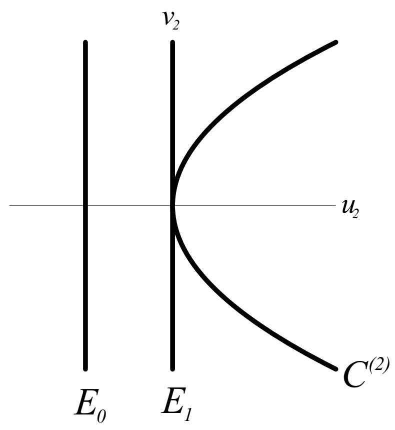

Figure 9.9: The Picture After The Second Blow-Up

For the third blow-up, make the substitution $(u_2, v_2) \rightsquigarrow (u_3v_3, v_3)$. This gives the total transform

$$F_3 = v_3(A_2v_3 + B_4u_3^3v_3^6 + \Omega u_3 + \cdots) \ ,$$

where the terms in the tail are multiples of monomials of the form $u^{m+n-5}v^{m+3n-6}$. For monomials above the Newton Polygon $m + 3n - 6 > 0$. The exceptional divisor E_2 is the line $v_3 = 0$ counted once. The image of E_1 is the line $u_3 = 0$. The proper transform $C^{(3)}$ is

$$\{A_2v_3 + B_4u_3^3v_3^6 + \Omega u_3 + \cdots = 0\} \ .$$

This is always smooth close to the origin since $A_2 \neq 0$. However E_2, the image of E_1 and the proper transform $C^{(3)}$ all meet in a single point. They do not form a normal crossing divisor. The schematic diagram can be seen in Figure 9.10. It is necessary to blow-up at least once more.

First, we need to check that nothing has been missed by the chosen chart. Make the substitution $(u_2, v_2) \rightsquigarrow (u_3, u_3v_3)$ this gives the total transform

$$\tilde{F}_3 = u_3(A_2u_3v_3^2 + B_4u_3^6v_3^4 + \Omega + \cdots) \ .$$

The proper transform does not pass through the origin, and so was not missed by the original choice of chart.

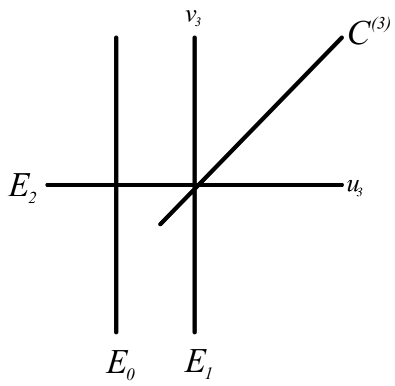

Figure 9.10: The Picture After The Third Blow-Up

For the fourth blow-up, make the substitution $(u_3, v_3) \rightsquigarrow (u_4, u_4v_4)$. This gives the total transform

$$F_4 = u_4(A_2v_4 + B_4u_4^8v_4^6 + \Omega + \cdots) ,$$

where the terms in the tail are multiples of monomials of the form $u^{2m+4n-12}v^{m+3n-6}$. For monomials above the Newton Polygon, we have $2m+4n-12 > 0$. The exceptional divisor E_3 is the line $u_4 = 0$ counted once. The image of E_2 is the line $v_4 = 0$. The image of E_1 does not appear in this chart. The proper transform $C^{(4)}$ is

$$\{A_2v_4 + B_4u_4^8v_4^6 + \Omega + \cdots = 0\} .$$

The proper transform $C^{(4)}$ along with all of the E_i form a normal crossing divisor. The schematic diagram can be seen in Figure 9.11. Since $A_2 \neq 0$, this is a smooth curve close to $(u_4, v_4) = (0, -\Omega/A_2)$. By the Implicit Function Theorem, we can parametrise the proper transform $C^{(4)}$ by writing v_4 as a function of u_4. Looking for a power series solution, we find that

$$v_4(u_4) = -\frac{\Omega}{A_2} - \frac{B_4\Omega^6}{A_2^7}u_4^8 + \cdots .$$

We need to check that this chosen chart has not missed anything. Making the substitution $(u_3, v_3) \rightsquigarrow (u_4v_4, v_4)$, the total transform is

$$\tilde{F}_4 = v_4(A_2 + B_4u_4^3v_4^8 + \Omega u_4 + \cdots) .$$

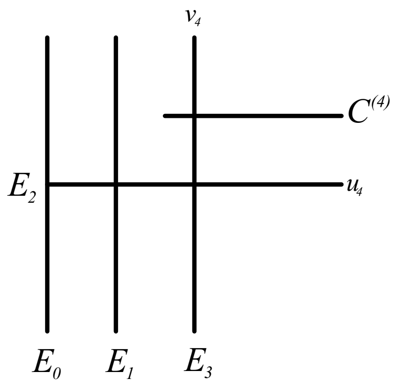

Figure 9.11: The Picture After The Fourth Blow-Up

Nothing passes through the origin, and so we conclude that the original choice of chart did not lose any branches during the blow-up.

The blow-down map can be calculated. We have $(u_4, v_4) \mapsto (u_4^2 v_4, u_4^5 v_4^3)$. Thus

$$(u_4, v_4(u_4)) \mapsto \left(-\frac{\Omega}{A_2} u_4^2 + \cdots, -\frac{\Omega^3}{A_2^3} u_4^5 + \cdots \right).$$

Using k-determinacy, we find that the image is a rhamphoid cusp, i.e. a singular curve which is locally diffeomorphic to $\{(x, y) \in \mathbb{R}^2 : x^2 - y^5 = 0\}$. □

Remark 9.5.12 Following on from Remark 9.5.10 on page 117 we see that the normal form for a $D_7^{\pm}$ singularity is $v(u^2 \pm v^5)$. This is a line and a rhampoid cusp which meet at the origin. The limiting tangent to the rhampoid cusp at its singular being transverse to the line. This is exactly the situation in Proposition 9.5.11.

9.6 Examples and pictures

Here we give examples and plots of certain cases.

Example. Consider a surface

$$\mathbf{X}(u, v) = (u, v, u^2 + u^2 v + uv^2 + uv^3 - v^4) .$$

This has an ordinary cusp of Gauß at the origin ($b_2^2 - 4c_4 \neq 0$). Also we have $b_2^2 - 3c_4 < 0$. The repeated A-direction set is locally diffeomorphic to a tacnode (cf. Proposition 9.5.7). In Figure 9.12 the repeated A-direction set (RADC) is made up of two branches, the A-parabolic set (APC) is a smooth curve, and the parabolic set (EPC) is also a smooth curve. The A-parabolic curve is tangent to the parabolic curve at the cusp of Gauß and meets it transversely away from the cusp of Gauß (cf. Proposition 9.4.1).

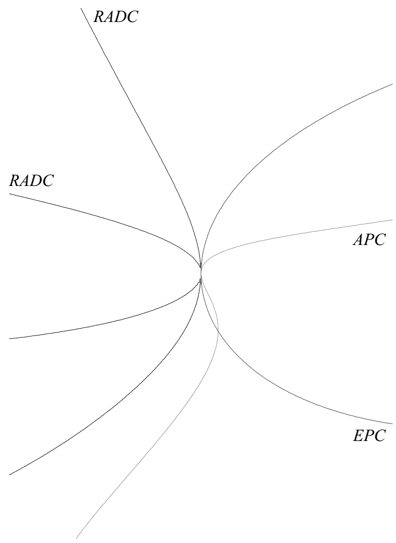

Figure 9.12: Some of the special curves in the uv-parameter plane

Example. Consider the surface is given by

$$\mathbf{X}(u,v) = \left(u, v, \frac{1}{2}(u^2+v^2) + \frac{1}{6}(u^3 - 3uv^2)\right).$$

The origin is an elliptically curved point. The origin is an affine umbilic. The parabolic curve is an oval, and has three ordinary cusps of Gauß. The affine parabolic curve stays inside the elliptic region. The three curves can be seen in Figure 9.13.

Example. Consider the surface is given by

$$\mathbf{X}(u,v) = \left(u, v, \frac{1}{2}(u^2-v^2) + \frac{1}{6}(u^3 + 3uv^2)\right).$$

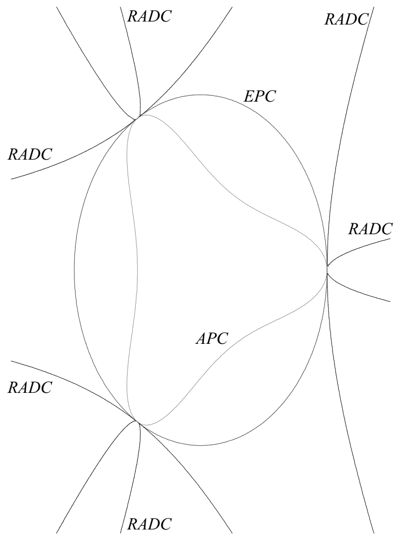

Figure 9.13: Some of the special curves in the uv-parameter plane

The origin is a hyperbolically curved point. The origin is an affine umbilic, the repeated A-direction set has a transverse crossing at this point. This picture shows many phenomena. The repeated A-direction set and the affine parabolic curves meet twice away from the parabolic curve; both curves are smooth and tangent at these places (cf. Proposition 9.2.1 on page 103). The affine parabolic set is tangent to the parabolic set at an ordinary cusp of Gauß. The repeated A-direction set is also tangent there, and is locally diffeomorphic to a tacnode. The three curves can be seen in Figure 9.14.

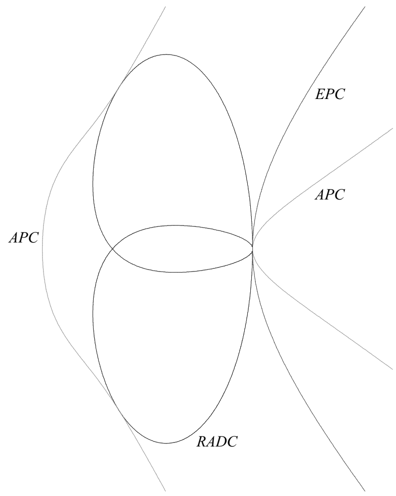

Figure 9.14: Some of the special curves in the uv-parameter plane

Chapter 10

Affine Sectional Curvature

10.1 Motivation

Here we consider an analogue of the Euclidean sectional curvature. Given a smooth surface $\mathbf{X} : U \to \mathbb{R}^3$, and a point $p \in \mathbf{X}(U)$, the tangent plane to $\mathbf{X}$ at p is given by $T_p\mathbf{X} = \langle \mathbf{X}_u, \mathbf{X}_v \rangle$. The Euclidean normal vector $\mathbf{N}$ is the unit vector based at p and perpendicular to $T_p\mathbf{X}$. Given some tangent vector $\mathbf{v} \in T_p\mathbf{X}$ we consider the plane $\langle \mathbf{v}, \mathbf{N} \rangle$, and its intersection with $\mathbf{X}(U)$. The plane curve curvature, at p, of this intersection is the Euclidean sectional curvature of $\mathbf{X}$ at p in the direction $\mathbf{v}$.
We state the following well known proposition:

Proposition 10.1.1 *Let $\kappa_p(\theta)$ be the Euclidean sectional curvature of a surface $\mathbf{X}$ at $p \in \mathbf{X}$ in the direction $\mathbf{v} \in T_p\mathbf{X}$ where $\mathbf{v}$ makes an angle θ with some fixed vector in the tangent plane $T_p\mathbf{X}$. Furthermore, let $H_p(\mathbf{X})$ be the Euclidean mean curvature of $\mathbf{X}$ at p. Then*

$$\frac{1}{\pi} \int_0^\pi \kappa(\theta)\ d\theta = H_p(\mathbf{X}) \ .$$

Proof It is well known (see [6]) that if we take principal coordinates and allow the fixed vector to be one of the principal directions, say corresponding to the first principal curvature κ_1 we see that

$$\kappa_p(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \ .$$

Integration gives the required result. □

Remark 10.1.2 The Euclidean sectional curvature can be found by looking at the restriction of the family of distance squared functions $F : \mathbb{R}^3 \times U \to \mathbb{R}$ to the plane spanned by $\mathbf{v}$ and the Euclidean normal vector. We look for the point $\mathbf{x}$ in the plane such that F has an $A_{\geq 2}$ singularity at p. This point $\mathbf{x}$ will be the centre of curvature of the cross sectional plane curve. Hence the reciprocal of the distance from p to $\mathbf{x}$ gives the curvature of the plane section curve at p. See [4] for more details on the family of Euclidean distance squared functions and the geometrical interpretation of the singularities.

10.2 The affine case

Here we prove an analogue of Proposition (10.1.1). The key here is to use Remark 10.1.2. If we take planar cross sections and then calculate the affine curvature of the plane cross sections using the standard planar affine curvature we do not get an analogous result. First we consider the following

Definition 10.2.1 *Let P denote the plane spanned by $\mathbf{A} \in T_p\mathbb{R}^3$ and some non-zero vector $\mathbf{v} \in T_p\mathbf{X}$. We define the affine sectional curvature of $\mathbf{X}$ at p in the direction $\mathbf{v}$ to be the reciprocal of the affine distance from p to the $A_{\geq 2}$ point of the restriction of the family of affine distance functions $\Delta : \mathbb{R}^3 \times U \to \mathbb{R}$ to the plane P and the intersection curve.*

Proposition 10.2.2 *Consider a smooth surface $\mathbf{X}$ and a point $p \in \mathbf{X}$, such that p is not an (Euclidean) parabolic point. Let J be the 2×2 matrix where*

$$J = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Let S be the affine shape operator matrix, i.e. the matrix Then the affine sectional curvature of $\mathbf{X}$ at p in the direction $\mathbf{v}$, provided $\mathbf{v}$ is not a (Euclidean) asymptotic *direction, is given by*

$$\mu_p(\mathbf{v}) = \frac{(S\mathbf{v})^\top J\mathbf{v}}{\mathbf{v}^\top J\mathbf{v}} \ . \tag{10.1}$$

Proof Let $p = \mathbf{X}(0,0)$. Consider the plane $P := \langle \mathbf{v}, \mathbf{A}\rangle$ containing the affine normal $\mathbf{A}$ at p and some non-zero vector $\mathbf{v} \in T_p\mathbf{X}$. Let $\mathbf{v} := \alpha\mathbf{X}_u + \beta\mathbf{X}_v$, say. Let $I \subseteq \mathbb{R}$ be an open interval containing 0 and $\gamma : I \to U$ be given by $\gamma(t) = (u(t), v(t))$ where $\gamma(0) = (0,0)$ and $\dot\gamma(0) = (\alpha, \beta)$, such that $(\mathbf{X} \circ \gamma)(t)$ parametrises $\{\mathbf{X}(U) \cap P\}$.

We wish to consider the restriction of the family of affine distance functions $\Delta : \mathbb{R}^3 \times U \to \mathbb{R}$ to the plane P and the curve $(\mathbf{X} \circ \gamma)(t)$. Doing this gives the family $\tilde\Delta : P \times I \to \mathbb{R}$. We wish to calculate $\mathbf{x} \in P$ such that $\tilde\Delta(\mathbf{x}, t)$ has an $A_{\geq 2}$ singularity at $t = 0$. Notice that $\tilde\Delta : P \times I \to \mathbb{R}$ is a two-parameter family of functions from the real line to the real line.

Let $F := [\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]$, so that $\tilde\Delta F = [\mathbf{x} - \mathbf{X}, \mathbf{X}_u, \mathbf{X}_v]$ by definition. We can calculate the derivatives of Δ implicitly, we see that

$$\tilde\Delta_t F + \tilde\Delta F_t = [\mathbf{x} - \mathbf{X}, \dot u\mathbf{X}_{uu} + \dot v\mathbf{X}_{uv}, \mathbf{X}_v] + [\mathbf{x} - \mathbf{X}, \mathbf{X}_u, \dot u\mathbf{X}_{uv} + \dot v\mathbf{X}_{vv}] \ . \tag{10.2}$$

When $t = 0$ we can write $\mathbf{x} - \mathbf{X} = \lambda_1\mathbf{v} + \lambda_2\mathbf{A}$ for suitable $\lambda_i \in \mathbb{R}$. Substituting this into Equation (10.2) we see that for $t = 0$

$$\tilde\Delta_t = -\frac{\lambda_1}{F}(\alpha^2 L + 2\alpha\beta M + \beta^2 N)$$

since $\dot u(0) = \alpha$ and $\dot v(0) = \beta$. Assuming that p is not a parabolic point and that $(\alpha : \beta)$ is not an asymptotic direction of $\mathbf{X}$ at p we conclude that $\tilde\Delta_t(\mathbf{x}, 0) = 0$ if, and only if, $\lambda_1 = 0$. Thus $\tilde\Delta(\mathbf{x}, t)$ has an $A_{\geq 1}$ singularity at $t = 0$ if, and only if, $\mathbf{x} = \mathbf{X}(0,0) + \lambda\mathbf{A}(0,0)$ for some $\lambda \in \mathbb{R}$.

Next we consider $\tilde\Delta_{tt}$. Implicit differentiation leads to some lengthy expressions. Solving $\tilde\Delta_t = \tilde\Delta_{tt} = 0$ with $\mathbf{x} - \mathbf{X} = \lambda\mathbf{A}$ gives

$$\lambda = \frac{\dot u^2 L + 2\dot u\dot v M + \dot v^2 N}{[\dot u\mathbf{X}_{uu} + \dot v\mathbf{X}_{uv}, \mathbf{X}_v, \dot u\mathbf{A}_u + \dot v\mathbf{A}_v] + [\mathbf{X}_u, \dot u\mathbf{X}_{uv} + \dot v\mathbf{X}_{vv}, \dot u\mathbf{A}_u + \dot v\mathbf{A}_v]} \ .$$

Making the substitutions $\mathbf{A}_u = a\mathbf{X}_u + b\mathbf{X}_v$ and $\mathbf{A}_v = c\mathbf{X}_u + d\mathbf{X}_v$ gives

$$\lambda = \frac{\dot u^2 L + 2\dot u\dot v M + \dot v^2 N}{\dot u(a\dot u + c\dot v)L + (\dot u(b\dot u + d\dot v) + \dot v(a\dot u + c\dot v))M + \dot v(b\dot u + d\dot v)N} \ .$$

Assuming that p is not a parabolic point, and $\mathbf{v}$ is not an Euclidean asymptotic direction, $\tilde{\Delta}(\mathbf{x}, t)$ has an $A_{\geq 2}$ singularity at $t = 0$ if, and only if, $\mathbf{x} - \mathbf{X} = \lambda_2 \mathbf{A}$ where

$$\lambda = \frac{\alpha^2 L + 2\alpha\beta M + \beta^2 N}{\alpha(a\alpha + c\beta)L + (\alpha(b\alpha + d\beta) + \beta(a\alpha + c\beta))M + \beta(b\alpha + d\beta)N} \ .$$

From the definition given of affine sectional curvature, the reciprocal $1/\lambda$ gives the sectional curvature of $\mathbf{X}$ at p in the direction $\alpha\mathbf{X}_u + \beta\mathbf{X}_v$. Simplification shows

$$\frac{1}{\lambda} = \frac{(S\mathbf{v})^\top J\mathbf{v}}{\mathbf{v}^\top J\mathbf{v}} \ .$$

□

From Proposition 5.3.1 on page 60 we see the matrix of the bilinear form h, with respect to the basis $\{\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}\}$, is proportional to J. That is, $\mathbf{v}^\top J\mathbf{w} \propto h(\mathbf{v}, \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in T_p\mathbf{X}$. In fact

$$|LN - M^2|^{1/4}(\mathbf{v}^\top J\mathbf{w}) = h(\mathbf{v}, \mathbf{w}) \ .$$

It follows that

$$\mu_p(\mathbf{v}) = \frac{h(S\mathbf{v}, \mathbf{v})}{h(\mathbf{v}, \mathbf{v})} \ .$$

Locally the affine sectional curvature is a map $U \times \mathbb{RP}^1 \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The global structure, say for a compact surface, is quite different. Consider the tangent plane $T_p\mathbf{X}$. We define the projectivised tangent space $\mathbb{P}(T_p\mathbf{X}) = T_p\mathbf{X} \ / \sim$ where for all $\mathbf{v}_1, \mathbf{v}_2 \in T_p\mathbf{X}$ we have $\mathbf{v}_1 \sim \mathbf{v}_2$ if, and only if, there exists a non-zero $\lambda \in \mathbb{R}$ such that $\mathbf{v}_1 = \lambda\mathbf{v}_2$. The tangent bundle of $\mathbf{X}$ is given by

$$T\mathbf{X} = \{(p, \mathbf{v}) : p \in \mathbf{X} \text{ and } \mathbf{v} \in T_p\mathbf{X}\} \ .$$

Hence, we define the projectivised tangent bundle as

$$\mathbb{P}(T\mathbf{X}) = \{(p, \mathbf{v}) : p \in \mathbf{X} \text{ and } \mathbf{v} \in \mathbb{P}(T_p\mathbf{X})\} \ .$$

In a global setting, the affine sectional curvature is a function $\mu : \mathbb{P}(T\mathbf{X}) \to \overline{\mathbb{R}}$, Equally $\mu : \mathbb{P}(T\mathbf{X}) \to S^1$ where S^1 is the circle. In what follows, we may consider the domain of definition locally. This allows us to study a product as opposed to a projectivised fibre bundle.

It is clear that the derivatives of $\mathbf{A}$ fall in the tangent plane to $\mathbf{X}$. The zeros of affine sectional curvature tell us when second order derivatives of $\mathbf{A}$ fall in the tangent plane. We have the following

Theorem 10.2.3 *Away from asymptotic directions $\mu_p(\mathbf{v}) = 0$ if, and only if, $D^2_{\mathbf{v}}\mathbf{A} := D_{\mathbf{v}}(D_{\mathbf{v}}\mathbf{A})$ lies in the tangent plane to $\mathbf{X}$ at p.*

Proof We know that $D_{\mathbf{v}}\mathbf{A} = -S\mathbf{v}$, and so $-D^2_{\mathbf{v}}\mathbf{A} = \nabla_{\mathbf{v}}(S\mathbf{v}) + h(\mathbf{v}, S\mathbf{v})\mathbf{A}$. It follows that $D^2_{\mathbf{v}}\mathbf{A} \in T_p\mathbf{X}$ if, and only if, $h(\mathbf{v}, S\mathbf{v}) = 0$, i.e. if, and only if, $h(S\mathbf{v}, \mathbf{v}) = 0$.

□

Theorem 10.2.4 *Let $\mathbf{v} \in \mathbb{RP}^1$ and let $\mathcal{H}_p(\mathbf{X})$ be the affine mean curvature of $\mathbf{X}$ at an elliptic point p, i.e. the mean average of the eigenvalues of the affine shape operator matrix S. If μ_p is as in Equation (10.1) then*

$$\frac{1}{\pi}\int_{\mathbb{RP}^1} \mu_p(\mathbf{v})\ d\mathbf{v} = \mathcal{H}_p(\mathbf{X})\ .$$

Proof Let $\mathbf{X}$ be in Pick normal form, as in Equation (5.3) on page 62, we prove the equality by explicit calculation: The shape operator matrix is given by

$$S(0,0) = \begin{pmatrix} \frac{\sigma^2}{2} - 6a_0 - a_2 & -\frac{3}{2}(a_1 + a_3) \\ -\frac{3}{2}(a_1 + a_3) & \frac{\sigma^2}{2} - a_2 - 6a_4 \end{pmatrix}.$$

It follows that the mean curvature is given by

$$\mathcal{H}_0(\mathbf{X}) = \frac{\sigma^2}{2} - 3a_0 - a_2 - 3a_4\ .$$

Since $\mathbb{RP}^1$ can be identified with the unit circle S^1 we can put $\mathbf{v} = \cos\theta\mathbf{X}_u + \sin\theta\mathbf{X}_v$. Calculating the sectional curvature gives

$$\mu_0(\theta) = \frac{\sigma^2}{2} - 6(a_0\cos^2\theta + a_4\sin^2\theta) - \frac{3}{2}(a_1 + a_3)\sin 2\theta - a_2\ .$$

The result follows by direct integration. □

In order to give a similar formula over the hyperbolic region, more delicate analysis is required. Notice that there are two distinct asymptotic directions at a hyperbolic point and so there are two simple zeros of $\mathbf{v}^\top J\mathbf{v}$.

Theorem 10.2.5 *Let $\mathbf{v} \in \mathbb{RP}^1$ and let $\mathcal{H}_p(\mathbf{X})$ be the affine mean curvature of $\mathbf{X}$ at a hyperbolic point p, i.e. the mean average of the eigenvalues of the affine shape operator matrix S, then we have the following equality involving the principal value of an integral:*

$$\frac{1}{\pi}\ p.v \int_{\mathbb{RP}^1} \mu_p(\mathbf{v})\ d\mathbf{v} = \mathcal{H}_p(\mathbf{X})\ .$$

Proof Let us take our surface in Pick normal form, as in Equation (5.2) on page 62.

In this basis the 2×2 matrix $J(0,0)$ has $+1$ then -1 along the leading diagonal, with zeros in the other two places. The affine shape operator matrix at the origin is

$$S(0,0) = \begin{pmatrix} \frac{\sigma^2}{2} - 6a_0 + a_2 & -\frac{3}{2}(a_1 - a_3) \\ \frac{3}{2}(a_1 - a_3) & \frac{\sigma^2}{2} + a_2 - 6a_4 \end{pmatrix}.$$

The asymptotic directions are $(1 : \pm 1)$. The function $\mu_p(\mathbf{v})$ has two simple poles in these directions. The non-asymptotic directions can be parametrised by

$$\begin{aligned} \mathbf{v}_1(t) &= (\cosh t : \sinh t)\ , \\ \mathbf{v}_2(t) &= (\sinh t : -\cosh t)\ , \end{aligned}$$

where $t \in \mathbb{R}$. These parametrise the same directions as $(\cos\theta : \sin\theta)$ where we take $-3\pi/4 < \theta < -\pi/4$ and $-\pi/4 < \theta < \pi/4$. These directions are all of $\mathbb{RP}^1$ with the exception of the asymptotic directions. We may now integrate, but we must take a normalising factor

$$\begin{aligned} I_r &:= \frac{1}{4r}\int_{-r}^{r} \mu_p(\mathbf{v}_1(t)) + \mu_p(\mathbf{v}_2(t))\ dt\ , \\ I_r &= \frac{1}{4r}\left[3(a_3 - a_1)\cosh^2 t + 2\left(\frac{\sigma^2}{2} - 3a_0 + a_2 - 3a_4\right)t\right]_{-r}^{r}\ . \end{aligned}$$

The limit of I_r as $r \to \infty$ is easy to calculate, since the final expression is independent of r before the limit is taken! It follows that

$$\lim_{r\to\infty} I_r = \frac{\sigma^2}{2} - 3a_0 + a_2 - 3a_4 \ .$$

Direct computation shows that this is also the mean of the eigenvalues of $S(0,0)$. □

Since there are no real asymptotic directions over the elliptic region and $\mathbb{RP}^1$ is compact, the affine sectional curvature function must have extrema.

Theorem 10.2.6 *At an elliptic point which is not an affine umbilic the affine sectional curvature $\mu_p : \mathbb{RP}^1 \to \mathbb{R}$ has a turning point at $\mathbf{v} \in \mathbb{RP}^1$ if, and only if, $\mathbf{v}$ is an affine principal direction of $\mathbf{X}$ at p. The value of $\mu_p(\mathbf{v})$ at such a point is the corresponding affine principal curvature.*

In order to prove Theorem 10.2.6 we need the following

Lemma 10.2.7 *Let A and B be two $n\times n$ matrices with B either positive or negative definite, furthermore let $\mathbf{v}$ be an $n \times 1$ vector $(v_1, \ldots, v_n)^\top$. Then the function*

$$f(\mathbf{v}) := \frac{\mathbf{v}^\top A\mathbf{v}}{\mathbf{v}^\top B\mathbf{v}}$$

has the property that $(\nabla f)(\mathbf{v}) = 0$ if, and only if, $\mathbf{v}$ is a relative eigenvector of the two symmetric matrices $A + A^\top$ and $B + B^\top$, with relative eigenvalue $f(\mathbf{v})$.

Proof [Lemma 10.2.7] Note that $\mathbf{v}^\top A\mathbf{v} = f(\mathbf{v})(\mathbf{v}^\top B\mathbf{v})$. Let $e_i := \partial\mathbf{v}/\partial v_i$, then

$$e_i^\top A\mathbf{v} + \mathbf{v}^\top Ae_i = \frac{\partial f}{\partial v_i}(\mathbf{v}^\top B\mathbf{v}) + f(\mathbf{v})(e_i^\top B\mathbf{v} + \mathbf{v}^\top Be_i) \ .$$

The expressions $\mathbf{v}^\top Ae_i$ and $\mathbf{v}^\top Be_i$ are real numbers, so $\mathbf{v}^\top Ae_i = (\mathbf{v}^\top Ae_i)^\top$ and $\mathbf{v}^\top Be_i = (\mathbf{v}^\top Be_i)^\top$. It follows that

$$e_i^\top(A + A^\top)\mathbf{v} = \frac{\partial f}{\partial v_i}(\mathbf{v}^\top B\mathbf{v}) + f(\mathbf{v})(e_i^\top(B + B^\top)\mathbf{v}) \ . \qquad (10.3)$$

If this is to hold for all $1 \le i \le n$, then in matrix notation

$$(A + A^\top)\mathbf{v} = \nabla f(\mathbf{v})(\mathbf{v}^\top B \mathbf{v}) + f(\mathbf{v})(B + B^\top)\mathbf{v} \ .$$

If $\nabla f(\mathbf{v}) = 0$ then $\mathbf{v}$ is a relative eigenvector of $A + A^\top$ and $B + B^\top$ with relative eigenvalue $f(\mathbf{v})$. If $\mathbf{v}$ is a relative eigenvector of $A + A^\top$ and $B + B^\top$ with eigenvalue $f(\mathbf{v})$ then because B is either positive or negative definite it follows that $\nabla f(\mathbf{v}) = 0$. $\square$

Proof [Theorem 10.2.6] For simplicity, let $p = \mathbf{X}(0, 0)$. Over the elliptic region it is true that $\mathbf{v}^\top J \mathbf{v} \neq 0$ for any $\mathbf{v} \in \mathbb{RP}^1$. Hence Lemma 10.2.7 can be applied. By Lemma 10.2.7 we see that $\mu_0 : \mathbb{RP}^1 \to \mathbb{R}$ has a stationary point at $\mathbf{v} \in \mathbb{RP}^1$ if, and only if,

$$(S^\top J + JS - 2\mu_0(\mathbf{v})J)\mathbf{v} = \mathbf{0} \ .$$

Taking the Pick normal form in Equation (5.3) we see that $S(0, 0)$ is a symmetric matrix and $J(0, 0)$ is the 2×2 identity matrix. Thus $\mu_0 : \mathbb{RP}^1 \to \mathbb{R}$ has a stationary point at $\mathbf{v} \in \mathbb{RP}^1$ if, and only if,

$$2(S - \mu_0(\mathbf{v})E)\mathbf{v} = \mathbf{0} \ ,$$

where E denotes the 2×2 identity matrix. Hence, $\mathbf{v}$ is an eigendirection of the affine shape operator matrix with eigenvalue $\mu_0(\mathbf{v})$. Thus $\mathbf{v}$ is an affine principal direction with affine principal curvature $\mu_0(\mathbf{v})$. The affine shape operator matrix S has distinct eigendirections and eigenvectors if, and only if, $4(a_0 - a_4)^2 + (a_1 + a_3)^2 > 0$. This occurs at all elliptic points, except of course affine umbilic points. Thus, the zeros of $\nabla \mu_p(\mathbf{v})$ must be isolated over $\mathbb{RP}^1$ and so give simple turning points. $\square$

Theorem 10.2.8 *At a hyperbolic point which is not an affine umbilic the affine sectional curvature $\mu_p : \mathbb{RP}^1 \to \mathbb{R}$ has a turning point at a non-asymptotic $\mathbf{v} \in \mathbb{RP}^1$ if, and only if, the non-asymptotic $\mathbf{v}$ is an affine principal direction of $\mathbf{X}$ at p. The value of $\mu_p(\mathbf{v})$ at such a point is the corresponding affine principal curvature.*

Proof This is analogous to the proof of Theorem 10.2.6. □

Over the hyperbolic region there are two distinct real asymptotic directions. In such directions $\mu_p(\mathbf{v})$ is generically undefined. It follows that $\mu_p : \mathbb{RP}^1 \to \mathbb{R}$ is unbounded for most hyperbolic points. The exception is when the numerator and denominator in the definition of affine sectional curvature are both zero. This happens when $(S\mathbf{v})^\top J\mathbf{v}$ and $\mathbf{v}^\top J\mathbf{v}$ share a common solution $\mathbf{v} \in \mathbb{RP}^1$. In general, we expect to have curves along which $\mathbf{v}^\top J\mathbf{v}$ and $(S\mathbf{v})^\top J\mathbf{v}$ have a common root.

Proposition 10.2.9 *The quadratic forms $\mathbf{v}^\top J\mathbf{v}$ and $(S\mathbf{v})^\top J\mathbf{v}$ have a common root at a hyperbolic surface point if, and only if, the point is a repeated A-direction point.*

Proof The only zeros of $\mathbf{v}^\top J\mathbf{v}$ are asymptotic directions. Assume that our point is a repeated A-direction point, then by the corollary of Proposition 5.6.8 on page 64 the unique affine principal direction will be asymptotic. Let $\mathbf{v}_0 \in \mathbb{RP}^2$ be the unique affine principal direction which is also asymptotic, then by definition $S\mathbf{v}_0 = \lambda\mathbf{v}_0$ for some $\lambda \in \mathbb{R}$. Trivially we have $\mathbf{v}_0^\top J\mathbf{v}_0 = 0$, but in addition $(S\mathbf{v}_0)^\top J\mathbf{v}_0 = \lambda(\mathbf{v}_0^\top J\mathbf{v}_0) = 0$.

Next assume that for some $\mathbf{v}_0 \in \mathbb{RP}^1$ we have $\mathbf{v}_0^\top J\mathbf{v}_0 = (S\mathbf{v}_0)^\top J\mathbf{v}_0 = 0$. Notice that $\mathbf{v}_0^\top J\mathbf{v}_0 = 0$ if, and only if, $\mathbf{v}_0$ is asymptotic. Let us therefore assume that $\mathbf{v}_0$ is asymptotic. Define a linear map $f_p : T_p\mathbf{X} \to \mathbb{R}$ given by $f_p(\mathbf{w}) := \mathbf{w}^\top J\mathbf{v}_0$. Since $\det(J) \neq 0$ and $\mathbf{v}_0 \neq \mathbf{0}$ it follows that $\ker(f_p) = \langle \mathbf{v}_0 \rangle$. Hence $(S\mathbf{v}_0)^\top J\mathbf{v}_0 = 0$ if, and only if, $S\mathbf{v}_0 \in \ker(f_p)$, i.e. if, and only if, $S\mathbf{v}_0 \in \langle \mathbf{v}_0 \rangle$, i.e. if, and only if, $\mathbf{v}_0$ is an affine principal direction. The corollary of Proposition 5.6.8 shows that an affine principal direction is asymptotic if, and only if, it is repeated. □

At repeated A-direction points, the asymptotic direction which is affine principal will give a well defined affine sectional curvature. In the other asymptotic direction μ_p will have a simple pole.

It is of interest to discover what happens when the two repeated A-direction curves meet, i.e. at an affine umbilic. In this case the quadratic forms $\mathbf{v}^\top J\mathbf{v}$ and

$(S\mathbf{v})^\top J\mathbf{v}$ have the same roots. We see that $\mu_p : \mathbb{RP}^1 \to \mathbb{R}$ is well defined at an affine umbilic - it is equal to the unique affine principal curvature.

Theorem 10.2.10 *The affine sectional curvature function $\mu_p : \mathbb{RP}^1 \to \mathbb{R}$ is constant if, and only if, p is an affine umbilic. Furthermore, the affine sectional curvature is equal to the unique affine principal curvature at such points.*

Proof Note that $\mu_p(\mathbf{v}) = \lambda$ for all $\mathbf{v}$ if, and only if, $(S\mathbf{v})^\top J\mathbf{v} = \lambda\mathbf{v}^\top J\mathbf{v}$ for all $\mathbf{v}$, i.e. if, and only if, $\mathbf{v}^\top(S^\top J)\mathbf{v} = \lambda\mathbf{v}^\top J\mathbf{v}$ for all $\mathbf{v}$, i.e. if, and only if, $S^\top J = \lambda J$ for all $\mathbf{v}$, i.e. if, and only if, $S = \lambda E$. □

To follow the Euclidean analogue, we make the following

Definition 10.2.11 *If $\mu_p(\mathbf{v}) = 0$ at a surface point p then the direction $\mathbf{v}$ is called an affine asymptotic direction of the surface at p.*

We have seen from Proposition 10.2.9 that the forms $\mathbf{v}^\top J\mathbf{v}$ and $(S\mathbf{v})^\top J\mathbf{v}$ have a common root at point p if, and only if, p is a point with repeated affine principal directions. Thus, away from these points, the affine asymptotic directions are given by solutions to $(S\mathbf{v})^\top J\mathbf{v} = 0$.

Chapter 11

Generic 1-Parameter Family Transitions

Here we consider the generic one-parameter transitions of the family of affine height functions. This is done by looking at the standard two-parameter family of affine height functions but defined over a one-parameter family of surfaces. We look at the A_3, A_4, and D_4 cases here. These are the only interesting simple singularities which have miniversal deformations (see [1]) of dimension less than or equal to three. A one-parameter family of surfaces gives a three-parameter family of affine height functions, and so we expect A_3, A_4, and D_4 singularities to be versally unfolded by the three-parameter family of affine height functions.

11.1 Basics of unfoldings

In what follows we only consider functions from a surface, and so we only consider function in two variables. The reader is referred to [4] for unfoldings of functions from the line to the line, and to [1] for the more general theory.

Definition 11.1.1 *Let $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ be a function germ. A function germ $F : \mathbb{R}^2 \times \mathbb{R}^k, 0 \to \mathbb{R}$ is called a k-parameter potential unfolding of f if $F((u,v),0) = f(u,v)$ for all (u,v) close to 0.*

The term potential is used because the actual value of the function is not the important thing, only its singularities are considered.

Definition 11.1.2 *Let $F : \mathbb{R}^2 \times \mathbb{R}^k, 0 \to \mathbb{R}$ be a k-parameter unfolding of the germ $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$. The big-bifurcation set of F, denoted by $\widetilde{\mathcal{B}}_F$, is*

$$\{(u, v, x_1, \ldots, x_k) \in \mathbb{R}^2 \times \mathbb{R}^k : F_u = F_v = F_{uu}F_{vv} - F_{uv}^2 = 0\} \ .$$

Definition 11.1.3 *Let $F : \mathbb{R}^2 \times \mathbb{R}^k, 0 \to \mathbb{R}$ be a k-parameter unfolding of the germ $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$. Let $\pi : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}^k$ be the canonical projection. The bifurcation set of F, denoted by $\mathcal{B}_F$, is given by the restriction of π to $\widetilde{\mathcal{B}}_F$.*

$$\mathcal{B}_F = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : \ \exists \ (u, v) \in \mathbb{R}^2 \ s.t. \ F_u = F_v = F_{uu}F_{vv} - F_{uv}^2 = 0\} \ .$$

Example. The standard normal form of an $A_3^{\pm}$ singularity is $\phi(u, v) := u^2 \pm v^4$. A basis for the local algebra is given by $\{v, v^2\}$. A miniversal unfolding is then $\Phi : \mathbb{R}^2 \times \mathbb{R}^2, 0 \to \mathbb{R}$ given by

$$\Phi((u, v), (x, y)) = u^2 \pm v^4 + xv + yv^2 \ .$$

The big-bifurcation set $\widetilde{\mathcal{B}}_\Phi$ is a smooth curve in $\mathbb{R}^3$ given by

$$(u, v, x, y) = (0, v, \pm 8v^3, \mp 6v^2) \ .$$

The bifurcation set $\mathcal{B}_\Phi$ is a cusp in $\mathbb{R}^2$ given by

$$(x, y) = (\pm 8v^3, \mp 6v^2) \ .$$

Definition 11.1.4 *Consider two unfoldings $F, G : \mathbb{R}^2 \times \mathbb{R}^k, 0 \to \mathbb{R}$. If there exist map germs*

$$\begin{aligned} A : \mathbb{R}^2 \times \mathbb{R}^k, 0 &\to \mathbb{R}^2, 0 \ , \\ B : \mathbb{R}^k, 0 &\to \mathbb{R}^k, 0 \ , \\ C : \mathbb{R}^k, 0 &\to \mathbb{R} \end{aligned}$$

with $B : \mathbb{R}^k, 0 \to \mathbb{R}^k, 0$ and $A(u, 0) : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ diffeomorphism germs, and if $F(u, x) = G(A(u, x), B(x)) + C(x)$ then F and G are said to be equivalent unfoldings.

Proposition 11.1.5 *If* $F : \mathbb{R}^2 \times \mathbb{R}^k, 0 \to \mathbb{R}$ *and* $G : \mathbb{R}^2 \times \mathbb{R}^k, 0 \to \mathbb{R}$ *are equivalent unfoldings then* $\widetilde{\mathcal{B}}_F$ *is locally diffeomorphic to* $\widetilde{\mathcal{B}}_G$.

Proof Let F have variables u_1 and u_2, with unfolding parameters $x_1, \ldots, x_k$. Let G have variables v_1 and v_2 with unfolding parameters $y_1, \ldots, y_k$. Since they are equivalent unfoldings, there exist map germs as in Definition 11.1.4. As place holders, let $v_i = A_i(u, x)$ for $1 \leq i \leq 2$ and $y_j = B_j(x)$ for $1 \leq j \leq k$. We have

$$F(u, x) = G(A(u, x), B(x)) + C(x) \ .$$

To compute the big-bifurcation sets, we must differentiate this identity. If ∇F denotes the column vector whose entries are the first order partial derivative of F with respect to u_1 and u_2, ∇G denotes the column vector whose entries are the first order partial derivative of G with respect to v_1 and v_2, and J_A is the 2×2 Jacobian matrix of first order partial derivatives of the A_i with respect to the u_j we find

$$\nabla F = J_A \ \nabla G \ .$$

We know that $A(u, 0) : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ is a diffeomorphism germ, so $\det(J_A)$ is non-zero for all (u, x) close to 0. Thus ∇F is the zero vector if, and only if, ∇G is the zero vector. Next consider the Hessian matrices.

Let $\mathcal{H}_F$ denote the 2×2 Hessian matrix of second order partial derivatives of F with respect to u_1 and u_2. Let $\mathcal{H}_G$ denote the 2×2 Hessian matrix of second order partial derivatives of G with respect to v_1 and v_2. Assuming that ∇F and ∇G are both the zero vector we find that

$$\mathcal{H}_F = J_A \ \mathcal{H}_G \ J_A^\top \ .$$

It follows that $\det(\mathcal{H}_F) = 0$ if, and only if, $\det(\mathcal{H}_G) = 0$.

It now follows that $(u, x) \in \widetilde{\mathcal{B}}_F$ if, and only if, $(A(u, x), B(x)) \in \widetilde{\mathcal{B}}_G$. Thus the map germ $(A, B) : \mathbb{R}^2 \times \mathbb{R}^k, 0 \to \mathbb{R}^2 \times \mathbb{R}^k, 0$ takes $\widetilde{\mathcal{B}}_F$ onto $\widetilde{\mathcal{B}}_G$. It is also a diffeomorphism germ by assumption since F and G are equivalent unfoldings. □

Proposition 11.1.6 *If $F : \mathbb{R}^2 \times \mathbb{R}^k, 0 \to \mathbb{R}$ and $G : \mathbb{R}^2 \times \mathbb{R}^k, 0 \to \mathbb{R}$ are equivalent unfoldings then $\mathcal{B}_F$ is locally diffeomorphic to $\mathcal{B}_G$.*

Proof This follows from the previous argument and the fact that $B : \mathbb{R}^k, 0 \to \mathbb{R}^k, 0$ is a diffeomorphism germ. □

Remark 11.1.7 In the example given on page 140 we had a smooth curve for the big-bifurcation set. Proposition 11.1.5 shows that any other two-parameter unfolding equivalent to that one must have a smooth curve for the big-bifurcation set close to the origin.

11.2 The geometry

We shall consider the family of affine height function. Let $U \subseteq \mathbb{R}^2$ be an open simply connected domain, and $\mathbf{X} : U \to \mathbb{R}^3$ an immersion. The family of affine height functions is given by $H : U \times S^2 \to \mathbb{R}$ such that

$$H(\mathbf{u}, \mathbf{x}) := \frac{[\mathbf{x}, \mathbf{X}_u, \mathbf{X}_v]}{[\mathbf{X}_u, \mathbf{X}_v, \mathbf{A}]} ,$$

where $\mathbf{x}$ is chosen so that $||\mathbf{x}|| = 1$. In Chapter 7 we saw that away from (Euclidean) parabolic points $H_u = H_v = 0$ if, and only if, $\mathbf{x}$ is in the direction of the affine normal line. Furthermore, if we also impose the condition that $H_{uu}H_{vv} - H_{uv}^2 = 0$ then the base point must be an affine parabolic point. Thus, the big-bifurcation set $\widetilde{\mathcal{B}}_H$ is the set of affine normals at affine parabolic points.

Proposition 11.2.1 *Consider the map germ $F : U \times S^2, 0 \to \mathbb{R}^3, 0$ given by*

$$F(u, v, x, y) = (H_u, H_v, H_{uu}H_{vv} - H_{uv}^2) .$$

The germ F has 0 as a regular point and 0 as a regular value if, and only if, the affine parabolic curve is smooth close to $\mathbf{X}(0, 0)$.

Proof The proof is by direct computation using the Maple computer algebra package. Consider a surface in Pick normal form (see page 62). There are two families of surfaces having an affine parabolic point at the origin: in the elliptic case they depend on whether $\sigma^2 - 2a_2 - 12a_4$ is zero or non-zero, and in the hyperbolic case on whether $\sigma^2 + 2a_2 - 12a_4$ is zero or non-zero (cf. the expressions for the affine shape operator in Equations (5.4) and (5.5) on page 64). From now on we assume that the origin is an affine parabolic point.

The Jacobian matrix for F is a 3×4 matrix, and to have rank less than three we need all four 3×3 minors to be singular. This gives four conditions on the Pick normal form coefficients. Let $p : U, 0 \to \mathbb{R}, 0$ be the defining equation for the affine parabolic curve, i.e. $p := ad - bc$. If we compute p_u and p_v we find that imposing the condition that all four 3×3 minors of J_F are singular gives $p_u(0,0) = p_v(0,0) = 0$.

We can solve the equations $p_u(0,0) = p_v(0,0) = 0$ in terms of the Pick normal form coefficients. When we do so we find there are four families of such solutions (depending on whether certain expressions are zero or non-zero in either case). Imposing these families of solutions on J_F yields, in each case, a 3×4 matrix with rank less than three.

□

Remark 11.2.2 Considering a k-parameter family of surfaces $\mathbf{X} : U \times \mathbb{R}^k \to \mathbb{R}^3$, gives a $(k+2)$-parameter family of affine height functions $H : U \times S^2 \times \mathbb{R}^k \to \mathbb{R}$. Proposition 11.2.1 shows that the affine parabolic curve of $\mathbf{X}(u, v, \mathbf{0})$ is smooth if, and only if, the big-bifurcation set of $H(u, v, x, y, \mathbf{0})$ is smooth.

Corollary 11.2.3 *Remark 11.1.7 and Proposition 11.2.1 show that any two-parameter family of affine height functions giving an unfolding of an A_3 that is equivalent to the one in the example on page 140 must have a smooth affine parabolic curve close to the origin.*

11.3 Technical notions

Here we recall the idea of a stratification, and define what it means for a function on a stratified set to be generic. The following may be found in [16].

Definition 11.3.1 (C^k stratification) *Let Z be a closed subset of a differentiable manifold M of class C^k. A C^k stratification of Z is a filtration by closed subsets*

$$Z = Z_d \supset Z_{d-1} \supseteq \cdots \supseteq Z_1 \supseteq Z_0$$

such that each difference $Z_i - Z_{i-1}$ is a differentiable submanifold of M of class C^k and dimension i, or is empty. Each connected component of $Z_i - Z_{i-1}$ is called a stratum of dimension i. Thus Z is a disjoint union of the strata, denoted $\{X_\alpha\}_{\alpha \in A}$.

Definition 11.3.2 (Frontier condition) *A stratification $Z = \bigcup_{\alpha \in A} X_\alpha$ is said to satisfy the frontier condition if: for all $(\alpha, \beta) \in A \times A$ such that $X_\alpha \cap \overline{X_\beta}$ is non-empty, one has $X_\alpha \subseteq \overline{X_\beta}$. As the strata are disjoint this means that $X_\alpha = X_\beta$ or $X_\alpha \subset \overline{X_\beta} - X_\beta$*

Definition 11.3.3 (Locally finite stratification) *One says that a stratification is locally finite if the number of strata is locally finite.*

Definition 11.3.4 (Whitney's condition (a)) *Take two adjacent strata X and Y, i.e. two C^1 manifolds of M such that $Y \subset \overline{X} - X$. The pair (X, Y) is said to satisfy Whitney's condition (a) at $y \in Y$, or to be (a)-regular at y if: for all sequences $\{x_i\} \in X$ with limit y such that, in a local chart at y, $\{T_{x_i}X\}$ tends to τ, $T_yY \subseteq \tau$.*

Definition 11.3.5 (Whitney's condition (b)) *Take two adjacent strata X and Y, i.e. two C^1 manifolds of M such that $Y \subset \overline{X} - X$. The pair (X, Y) is said to satisfy Whitney's condition (b) at $y \in Y$, or to be (b)-regular at y if: given sequences $\{x_i\} \in X$ and $\{y_i\} \in Y$ with limit y such that, in a local chart at y, $\{T_{x_i}X\}$ tends to τ and the lines $\overline{x_i y_i}$ tend to λ, then $\lambda \in \tau$.*

Definition 11.3.6 (Whitney stratification) *When $Z = \bigcup_{\alpha \in A} X_\alpha$ is a locally finite stratification such that all pairs of adjacent strata satisfy the frontier condition and are (b)-regular at all points, we say we have a Whitney stratification of Z.*

Theorem 11.3.7 (Whitney 1965) *Every analytic variety (in $\mathbb{R}^n$ or $\mathbb{C}^n$) admits a Whitney stratification whose strata are analytic (hence C^∞) manifolds.*

Hironaka proved that the same is true of every subanalytic set (in particular every semialgebraic set). See [16] for references.

Next we introduce the idea of a generic function on a semialgebraic set, the reader is referred to [3].

Let Z be a semialgebraic Whitney stratification of some open neighbourhood U of the origin $0 \in \mathbb{R}^n$, with 0 being a stratum. If $h : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ is the germ of a submersion at 0 we wish to consider the way in which the level sets of h meet our stratification. We have the following

Definition 11.3.8 (Generalised transverse) *The function h is generalised transverse to Z if given two adjacent strata X and Y and any sequence $\{x_i\} \in X$ with limit $y \in Y$ such that, in a local chart at y, $\{T_{x_i}X\}$ tends to τ then $dh : \tau \to \mathbb{R}$ has maximal rank at y. (So the smooth hypersurface $h^{-1}(h(y))$ meets τ transversally.)*

Definition 11.3.9 (Generic function) *A function h is said to be generic on Z, close to 0, if it is generalised transverse to Z, close to 0.*

Clearly if h is generalised transverse to Z at 0 then h is generalised transverse to Z in some neighbourhood of 0.

11.4 The non-versal A_3 case

Consider a function germ $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ with an $A_3^\pm$ singularity. This function germ is $\mathcal{A}$-equivalent to one of the normal forms $g_\pm(u, v) = u^2 \pm v^4$. The set $\{v, v^2\}$ can be chosen as a basis for the local algebra of this normal form. Thus miniversal unfoldings of the normal forms are given by $G_\pm : \mathbb{R}^2 \times \mathbb{R}^2, 0 \to \mathbb{R}, 0$ where

$$G_\pm = u^2 \pm v^4 + xv + yv^2 \ .$$

We need at least a two-parameter unfolding of an $A_3^\pm$ singularity if there is any hope of it being versal.

In this section, we consider a one-parameter family of surfaces $\mathbf{X} : U \times I \to \mathbb{R}^3$, with family parameter $t \in I$. This gives a three-parameter family of affine height functions $H : U \times S^2 \times I \to \mathbb{R}$. We assume that for $\mathbf{x}$ in the direction of $\mathbf{A}(0,0)$ and $t = 0$, the function $H : \mathbb{R}^2 \to \mathbb{R}$ has an $A_3^{\pm}$ singularity at the origin. Finally, we assume that the affine parabolic curve is singular at the origin for $t = 0$.

Proposition 11.4.1 *For $t = 0$, the family of affine height functions $H|_{t=0} : U \times S^2 \to \mathbb{R}$ cannot be a versal unfolding of an $A_3^{\pm}$.*

Proof The big-bifurcation sets of the unfoldings $G_{\pm}$ are smooth. By Proposition 11.2.1 the affine parabolic curve of a surface whose family of affine height functions versally unfolded an $A_3^{\pm}$ would have to be smooth. This implies that the restricted family $H|_{t=0}$ cannot be versal. □

Let us now assume that the whole family $H : U \times S^2 \times I \to \mathbb{R}$ does versally unfold an $A_3^{\pm}$. In what follows we have a situation where projection of the bifurcation set onto the t-parameter does not give a generic function: the t-constant sections will not be generic sections. The first level of degeneracy is when the projection yields a Morse function (see [4]). The following results allow us to know when this projection does indeed yield a Morse function. The conditions for a function to be Morse on $\mathcal{B}_H$ can be found in [3], on page 144.

It is important to note that, in the three-parameter versal $A_3^{\pm}$ case, the big-bifurcation set is smooth and the bifurcation set is diffeomorphic to a cuspidal edge; i.e. diffeomorphic to $(X, Y) \mapsto (X^2, X^3, Y)$.

Proposition 11.4.2 *Let $f : S^2 \times I \to \mathbb{R}$ be a smooth function and $\pi : U \times S^2 \times I \to S^2 \times I$ be the canonical projection. If the restricted composite $f \circ \pi|_{\widetilde{\mathcal{B}}_H} : \widetilde{\mathcal{B}} \to \mathbb{R}$ is Morse, then $f : S^2 \times I \to \mathbb{R}$ is Morse on $\mathcal{B}_H$.*

Proof We find the conditions for the restricted composite $f \circ \pi|_{\widetilde{\mathcal{B}}_H}$ to be Morse, and show that this implies that the function f is Morse.

Consider a function $f : S^2 \times I \to \mathbb{R}$ given by

$$f(x, y, t) := \alpha_0 x + \alpha_1 y + \alpha_2 t + \beta_0 x^2 + \beta_1 xy + \beta_2 xt + \beta_3 y^2 + \beta_4 yt + \beta_5 t^2 + \cdots .$$

The big-bifurcation set is given by a manifold germ which is $\mathcal{A}$-equivalent to $(X, Y) \mapsto (X^2, X^3, Y)$. Thus

$$(f \circ \pi|_{\widetilde{\mathcal{B}}_H})(X, Y) = \alpha_2 Y + \alpha_0 X^2 + \beta_5 Y^2 + \cdots \ .$$

This is a Morse function if, and only if, $\alpha_2 = 0$ and $\alpha_0 \beta_5 \neq 0$.

Let us assume that $\alpha_2 = 0$ and $\alpha_0 \beta_5 \neq 0$. There is a single one-dimensional stratum for the cuspidal edge, i.e. the t-axis, and two two-dimensional strata. For the function $f : S^2 \times I \to \mathbb{R}$ to be Morse on $\mathcal{B}_H$ we need the restriction of f to each of the strata to have isolated non-degenerate singularities. Clearly, our assumptions show that f is Morse on the two two-dimensional strata. We should consider the one-dimensional stratum. We have

$$f(0, 0, t) = \alpha_2 t + \beta_5 t^2 + \cdots \ .$$

This is singular for $t = 0$ if, and only if, $\alpha_2 = 0$. This is a non-degenerate singularity if, and only if, $\beta_5 \neq 0$. These conditions are satisfied by assumption. □

We are interested in when the projection to the t-parameter yields a Morse function. We have seen that it is enough to check that $f \circ \pi|_{\widetilde{\mathcal{B}}_H} : \widetilde{\mathcal{B}}_H \to I$ is Morse. This may not be directly possible. To simplify the problem, consider the following

Proposition 11.4.3 *Consider the canonical projection $\pi : U \times S^2 \times I \to S^2 \times I$. The restricted composite $f \circ \pi|_{\widetilde{\mathcal{B}}_H} : \widetilde{\mathcal{B}}_H \to I$ is suspension contact equivalent to the mapping $F_0 : U \times S^2 \times \{0\} \to \mathbb{R}^3$ given by*

$$F_0(\mathbf{u}, \mathbf{x}) = (H_u, H_v, H_{uu}H_{vv} - H_{uv}^2)|_{t=0} \ .$$

In order to prove this, we need the following

Lemma 11.4.4 (J. W. Bruce [2]) *Let $(Y, 0)$ be the germ of a smooth manifold and $(M_i, 0)$ be germs of submanifolds, $i = 1, 2$, with $T_0M_1 \subset T_0M_2$ or $T_0M_2 \subset T_0M_1$. If $f_i : (Y, 0) \to (P_i, 0)$ are germs of submersions with $(f_i^{-1}(0), 0) = (M_i, 0)$ then $f_1 : (M_2, 0) \to (P_1, 0)$ and $f_2 : (M_1, 0) \to (P_2, 0)$ are suspension contact equivalent. That is after adding a trivial factor $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ to f_1 or f_2 (say $\tilde{f}_2(x) = (f_2(x), u)$) f_1 and $\tilde{f}_2$ become contact equivalent.*

Proof [Of Proposition 11.4.3] In the language of Lemma 11.4.4, let $Y = U \times S^2 \times I$, $P_1 = I$, $P_2 = \mathbb{R}^3$, $f_1(\mathbf{u}, \mathbf{x}, t) = t$, and $f_2(\mathbf{u}, \mathbf{x}, t) = (H_u, H_v, H_{uu}H_{vv} - H_{uv}^2)$, so that $M_1 = U \times S^2 \times \{0\}$, and $M_2 = \widetilde{\mathcal{B}}_H$.

We need to show that $T_0M_2 \subset T_0M_1$. Let the 3×5 Jacobian matrix of f_2 have columns $\{c_1, \ldots, c_5\}$, these must have rank three since $\widetilde{\mathcal{B}}_H$ is smooth. The four columns $\{c_1, \ldots, c_4\}$ are the columns of the 3×4 Jacobian matrix of F_0. These four columns must have rank less than three since $\widetilde{\mathcal{B}}_H|_{t=0}$ is singular. It follows that c_5 must be linearly independent of $\{c_1, \ldots, c_4\}$; any solution to the vector equation

$$\alpha c_1 + \beta c_2 + \gamma c_3 + \delta c_4 + \varepsilon c_5 = \mathbf{0}$$

must have $\varepsilon = 0$. It follows that any tangent vector to $\widetilde{\mathcal{B}}_H$ must have its final component zero, i.e. must be contained in $U \times S^2 \times \{0\}$. It follows that $T_0M_2 \subset T_0M_1$.

Lemma 11.4.4 tells us that $f_1 : M_2 \to P_1$ and $f_2 : M_1 \to P_2$ are suspension contact equivalent. This means that the restricted projection $f \circ \pi|_{\widetilde{\mathcal{B}}_H} : \widetilde{\mathcal{B}}_H \to I$ is suspension contact equivalent to the mapping $F_0 : U \times S^2 \times \{0\} \to \mathbb{R}^3$. □

Recall that $\Sigma^{1,0}$ points of a map are corank one critical points such that the restriction of the map to the critical set is a submersion.

Proposition 11.4.5 *The map $F_0 : U \times S^2 \times \{0\} \to \mathbb{R}^3$ has only $\Sigma^{1,0}$ points if, and only if, $f \circ \pi|_{\widetilde{\mathcal{B}}_H} : \widetilde{\mathcal{B}}_H \to I$ has only Morse singularities.*

Proof Consider two map germs $\Phi : \mathbb{R}^2 \times \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ and $\phi : \mathbb{R}^2, 0 \to \mathbb{R}, 0$. Assume that Φ and ϕ are suspension contact equivalent. It follows by definition that

$$\Phi(u, v, x, y) \overset{\mathcal{K}}{\sim} (\phi(u, v), x, y) \ .$$

we find that $\Sigma^1_\Phi = \Sigma^1_\phi \times \mathbb{R}^2$. Next, consider $\Phi|_{\Sigma^1_\Phi}$ and $\phi|_{\Sigma^1_\phi}$. We have

$$\begin{aligned} \Phi|_{\Sigma^1_\Phi} &: \Sigma^1_\phi \times \mathbb{R}^2, 0 \to \mathbb{R}^3, 0 \ , \\ \phi|_{\Sigma^1_\phi} &: \Sigma^1_\phi, 0 \to \mathbb{R}, 0 \ . \end{aligned}$$

It follows that $\phi|_{\Sigma^1_\phi}$ is regular if, and only if, $\Phi|_{\Sigma^1_\Phi}$ is regular. □

Propositions 11.4.2, 11.4.3, and 11.4.5 show that if $F_0 : U \times S^2 \times \{0\} \to \mathbb{R}^3$, where

$$F_0(\mathbf{u}, \mathbf{x}) = (H_u, H_v, H_{uu}H_{vv} - H_{uv}^2)|_{t=0}$$

has only $\Sigma^{1,0}$ points then the function $f : S^2 \times I \to \mathbb{R}$ given by $(x, y, t) \mapsto t$ will be Morse.

Proposition 11.4.6 *The map $F_0 : U \times S^2 \times \{0\} \to \mathbb{R}^3$ has $\Sigma^{1,0}$ points if, and only if, the affine parabolic curve undergoes a Morse transition.*

Proof The proof is by direct computation. Here we sketch the steps of the proof while omitting the actual expressions. Recall from the proof of Proposition 11.2.1 on page 142 that if $D := H_{uu}H_{vv} - H_{uv}^2$, then the Jacobian of F_0 is given by

$$J_{F_0} = \begin{pmatrix} H_{uu} & H_{uv} & H_{ux} & H_{uy} \\ H_{uv} & H_{vv} & H_{vx} & H_{vy} \\ D_u & D_v & D_x & D_y \end{pmatrix} .$$

Let $p : U \times I \to \mathbb{R}$ be the defining equations of the affine parabolic curves. Assume that $p(0,0,0) = 0$ so that the origin is an affine parabolic point for $t = 0$. Since we are considering the non-versal A_3 we have $p_u(0,0,0) = p_v(0,0,0) = 0$, i.e. the affine parabolic curve is singular at the origin for $t = 0$. The condition for a Morse singularity at the origin for $t = 0$ is $(p_{uu}p_{vv} - p_{uv}^2)(0,0,0) \neq 0$. The condition for a Morse transition is then $p_t(0,0,0) \neq 0$.

In the proof of Proposition 11.2.1 we also saw that the last two columns of J_{F_0} always have rank two away from Euclidean parabolic points. Let J_{F_0} have columns $\{c_1, \ldots, c_4\}$, then the singular points of F_0 are given by $[c_1, c_3, c_4] = [c_2, c_3, c_4] = 0$.

Let us define a map $G : U \times S^2 \to \mathbb{R}^2$ where

$$G(u, v, x, y) := ([c_1, c_3, c_4], [c_2, c_3, c_4]) .$$

The singular points of F_0 are then given by $G^{-1}(0)$. We can compute the 2×4 Jacobian matrix J_G, and we find that generically all of the 2×2 minors have non-zero determinant. In fact, the Jacobian matrix fails to have maximal rank if, and only if, the affine parabolic curve has a degenerate singularity at the origin for $t = 0$. This means that generically we can use any two of $\{u, v, x, y\}$ to parametrise $G^{-1}(0)$.

Let us assume that the last two columns of J_G are linearly independent so that we may write x and y as functions of u and v, i.e. write $x = x(u, v)$ and $y = y(u, v)$ such that $G(u, v, x(u, v), y(u, v)) = 0$. We need only know the jets of $x(u, v)$ and $y(u, v)$ up to some order. This is done by making a power series substitution and then comparing all coefficients to zero.

Next we consider $F_0(u, v, x(u, v), y(u, v)) : U \to \mathbb{R}^3$, and the 2×3 Jacobian matrix. we find that the Jacobian matrix has maximal rank if, and only if, the affine parabolic curve has a Morse singularity at the origin when $t = 0$. Moreover, we have a Morse transition if, and only if, the family of three parameter family of affine height functions is a versal unfolding; which it is by assumption in the A_3 non-versal case.

In the case where the last two columns of J_G are not linearly independent we can similarly find two columns which are, and write those variables as functions of the others. □

Corollary 11.4.7 *If the affine parabolic curve undergoes a Morse transition then the affine Gauß map will undergo a lips or beaks transition.*

Example. Consider a one-parameter family of surfaces, $\mathbf{X} : U \times I \to \mathbb{R}^3$ say, in Pick normal form, all with elliptic points at the origin,

$$\mathbf{X}(u, v, t) = \left(u, v, \frac{1}{2}(u^2 + v^2) + \frac{\sigma(t)}{6}(u^3 - 3uv^2) + \sum_{i=0}^{4} a_i(t)u^{4-i}v^i + \cdots \right).$$

It is possible to compute the conditions on the Pick coefficients for $H(u, v, 0, 0, 0) = u^2 + v^4 + O(5)$. Let us assume this to be the case.

For a non-versal A_3, we require that $H(u, v, x, y, 0)$ is not a versal unfolding of the A_3^+. This is so if, and only if, $\sigma(0) = 0$. In this case, the affine parabolic curve at the origin, for $t = 0$, is singular if, and only if, $\sigma(0) = 0$. This was of course predicted in Proposition 11.2.1 on page 142. Let us assume that $\sigma(0) = 0$.

We want $H(u, v, x, y, t)$ to be a versal unfolding of the A_3^+. This is so if, and only if, $\dot{a}_2(0) + 6\dot{a}_4(0) \neq 0$. We assume this to be non-zero.

Let the family of affine parabolic curves have equations $p(u, v, t) = 0$. Given our assumptions, the affine parabolic curve, for $t = 0$, has a Morse singularity at the origin if, and only if, $(p_{uu}p_{vv} - p_{uv}^2)(0, 0, 0) \neq 0$, i.e. if, and only if, $a_2(0) + 3a_3^2(0) \neq 0$. In addition, the family undergoes a Morse transition if, and only if, $p_t(0, 0, 0) \neq 0$, i.e. if, and only if, $\dot{a}_2(0) + 6\dot{a}_4(0) \neq 0$. This is also the versality condition. The Morse transition is automatic.

The sign of $(p_{uu}p_{vv} - p_{uv}^2)(0, 0, 0)$ distinguishes between an elliptic Morse transition, i.e. something of the form $\{(u, v) : u^2 + v^2 + h(t) = 0\}$ where $\dot{h}(0) \neq 0$, and a hyperbolic Morse transition, i.e. something of the form $\{(u, v) : u^2 - v^2 + h(t) = 0\}$ where $\dot{h}(0) \neq 0$. Now we give two examples of one-parameter families. A family with an elliptic Morse transition is given by

$$\mathbf{X}(u, v) = \left(u, v, \frac{1}{2}(u^2 + v^2) + \frac{1}{3}v^4 + (t - 2)u^2v^2 + \frac{1}{3}u^6 + 4u^2v^4 + \frac{2}{15}v^6\right) .$$

As t passes through zero, the affine parabolic curve (A.P.C.) undergoes an elliptic Morse transition, the affine Gauß map (A.G.M.) undergoes a “lips” transition. These can be seen in Figure 11.1 on page 151.

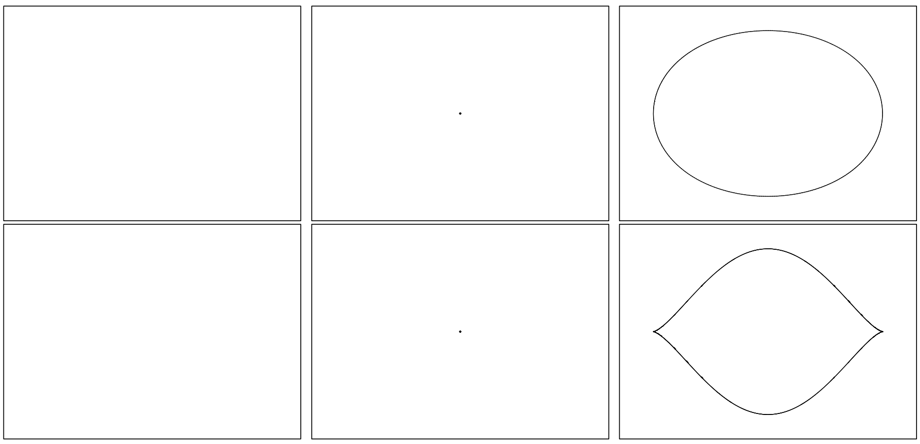

Figure 11.1: Elliptic Morse. Above: A.P.C. Below: A.G.M.

A family with a hyperbolic Morse transition is given by

$$\mathbf{X}(u, v) = \left(u, v, \frac{1}{2}(u^2 + v^2) - \frac{1}{2}u^4 + (1 + t)u^2v^2 - \frac{1}{6}v^4 + \frac{11}{15}u^6\right) .$$

As t passes through zero, the affine parabolic curve (A.P.C.) undergoes a hyperbolic Morse transition, the affine Gauß map (A.G.M.) undergoes a "beaks" transition. These can be seen in Figure 11.2 on page 152.

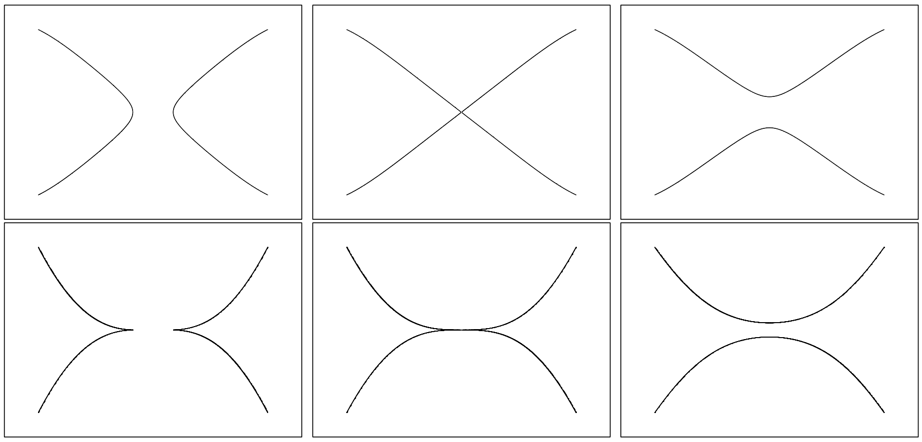

Figure 11.2: Hyperbolic Morse. Above: A.P.C. Below: A.G.M.

11.5 The versal A_4 case.

Consider a function germ $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ with an A_4 singularity. This function germ is $\mathcal{A}$-equivalent to the normal form $g(u,v) = u^2 + v^5$. The set $\{v, v^2, v^3\}$ can be chosen as a basis for the local algebra of this normal form. Thus a miniversal unfolding of the normal form is given by $G : \mathbb{R}^2 \times \mathbb{R}^3, 0 \to \mathbb{R}, 0$ where

$$G = u^2 + v^5 + xv + yv^2 + zv^3 \ .$$

We need at least a three-parameter unfolding of an A_4 singularity if there is any hope of it being versal. For this reason, we consider a one-parameter family of surfaces.

Let $\mathbf{X} : U \times I \to \mathbb{R}^3$ be a smooth one-parameter family of surface parametrisations, with family parameter $t \in I$. This gives rise to a three-parameter family of affine height functions $H : U \times S^2 \times I \to \mathbb{R}$, where $H(\mathbf{u}, \mathbf{x}, t_0)$ is the normal two-parameter family of height functions coming from the surface $\mathbf{X}(\mathbf{u}, t_0)$.

Let us assume that for $t = 0$, the family of affine height functions has an A_4 singularity at $\mathbf{u} = (0,0)$ in the direction $\mathbf{A}(0,0)$. Furthermore, let us also assume that the family $H : U \times S^2 \times I \to \mathbb{R}$ versally unfolds this singularity.

Since the family is versal, it must be equivalent, as a potential unfolding, to the unfolding of the normal form normal form, i.e. to $G : \mathbb{R}^2 \times \mathbb{R}^3, 0 \to \mathbb{R}, 0$ given above. The big-bifurcation set $\widetilde{\mathcal{B}}_H$ must be diffeomorphic to $\widetilde{\mathcal{B}}_G$ close to the origin. The bifurcation set $\mathcal{B}_H$ must be diffeomorphic to $\mathcal{B}_G$ close to the origin.

First, we prove some facts about the unfolding of the normal form.

Proposition 11.5.1 *The big-bifurcation set $\widetilde{\mathcal{B}}_G$ of the following unfolding is smooth*

$$G(\mathbf{u}, \mathbf{x}) = u^2 + v^5 + xv + yv^2 + zv^3 \ .$$

Proof Solving $G_u = G_v = G_{uu}G_{vv} - G_{uv}^2 = 0$ gives

$$(u, v, x, y, z) = (0, v, 15v^4 + 3zv^2, -10v^3 - 3zv, z) \ .$$

This is clearly always smooth. □

Proposition 11.5.2 *Let $\pi : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ be the canonical projection $(\mathbf{u}, \mathbf{x}) \mapsto \mathbf{x}$, and let $g : \mathbb{R}^3 \to \mathbb{R}$. The function g is generic on $\mathcal{B}_G$ if, and only if, the restricted composite $g \circ \pi|_{\widetilde{\mathcal{B}}_G} : \widetilde{\mathcal{B}}_G \to \mathbb{R}$ is a submersion.*

Proof The bifurcation set $\mathcal{B}_G$ is parametrised by $(15v^4 + 3zv^2, -10v^3 - 3zv, z)$. The singular locus of this set is parametrised by $(-15v^4, 20v^3, -10v^2)$. The self intersection locus is given by $(5v^4, 0, -10v^2/3)$. At the swallow tail point, i.e. at $(0,0,0)$, there is a unique limiting tangent direction to these strata. That direction is $(0 : 0 : 1)$. Therefore, a function $g : \mathbb{R}^3 \to \mathbb{R}$ given by $g(x, y, z) = ax + by + cz + \cdots$ is generic on $\mathcal{B}_G$ if, and only if, $c \neq 0$.

Recall the parametrisation of $\widetilde{\mathcal{B}}_G$, we had

$$(u, v, x, y, z) = (0, v, 15v^4 + 3zv^2, -10v^3 - 3zv, z) \ .$$

It follows that the restricted composite $g \circ \pi|_{\widetilde{\mathcal{B}}_G} : \widetilde{\mathcal{B}}_G \to \mathbb{R}$ is given by

$$(v, z) \mapsto a(15v^4 + 3zv^2) - b(10v^3 + 3zv) + cz + \cdots \ .$$

This latter function is a submersion if, and only if, $c \neq 0$. □

Proposition 11.5.3 *Consider the family of affine height functions with a three-parameter versal unfolding of an* A_4. *Let* $\pi_2 : U \times S^2 \times I \to S^2 \times I$ *be the canonical projection, and* $g : S^2 \times I \to \mathbb{R}$ *some function. Then* g *is generic on* $\mathcal{B}_H$ *(*H *as above) if, and only if,* $g \circ \pi_2|_{\tilde{\mathcal{B}}_H}$ *is a submersion.*

In order to prove this proposition let us consider the normal form for an A_4, and the miniversal unfolding $G = u^2 + v^5 + xv + yv^2 + zv^3$. Let $\pi_1 : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ be the canonical projection. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be some function. Since G and H are equivalent unfoldings we may construct the following diagram:

$$\begin{array}{ccccc} \mathbb{R}^2 \times \mathbb{R}^3 & \xrightarrow{\pi_1} & \mathbb{R}^3 & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle (A,B)}\downarrow & & {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ U \times S^2 \times I & \xrightarrow{\pi_2} & S^2 \times I & \xrightarrow{g} & \mathbb{R} \end{array}$$

for suitable A and B, see Definition 11.1.4 on page 140. We have the following

Lemma 11.5.4 *Consider the normal form for an* A_4, *and the miniversal unfolding* $G = u^2 + v^5 + xv + yv^2 + zv^3$. *Let* π_1 *and* f *be as above and* H, π_2, *and* g *be as in Proposition 11.5.3. Then*

1. f *is generic on* $\mathcal{B}_G$ *if, and only if,* g *is generic on* $\mathcal{B}_H$.
2. $f \circ \pi_1|_{\tilde{\mathcal{B}}_G}$ *is a submersion if, and only if,* f *is generic on* $\mathcal{B}_G$.
3. $f \circ \pi_1|_{\tilde{\mathcal{B}}_G}$ *is a submersion if, and only if,* $g \circ \pi_2|_{\tilde{\mathcal{B}}_H}$ *is a submersion.*

Proof [Of Lemma] To prove 1, let us consider the following diagram of differentials:

$$\begin{array}{ccc} T_0\mathbb{R}^3 & \xrightarrow{d_0 f} & T_0\mathbb{R} \\ {\scriptstyle d_0 B}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ T_0(S^2 \times I) & \xrightarrow{d_0 g} & T_0\mathbb{R} \end{array}$$

The function $f : \mathbb{R}^3 \to \mathbb{R}$ is generic on $\mathcal{B}_G$ if, and only if, the zero level of $\{f = 0\}$ is transverse to the unique limiting tangent direction of the singular strata of $\mathcal{B}_G$. Let $\mathbf{v} \in T_0\mathbb{R}^3$ be a non-zero vector in this unique direction. The map f is generic if, and only if, $\mathbf{v} \notin \ker(d_0 f)$. Clearly the function $g : S^2 \times I \to \mathbb{R}$ is generic on $\mathcal{B}_H$ if, and only if, $d_0B(\mathbf{v}) \notin \ker(d_0 g)$. Next, we find that $d_0 f(\mathbf{v}) = (d_0 g \circ d_0 B)(\mathbf{v})$. Thus $\mathbf{v} \notin \ker(d_0 f)$ if, and only if, $\mathbf{v} \notin \ker(d_0 g \circ d_0 B)$. Moreover, $\mathbf{v} \notin \ker(d_0 g \circ d_0 B)$ if, and only if, $d_0 B(\mathbf{v}) \notin \ker(d_0 g)$.
It follows that f is generic on $\mathcal{B}_G$ if, and only if, g is generic on $\mathcal{B}_H$.

Statement 2 is proved in Proposition 11.5.2 on page 153.
To prove 3, let us consider the following diagram:

$$\begin{array}{ccccc} \mathbb{R}^2 \times \mathbb{R}^3 & \xrightarrow{\pi_1} & \mathbb{R}^3 & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle (A,B)}\downarrow & & {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ U \times S^2 \times I & \xrightarrow{\pi_2} & S^2 \times I & \xrightarrow{g} & \mathbb{R} \end{array}$$

From this diagram, we find that $f \circ \pi_1 = g \circ \pi_2 \circ (A, B)$. Since (A, B) is a diffeomorphism germ at $(0, 0)$, it follows that $f \circ \pi_1$ is a submersion if, and only if, $g \circ \pi_2$ is a submersion. □

Proposition 11.5.5 *Consider the map $F : U \times S^2 \times I \to \mathbb{R}^3$ given by*

$$F(\mathbf{u}, \mathbf{x}, t) = (H_u, H_v, H_{uu}H_{vv} - H_{uv}^2)$$

and the canonical projection $g : S^2 \times I \to I$ given by $(x, y, t) \mapsto t$. The restricted map $F_0 : U \times S^2 \times \{0\} \to \mathbb{R}^3$ given by $(u, v, x, y) \mapsto (H_u, H_v, H_{uu}H_{vv} - H_{uv}^2)|_{t=0}$ is a submersion if, and only if, the restricted composite $g \circ \pi|_{\widetilde{\mathcal{B}}_H} : \widetilde{\mathcal{B}}_H \to I$ is a submersion.

Proof The map F_0 is a submersion if, and only if, the Jacobian matrix of the map has maximal rank, i.e. rank three.

Since $\widetilde{\mathcal{B}}_H$ is smooth, tangent vectors of $\widetilde{\mathcal{B}}_H$ are kernel vectors of the Jacobian matrix of F. For $1 \leq i \leq 5$, let c_i denote the i^{th} column of the Jacobian matrix of F.

Let $(\alpha, \beta, \gamma, \delta, \varepsilon)$ be a kernel vector of the Jacobian matrix of F. Then

$$\alpha c_1 + \beta c_2 + \gamma c_3 + \delta c_4 + \varepsilon c_5 = \mathbf{0} \ . \tag{11.1}$$

The restricted composite $g \circ \pi|_{\widetilde{\mathcal{B}}_H} : \widetilde{\mathcal{B}}_H \to \mathbb{R}$ is a submersion if, and only if, Equation (11.1) has a solution with $\varepsilon \neq 0$.

Assume that $g \circ \pi|_{\widetilde{\mathcal{B}}_H}$ is a submersion, then there exists a solution of Equation (11.1) which has $\varepsilon \neq 0$. Thus c_5 can be written as a linear combination of the other c_i. Since $\widetilde{\mathcal{B}}_H$ is smooth $\{c_1, \ldots, c_5\}$ have rank three, but c_5 can be written as a linear combination of the other c_i, so $\{c_1, \ldots, c_4\}$ must also have rank three. This implies that F_0 must be a submersion.

Next assume that F_0 is a submersion. This implies that $\{c_1, \ldots, c_4\}$ have rank three. Every vector in $\mathbb{R}^3$ can be written as a linear combination of them. Therefore, c_5 is a linear combination of them. Therefore, there must exist a solution to Equation (11.1) with $\varepsilon \neq 0$. This implies that $g \circ \pi|_{\widetilde{\mathcal{B}}_H}$ must be a submersion. □

Proposition 11.5.6 *If H is a versal unfolding of the A_4 singularity, giving a locally smooth affine parabolic curve for $t = 0$, then the projection along the t-parameter is a generic section of the bifurcation set of H.*

Proof In Proposition 11.5.1 on page 153 we saw that the big-bifurcation set of the standard miniversal unfolding was smooth, and so by Proposition 11.1.5 on page 141, any three-parameter versal unfolding of an A_4 will also have a smooth big-bifurcation set. In proposition 11.5.2 on page 153 we saw that the function $g : U \times I \to \mathbb{R}$ is generic on the bifurcation set of H if, and only if, the restricted composite $g \circ \pi|_{\widetilde{\mathcal{B}}_H} : \widetilde{\mathcal{B}}_H \to \mathbb{R}$ is a submersion. In Proposition 11.5.5 on page 155 we saw that this was true if, and only if, the map $F : U \times S^2 \times I \to \mathbb{R}^3$ given by $F(\mathbf{u}, \mathbf{x}, t) = (H_u, H_v, H_{uu}H_{vv} - H_{uv}^2)$, restricted to the $t = 0$ section is a submersion. In Proposition 11.2.1 on page 142 we saw that this was true if, and only if, the affine parabolic curve for $t = 0$ was smooth. □

Example. Consider the one-parameter family of surfaces $\mathbf{X} : U \times I \to \mathbb{R}^3$ given by

$$\mathbf{X}((u,v),t) = \frac{1}{2}(u^2+v^2) + u^4 + tuv^3 + u^4v + u^3v^2 + \frac{3}{8}u^2v^4 \ .$$

One can check that $H(u,v,0,0,0)$ is $\mathcal{A}$-equivalent to u^2+v^5 and that $H(\mathbf{u},\mathbf{x},t)$ is a versal unfolding of the given singularity.

Example.

The Swallow tail transition can be seen in Figure 11.3. This occurs in the image of the affine Gauß map. The affine parabolic curve remains smooth throughout.

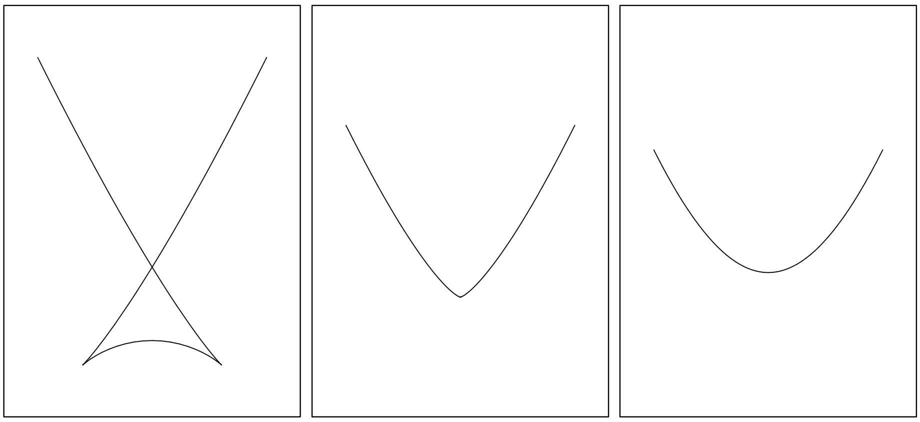

Figure 11.3: The Swallow Tail Transition.

11.6 The $D_4^\pm$ transitions

Consider a function germ $f : \mathbb{R}^2,0 \to \mathbb{R},0$ with a $D_4^\pm$ singularity. This function germ is $\mathcal{A}$-equivalent to the normal form $g_\pm(u,v) = u^3 \pm uv^2$. The set $\{u,v,u^2\}$ can be chosen as a basis for the local algebra of these normal forms. Thus miniversal unfoldings of the normal forms are given by $G_\pm : \mathbb{R}^2 \times \mathbb{R}^3, 0 \to \mathbb{R}, 0$ where

$$G_\pm = u^3 \pm uv^2 + au + bv + cu^2 \ .$$

We need at least a three-parameter unfolding of a $D_4^\pm$ singularity if there is any hope of it being versal. For this reason, we consider a one-parameter family of surfaces.

11.6.1 The D_4^+ case

Here we consider the singularity with u^3+uv^2 as its most familiar normal form. The set of (u, v) such that $u^3+uv^2=0$ comprises three lines through the origin, one real and two complex. For simplicity of calculation, we shall consider the slightly different normal form of u^3+v^3. The set $\{u, v, uv\}$ can be chosen as a basis for the local algebra. Thus miniversal unfolding of the normal form is given by $G : \mathbb{R}^2 \times \mathbb{R}^3, 0 \to \mathbb{R}, 0$ where

$$G = u^3 + v^3 + \alpha u + \beta v + \tau uv \ .$$

We shall adopt the method used by Bruce, Giblin, and Tari in [5]. When we consider sections of the bifurcation set in order to find the evolution of the affine Gauß map, we have the problem of a smooth modulus. There is no discrete classification of the image of the Gauß map. The equivalence of natural stratified equivalence, instead of local diffeomorphism, gives two distinct types of generic function on the bifurcation set of the D_4^+.

We have the geometric unfolding $H : U \times S^2 \times I \to \mathbb{R}$, and the miniversal unfolding $G : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}$. If H is a versal unfolding (and hence miniversal) of the D_4^+, then G and H must be equivalent as unfoldings. There must exists map germs as in Definition 11.1.4 on page 140.

We wish to find a certain order jet of $B : \mathbb{R}^3, 0 \to S^2 \times I, 0$. If we know the condition for a function to be generic on the standard bifurcation set, and we know the diffeomorphism between the standard and the geometric bifurcation sets, then we know the condition for a function to be generic on the geometric bifurcation set.

Consider two versal unfoldings of a D_4^+, namely $F : \mathbb{R}^2 \times \mathbb{R}^3, 0 \to \mathbb{R}$ and the standard one $G : \mathbb{R}^2 \times \mathbb{R}^3, 0 \to \mathbb{R}, 0$. Let F have variables x and y, with unfolding parameters a, b, and t. Let G have variables u and v, with unfolding parameters α, β, and τ. We know that

$$G(u, v, \alpha, \beta, \tau) = F(A(u, v, \alpha, \beta, \tau), B(\alpha, \beta, \tau)) + C(\alpha, \beta, \tau) \ , \tag{11.2}$$

where $x = A_1(\mathbf{u}, \mathbf{a}), y = A_2(\mathbf{u}, \mathbf{a}), a = B_1(\mathbf{a}), b = B_2(\mathbf{a})$, and $t = B_3(\mathbf{a})$.

Applying the chain rule to Equation (11.2), we find that

$$\begin{aligned}
\frac{\partial G}{\partial \alpha} &= \frac{\partial F}{\partial x}\frac{\partial A_1}{\partial \alpha} + \frac{\partial F}{\partial y}\frac{\partial A_2}{\partial \alpha} + \frac{\partial F}{\partial a}\frac{\partial B_1}{\partial \alpha} + \frac{\partial F}{\partial b}\frac{\partial B_2}{\partial \alpha} + \frac{\partial F}{\partial c}\frac{\partial B_3}{\partial \alpha} + \frac{\partial C}{\partial \alpha}\,, \\
\frac{\partial G}{\partial \beta} &= \frac{\partial F}{\partial x}\frac{\partial A_1}{\partial \beta} + \frac{\partial F}{\partial y}\frac{\partial A_2}{\partial \beta} + \frac{\partial F}{\partial a}\frac{\partial B_1}{\partial \beta} + \frac{\partial F}{\partial b}\frac{\partial B_2}{\partial \beta} + \frac{\partial F}{\partial c}\frac{\partial B_3}{\partial \beta} + \frac{\partial C}{\partial \beta}\,, \\
\frac{\partial G}{\partial \tau} &= \frac{\partial F}{\partial x}\frac{\partial A_1}{\partial \tau} + \frac{\partial F}{\partial y}\frac{\partial A_2}{\partial \tau} + \frac{\partial F}{\partial a}\frac{\partial B_1}{\partial \tau} + \frac{\partial F}{\partial b}\frac{\partial B_2}{\partial \tau} + \frac{\partial F}{\partial c}\frac{\partial B_3}{\partial \tau} + \frac{\partial C}{\partial \tau}\,.
\end{aligned}$$

We know that $\partial G/\partial\alpha = u$, $\partial G/\partial\beta = v$, and $\partial G/\partial\tau = uv$. Thus, we compute the u, v, and uv coefficients on the right hand side. To do this, we first compute a certain order jet of the diffeomorphism which takes $F(x, y, 0, 0, 0)$ onto $u^3 + v^3$. The jet is sufficient since we will compare coefficients, and so a sufficiently high order jet will yield the same result as the whole diffeomorphism.

The method here is to impose the conditions that the cubic part of $F(x, y, 0, 0, 0)$ has factors $p_i x + q_i y$ for $1 \leq i \leq 3$, where $p_i \in \mathbb{R}$ for all $1 \leq i \leq 3$, $q_2 \in \mathbb{C}$ with $q_2 \neq \overline{q_2}$, and $q_3 = \overline{q_2}$. This gives a zero level set of three lines through the origin, one real and two complex, but with F still real. This means that F will have D_4^+. If F is the geometric unfolding coming from the family of affine height functions, this further imposes conditions on the Pick normal form coefficients.

There is, of course, an affine transformation taking three distinct lines onto any three distinct lines. Thus, a linear transformation in the source and a scaling in the target are all that are needed to take the cubic part $\sum_{i=1}^{3} p_i x + q_i y$ onto $u^3 + v^3$. Applying this to the geometric unfolding, we find that there are no u, v, or uv coefficients in the partial derivatives of A_1, A_2, or C.

In the geometric case, we need to compute the coefficients in

$$\begin{aligned}
\frac{\partial G}{\partial \alpha} &= \frac{\partial F}{\partial a}\frac{\partial B_1}{\partial \alpha} + \frac{\partial F}{\partial b}\frac{\partial B_2}{\partial \alpha} + \frac{\partial F}{\partial c}\frac{\partial B_3}{\partial \alpha}\,, \\
\frac{\partial G}{\partial \beta} &= \frac{\partial F}{\partial a}\frac{\partial B_1}{\partial \beta} + \frac{\partial F}{\partial b}\frac{\partial B_2}{\partial \beta} + \frac{\partial F}{\partial c}\frac{\partial B_3}{\partial \beta}\,, \\
\frac{\partial G}{\partial \tau} &= \frac{\partial F}{\partial a}\frac{\partial B_1}{\partial \tau} + \frac{\partial F}{\partial b}\frac{\partial B_2}{\partial \tau} + \frac{\partial F}{\partial c}\frac{\partial B_3}{\partial \tau}\,.
\end{aligned}$$

The left hand side is known to us, the right hand side can be calculated. Let $\chi(\partial F/\partial a)$ be the column vector whose first entry is the u coefficient of $\partial F/\partial a$, whose second entry is the v coefficient of $\partial F/\partial a$, and whose third entry is the uv coefficient of $\partial F/\partial a$. Then

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \chi\left(\frac{\partial F}{\partial a}\right)\frac{\partial B_1}{\partial \alpha} + \chi\left(\frac{\partial F}{\partial b}\right)\frac{\partial B_2}{\partial \alpha} + \chi\left(\frac{\partial F}{\partial c}\right)\frac{\partial B_3}{\partial \alpha} \ , \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= \chi\left(\frac{\partial F}{\partial a}\right)\frac{\partial B_1}{\partial \beta} + \chi\left(\frac{\partial F}{\partial b}\right)\frac{\partial B_2}{\partial \beta} + \chi\left(\frac{\partial F}{\partial c}\right)\frac{\partial B_3}{\partial \beta} \ , \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &= \chi\left(\frac{\partial F}{\partial a}\right)\frac{\partial B_1}{\partial \tau} + \chi\left(\frac{\partial F}{\partial b}\right)\frac{\partial B_2}{\partial \tau} + \chi\left(\frac{\partial F}{\partial c}\right)\frac{\partial B_3}{\partial \tau} \end{aligned}$$

For brevity, let us write F_a for $\partial F/\partial a$, etc, I_3 for the 3×3 identity matrix, and J_B for the 3×3 Jacobian matrix of $B : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0$. We now have the matrix equation

$$I_3 = (\chi(F_a) \mid \chi(F_b) \mid \chi(F_c)) \ J_B \ .$$

We can compute $(\chi(F_a) \mid \chi(F_b) \mid \chi(F_c))$ without difficulty. Knowing that it is the inverse matrix of J_B allows us to compute J_B. Of course $B : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ is the diffeomorphism which takes the bifurcation set of G onto the bifurcation set of F.

Consider a one-parameter family of surfaces $\mathbf{X} : U \times I \to \mathbb{R}^3$ given in Pick normal form at elliptic points, we have

$$\mathbf{X}((u,v),t) = \left(u, v, \frac{1}{2}(u^2+v^2) + \frac{\sigma(t)}{6}(u^3 - 3uv^2) + \sum_{i=0}^{4} a_i(t)u^{4-i}v^i + \cdots \right) .$$

The conditions for there to be no quadratic part of the height function at the origin in the direction of the affine normal when $t = 0$ are

$$\begin{aligned} a_1(0) + a_3(0) &= 0 \ , \\ \sigma(0)^2 - 12a_0(0) - 2a_2(0) &= 0 \ , \\ \sigma(0)^2 - 2a_2(0) - 12a_4(0) &= 0 \ . \end{aligned}$$

After following the method outlined above, assuming that the height function for $t = 0$, at the origin, in the direction of the affine normal has a D_4^+ singularity, we find that the determinant of $(\chi(F_a) \mid \chi(F_b) \mid \chi(F_c))$ is zero if, and only if, the family is non-versal. This is clear, a bifurcation set of an unfolding of a D_4^+ is diffeomorphic to the bifurcation set of a versally unfolded D_4^+ if, and only if, it is versal itself. Notice that the versal condition is dependent on $\sigma(t)$, the $a_i(t)$, the $b_j(t)$, and their first order derivatives, all for $t = 0$.

Example. Consider a one-parameter family of surfaces in Pick normal forms, all with an elliptic point at the origin. We can compute the conditions on the Pick coefficients so that $H(u, v, 0, 0, 0) = u^3 + v^3 + O(4)$. In such a case, H is a versal unfolding if, and only if, $\dot{a}_1(0) + \dot{a}_3(0) \neq 0$. Adopting the method above, we compute the Jacobian matrix of B. This third component of B then has 1-jet

$$\frac{4\sigma}{3}(\sigma a_3 + 5b_5 + 1)\alpha + \frac{\sigma}{3}(8 + 9\sigma a_3 + 20b_5)\beta - \frac{3}{2}(\dot{a}_1 + \dot{a}_3)\tau \Big|_{t=0} .$$

The kernel vector of B_3 is

$$\left(\frac{4\sigma}{3}(\sigma a_3 + 5b_5 + 1), \frac{\sigma}{3}(8 + 9\sigma a_3 + 20b_5), -\frac{3}{2}(\dot{a}_1 + \dot{a}_3) \right) \Big|_{t=0} .$$

Regarding this as a point in the real projective plane, with coordinates $(\lambda_1 : \lambda_2 : \lambda_3)$, as in [5], we divide the plane into four regions by the three lines $\lambda_i = 0$. The regions containing the points $(1 : -1 : -1)$, $(1 : 1 : 1)$ give one generic function, and the regions containing the points $(1 : 1 : -1)$, $(1 : -1 : 1)$ give the other. Assuming versality, i.e. $\dot{a}_1(0) + \dot{a}_3(0) \neq 0$, we have the point

$$\left(\frac{8\sigma^2 a_3 + 40\sigma b_5 + 8\sigma}{9(\dot{a}_1 + \dot{a}_3)} : \frac{18\sigma^2 a_3 + 40\sigma b_5 + 16\sigma}{9(\dot{a}_1 + \dot{a}_3)} : 1 \right)$$

It follows that $(\sigma a_3 + 5b_5 + 1)(9\sigma a_3 + 20b_5 + 8) > 0$ corresponds to the first of these cases, and $(\sigma a_3 + 5b_5 + 1)(9\sigma a_3 + 20b_5 + 8) < 0$ the other. The family of surfaces

$$\mathbf{X}(u, v, t) = \left(u, v, \frac{1}{2}(u^2 + v^2) + tuv^3 + u^4 v - 2u^3 v^2 - 2u^2 v^3 + uv^4 \right)$$

has $(\sigma a_3 + 5b_5 + 1)(9\sigma a_3 + 20b_5 + 8) > 0$. We can see the transitions of the A-parabolic set, and its image under the affine Gauß map in figure 11.4. The family of surfaces

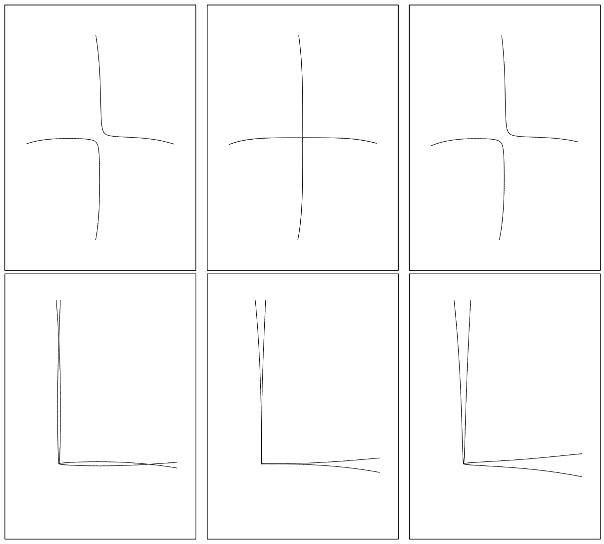

Figure 11.4: D_4^+, case 1. Above: A.P.C. Below: A.G.M.

$$\mathbf{X}(u,v,t) = \left(u, v, \frac{1}{2}(u^2+v^2) + tuv^3 - \frac{1}{2}u^4v - 2u^3v^2 + u^2v^3 + uv^4 - \frac{3}{10}v^5\right)$$

has $(\sigma a_3 + 5b_5 + 1)(9\sigma a_3 + 20b_5 + 8) < 0$. We can see the transitions of the A-parabolic set, and its image under the affine Gauß map in figure 11.5.

11.6.2 The D_4^- case

Consider the D_4^- case, this singularity has normal for $u^3 - uv^2$. A miniversal deformation is given by

$$G(u,v,\alpha,\beta,\tau) = u^3 - uv^2 + \alpha u + \beta v + \tau u^2 \ .$$

Proposition 11.6.1 *Consider a generic one-parameter family of surfaces. Let the three-parameter family of affine height functions be a versal unfolding of a D_4^-. Then the t-constant sections of the bifurcation set are generic sections.*

Proof First, consider G above and some other miniversal unfolding of a D_4^-, say F. Since F and G are both miniversal unfoldings we can find map germs $\tilde{A} : \mathbb{R}^2_{u,v} \times \mathbb{R}^3_{a,b,t}, 0 \to \mathbb{R}^2_{x,y}, 0$, $\tilde{D} : \mathbb{R}^3_{a,b,t}, 0 \to \mathbb{R}^3_{\alpha,\beta,\tau}, 0$, and $\tilde{C} : \mathbb{R}^3_{a,b,t}, 0 \to \mathbb{R}$, where

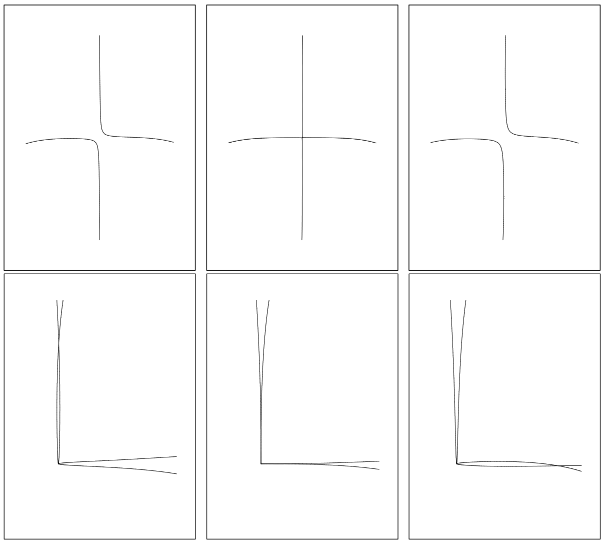

Figure 11.5: D_4^+, case 2. Above: A.P.C. Below: A.G.M.

$\tilde{D}$ is a diffeomorphism germ, $\tilde{A}(u, v, 0, 0, 0) : \mathbb{R}^2_{u,v}, 0 \to \mathbb{R}^2_{x,y}, 0$ is a diffeomorphism germ, $\tilde{C}$ is smooth, such that the following identity holds:

$$G(u, v, \tilde{D}(a, b, t)) = F(\tilde{A}(u, v, \alpha, \beta, \tau), a, b, t) + C(\tilde{D}(a, b, t)) \ .$$

Introducing new map germs A, D, and C, this can be rewritten as

$$G(u, v, D(a, b, t)) = F(A(u, v, a, b, t), a, b, t) + C(a, b, t) \ . \tag{11.3}$$

Consider the function $f : \mathbb{R}^3_{a,b,t} \to \mathbb{R}$ given by $f(a, b, t) = t$. We require that this function is generic on $\mathcal{B}_F$. Consider the image of the $t = 0$ plane under the diffeomorphism D, we get $(D_1(a, b, 0), D_2(a, b, 0), D_3(a, b, 0)) \subset \mathbb{R}^3_{\alpha,\beta,\tau}$. This surface has a tangent plane spanned by

$$D_a := \left(\frac{\partial D_1}{\partial a}, \frac{\partial D_2}{\partial a}, \frac{\partial D_3}{\partial a}\right) \quad \text{and} \quad D_b := \left(\frac{\partial D_1}{\partial b}, \frac{\partial D_2}{\partial b}, \frac{\partial D_3}{\partial b}\right) .$$

The bad direction, i.e. the limiting direction of the singular strata to $\mathcal{B}_G$, is $(0 : 0 : 1)$. Since D is a diffeomorphism, and the plane $t = 0$ is smooth, it follows that $D_a \neq \mathbf{0}$

and $D_b \neq \mathbf{0}$. Thus f is generic if, and only if, $(0,0,1) \notin \operatorname{span}(D_a, D_b)$, i.e.

$$\begin{vmatrix} \partial D_1/\partial a & \partial D_2/\partial a & \partial D_3/\partial a \\ \partial D_1/\partial b & \partial D_2/\partial b & \partial D_3/\partial b \\ 0 & 0 & 1 \end{vmatrix} \neq 0 \; .$$

Now we calculate the $\partial D_i/\partial a$ and $\partial D_j/\partial b$, for $1 \leq i \leq j \leq 2$. To do this, we simply apply the chain rule to Equation (11.3). Let $\mathbf{X} : \mathbb{R}^2_{x,y} \times I_t \to \mathbb{R}^3$ be a one-parameter family of surfaces, and $F : \mathbb{R}^2_{x,y} \times S^2_{a,b} \times I_t \to \mathbb{R}$ the three-parameter family of affine height functions. Let $\mathbf{X}$ be given in Pick normal form.

Differentiating Equation (11.3) by a, we find that

$$\frac{\partial D_1}{\partial a}u + \frac{\partial D_2}{\partial a}v + \frac{\partial D_3}{\partial a}u^2 = \frac{\partial F}{\partial x}\frac{\partial A_1}{\partial a} + \frac{\partial F}{\partial y}\frac{\partial A_2}{\partial a} + \frac{\partial F}{\partial a} + \frac{\partial C}{\partial a} \; . \tag{11.4}$$

Evaluating at $u = v = a = b = t = 0$ gives $F_x = F_y = F_a = 0$, hence $C_a = 0$. Taking the derivatives with respect to b and t, then evaluating at $u = v = a = b = t = 0$ shows that $C_b = C_t = 0$ when $u = v = a = b = t = 0$.

We can also evaluate at $a = b = t = 0$ and compare coefficients since the D_i depend only on a, b, and t. First, we compare the u coefficients in Equation (11.4). Direct computation shows that $F_x(x, y, 0, 0, 0) \in \mathfrak{m}^2$, as too is $F_y(x, y, 0, 0, 0)$. Since $x = A_1(u, v, 0, 0, 0)$ and $y = A_2(u, v, 0, 0, 0)$ and the $A_i(u, v, 0, 0, 0)$ both begin with (linearly independent) linear terms in u and v, it follows that $F_x(A_1, A_2, 0, 0, 0)$ and $F_y(A_1, A_2, 0, 0, 0)$ begin with quadratic terms in u and v. Their u coefficient is zero. The u coefficient of F_a can be found. we find that $F_a = -x + \cdots$, and so the u coefficient of F_a is minus the u coefficient of $A_1(u, v, 0, 0, 0)$, i.e. $-\partial A_1/\partial u(0, 0, 0, 0, 0)$. Moreover, C_a depends on only a, b, and t, so has zero u coefficient. Equation (11.4) shows that

$$\frac{\partial D_1}{\partial a} = \frac{\partial A_1}{\partial u}$$

for $u = v = a = b = t = 0$. Similar methods show that the v coefficients of F_x, F_y, and C_a are zero. Equation (11.4), for $u = v = a = b = t = 0$, then gives

$$\frac{\partial D_2}{\partial a} = \frac{\partial A_2}{\partial u} \; .$$

Differentiating Equation (11.3) with respect to b and employing the same methods, for $u = v = a = b = t = 0$, gives

$$\frac{\partial D_1}{\partial b} = \frac{\partial A_1}{\partial v} \quad \text{and} \quad \frac{\partial D_2}{\partial b} = \frac{\partial A_2}{\partial v}$$

Since $A(u, v, 0, 0, 0) : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ is a diffeomorphism germ, it follows that

$$\frac{\partial D_1}{\partial a}\frac{\partial D_2}{\partial b} - \frac{\partial D_1}{\partial b}\frac{\partial D_2}{\partial a} \neq 0$$

for $u = v = a = b = t = 0$. □

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