Oscillating Möbius Sequences

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Abstract

An exploration of Möbius Sequences of Real Numbers, and the necessary and sufficient conditions for such sequences to oscillate. It is shown that all such conditions factorise into a product of conditions of degree 2. This is illustrated by plotting the different types of sequence on a new 3-D graphic, where nested families of elliptic paraboloids are formed.

1 Introduction: What is a Möbius Sequence?

A Möbius Sequence is an iterative sequence where successive terms are defined by

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \qquad n = 0, 1, 2, \ldots$$

Once a, b, c, d and x_0 are given and if they are real, the whole sequence is determined and consists solely of real numbers. Here, the values $c = 0$ and $ad - bc = 0$ will be excluded, as these either cause the function to collapse to a linear equation or to a constant. For convenience, we also define $x_{n+1} = \infty$ when $x_n = -\frac{d}{c}$, and $x_{n+1} = \frac{a}{c}$ when $x_n = \infty$.

A Möbius Sequence can have up to 2 fixed points, at

$$x_0 = \frac{(a-d) \pm \sqrt{(d-a)^2 + 4bc}}{2c} \tag{1}$$

where $x_0 = x_1 = x_2 = x_3$, etc. Otherwise, any sequence will always oscillate with the same period, or converge to the same limit, whatever value of x_0 we choose.

An oscillating sequence is a sequence where the output will eventually repeat. The period of oscillation, also known as the cycle length, is the number of outputs between each repeat.

In this article I attempt to isolate necessary and sufficient conditions for a Möbius Sequence to oscillate with any given period, and then look at ways of displaying such conditions. Indeed, we will see that the set of points in (a, b, d)-space corresponding the the set of sequences of order n forms a set of $\frac{1}{2}\phi(n)$ elliptic paraboloids, which nest inside each other.

2 Oscillating Möbius Sequence

If a sequence is to oscillate with period n, then $x_0 = x_n$. Therefore, we can find general conditions for any oscillating cycle, simply by repeatedly applying the iterative formula and equating the output to x_0. For example, if a sequence is to oscillate with period 2, then

$$x_0 = x_2 = \frac{ax_1 + b}{cx_1 + d} = \frac{a(\frac{ax_0+b}{cx_0+d}) + b}{c(\frac{ax_0+b}{cx_0+d}) + d}$$

Expanding, simplifying, and collecting like terms gives:

$$(a+d)[cx_0^2 + (d-a)x_0 - b] = 0$$

The right hand bracket solves to give the condition in (1). We can therefore conclude that a Möbius sequence will oscillate with period two if, and only if, $a + d = 0$.

However, using algebraic brute force in this way is going to be messy (though theoretically possible) if we start to look at larger periods of oscillation. To refine this method, we can define the Matrix M such that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad x_{n+1} = \frac{ax_n + b}{cx_n + d}$$

This is very useful, as we can use repeated Matrix multiplication, which can be done very quickly on a computer, instead of repeated algebraic substitution. Clearly if we define

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = M \begin{pmatrix} x_n \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_n \\ 1 \end{pmatrix} = \begin{pmatrix} ax_n + b \\ cx_n + d \end{pmatrix}$$

then $\frac{w_1}{w_2} = x_{n+1}$. This can be extended to show that if

$$M^n \begin{pmatrix} x_0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

then $x_n = \frac{v_1}{v_2}$.

However, as an oscillating sequence will have $x_0 = x_n$, for a sequence of period n, then we can say that

$$M^n \begin{pmatrix} x_0 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} x_n \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ 1 \end{pmatrix}$$

where λ is an arbitrary real constant. This leads to the following important result:

Result 1 *A Möbius Sequence oscillates with period n if and only if $M^n = \lambda I$, where I is the identity matrix, λ is a real number, and M^k is not scalar for any k in $0 < k < n$.*

The ease of this method can be demonstrated thus, with the example $n = 2$:

$$M^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus

$$\lambda = a^2 + bc = bc + d^2$$

and

$$ac + cd = ab + bd = 0.$$

These simplify to give

$$a^2 = d^2$$

and

$$c(a+d) = b(a+d) = 0$$

This suggests two options: either $a + d = 0$, or $b = c = 0$ and $a = d$. However, we can eliminate the second option as we restricted the definition of Möbius sequences to exclude the possibility that $c = 0$. Hence a two-cycle will occur if and only if $a = -d$. This is as expected from the repeated substitution method given above.

This example shows how it is possible to derive necessary and sufficient conditions for a Möbius sequence to oscillate with any period n: take M^n, equate the top-left and bottom-right entries, and then equate the top-right and bottom-left entries with zero.

However, caution must be exercised: the conditions for any period n where n is a composite (i.e. non-prime) number, will also include the conditions for any period k where $1 < k < n$ and $k|n$. For example, if $x_0 = x_2$, then $x_0 = x_4 = x_6$ also. Therefore, the necessary conditions for a sequence to have period 2 will need to be removed from the conditions for a sequence to oscillate with period 4 or 6, as these will not represent the "true" conditions.

3 Plotting in 3 Dimensions

Using the method described in the section above, the following results can be produced:

n	Condition	e.g. $(a,b,c,d) =$
2	$a + d = 0$	$(1,1,1,-1)$
3	$a^2 + ad + bc + d^2 = 0$	$(1,-3,1,1)$
4	$a^2 + 2bc + d^2 = 0$	$(2,-4,1,2)$
5	$a^4 + a^3d + 3a^2bc + a^2d^2 + b^2c^2 + 4abcd + 3bcd^3 + ad^3 + d^4 = \frac{1}{4}(2a^2 + (3+\sqrt{5})bc + (1-\sqrt{5})ad + 2d^2)(2a^2 + (3-\sqrt{5})bc + (1+\sqrt{5})ad + 2d^2) = 0$	$(0, 3+\sqrt{5}, -2, 2)$
6	$a^2 + d^2 + 3bc - ad = 0$	$(3,-1,3,3)$
8	$a^4 + 4a^2bc + 2b^2c^2 + 4abcd + 4bcd^2 + d^4 = (a^2 + 2bc + d^2 + \sqrt{2}(ad - bc))(a^2 + 2bc + d^2 - \sqrt{2}(ad - bc)) = 0$	$(2, 2\sqrt{2}-4, 1, 0)$

Table 1: Conditions necessary and sufficient for various cycles of length n. Examples have been given of sequences that satisfy the conditions.

While we have been successful in obtaining our initial target: the isolation of necessary and sufficient conditions for a sequence to oscillate with a given length of period. However, the equalities resulting are rather unwieldy. In an attempt to make them easier to understand, the following method of plotting them in 3 dimensions is now outlined:

If you recall the equation

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \qquad n = 0, 1, 2, \ldots$$

we stated that $c \neq 0$. Hence we can divide the top and bottom of the function by c without changing its output. Thus

$$x_{n+1} = \frac{\frac{a}{c}x_n + \frac{b}{c}}{x_n + \frac{d}{c}}, \qquad n = 0, 1, 2, \ldots$$

We can now relabel the variables $\frac{a}{c}, \frac{b}{c}$, and $\frac{d}{c}$ with their original letters, and refer to the 'standardised' function

$$x_{n+1} = \frac{ax_n + b}{x_n + d}, \qquad b \neq ad, \qquad n = 0, 1, 2, \ldots$$

having shown that we obtain precisely the same output as with the original.

What is more, we can take the equations in Table 1 and reduce them to only three variables: a, b, and d. This allows us to plot the sets of points that satisfy each equation in three dimensional space, with each point representing a specific Möbius sequence. These sets will form surfaces, which we can analyse more easily than the original algebraic equations (see Figure 1). Each surface represents a family of sequences, each with the same characteristic behaviour.

We can also plot the saddle-shape $b = ad$, which is is set of "degenerate" sequences, where the output of the sequence is the constant a whatever the value of x_0.

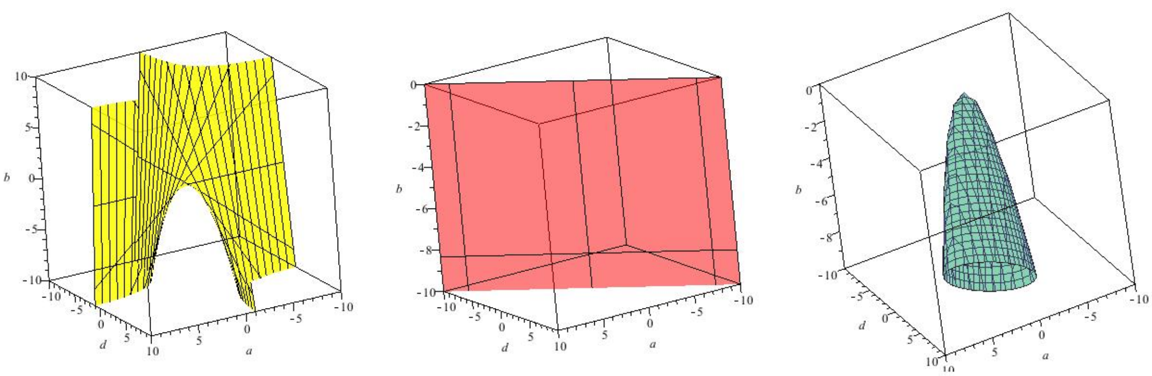

Figure 1: From left to right: the "degenerate" set, where $b = ad$; the set of 2-cycles, where $a = -d$; the set of 3-cycles.

4 The shapes of the 3-D Surfaces

The set of sequences that form a 2-cycle (i.e. oscillate with period 2) form a plane in $a+d = 0$. Experimental observation appears to show that each surface representing a oscillating sequence of period n, for $n \geq 3$, appears to form one or more elliptic paraboloid(s).

We can prove that this is indeed the case by considering the eigenvalues of M. For a sequence to oscillate with a period of three, or greater, then it can be shown that M cannot have any real eigenvalues. (If it did then one of the eigenvectors would act as a limit to which the sequence could converge. The one exception is when the eigenvalues are 1 and -1; this forces the sequence to oscillate with period two.)

As M has no real eigenvalues, then it must have two distinct complex eigenvalues, say λ and μ. If the sequence is oscillating with period n ($n \geq 3$), then we know that M^n is a scalar matrix. The eigenvalues of any scalar matrix are equal, and as the eigenvalues of M^n are λ^n and μ^n, $\lambda^n = \mu^n$. It follows that $(\lambda/\mu)^n = 1$. As the sequence does not oscillate for any lower value of n, we can say that $(\lambda/\mu)^k \neq 1$ for $1 \leq k < n$.

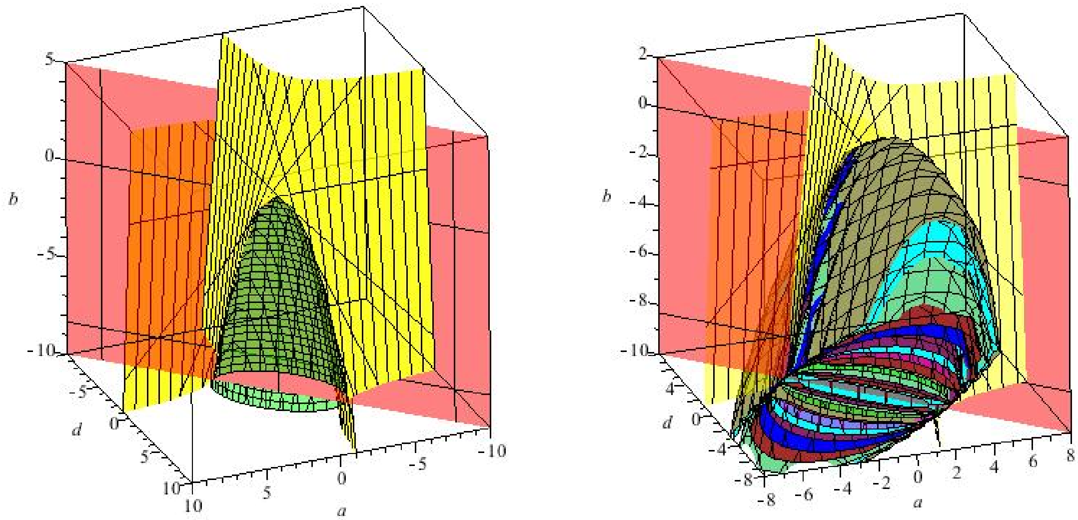

Figure 2: The degenerate set and Period 2: with Period 3; and with Periods 3 to 10, showing how the paraboliods nest inside each other.

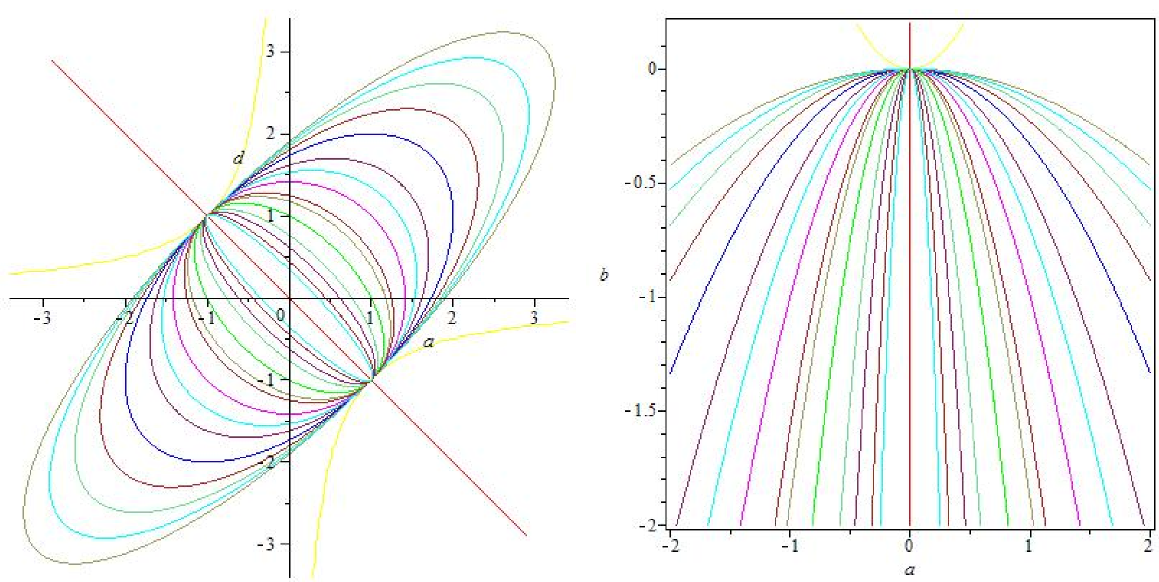

Figure 3: Sections through the surfaces for Periods 2 to 10 with the degenerate set: through the plane $b = -1$ (left) and $a = d$ (right).

This means that λ/μ is a primitive n^{th} root of unity, so

$$\lambda/\mu = e^{(2\pi ir/n)} = e_r$$

say, where r is any integer co-prime to n.

As the sum of the eigenvalues of a matrix is equal to its trace,

$$a + d = \lambda + \mu = \mu(1 + e_r)$$

$$\mu = \frac{a + d}{1 + e_r}$$

Also, the product of the eigenvalues is equal to the determinant:

$$ad - bc = \lambda\mu = e_r\mu^2 = \frac{e_r(a + d)^2}{(1 + e_r)^2}$$

$$\frac{(a + d)^2}{ad - bc} = \frac{(1 + e_r)^2}{e_r} = e_r + 2 + \frac{1}{e_r} = 2(\Re(e_r) + 1)$$

For each r co-prime to n, the above equation gives a necessary and sufficient condition for M^n, and no lower power, to be scalar.

Fixing $c = 1$, we can rewrite the condition as

$$(a + d)^2 = \alpha(ad - b)$$

where $\alpha = 2(\Re(e_r) + 1)$. Further, we know that for $n > 2$, $0 < \alpha < 4$, as $-1 < \Re(e_r) < 1$.

For ease of understanding, we can now change the coordinates using the substitution $a = p + q, d = p - q$. This has the effect of rotating the the surfaces by 45 degrees so that the ellipses in figure (3) lie along the p, q coordinate axes, with a minor change in scale which doesn't alter the shape of the surfaces. This gives:

$$(4 - \alpha)p^2 + \alpha q^2 = -b\alpha \tag{2}$$

A cross section through the (p, q)-plane is obtained by fixing b constant, and it can seen clearly that an ellipse is formed for $b < 0$. One ellipse is formed for each value of α. It can be shown that any oscillating sequence with $n > 2$ must have $bc < 0$, so the ellipse will always be a real ellipse. [If $(d - a)^2 + 4bc \geq 0$, and $a + d \neq 0$, a Möbius sequence will converge, from which it follows that for a sequence to oscillate with period $n > 2$, $4bc < -(d - a)^2$, from which the more general condition $bc < 0$ is derived.]

If we take a cross section through the (q, b)-plane (by fixing p constant), a parabola of the form $q^2 + b = C$, where C is a constant is produced. For q constant, a similar parabola is produced in the (p, b)-plane.

This shows that the surfaces produced, when plotting oscillating Möbius sequences in (a, b, d)-space, are always elliptic paraboloids for oscillations of length greater than or equal to 3.

As $\Re(e_r) = \Re(e_{n-r})$, we can further say that the number of distinct values of $\Re(e_r)$ for n is equal to $\frac{1}{2}\phi(n)$. This means that there will be $\frac{1}{2}\phi(n)$ different paraboloids produced for n, as there will be $\frac{1}{2}\phi(n)$ different solutions to equation (2). This is equivalent to saying that there will be $\frac{1}{2}\phi(n)$ distinct conditions of degree two, which when taken together give a single condition of degree $\phi(n)$, as expected.

Moreover, as $(p, q) = (0, \pm\sqrt{-b})$ satisfies (2), it follows that all the elliptic paraboloids will pass through $(a, b, d) = (\pm\sqrt{-b}, b, \mp\sqrt{-b})$. This apparent contradiction is overcome by remembering that the values $ad - bc = 0$ were excluded at the start as they cause the output of the iterative function to be a constant. Indeed, it can be clearly seen from Figures (2) and (3) that all the paraboloids intersect with each other only when they also intersect with the set of "degenerative" sequences, i.e. in the line $(a = t, b = -t^2, d = -t)$, where t is a parameter.

5 Conclusion

We have shown that necessary and sufficient conditions can be found for a Möbius sequence to oscillate with any chosen length of period. Any such conditions can be factorised into the product of $\frac{1}{2}\phi(n)$ degree two conditions. This is illustrated by the fact that when the sequences corresponding to an oscillation length are plotted in (a, b, d)-space, a set of $\frac{1}{2}\phi(n)$ elliptic paraboloids are produced. All such paraboloids will intersect each other in the line $(t, -t^2, -t)$.

6 Further Reading

J.W.Bruce, P.J.Giblin, and P.J.Rippon: *Microcomputers and Mathematics*, Cambridge University Press, 1990.
Ke Chen, Peter Giblin, and Alan Irving: *Mathematical Explorations with MATLAB*, Cambridge University Press, 1999.

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