Bitangent planes of surfaces and applications to thermodynamics

Peter Giblin [a], Graham Reeve [b],*

[a] *Department of Mathematical Sciences, The University of Liverpool, Liverpool L69 7ZL, United Kingdom*
[b] *School of Mathematics, Computer Science and Engineering, Liverpool Hope University, Liverpool L16 9JD, United Kingdom*

A B S T R A C T

The classical van der Waals equation, applied to one or two mixing fluids, and the Helmholtz (free) energy function A yield, for fixed temperature T, a curve in the plane $\mathbb{R}^2$ (one fluid) or a surface in 3-space $\mathbb{R}^3$ (binary fluid). A line tangent to this curve in two places (bitangent line), or a set of planes tangent to this surface in two places (bitangent planes) have a thermodynamic significance which is well documented in the classical literature. Points of contact of bitangent planes trace 'binodal curves' on the surface in $\mathbb{R}^3$. The study of these bitangents is also classical, starting with D.J. Korteweg and J.D. van der Waals at the end of the 19th century, but continuing into modern times. In this paper we give a summary of the thermodynamic background and of other mathematical investigations and then present a new mathematical approach which classifies a wide range of situations in $\mathbb{R}^3$ where bitangents occur. In particular, we are able to justify many of the details in diagrams of binodal curves observed by Korteweg and others, using techniques from singularity theory.

© 2023 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

The origins of this work are twofold. Firstly, on the mathematical side, Giblin and Zakalyukin ([10,11]) have studied so-called *centre symmetry set (CSS)* of a pair of smooth surfaces in real euclidean 3-space (or two disjoint regions on a single surface). A part of this work ([10, §4], [11, §3]) concerns two local surface patches M, N having a common tangent plane: for the CSS we consider all pairs of *parallel* tangent planes to M and N and the 'chords' joining their points of contact. This 2-parameter family of chords has, in suitable circumstances a real local envelope, that is a real surface S tangent to all the chords (regarded here as infinite straight lines). The generally singular surface S is the CSS of M and N. Within this family of chords there may be a 1-parameter family for which the corresponding parallel planes actually coincide—we say the plane is a 'bitangent plane'—and the corresponding chords play a special role within the CSS. The contact points of the bitangent planes trace *binodal curves* [19] on M and N (also called *coexistence curves* [3] and *conodal curves* or *connodal curves* [26,25]). The local structure of these curves will depend on the geometrical properties of M and N. The straight line joining points of contact of bitangent planes is called a *tie line* or *bitangent chord*, and these lines form a ruled surface.

---

* Corresponding author.
    E-mail addresses: pjgiblin@liv.ac.uk (P. Giblin), reeveg@hope.ac.uk (G. Reeve).

In part we are responding to a suggestion on page 85 of [25]: 'It would be interesting to compare Korteweg's method of continuous deformation of surfaces with the methodology of catastrophe theory'. But the major part of our mathematical contribution is to give a complete list of normal forms of singularities up to codimension 1 (generic 1-parameter families of surfaces) occurring in the bitangent plane context, extending the lists in [10] and [23].

Secondly, the work of D.J. Korteweg on thermodynamic stability of mixtures of two fluids, as recounted in [25], necessitates a study of bitangent planes of surfaces. For Korteweg, the surface in question is the isothermal surface defined by the 'Helmholtz (free) energy function' $A(V, x)$. In Section 3 we shall briefly describe this function, and give some details of the role that bitangent planes play, first describing the simpler situation where a surface is replaced by a plane curve and bitangent planes by bitangent lines. References for Helmholtz free energy include [3,4].

In this article we shall not explicitly study thermodynamic stability or the consequent need for surfaces which are convex (see for example [24]) but instead place the study of binodal curves and related surface geometry in the context of modern singularity theory, combining geometrical information which is invariant to affine transformations of 3-space with more qualitative results invariant to local diffeomorphisms of the ambient space. The latter results are obtained by reducing families of functions to normal forms using appropriate equivalence relations—these preserve essential structure such as cusps, but do not necessarily preserve all geometrical features. In fact it is the ability to use appropriate equivalence relations on families of functions, and then to invoke the tools of singularity theory to reduce to normal forms, which distinguishes our work from that of Korteweg, and also from the work of the authors mentioned in §2. From a singularity point of view, this work also extends the work of [10,11] to include a classification of centre symmetry sets for 1-parameter of surfaces in 3-space in the vicinity of one the surfaces.

In more detail, the structure of this article is as follows. In §2 we briefly describe some other work in the same area as our article. In §3 we give a sketch of the thermodynamic background, starting with the simpler situation of a single fluid and continuing with the two fluid case. In §4 we give two mathematical approaches to the study of binodal curves, the first a direct approach which allows us to describe the local geometry of the surface patches in the various cases, and the second a more general method via 'generating functions' which leads to normal forms and in addition to a more precise description of singularities for generic 1-parameter families of surfaces. The local geometry is further explained in §5. In §6 we use the normal forms method to make pictures of the binodal curves and of the ruled surface of tie lines which preserve singularities up to local diffeomorphism, but not the local geometry of the surfaces and curves such as curvature and inflexions. Calculations and sample proofs are in §7 and we make some concluding remarks in §8.

We restrict ourselves in this article to the 'bilocal' case, that is bitangent planes having contact with two separated surface patches, which could nevertheless be part of the same larger surface. The 'local' case, typified by bitangent planes both of whose contact points are in an arbitrarily small neighbourhood of a cusp of Gauss (plait point, godron) on a single surface patch (see for example [1]), gives a number of additional cases and we hope to cover these elsewhere.

2. Other work in this area

A.N. Varchenko in [27] considers a different thermodynamic potential from the Helmholtz potential, one which is a function of pressure P, temperature T and proportion of two fluids x. (The main results are for a 2-component mixture with a single variable x, but there is also a general treatment for larger numbers of components.) For a fixed pressure and temperature the thermodynamic potential of each homogeneous phase has its own graph as a function of x; these graphs are convex downwards and generally intersect. For a given x thermodynamic stability requires that the actual value of the potential follows the convex hull of the homogeneous phase graphs. Varchenko therefore studies the evolution of these convex hulls of graphs in the plane, varying in a 2-parameter family (parameters P and T).

P.H.E. Meijer in [19] investigates the evolution of tie lines and binodal curves by means of differential equations, and his article contains a clear statement of the underlying physics. In particular he identifies the evolution of binodal curves which we call 'lips' in §5 (he gives the typical 'lips' diagram the equally descriptive name 'Napoleon's hat').

Ricardo Uribe-Vargas in [26] undertakes a detailed investigation of the behaviour of binodal curves close to cusps of Gauss and their relationship with other curves such as the parabolic curve. In this case the binodal and the parabolic curves are tangent at the cusp of Gauss and this creates a region between them in which the surface is convex but not thermodynamically stable on account of the double tangencies. The author also studies the evolution of these curves, and others, during deformations of the surface.

In an unpublished Master's thesis [20] (see also [7]), W.E. Olsen, a student of Daniel Dreibelbis, studies bitangent lines and planes to two surface patches M, N by considering the corresponding subsets of $M \times N$ and finds geometrical conditions for singularities of their projections to the two factors. He also gives examples of 1-parameter deformations of the surfaces and considers the more degenerate case of double-points, that is points of intersection of M and N.

Determination of plait points (cusps of Gauss, godrons) on surfaces connected with thermodynamic equilibrium continues to be of practical interest; see for example [17].

The situation considered in this article, that is where we consider singularities in the vicinity of one of the surfaces, is also called 'on shell', and various applications for this have been explored including for example in semiclassical physics, see [5,6].
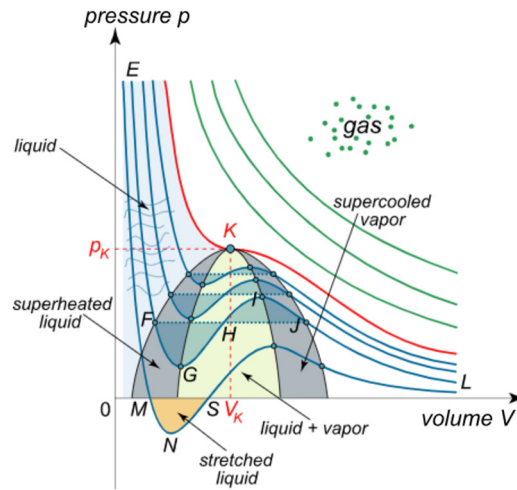
Fig. 1. Van der Waals isotherms in the PV-plane (reproduced with permission from math24.net [18]). (The outer boundary through F, K, J is the binodal curve and the inner boundary through G, K, I is the spinodal curve. For interpretation of the colours in the figure, the reader is referred to the web version of this article.)

3. Thermodynamic background

3.1. One fluid

For a fixed temperature T, the Van der Waals equation

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT, \tag{1}$$

where a, b are constants depending on the substance and R is an absolute constant, describes the relation between pressure P and molar volume V of a single fluid. In the (P, V)-plane, this dependence is represented as a family of isotherms, that is curves of constant temperature T; see Fig. 1.

For a given system, if we change the volume, or the pressure, of the fluid under some isothermal process (where the temperature stays constant) the pressure will be given by following one of these isotherms. Typically an isothermal process occurs when a system is in contact with an outside thermal reservoir, and a change in the system occurs slowly enough to allow the system to be continuously adjusted to the temperature of the reservoir through heat exchange.

For temperatures below the so called critical temperature (the red (thicker) curve through K in Fig. 1), the isotherms have an undulating shape. A portion of such an isotherm between its local minimum and local maximum has a positive derivative $P_V > 0$, which corresponds to an *unstable* state of the substance. (Here and below we use suffices to denote partial derivatives.) Any small positive perturbation of the pressure $dP > 0$ causes an increase in the fluid volume (since $dV > 0$), which in theory would lead to an explosive expansion of the fluid. If the curve slopes downwards, that is $P_V < 0$, then increasing the pressure would result in a decrease in volume ($dV < 0$). In reality, a liquid-gas phase transition occurs in this part of the PV-diagram. This transition is accompanied by a significant change in the volume V at constant pressure P and constant temperature T. Such a transition is represented by a straight horizontal line, called a *tie line*, in the PV-diagram (the segment line FJ in Fig. 1). For example, an isothermal process would follow the tie line FHJ, rather than $FGHIJ$, along which the two phases liquid/vapour coexist in varying proportions in the container at this moment.

The location of the horizontal section FHJ is determined from thermodynamic considerations, resulting in the so-called Maxwell equal area rule. This states that the areas of the curved shapes FGH and HIJ should be equal (see [3], page 238 for details).

The set of start and end points of the horizontal segment lines at different temperatures form a curve, which is called the *binodal* curve. The curve passing through the local minima and maxima of all the isotherms is called the *spinodal* curve. The unstable states (for which $P_V > 0$) are within the region bounded by the spinodal curve. The part of the diagram between the spinodal and binodal curves in principle satisfies the stability criterion $P_V < 0$; the states in this region are called *metastable* states. The left half of the indicated region represents the superheated liquid, and the right half corresponds to the supercooled vapour. Superheated liquid is familiar to anyone who has heated a liquid in a microwave to above boiling point, only to have it suddenly boil after inserting a spoon or some other nucleation site.

The Helmholtz energy A is a *thermodynamic potential* that measures the useful work obtainable from a closed thermodynamic system. For a single fluid it is defined by

$$A = -RT \ln(V - b) - \frac{a}{V^2} \text{ so that } A_V = -\frac{RT}{V - b} + \frac{a}{V^2} = -P \tag{2}$$
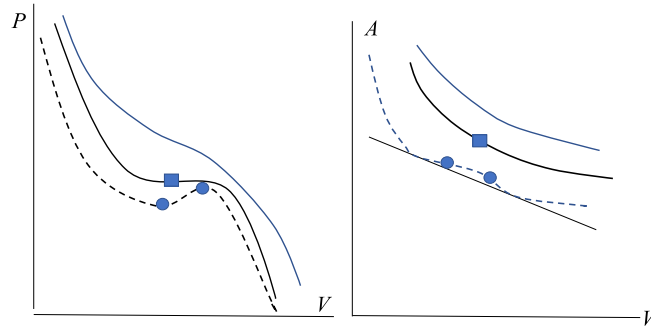
Fig. 2. The PV curve versus the AV curve where A is the Helmholtz energy, $A_V = -P$, for three values of the temperature, increasing from top to bottom. The middle curve is the critical PV curve, with a horizontal inflexion (at the square), corresponding to an 'undulation' on the AV curve—a 'very flat' point where the tangent line has 4-point contact with the curve. Decreasing the temperature 'resolves' this undulation to a curve with increasing (negative) gradient and increasing temperature to a curve with two inflexions, marked by dots. This permits the formation of a double tangent. All the AV curves must be decreasing as V increases, since $P > 0$.

from Van der Waal's equation (1). See Fig. 2 and §3.2 below for further discussion. The existence of maxima and minima on the PV curve implies inflexions, given by $A_{VV} = 0$ on the AV curve and hence a double tangent. In fact the conditions on the contact points of the double tangent line can be interpreted as saying the pressure P and the 'thermodynamic potential' μ are the same at these points. We give more details of this in §3.2.

3.2. Binary fluids

Thermodynamic theory has a concept of 'internal energy' E, which is a function of the 'extensive variables' $S, V, N_1, ...N_r$. Here S is entropy, V is molar volume and the N_i are measures of the molar quantities of the substances present. From this function other 'intensive variables' are determined, in fact these can be regarded as definitions: Temperature $T = E_S$, pressure $P = -E_V$ and 'chemical potentials' $\mu_i = E_{N_i}$. Let us assume the N_i are constant, so that E is a function of S and V. By means of 'Legendre transformations'—essentially representing surfaces by their duals—E can be regarded, at any rate locally, as a function of any pair of variables among S, V, P, T. Helmholtz energy A is defined in this context by $A = E - TS$ and it measures the amount of energy needed to create a system in the absence of changes in temperature and volume, when account is taken of spontaneous transfer of energy from the environment.

In general, given a function $f(x, y)$ where $x = x(a, b)$ and $y = y(a, b)$ then the chain rule says, for example, that $f_b|a = f_x|y \times x_b|a + f_y|x \times y_b|a$, where $f_b|a$ for example says that f is regarded as a function of a and b and a is held constant in the differentiation with respect to b. Applying this to $f = E, x = S, y = V, a = T$ and $b = V$ gives, after rearrangement and using $E_S|V = T$, $E_V|S = E_V|T - TS_V|T$. We then obtain the fact that the Helmholtz energy $A = E - TS$ has the important property

$$-P = E_V|S = (E - TS)_V|T = A_V|T. \tag{3}$$

On the other hand we can find P from the Van der Waals equation (1) which, for two fluids in the proportion $x : 1 - x$, becomes

$$\left(P + \frac{a(x)}{V^2} \right) (V - b(x)) = RT,$$

where a and b now depend on x. Keeping x fixed and solving this for P then integrating with respect to V gives

$$P = \frac{RT}{V - b(x)} - \frac{a(x)}{V^2}, \quad A = -RT \ln(V - b(x)) - \frac{a(x)}{V} + \text{ function of } T.$$

This function of T is known as the 'entropy mixing term' and for two fluids it is $-R(x \ln x + (1 - x) \ln(1 - x))$ (see for example [4, p.432]). It is also called the 'Gibbs mixing term' in [25, p.49]. Hence

$$A(V, x) = -RT \ln(V - b(x)) - \frac{a(x)}{V} + RT(x \ln x + (1 - x) \ln(1 - x)).$$

This is the family of isothermal (constant T) surfaces in (AVx)-space which is the generalisation of the isothermal AV curves in Fig. 2. We recover the plane curve case (2) by substituting $x = 1$, remembering that $(1 - x) \ln(1 - x) \to 0$ as $x \to 1$.

A normal vector to the isothermal surface at $u = (V_0, x_0, A(V_0, x_0))$ is $(A_V(V_0, x_0), A_x(V_0, x_0), -1)$. Writing down the equation of the tangent plane at u and substituting $V = x = 0$ shows that the intercept of the tangent plane at u with the

A-axis is $A(V_0, x_0) - V_0 A_V(V_0, x_0) - x_0 A_x(V_0, x_0)$. The condition for two tangent planes, at u_1 and u_2 say, to be identical is therefore that $A_V(V_0, x_0)$, $A_x(V_0, x_0)$ and this intercept should coincide for $u = u_1$ and $u = u_2$. According to ([25, p.50]) this can be interpreted as saying that the two points u_1 and u_2 share the same pressure P and the same chemical potentials μ_1, μ_2 of the two components.

For our mathematical investigation we are interested in studying the ruled surface of tie lines joining points of contact of bitangent planes (called the *criminant* surface below) to the surfaces M and N and reducing the configuration of M, N and this ruled surface to 'normal forms' under local diffeomorphism in order to examine its basic structure.

4. The mathematical setup

We consider two surface patches M, N in $\mathbb{R}^3$ with coordinates (x, y, z), having a common tangent plane $z = 0$ at the points $(0, 0, 0) \in M, (0, 1, 0) \in N$. The surfaces will be defined locally by $M : \{(u, v, f(u, v))\}; N : \{(x, 1 + y, g(x, y))\}$ where u, v, x, y are small and f, g have no constant or degree 1 terms, that is they have zero 1-jets. We expand the functions f, g by Taylor series about the base points $(0, 0)$:

$$f(u, v) = f_{20}u^2 + f_{11}uv + f_{02}v^2 + f_{30}u^3 + f_{21}u^2v + f_{12}uv^2 + f_{03}v^3 + + f_{40}u^4 + \ldots$$
$$g(x, y) = g_{20}x^2 + g_{11}xy + g_{02}y^2 + g_{30}x^3 + g_{21}x^2y + g_{12}xy^2 + g_{03}y^3 + \ldots \qquad (4)$$

where the subscripts ij indicate that the corresponding monomial is $u^i v^j$ or $x^i y^j$.

Sometimes we have to consider a *generic family of surfaces* in which case the local parametrizations will vary according to a local parameter τ say, taken as close to 0. In that case we call the surfaces in the family M_τ and N_τ and write them locally as

$$f(u, v, \tau) = f_{20}u^2 + f_{11}uv + f_{02}v^2 + f_{30}u^3 + f_{21}u^2v + f_{12}uv^2 + f_{03}v^3 + + f_{40}u^4 + \ldots$$
$$+ \tau(f_{031}v^3 + f_{121}uv^2 + \ldots) + \tau^2(f_{032}v^3 + f_{122}uv^2 + \ldots) + \ldots$$
$$g(x, y, \tau) = g_{20}x^2 + g_{11}xy + g_{02}y^2 + g_{30}x^3 + g_{21}x^2y + g_{12}xy^2 + g_{03}y^3 + \ldots \qquad (5)$$
$$+ \tau(g_{101}x + g_{011}y + g_{201}x^2 + g_{111}xy + g_{021}y^2 + g_{301}x^3 + \ldots)$$
$$+ \tau^2(g_{102}x + g_{012}y + g_{202}x^2 + g_{112}xy + \ldots) + \ldots$$

where the subscripts ijk indicate that the corresponding monomial is $\tau^k u^i v^j$ or $\tau^k x^i y^j$, and for brevity, the subscripts ij are used when k is zero. Here the 2-jet of f has been retained by applying a smooth family of affine transformations. A point of M_τ is $(u, v, f(u, v, \tau))$ and a point of N_τ is $(x, y + 1, g(x, y, \tau))$. When $\tau = 0$ we recover the base surfaces $M_0 = M$ and $N_0 = N$ of the family.

4.1. Equations defining binodal curves

To make explicit calculations of binodal curves using jets we can use the inverse image of $(0, 0, 0)$ under the map $H = (H_1, H_2, H_3) : (\mathbb{R}^4, 0) \to (\mathbb{R}^3, 0)$ given by

$$H(u, v, x, y) = (f_u - g_x, f_v - g_y, (x - u)f_u + (1 + y - v)f_v + f - g), \qquad (6)$$

where subscripts representing *variables*, as opposed to subscripts representing coefficients as in (4) and (6), stand for partial derivatives. Functions are evaluated at (u, v) or (x, y) as appropriate. Vanishing of the first two components of H says that the normal vectors to M and N are parallel and vanishing of the third says that the line joining the points (u, v, f) and $(x, 1 + y, g)$ is perpendicular to the normal to M. Altogether $H^{-1}(\mathbf{0})$ selects the points of M and N for which tangent planes coincide. Projecting $H^{-1}(\mathbf{0})$ to the first two coordinates gives the binodal curve on M and projecting to the last two coordinates gives the binodal curve on N. Studying these projections can give geometrical information on the binodal curves; an example is given below and in §5 we shall show how this direct method can be used to obtain such information.

Proposition 4.1. *Suppose that*

(i) The surface N is parabolic at $(0, 1, 0)$, but does not have a cusp of Gauss at that point. This amounts to saying that the unique asymptotic direction at N is not a root of the cubic terms of g,

(ii) The unique asymptotic direction for N at $(0, 1, 0)$ is not along the tie line $x = 0$, that is x is not a factor of the quadratic terms of g,

(iii) The origin is not a parabolic point on M, that is $f_{11}^2 - 2f_{20}f_{02} \neq 0$ in (4).

Then the binodal curve on M at the origin has an ordinary cusp. In addition, the limiting tangent to this cusp is conjugate to the tie line.

An ordinary cusp is locally diffeomorphic to $\{(t^2, t^3)\}$ for small t. Writing the quadratic terms of g as $(ax + by)^2$, (i) says that $a^3 g_{03} - a^2 b g_{12} + ab^2 g_{21} - b^3 g_{30} \neq 0$ and (ii) says that $b \neq 0$.

Remark 4.2. The conclusion of the proposition above is essentially Korteweg's assertion called K2 in [25, p.76]; proofs by Korteweg and others are in [12–16].

The proof of Proposition 4.1 amounts to using the jets of H as in (6) to solve for x and y as functions of u. The 1-jet of the second component H_2 of H at $u = v = x = y = 0$ is

$$f_{11} u + 2 f_{02} v - 2abx - b^2 y$$

so that using $b \neq 0$ we can use $H_2 = 0$ to solve locally for $y = y(u, v, x)$. We can then substitute for y in $H_1 = 0$ and $H_3 = 0$; these then have leading terms in u, v, x given by

$$-(af_{11} - 2bf_{20})u/b - (2af_{02} - bf_{11})v/b, \text{ and } f_{11} u + 2 f_{02} v, \text{ respectively.}$$

The second of these can be solved locally for $u(v, x)$ or $v(u, x)$ since f_{11}, f_{02} are not both zero. In either case we substitute in $H_1 = 0$ and solve for $u(x)$ and $v(x)$. These have the form $u(x) = px^2 + qx^3 + \dots$ and $v(x) = rx^2 + sx^3$ where

$$ps - qr = \frac{6(a^3 g_{03} - a^2 b g_{12} + ab^2 g_{21} - b^3 g_{30})}{b^6(f_{11}^2 - 4 f_{20} f_{02})},$$

which is nonzero by assumption. This proves that the curve in M (strictly the projection of this curve on to the tangent plane at the origin) has an ordinary cusp. Finally the limiting tangent direction (p, r) of the cusp is a nonzero multiple of $(-2 f_{02}, f_{11})$, which is a vector conjugate to the tie line direction $(0, 1)$ in the tangent plane. □

4.2. Generating families

The direct method above is hard to apply to the situation where the surfaces M and N vary in a 1-parameter family M_τ, N_τ, though Korteweg, as reported in [25], produced many interesting diagrams of the evolution of binodal curves and other surface features such as parabolic and flecnodal curves. We shall adopt here a more general approach, based on the idea of generating families.

Following the method used in [11, p.44-45], [23], we use the generating family, for each fixed τ near 0,

$$\mathcal{F}(n, u, v, x, y, \lambda, q, \tau) = \langle \lambda(u, v, f(u, v, \tau)) + \mu(x, y + 1, g(x, y, \tau)) - q, n \rangle.$$

Here, the angle brackets $\langle , \rangle$ denote inner product in 3-space, λ, μ are barycentric coordinates on a real line (that is $\lambda + \mu = 1$), $n = (n_1, n_2, n_3)$ and $q = (q_1, q_2, q_3)$ are vectors in $\mathbb{R}^3$ and u, v, x, y, τ are as above. For a single generic surface the family parameter τ will be absent.

The *criminant* of $\mathcal{F}$, for a fixed τ, is the ruled surface created by the tie lines and is given by

$$\Delta \mathcal{F} = \left\{ q : \text{for some } (n, u, v, x, y, \lambda), (\mathcal{F} =) \frac{\partial \mathcal{F}}{\partial u} = \frac{\partial \mathcal{F}}{\partial v} = \frac{\partial \mathcal{F}}{\partial x} = \frac{\partial \mathcal{F}}{\partial y} = \frac{\partial \mathcal{F}}{\partial \lambda} = \frac{\partial \mathcal{F}}{\partial n} = 0 \right\},$$

where $\frac{\partial \mathcal{F}}{\partial n}$ stands for $\left(\frac{\partial \mathcal{F}}{\partial n_1}, \frac{\partial \mathcal{F}}{\partial n_2}, \frac{\partial \mathcal{F}}{\partial n_3} \right)$. In the definition of $\Delta \mathcal{F}$ the term '$\mathcal{F} =$' is bracketed since $\mathcal{F}$ is automatically equal to zero given the three equations $\partial \mathcal{F}/\partial n = 0$. Suppose $\lambda \neq 0$ and $\lambda \neq 1$. Then $\Delta \mathcal{F}$ is the set of points $q \in \mathbb{R}^3$ which are on a straight line joining a point of M_τ to a point of N_τ (from $\partial \mathcal{F}/\partial n = 0$), and the line joining these points lies in a bitangent plane to M_τ and N_τ with common normal n (from the other conditions). When $\lambda = 0$, q is a point of N_τ (and n is parallel to the normal to N_τ there and perpendicular to the line joining the two points of M_τ and $N\tau$); likewise $\lambda = 1$ gives points of M. The surfaces M_τ and N_τ are usually called the *redundant components* of $\Delta \mathcal{F}$. The closure of the part of $\Delta \mathcal{F}$ for $\lambda \neq 0, 1$ consists exactly of the (infinite) straight lines joining points which share a common tangent plane, that is the ruled surface containing these lines. The projection to (u, v) or to (x, y) gives the binodal curve in M_τ or N_τ respectively.

Remark 4.3. The *caustic* $\Sigma \mathcal{F}$ is the set of q for which first partial derivatives with respect to u, v, x, y are 0 and the 4×4 matrix of second partial derivatives is singular. The *centre symmetry set* is the union of $\Delta \mathcal{F}$ and $\Sigma \mathcal{F}$. It is discussed in detail in [10,11].

The point of these definitions is that up to local diffeomorphism in $\mathbb{R}^3$ the sets $\Delta \mathcal{F}$ and $\Sigma \mathcal{F}$ are invariant under appropriate changes of coordinates (to be introduced below), allowing us to reduce the family $\mathcal{F}$ to a normal form and deduce the local structure from that. For geometrical information such as conjugate and asymptotic directions we still need to make explicit calculations, as in §4.1.

We shall make use also of *stabilisation* which means that, if a family contains a nondegenenerate quadratic form in 'extra' variables not occurring elsewhere in the family, then this form can be removed without affecting the diffeomorphism type of the criminant (or the caustic). This is because the zero partial derivatives with respect to those extra variables ensures that their values are zero.

In order to study the binodal curves we need to work in a neighbourhood of a point of M or N, that is with λ as above close to 1 or 0. In principle any base value $\lambda = \lambda_0$ can be chosen and then we write $\lambda = \lambda_0 + \varepsilon$. The 'base point' on the second coordinate axis in $\mathbb{R}^3$ is then $(0, 1 - \lambda_0, 0)$ and to work with small coordinates we write $q = (q_1, q_2, q_3) = (q_1, \widetilde{q_2} + 1 - \lambda_0, q_3)$. For the following proposition we take $\lambda_0 = 0$ so that we are 'working near to N' and $q = (q_1, \widetilde{q_2} + 1, q_3)$. Thus the overall base point from which we expand our functions is given by $x = y = u = v = 0, q_1 = \widetilde{q_2} = q_3 = 0, \varepsilon = 0$ ($\lambda = 0$), $n_1 = n_2 = 0, n_3 = 1$.

Proposition 4.4. *Using stabilisation, the family $\mathcal{F}$ can be reduced, near to the surface N, to the following family, where $\widetilde{q_2} = q_2 - 1$, $q = (q_1, q_2, q_3)$.*

$$\Phi(u, v, \varepsilon, q, \tau) = \varepsilon f(u, v, \tau) + (1 - \varepsilon) g\left(\frac{q_1 - \varepsilon u}{1 - \varepsilon}, \frac{\widetilde{q_2} - \varepsilon v + \varepsilon}{1 - \varepsilon}, \tau\right) - q_3,$$

in variables $(u, v) \in \mathbb{R}^2$ and parameters $\varepsilon \in \mathbb{R}, q \in \mathbb{R}^3$ in a neighbourhood of $\varepsilon = 0, q_1 = \widetilde{q_2} = q_3 = 0, \tau = 0$.

Proof. Writing the family $\mathcal{F}$ in the coordinate form we get

$$\mathcal{F} = An_1 + Bn_2 + Cn_3$$

where

$$A = \varepsilon u + (1 - \varepsilon)x - q_1,$$
$$B = \varepsilon v + (1 - \varepsilon)(y + 1) - \widetilde{q_2} - 1$$

and

$$C = \varepsilon f(u, v) + (1 - \varepsilon)g(x, y) - q_3$$

For ε small the functions A and B are regular with respect to x and y and so can be chosen as the coordinate functions instead of x and y, that is

$$x = \frac{A + q_1 - \varepsilon u}{1 - \varepsilon}, \quad y = \frac{B + \widetilde{q_2} - \varepsilon(v - 1)}{1 - \varepsilon}.$$

So in the new coordinates we have

$$\mathcal{F} = An_1 + Bn_2 + C(A, B, u, v, \varepsilon, q)n_3$$

where the function C does not depend on n_1 and n_2. Applying Hadamard's lemma to the function C we get

$$C(A, B, u, v, \varepsilon, q) = C(0, 0, u, v, \varepsilon, q) + A\varphi_1 + B\varphi_2,$$

where φ_1 and φ_2 are smooth functions in $A, B, u, v, \varepsilon, q$ which vanish at $A = B = \varepsilon = q_1 = \widetilde{q_2} = q_3 = 0$. (This is because, g having no linear terms, $\partial C / \partial A = \partial C / \partial B = 0$ at the base point.)

Now the function $\mathcal{F}$ takes the form

$$\mathcal{F} = A(n_1 + \varphi_1 n_3) + B(n_2 + \varphi_2 n_3) + C(0, 0, u, v, \varepsilon, q)$$

where the first two terms represent a non degenerate quadratic form in the independent variables $A, (n_1 + \varphi_1 n_3), B$ and $(n_2 + \varphi_2 n_3)$. Therefore, the function $\mathcal{F}$ is stably-equivalent to the function $\Phi = C(0, 0, u, v, \varepsilon, q)$ being the restriction of the function C to the subspace $A = B = 0$. This completes the proof. □

4.3. Space-time contact equivalence

To study the local structure of the criminant (ruled tie line) surface in the vicinity of the surface, and hence also the local structure of the binodal curve, we reduce the generating function Φ to a normal form up to an appropriate equivalence relation which preserves the surface up to local diffeomorphism. The relevant equivalence relation is the following notion of space-time equivalence, adapted for 1-parameter families of surfaces from [10], (see also [8,9]):

Definition 4.5. Two germs of families F_1 and F_2 with variables $u \in \mathbb{R}^2$, (time) parameter $\lambda \in \mathbb{R}$, (space) parameter $q \in \mathbb{R}^3$, and (family) parameter $\tau \in \mathbb{R}$ are called space-time contact equivalent if there exists a nonzero function $\phi(u, \lambda, q, \tau)$ and a diffeomorphism $\theta : \mathbb{R}^2 \times \mathbb{R}^{1+3+1} \to \mathbb{R}^2 \times \mathbb{R}^{1+3+1}$, of the form

$$\theta : (u, \lambda, q, \tau) \mapsto (U(u, \lambda, q, \tau), \Lambda(\lambda, q, \tau), Q(q, \tau), T(\tau))$$

such that $\phi F_1 = F_2 \circ \theta$.

Remark 4.6. Space-time contact equivalence allows us to reduce a generating function $\mathcal{F}$ to one of a *finite* list of cases, each of which produces a 'model' of the criminant and the binodal curves. Thus we obtain a finite list 'diagrams' or 'pictures' which represent the essential features of the various cases which arise. This representation does not, however, preserve all the geometrical properties of the surfaces and curves involved. Cusps are preserved on the binodal curves but inflexions are not. To study inflexions requires a different technique ('duals' of surfaces and curves) and we hope to pursue this elsewhere. Surface singularities such as cuspidal edges, swallowtail points, Whitney umbrellas and the like are preserved. This is what we mean by 'essential features'. There is more information about the geometry of the different cases in §5, and about *drawing* the criminant and binodal curves in §6.

4.4. Expanding the generating function

We now proceed by expanding the generating function as a power series, and consider the lowest degree terms. First redefine q_3 so that Φ (as in Proposition 4.4) becomes divisible by ε: $\widetilde{q}_3 = q_3 - g(q_1, \widetilde{q}_2, \tau)$, an allowable change of variable according to Definition 4.5. We denote by $\Phi_0(u, v, \varepsilon, \widetilde{q}_3) = \Phi(u, v, \varepsilon, 0, 0, \widetilde{q}_3, 0)$, the organising centre of the family, which has first few terms of the power series in ε at the origin as

$$\Phi_0 = -q_3 + (f_{02} v^2 + f_{11} uv + f_{20} u^2 + f_{30} u^3 + ...)\varepsilon$$
$$+ (g_{02} - 2g_{02} v - g_{11} u + g_{02} v^2 + g_{11} uv + g_{20} u^2)\varepsilon^2 + (g_{02} + g_{03} + ...)\varepsilon^3 + ...$$

where the dots denote higher degree terms.

Notation 4.7. *From this point we shall revert to the notation (q_1, q_2, q_3) for coordinates in $\mathbb{R}^3$, all of these quantities being understood to be close to 0.*

Consider the space W of function germs of the type $\mathcal{F} = -q_3 + \varepsilon \mathcal{H}(u, v, \varepsilon, q_1, q_2, \tau)$. Following [10,21] we show stability of the generating function inside this space W.

The transversality theorem implies that since the base points of the two surfaces already share a bitangent plane (one condition) at $\mu_0 = 0$ only one extra conditions can be imposed on the derivatives of the surfaces. It follows that there are three distinct generic singularity types that can occur on the surfaces. In particular we shall prove the following propositions.

Proposition 4.8 *(See Proposition 7.1). For a generic pair of surfaces M and N near $(0, 1, 0) \in N$ the generating family germ Φ is space-time contact equivalent to one of the following normal forms and is stable inside the space W. The Cases refer to §5 below.*

Case 1 $\widehat{B}_2 : \mathcal{F} = -q_3 + \varepsilon(u^2 \pm v^2 + \varepsilon + q_1)$

Case 2 $\widehat{B}_3 : \mathcal{F} = -q_3 + \varepsilon(u^2 \pm v^2 \pm \varepsilon^2 + q_2\varepsilon + q_1)$,

Case 3 $\widehat{C}_3 : \mathcal{F} = -q_3 + \varepsilon(u^3 + u\varepsilon + \varepsilon + q_2 u + q_1 \pm v^2)$.

When we consider 1-parameter families of surfaces we are permitted to impose one extra condition on the derivatives of the surfaces. This gives rise to the following six additional cases:

Proposition 4.9 *(See Proposition 7.3). For a generic one-parameter family of pairs of surfaces M_τ, N_τ near $(0, 1, 0) \in N_0$, in addition to the list of cases from Proposition 4.8 the affine generating family germ Φ is space-time contact equivalent to one of the following normal forms and is stable inside the space W.*

Case 2a $\widehat{B}_4 : \mathcal{F} = -q_3 + \varepsilon(u^2 \pm v^2 + \varepsilon^3 + \tau\varepsilon^2 + q_2\varepsilon + q_1)$,

Case 3a/3e $\widehat{C}_3^* : \mathcal{F} = -q_3 + \varepsilon(u^3 + u\varepsilon + \varepsilon + (\tau \pm q_2^2)u + q_1 \pm v^2)$,

Case 3b $\widehat{C}_4 : \mathcal{F} = -q_3 + \varepsilon(u^4 + \tau u^2 + u\varepsilon + \varepsilon + q_2 u + q_1 \pm v^2)$,

Case 3c $\widehat{C}_{3,1} : \mathcal{F} = -q_3 + \varepsilon(u^3 + u(\tau\varepsilon \pm \varepsilon^2) + \varepsilon + q_2 u + q_1 \pm v^2)$,

Case 3d $\widehat{F}_4 : \mathcal{F} = -q_3 + \varepsilon(u^3 + u\varepsilon \pm \varepsilon^2 + \tau\varepsilon + q_2 u + q_1 \pm v^2)$,
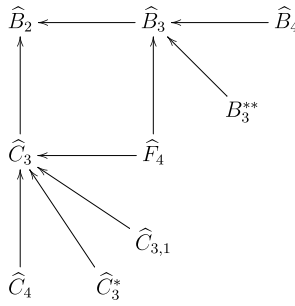
Fig. 3. An adjacency diagram of the various singularity types.

and the non-simple

Case 2b $\ \widehat{B}_3^{**}: \mathcal{F} = -q_3 + \varepsilon(u^2 \pm v^2 \pm \varepsilon^2 + a(q_1, q_2)\varepsilon \pm q_1^2 \pm q_2^2 + \tau)$

where $a(q_1, q_2)$ is a functional modulus.

See Fig. 3 for an adjacency diagram of these singularities.

Remark 4.10. The notation used for these normal forms varies in the literature (compare [10] and [22]). Some of these here are new classes, but we opt to follow the naming convention used in [22]. The ⌢ indicates that the singularity occurs in the vicinity of the surface N or N_0, which is the case for all singularities considered in this article. For the unstarred singularities, the numerical subscripts refer to the codimension of the singularity. In each case the signs of the $\pm$ are independent and correspond to different singularity types. Occasionally we distinguish two distinct sub-cases of $\widehat{C}_3^*$ as $\widehat{C}_3^{*+}$ or $\widehat{C}_3^{*-}$ (see Proposition 7.3). In all cases the sign of v^2 gives different singularity types but does not affect the criminant up to local diffeomorphism. The * indicates that the singularity of the same name fails to be versally unfolded in the 'standard' way by terms linear in q_i and τ. See Remark 7.5 for a note about the apparently missing $\widehat{B}_3^*$ which fails to occur in our geometrical context.

5. Geometrical properties

Here we consider the local geometry of M at $(0, 0, 0)$ and N at $(0, 1, 0)$, but the singularity that we refer to below is the one that occurs on N. We relate the different cases to the normal forms listed in Propositions 4.8 and 4.9. The tie line corresponding to the base points on M and N is the second coordinate axis in $\mathbb{R}^3$, joining the points $(0, 0, 0) \in M$ and $(0, 1, 0) \in N$.

The calculations for generic surfaces M, N (Proposition 4.8) are performed using the direct method of §4.1. For generic families M_τ, N_τ (Proposition 4.9) the direct method is used to analyse the geometry of the base surfaces M_0, N_0 but for the evolution of criminant and binodal curves for $\tau \neq 0$ we need the method of normal forms; see §6. The details of a sample reduction to normal form are given in §7.

Case 1: $\widehat{B}_2$. The simplest case, where M is not parabolic at $(0, 0, 0)$ and $g_{02} \neq 0$, so that the tie line is not an asymptotic direction on N at $(0, 1, 0)$. Both binodal curves are smooth.

Case 2: $\widehat{B}_3$. Suppose M is not parabolic at $(0, 0, 0)$ but we impose the single condition $g_{02} = 0$. We require $g_{03} \neq 0, g_{11} \neq 0$, that is the tie line is asymptotic at $(0, 1, 0) \in N$ but $(0, 1, 0)$ is not a flecnodal point of N; also N is not parabolic at $(0, 1, 0)$. Both binodal curves, in M and in N, are smooth and the tangent to the binodal curve in N is along the tie line. (If an asymptotic direction on M is along the tie line then also the tangent to the binodal curve in M is along this line.)

Case 2a: $\widehat{B}_4$. Suppose M_0 is not parabolic at $(0, 0, 0)$, but $g_{02} = g_{03} = 0, g_{04} \neq 0$, that is $(0, 1, 0)$ is a flecnodal point of N_0 with asymptotic direction along the tie line, but this does not have 5-point contact with N_0. As this is two differential conditions it is stable only in a generic 1-parameter family of surface pairs M_τ and N_τ, and for genericity here we also require $g_{11} \neq 0$ (equivalently, $(0, 1, 0)$ is not a parabolic point of N_0) and $g_{021} \neq 0$. Both binodal curves, in M_0 and in N_0, are smooth (hence this remains true for M_τ and N_τ) and the tangent to the binodal curve in N_0 is along the tie line. (If an asymptotic direction on M_0 is along the tie line then also the tangent to the binodal curve in M_0 is along this line.) The criminant in this case is a swallowtail surface but the binodal curve is the intersection of the criminant with N_0 and this curve is smooth as in Case 2. The criminant, that is the ruled surface of tie lines, is of course tangent to N_0 and N_τ along the binodal curve.

Case 2b: $\widehat{B}_3^{**}$. Suppose M_0 is not parabolic at $(0, 0, 0)$, but $g_{02} = g_{11} = 0$, thus $(0,1,0)$ is parabolic on N_0 and the asymptotic direction there is along the tie line. We assume also $g_{20} \neq 0, g_{03} \neq 0$, so that $(0, 1, 0)$ is not a flat umbilic on N_0 and also is

not a flecnodal point, which here is equivalent to saying that it is not a cusp of Gauss. Using the direct method of §4.1 it can be checked that the binodal curve on M_0 is either an isolated point or two curves having 3-point contact (that is having the same 2-jet), while on N_0 there is respectively an isolated point or a transverse crossing of two smooth curves. This method does not allow us to determine the behaviour as τ passes through 0 in a generic family. For the evolution of the binodal curve on M_τ see Case 3e. Examples suggest that the evolution on N_τ is a Morse transition: an isolated point will disappear or evolve into a smooth curve, while a crossing resolves into two smooth curves (e.g. by moving the asymptotic direction away from the tie line, giving Case 2a). Compare [2, Prop.2.7].

Case 3: $\widehat{C}_3$. Suppose M is parabolic at the origin, but not a cusp of Gauss, and the asymptotic direction at the origin is not along the tie line. Thus writing the quadratic terms of M as $(au + v)^2$, the linear form $au + v$ is not a factor of the cubic terms of M. Suppose also that $g_{02} \neq 0$, so that the tie line is not an asymptotic direction for N, that $(0, 1, 0)$ is not a parabolic point on N and that $f_{11}g_{02} \neq f_{02}g_{11}$. Then the binodal curve on M is smooth, that on N has an ordinary cusp (locally diffeomorphic to $\{(t^2, t^3)\}$) with the limiting tangent at the cusp not along the asymptotic direction at $(0, 1, 0)$.

Case 3a: $\widehat{C}_3^*$. Suppose M_0 is parabolic at $(0, 0, 0)$, N_0 is parabolic at $(0, 1, 0)$, neither has a cusp of Gauss, neither has the asymptotic direction along the tie line (that is f_{02} and g_{02} are both nonzero), and these asymptotic directions are not parallel (in the presence of the two previous conditions this is equivalent to $f_{11}g_{02} - f_{02}g_{11} \neq 0$). This is a symmetrical situation between M_0 and N_0, and occurs generically in a 1-parameter family. Here we distinguish two sub-cases as $\widehat{C}_3^{*+}$ and $\widehat{C}_3^{*-}$ (see Proposition 7.1). On M_0 and N_0 the binodal curve consists of two branches having exactly 3-point contact ($\widehat{C}_3^{*-}$), or else is an isolated point ($\widehat{C}_3^{*+}$). Writing the quadratic terms for M_0 as $(au + v)^2$ and those for N_0 as $(cx + y)^2$ the condition for two real branches is

$$(f_{30} - f_{21}a + f_{12}a^2 - f_{03}a^3)(g_{30} - g_{21}c + g_{12}c^2 - g_{03}c^3) > 0. \tag{7}$$

In a generic family two real branches separate as a 'beaks' transition and an isolated point evolves as a 'lips' transition. See §7. The criterion (7) above for a two real branches coincides with the condition $c_3 < 0$ for a beaks transition in §7. The criminant is locally diffeomorphic to a folded Whitney umbrella.

Case 3b: $\widehat{C}_4$. Suppose M_0 has a (nondegenerate) cusp of Gauss at $(0, 0, 0)$, N_0 is not parabolic at $(0, 1, 0)$, $g_{02} \neq 0$ (that is the tie line is not in an asymptotic direction) and $f_{11}g_{02} - f_{02}g_{11} \neq 0$. The binodal curve in M_0 is then smooth, with tangent the asymptotic direction which is also the tangent to the parabolic curve on M_0. The binodal curve in N_0 has a cusp locally diffeomorphic to (t^3, t^4), in a generic family this evolves as a 'swallowtail transition'. The extra feature in this case is a 'local binodal curve' on M_0: there are bitangent planes of M_0 whose contact points tend to coincidence on M_0, forming a smooth curve passing through the cusp of Gauss and tangent to the parabolic curve there. This 'local binodal curve' lies in the hyperbolic region of M_0, but the bilocal binodal curve defined by bitangent planes having contact points one on M_0 and the other on N_0 need not.

The relative positions of the local binodal curve, the bilocal binodal curve and the parabolic curve depend on the local geometry of M_0. In fact taking the surface M_0 to have local form $z = y^2 + f_{21}x^2y + f_{12}xy^2 + f_{03}y^3 + ... + f_{40}x^4 + ...$ the leading terms in the expressions for these three curves on M_0 are:

$$\text{Local: } y = -\frac{2f_{40}}{f_{21}}x^2, \quad \text{Bilocal : } y = -\frac{f_{21}}{2}x^2, \quad \text{Parabolic } y = -\frac{6f_{40}}{f_{21}}x^2.$$

A nondegenerate cusp of Gauss in these coordinates has $f_{21}^2 \neq 4f_{40}$. If $f_{40} < 0$ the cusp of Gauss is automatically elliptic ($f_{21}^2 > 4f_{40}$), the bilocal binodal curve is in the hyperbolic region of M_0 (locally $f_{21}x +$ h.o.t. in x, y is < 0 for hyperbolic points of M_0) and the local curve separates it from the parabolic curve. The situation if $f_{40} > 0$ is more complicated.

Case 3c: $\widehat{C}_{3,1}$. Suppose $(0, 0, 0)$ is a parabolic point of M_0 (not a cusp of Gauss), $(0, 1, 0)$ is not a parabolic point of N_0, the tie line is not in an asymptotic direction on N_0 (that is $g_{02} \neq 0$) and $f_{11}g_{02} = f_{02}g_{11}$. Then on N_0 the binodal curve has an ordinary cusp with the limiting tangent to the cusp parallel to the asymptotic direction on M_0. The binodal curve on M_0 is smooth with tangent also parallel to the asymptotic direction.

Case 3d: $\widehat{F}_4$. Suppose $(0, 0, 0)$ is a parabolic point of M_0 (not a cusp of Gauss), $(0, 1, 0)$ is not a parabolic point of N_0, the tie line is in an asymptotic direction on N_0 but not at a flecnodal point, that is $g_{02} = 0, g_{03} \neq 0$. Finally suppose $f_{11}g_{02} \neq f_{02}g_{11}$, that is $f_{02}g_{11} \neq 0$. Geometrically this is the same as (3c): on N_0 the binodal curve has an ordinary cusp with the limiting tangent to the cusp parallel to the asymptotic direction on M_0. The binodal curve on M_0 is smooth with tangent also parallel to the asymptotic direction. The criminant is locally diffeomorphic to an open swallowtail.

Case 3e: $\widehat{C}_3^*$. This is the same as Case 2b, with M_τ and N_τ reversed, and the singularity referring as always to the point $(0, 1, 0) \in N_0$. Thus N_0 is not parabolic at $(0, 1, 0)$, $f_{02} = f_{11} = 0$, while f_{20} and f_{03} are nonzero. On N_0 the binodal curve is an isolated point or two curves with 3-point contact and the evolution of the binodal curve as τ passes through zero is via respectively a lips or beaks transition.

5.1. Two proofs

There follow two sample proofs of the geometrical statements above.

Case 3a Since we are looking at the single surface N_0 we can use the forms (4) and because of the assumptions in this case we can write

$$f(u, v) = (au + v)^2 + f_{30}u^3 + f_{21}u^2v + f_{12}uv^2 + f_{03}v^3 + \dots$$
$$g(x, y) = (cx + y)^2 + g_{30}x^2 + g_{21}x^2y + g_{12}xy^2 + g_{03}y^3 + \dots$$

where $a \neq c$ since the asymptotic directions on M_0 and N_0 are not parallel. We shall also assume $a \neq 0$ and $c \neq 0$: these assumptions are not necessary to the argument and in any case are generic assumptions.

The conditions for the tangent planes at $(u, v, f(u, v)) \in M_0$ and $(x, y + 1, g(x, y)) \in N_0$ to be identical are (subscripts denoting partial derivatives)

$$
\begin{aligned}
\text{(i)} \quad & f_u(u, v, f(u, v)) = g_x(x, y, g(x, y)) \\
\text{(ii)} \quad & f_v(u, v, f(u, v)) = g_y(x, y, g(x, y))) \\
\text{(iii)} \quad & (x - u)f_u(u, v, f(u, v)) + (y + 1 - v)f_v(u, v, f(u, v)) \\
& \qquad\qquad + f_u(u, v, f(u, v)) - g(x, y, g(x, y)) = 0
\end{aligned}
\tag{8}
$$

The first two equations state that the normals to M_0 and N_0 are parallel, so that the tangent planes are parallel. The third equation states that the line joining the two points on M_0 and N_0 is perpendicular to the normal to M_0, and therefore also to the normal to N_0. The two tangent planes are therefore identical.

Solving (i) for u as a function of v, x, y and substituting in (ii) and (iii) gives equations with linear terms

$$\frac{2c(c - a)}{a}x + \frac{2(c - a)}{a}y \quad \text{and} \quad \frac{2c^2}{a}x + \frac{2c}{a}y$$

respectively. Because these are proportional we cannot use them to solve for x and y as functions of v but instead solve (ii) for x as a function $x(v, y)$ and substitute further in (iii). The result is an expression of the form

$$x = -\frac{y}{c} + \alpha v^2 + \beta y^2 + \dots$$

for certain values of α and β and an equation in v and y with 2-jet

$$\frac{3(a^3 f_{03} - a^2 f_{12} + a f_{21} - f_{30})}{a^2(a - c)}v^2 - \frac{3(c^3 g_{03} - c^2 g_{12} + c g_{21} - g_{30})}{c^2(a - c)}y^2.$$

Neither coefficient is zero since neither surface has a cusp of Gauss: the condition for this is that respectively $(au + v)$ and $(cx + y)$ should be a factor of the cubic terms of f and g. If the two coefficients of v^2 and y^2 have opposite sign then the locus in v, y has an isolated point, so the same applies to the locus in x, y which we are studying here, so we assume the signs are the same and write the 2-jet as $A^2 v^2 - B^2 y^2$ where $A > 0, B > 0$. The two branches in the v, y plane therefore have the form say

$$v_1 = \frac{B}{A}y + v_{12}y^2 + v_{13}y^3 + \dots \text{ and } v_2 = -\frac{B}{A}y + v_{22}y^2 + v_{23}y^3 + \dots.$$

These can then be substituted into $x(v, y)$ as above to obtain x as a function of y alone for the two branches of the binodal curve in the x, y plane. To show that the branches in the x, y plane have at least 3-point contact we do not need to calculate further for substituting for $v = v_1$ and $v = v_2$ yields

$$x = -\frac{y}{c} + \alpha\left(\frac{B}{A}y + v_{12}y^2\right)^2 + \beta y^2 + \dots \text{ and } x = -\frac{y}{c} + \alpha\left(-\frac{B}{A}y + v_{22}y^2\right)^2 + \beta y^2 + \dots$$

respectively, which clearly have the same 2-jet.

It takes a little more effort to show that the contact is exactly 3-point but calculating v_{12} and v_{22} and remembering the definitions of A and B the difference between the coefficients of y^3 in the branches of the binodal curve in the x, y plane comes to a nonzero multiple of

$$(a - c)(a^3 f_{03} - a^2 f_{12} + a f_{21} - f_{30})^2(c^3 g_{03} - c^2 g_{12} + c g_{21} - g_{30})$$

which is known to be nonzero. $\quad\square$

Cases 2b and 3e We shall use the notation of Case 2b but the singularity on N_0 for Case 3e is the same as that on M_0 for Case 3e and the calculation will find both. The calculation below is less detailed than Case 3a above.

From the conditions of Case 2b the functions f and g as in (4) take the form

$$f(u,v) = f_{20}u^2 + f_{11}uv + f_{02}y^2 + f_{30}u^3 + f_{21}u^2v + f_{12}uv^2 + f_{03}v^3 + \ldots$$

$$g(x,y) = g_{20}x^2 + g_{30}x^2 + g_{21}x^2y + g_{12}xy^2 + g_{03}y^3 + \ldots$$

where $f_{11}^2 - 4f_{20}f_{02}$, g_{20} and g_{03} are all nonzero. We shall also assume for the calculation that $f_{11} \neq 0$: this condition can be avoided by taking a different route.

The three equations (8) allow us to first express u as $u(v,x,y)$ using (i), then x as $x(v,y)$ using (ii); we end up with a relationship between v and y without linear terms, of the form

$$pv^2 + qvy + ry^2 + \text{ higher terms, where } q = \frac{g_{12}(f_{11}^2 - 4f_{20}f_{02})}{f_{11}g_{20}}, \quad r = 3g_{03}. \tag{9}$$

Here p is a much more complicated expression in the second and third order terms of f and g; however p is not in fact needed for the calculation here.

We assume that (9) has two distinct real branches, that is $p^2 - 4qr > 0$, otherwise the corresponding binodal curves in the (u,v) and (x,y) planes will be isolated points. The case of tangential branches is ruled out as non-generic. Write

$$y = y_i(v) = \alpha_i v + \beta_i v^2 + \ldots, \quad i = 1,2 \tag{10}$$

for these two branches, where $\alpha_1 \neq \alpha_2$. We want to deduce, from the locus (9) in the (v,y) plane, the corresponding loci in the (u,v) plane (for M_0) and in the (x,y) plane (for N_0).

For M_0, that is in the (u,v) plane, the expression for $u(v,y)$ has the form

$$-\frac{2f_{02}}{f_{11}}v + av^2 + bvy + cy^2 + \text{ higher terms, where } b = \frac{g_{12}(f_{11}^2 - 4f_{20}f_{02})}{f_{11}^2 g_{20}}, \quad c = \frac{3g_{03}}{f_{11}} \tag{11}$$

and a is more complicated but does not in fact enter the calculation. Thus the coefficients of v in the expressions for $u(v,y_1(v))$ and $u(v,y_2(v))$ will be equal, so the branches in the (u,v) plane will be tangent. The coefficients of v^2 in these two branches are $a + b\alpha_1 + c\alpha_1^2$ and $a + b\alpha_2 + c\alpha_2^2$. These are equal if and only if $(\alpha_1 - \alpha_2)(b + c(\alpha_1 + \alpha_2)) = 0$. But α_1 and α_2 satisfy the equation $p + q\alpha + r\alpha^2 = 0$, by (9), so $\alpha_1 + \alpha_2 = -q/r$, and equality of the 2-jets requires $qc = br$, which is easily checked to be true. Hence the branches in the (u,v) plane have (at least) 3-point contact. The condition for exactly 3-point contact is too complicated to reproduce here. This completed the proof for M_0.

For the binodal curve on N_0 let us write the two branches in the (v,y) plane, assumed real, as in (10) above. Now x as a function of v and y has linear term Av where $A = 4(f_{11}^2 - 4f_{20}f_{02})/2f_{11}g_{20}$. Thus the linear terms of the (x,y) curve, parametrised by v, are $(Av, \alpha_1 v)$ and $(Av, \alpha_2 v)$ The branches therefore form a transverse crossing since $\alpha_1 \neq \alpha_2$. This completes the calculation for N_0 and therefore for Cases 2b and 3e. □

6. Representation of the criminant surface and the binodal curves using normal forms for the generating functions $\mathcal{F}$

In §7 we present explicit calculations and derive the normal forms for Case 3 and Case 3a (see Propositions 4.8 and 4.9, and §5). In the present section we use the normal forms to draw the criminant surface and the binodal curves (see Figs. 4-6).

The reason for selecting these cases is that they are referred to as K2 and K3 by Korteweg in [13]; see also 'Second theorem' and 'Third theorem' in [25, p.76-77]. The 'second theorem' (K2) has already been noted in Remark 4.2. The 'third theorem' (K3) asserts that for our Case 3a, at the moment of transition $\tau = 0$ in the family of surfaces M_τ, N_τ, the binodal curves on both surfaces M and N consist of two 'real or imaginary branches' having the same tangent and curvature. The case of real branches we refer to as a 'beaks transition' ($\widehat{C}_3^{*-}$); imaginary branches give a 'lips transition', $\widehat{C}_3^{*+}$, where the real binodal curves are isolated points. As the family of surfaces evolves, the 'lips' becomes empty in one direction and in the other direction opens out in the manner shown in Fig. 7. However this figure suggests that the 'lips' curve has 4 inflexions. As noted in Remark 4.6, our methods do not preserve inflexions, but according to Korteweg's own calculations there are actually two inflexions, as in Fig. 8.

First we show how the normal form can be used to derive an explicit local parametrisation for the criminant surface, that is the ruled surface formed by the tie lines. We do the calculations for the Case 3 ($\widehat{C}_3$) but the other parametrizations can be derived similarly. We also include figures for the two Cases 3 and 3a.

The criminant $\Delta\mathcal{F}$ for a normal form $\mathcal{F}$ as in Proposition 4.8 is the set given by

$$\Delta\mathcal{F} = \left\{ (q_1, q_2, q_3) \mid \exists (u,v,\varepsilon), \mathcal{F} = \frac{\partial\mathcal{F}}{\partial u} = \frac{\partial\mathcal{F}}{\partial v} = \frac{\partial\mathcal{F}}{\partial\varepsilon} = 0 \right\}.$$
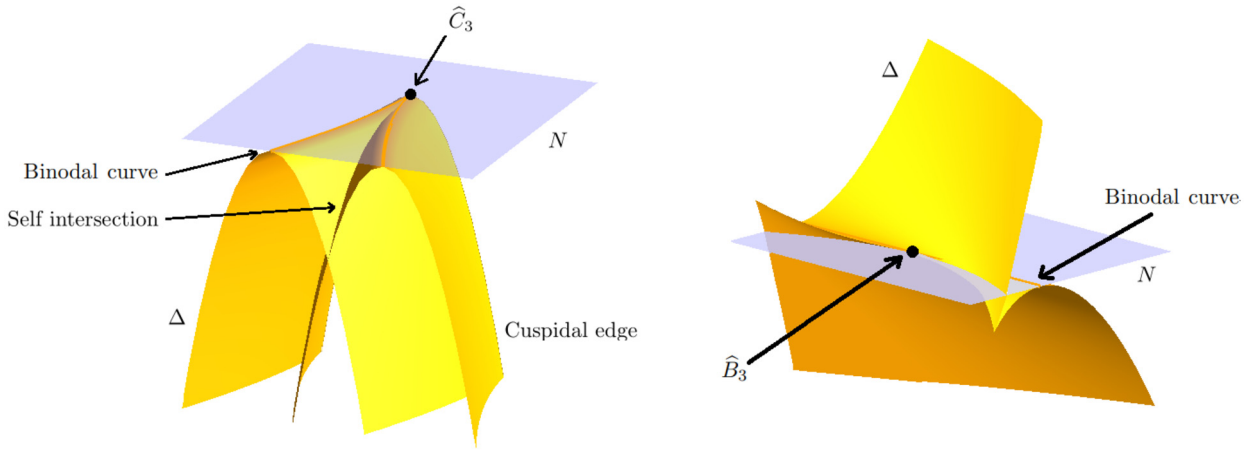
Fig. 4. The criminants and binodal curves of $\widehat{C}_3$ and $\widehat{B}_3$, drawn from the normal forms as in Proposition 4.8. In the left-hand figure the curved surface is called a 'folded Whitney umbrella'. Note that the surface N becomes a plane in this representation, so geometric information about the curvature of N is lost. See Remark 4.6 for information about what is lost and what is preserved by using normal forms.
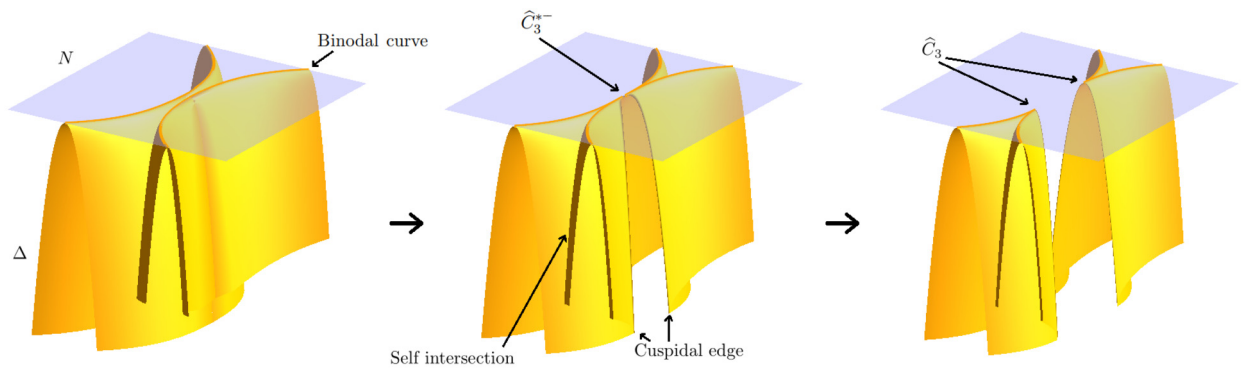

Fig. 5. The criminants and binodal curves of $\widehat{C}_3^{*-}$ as it evolves (left to right) in a generic 1-parameter family of surfaces, using the normal form as in Proposition 4.9.

The problem here is to turn this description as a zero set into a *local parametrization* close to the base point on N.

For the normal form

$$\widehat{C}_3 : \mathcal{F} = -q_3 + \varepsilon(u^3 + u\varepsilon + \varepsilon + q_2 u + q_1 \pm v^2)$$

we calculate the derivatives as

$$\frac{\partial \mathcal{F}}{\partial u} = \varepsilon(3u^2 + \varepsilon + q_2)$$

$$\frac{\partial \mathcal{F}}{\partial v} = \pm 2\varepsilon v,$$

$$\frac{\partial \mathcal{F}}{\partial \varepsilon} = u^3 + 2\varepsilon u + 2\varepsilon + q_2 u + q_1 \pm v^2.$$

The vanishing of $\frac{\partial \mathcal{F}}{\partial v} = \pm 2\varepsilon v$ means that either $\varepsilon = 0$ or $v = 0$ if $\varepsilon \neq 0$.

1. If $\varepsilon = 0$, then $\frac{\partial \mathcal{F}}{\partial u}$ is automatically zero and $\mathcal{F} = 0$ implies $q_3 = 0$. Then $\frac{\partial \mathcal{F}}{\partial \varepsilon} = 0$ gives that $q_1 = -u^3 - 2\varepsilon u - 2\varepsilon - q_2 u \mp v^2$ where q_2 and the other variables are arbitrary. So this component, called the redundant component, is given by the plane $q_3 = 0$ and it corresponds to the surface N.

2. If $\varepsilon \neq 0$ then $\frac{\partial \mathcal{F}}{\partial u} = 0$ gives that $q_2 = -3u^2 - \varepsilon$, and $\frac{\partial \mathcal{F}}{\partial \varepsilon}$ gives $q_1 = 2u^3 - \varepsilon u - 2\varepsilon$. Substituting these into $\mathcal{F} = 0$ gives $q_3 = -\varepsilon^2 u - \varepsilon^2$, so we have $(q_1, q_2, q_3) = 2u^3 - 2\varepsilon, -3u^2, -\varepsilon^2$ which is a folded Whitney umbrella.

The curve along which the folded Whitney umbrella of the criminant and the surface N are tangent is given by $(0, 2u^3, -3u^2)$. This corresponds to the binodal curve and has an ordinary cusp at the origin.
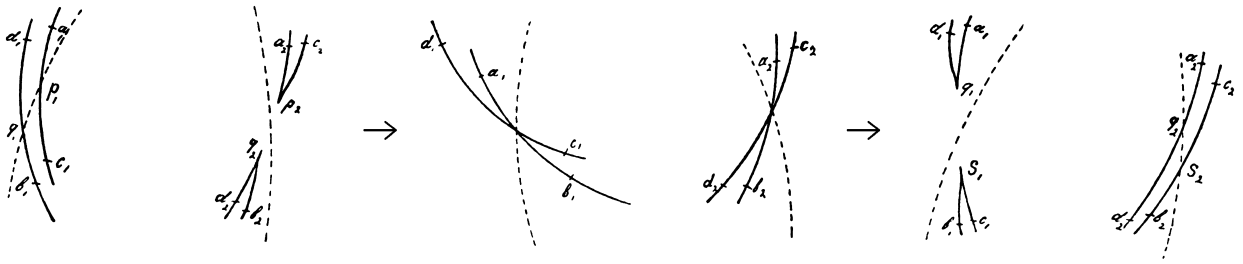
Fig. 6. The binodal curves during a beaks transition of Case 3a ($\widehat{C}_3^{*-}$) from Korteweg 1891 [15], also reproduced in [25, p.79]. The transitions taking place at both M and N are shown simultaneously. The centre pair in this diagram show two curves (full black lines) which should be tangential as well as crossing: they have 3-point contact in the manner of $y = \pm x^3$. The dashed lines represent the parabolic (spinodal) curve.
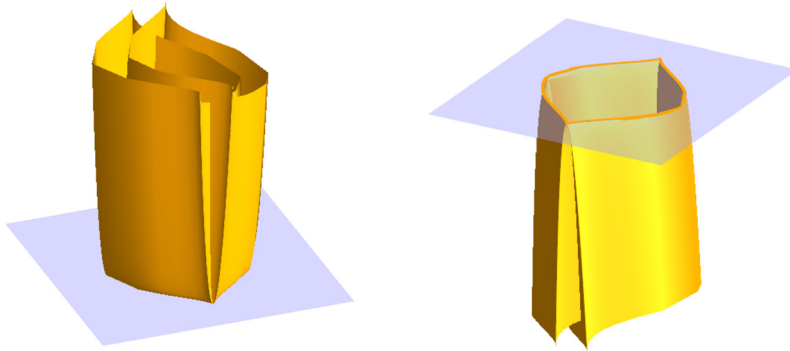

Fig. 7. Two views of the criminant of $\widehat{C}_3^{*+}$ for a small negative value of τ. The 'lips' figure can be seen on the right, but beware that inflexions are not preserved by our methods so the presence of 4 inflexions rather than 2 as in the next figure is not significant.
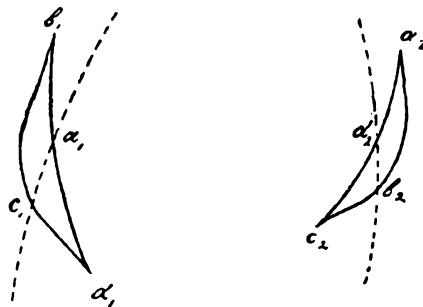

Fig. 8. A lips transition of the binodal curve in Case 3a (for a small negative value of τ in $\widehat{C}_3^{*+}$) from Korteweg 1891 [15]. (This figure is not reproduced in [25].) The binodal curves at both M and N are shown simultaneously. Note that our methods in this article cannot verify the presence of *inflexions* in this figure. Compare Remark 4.6.

7. Reduction to normal forms, calculations and proofs

For Case 3 we have the following proposition:

Proposition 7.1. *For a pair of surfaces M and N corresponding to Case 3, near $\lambda = 0$ the affine generating family germ Φ is space-time contact equivalent to the following normal form and is stable inside the space W:*

$$\widehat{C}_3 : \mathcal{F} = -q_3 + \varepsilon(u^3 + u\varepsilon + \varepsilon + q_2 u + q_1 \pm v^2).$$

Corollary 7.2. *If M has a parabolic point, the corresponding binodal curve on N has an ordinary cusp.*

For Case 3a, where both surfaces have parabolic points, the singularity $\widehat{C}_3$ fails to be versally unfolded and results in a singularity of type C_3^{*+} or C_3^{*-} depending on the sign of the expression
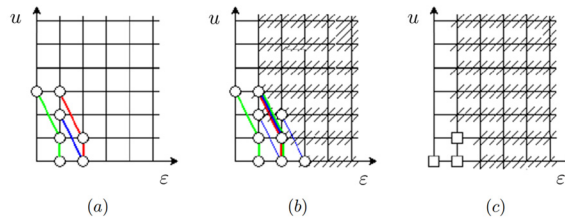
Fig. 9. The Newton diagram of $\widehat{C}_3$.

$$c_3 := -\left(f_{30} - \frac{f_{21}f_{11}}{2f_{02}} + \frac{f_{12}f_{11}^2}{4f_{02}^2} - \frac{f_{03}f_{11}^3}{8f_{02}^3} \right)\left(g_{30} - \frac{g_{21}g_{11}}{2g_{02}} + \frac{g_{12}g_{11}^2}{4g_{02}^2} - \frac{g_{03}g_{11}^3}{8g_{02}^3} \right).$$

In particular we have the following proposition:

Proposition 7.3. *For a generic one-parameter family of pairs of surfaces M_τ and N_τ in which at the moment $\tau = 0$ in the family, there exists a bitangent plane which is tangent to each surface at a parabolic point, are both parabolic, and share a near $\lambda = 0$ the affine generating family germ Φ is space-time contact equivalent to one of the following normal forms and is stable inside the space W:*

$$\widehat{C}_3^{*+} : \mathcal{F} = -q_3 + \varepsilon(u^3 + u\varepsilon + \varepsilon + (\tau + q_2^2)u + q_1 \pm v^2), \text{ if } c_3 > 0$$
$$\widehat{C}_3^{*-} : \mathcal{F} = -q_3 + \varepsilon(u^3 + u\varepsilon + \varepsilon + (\tau - q_2^2)u + q_1 \pm v^2), \text{ if } c_3 < 0.$$

The previous proposition together with explicit calculations from the normal form will prove:

Corollary 7.4. *The binodal curves on both M and N undergo a lips transition if $c_3 > 0$ and a beaks transition if $c_3 < 0$. (See (7) in §5 for a simplified version of $c_3 < 0$ when special coordinates are used.)*

Proof of Proposition 7.1. In this case since M is parabolic, we can make a change of coordinates $v = v' - \frac{uf_{11}}{2f_{02}}$ to reduce the part of the generating family that is quadratic in u and v and linear in ε to give

$$\Phi_0(u, v', \varepsilon) = -q_3 + \varepsilon\left(f_{02}v'^2 + \frac{8f_{02}^3 f_{30} - 4f_{02}^2 f_{11}f_{21} + 2f_{02}f_{11}^2 f_{12} - f_{03}f_{11}^3}{8f_{02}^3}u^3 + \ldots \right)$$
$$+ \left(g_{02}v'^2 + \frac{f_{02}g_{11} - f_{11}g_{02}}{f_{02}}uv' + \frac{4f_{02}^2 g_{20} - 2f_{02}f_{11}g_{11} + f_{11}^2 g_{02}}{4f_{02}^2}u^2 - 2g_{02}v' \right.$$
$$\left. - \frac{f_{02}g_{11} - f_{11}g_{02}}{f_{02}}u + g_{02}) \right)\varepsilon^2 + \ldots + (g_{02} + g_{03} + \ldots)\varepsilon^3 + \ldots$$

Using the lowest degree terms in u and v' that are linear in ε, a further change in coordinates (and dropping the prime $'$) can then remove any other terms divisible by $u^2\varepsilon$ or $v'\varepsilon$ to give the pre-normal form

$$\Phi_0(u, v, \varepsilon) = -q_3 + \varepsilon\left(f_{02}v^2 + \frac{8f_{02}^3 f_{30} - 4f_{02}^2 f_{11}f_{21} + 2f_{02}f_{11}^2 f_{12} - f_{03}f_{11}^3}{8f_{02}^3}u^3 + \ldots \right)$$
$$+ \left(g_{02} - \frac{f_{02}g_{11} - f_{11}g_{02}}{f_{02}}u \right)\varepsilon^2 + \ldots$$

In Case 3 we assume that each of the coefficients here is nonzero, and the vanishing of the various individual expressions $8f_{02}^3 f_{30} - 4f_{02}^2 f_{11}f_{21} + 2f_{02}f_{11}^2 f_{12} - f_{03}f_{11}^3$, $f_{02}g_{11} - f_{11}g_{02}$, g_{02}, and f_{02} gives the various sub-cases 3b, 3c, 3d and 3e respectively.

The family is infinitesimally space-time stable inside the space of functions W if any germ divisible by ε lies in the tangent space to the orbit restricted to $q = 0$

$$T_{\widehat{C}_3}\mathcal{F} = \mathcal{O}_{u,v,\varepsilon}\left\{ \mathcal{F}, \frac{\partial\mathcal{F}}{\partial u}, \frac{\partial\mathcal{F}}{\partial v} \right\} + \mathcal{O}_\varepsilon\left\{ \frac{\partial\mathcal{F}}{\partial\varepsilon} \right\}.$$

We can remove the monomials corresponding to the shaded region of Fig. 9 up to space-time contact equivalence as follows. We have mod $T_{\widehat{C}_3^*}\mathcal{F}$:

$$\mathcal{F} = \varepsilon(u^3 + \varepsilon u + \varepsilon + v^2) \equiv 0, \tag{12}$$

$$\frac{\partial \mathcal{F}}{\partial u} = 3u^2 \varepsilon + \varepsilon^2 \equiv 0, \tag{13}$$

$$\frac{\partial \mathcal{F}}{\partial v} = 2v\varepsilon \equiv 0, \tag{14}$$

$$\frac{\partial \mathcal{F}}{\partial \varepsilon} = u^3 + 2\varepsilon u + 2\varepsilon + v^2 \equiv 0. \tag{15}$$

We can multiply relation (15) by $-\frac{1}{2}\varepsilon$ to give

$$-\frac{1}{2}\varepsilon \frac{\partial \mathcal{F}}{\partial \varepsilon} = -\frac{1}{2}u^3 \varepsilon - \varepsilon^2 u - \varepsilon^2 - \frac{1}{2}v^2 \varepsilon \equiv 0. \tag{16}$$

Firstly the relation (14) implies that monomials divisible by $v\varepsilon$ belong to the tangent space to the orbit. The relations (16) and (12) then imply that monomials of the form $u^3 \varepsilon^{k_1}$ for some $k_1 \geq 0$ belong to the tangent space to the orbit. Using this fact, we can multiply (13) by u to show that monomials of the form $u\varepsilon^{k_2+2}$ for some $k_2 \geq 0$ also belong to the orbit. Now, multiplying relation (15) by functions in ε only, yield that ε^{k_3+2} for some $k_3 \geq 0$. Now relation (13) implies that monomials of the form $u^2 \varepsilon^{k_4}$ for some $k_4 \geq 0$ also belong. Finally, (13) can be used to obtain any monomial in the shaded region of Fig. 9. A basis for $Q = \varepsilon\mathbb{R}[[u, v, \varepsilon]]/T^*_{\widehat{C}_3}\Phi_0$ is generated by the three monomials $1, \varepsilon$ and εu.

Up to quadratic terms in q the function Φ can now be written as

$$\Phi(u, v, \varepsilon) = \Phi_0(u, v, \varepsilon) + \phi_1(u, v, \varepsilon)q_1 + \phi_2(u, v, \varepsilon)q_2 + \phi_3(u, v, \varepsilon)q_3 \tag{17}$$

for some functions ϕ_1, ϕ_2 and $\phi_3 = -1$.

The singularity $\widehat{C}_3$ is versally unfolded if the following matrix evaluated at the origin

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \frac{\partial \phi_1}{\partial \varepsilon} & \frac{\partial \phi_2}{\partial \varepsilon} & \frac{\partial \phi_3}{\partial \varepsilon} \\ \frac{\partial^2 \phi_1}{\partial \varepsilon \partial u} & \frac{\partial^2 \phi_2}{\partial \varepsilon \partial u} & \frac{\partial^2 \phi_3}{\partial \varepsilon \partial u} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ g_{11} & 2g_{02} & 0 \\ -2g_{20} & -g_{11} & 0 \end{pmatrix}, \tag{18}$$

has nonzero determinant. So if N is not a parabolic point, then the singularity is versally unfolded and has normal form:

$$\widehat{C}_3 : F = -q_3 + \varepsilon(u^3 + u\varepsilon + \varepsilon + q_2 u + q_1 \pm v^2)$$

When both M and N have parabolic points the singularity at each surface is of type $\widehat{C}_3^{*\pm}$ or type $\widehat{C}_3^*$. We now prove this and derive the necessary conditions for each type. $\square$

Proof of Proposition 7.3. To show versality and in order to classify the type of $\widehat{C}_3^{*\pm}$ we need to keep track of higher degree terms in q.

Decompose Φ as a power series in u, v by

$$\Phi(u, v, \varepsilon.q_1, q_2, \tau) = \sum_{i,j} \phi_{ij}(\varepsilon, q_1, q_2, \tau)u^i v^j$$

for some functions ϕ_{ij}.

Similarly to the previous case, substituting $u = \xi_1(u, v, q_1, q_2, \tau)$ and $v = \xi_2(u, v, q_1, q_2, \tau)$ for some functions ξ_1 and ξ_2, the higher degree terms can be removed, reducing the generating family Φ to the form

$$\tilde{\Phi} = -q_3 + \varepsilon(\tilde{\phi}_{00} + \tilde{\phi}_{10}u + \tilde{\phi}_{01}v + \tilde{\phi}_{20}u^2 + u^3 \pm v^2)$$

for some functions $\tilde{\phi}_{ij}$ in variables ε, q_1, q_2 and τ which vanish at the origin.

Solving as a power series reveals that the necessary functions are

$$\xi_1(u, v, \varepsilon, q_1, q_2, \tau) = \left(\frac{1}{\phi_{30}}\right)^{\frac{1}{3}} U - \frac{1}{3}\left(\frac{1}{\phi_{30}}\right)^{\frac{5}{3}} \phi_{40} U^2 - \frac{\phi_{30}\phi_{50} - \phi_{40}^2}{3\phi_{30}^3} U^3 + \dots$$

$$\xi_2(u, v, \varepsilon, q_1, q_2, \tau) = \left(\frac{1}{\phi_{02}}\right)^{\frac{1}{2}} V - \frac{\phi_{03}}{2\phi_{02}^2} V^2 - \frac{1}{2}\phi_{12}\left(\frac{1}{\phi_{02}}\right)^{\frac{3}{2}} UV - \frac{\phi_{21}}{2\phi_{02}} U^2 - \frac{4\phi_{02}\phi_{04} - 5\phi_{03}^2}{8\phi_{02}^{\frac{7}{2}}} V^3 + \dots$$

Now a further change of coordinates $U = u_1 - \frac{\tilde{\phi}_{20}}{3}$ and $V = v_1 - \frac{\tilde{\phi}_{01}}{2}$ gives

$$\widehat{\Phi} = -q_3 + \varepsilon(\widehat{\phi}_{00} + \widehat{\phi}_{10}u + u^3 \pm v^2)$$

for functions $\widehat{\phi}_{00}$ and $\widehat{\phi}_{10}$ not containing u, v where

$$\widehat{\phi}_{00}|_{\varepsilon=\tau=0} = 2g_{02}q_2 + g_{11}q_1 + h.o.t.$$

$$\widehat{\phi}_{10}|_{\varepsilon=\tau=0} = -2\frac{(f_{11}g_{20} - f_{20}g_{11})q_1}{f_{11}} - \frac{(f_{11}g_{11} - 4f_{20}g_{02})q_2}{f_{11}} + h.o.t.$$

Since the surface has a parabolic point, the linear terms in q_2 vanish and the necessary condition to be versally unfolded is that the terms

$$\frac{1}{2}\frac{\partial \widehat{\Phi}}{\partial u \partial q_2^2} = -\frac{24g_{02}{}^3g_{30} - 12g_{02}{}^2g_{11}g_{21} + 6g_{11}{}^2g_{12}g_{02} - 3g_{03}g_{11}{}^3}{2g_{11}{}^2g_{02}}$$

and $\frac{\partial^3 \widehat{\Phi}}{\partial u \partial \tau}$ are both nonzero, each of which provide independent conditions and so are avoided generically. Further spacetime contact transformations in parameters can reduce the generating family can be reduced to the normal form:

$$\widehat{C}_3^{*\pm} : \mathcal{F} = -q_3 + \varepsilon(u^3 + u\varepsilon + \varepsilon + (\tau \pm q_2^2)u + q_1 \pm v^2),$$

where the $\pm u^2$ term is independent and does not affect the diffeomorphism type of the Centre Symmetry Set.

The sign of the term $\pm q_2^2$ is determined by the product of the derivatives at $u = v = \varepsilon = q_1 = q_2 = \tau = 0$

$$\frac{\partial^3 \widehat{\Phi}}{\partial u^3}\frac{\partial^3 \widehat{\Phi}}{\partial u \partial q_2^2} = -\left(f_{30} - \frac{f_{11}f_{21}}{2f_{02}} + \frac{f_{11}{}^2f_{12}}{4f_{02}{}^2} - \frac{f_{03}f_{11}{}^3}{8f_{02}{}^3}\right)\left(g_{30} - \frac{g_{11}g_{21}}{2g_{02}} + \frac{g_{11}{}^2g_{12}}{4g_{02}{}^2} - \frac{g_{03}g_{11}{}^3}{8g_{02}{}^3}\right)$$

multiplied by a positive factor.

If this expression is negative then there is a lips bifurcation, and if positive then a beaks bifurcation on the binodal. The expression is zero if and only if one of the surfaces has a cusp of Gauss which is not true for this case. $\quad\square$

Remark 7.5. A note about $\widehat{B}_3$. The condition here is, as Case 2 above, $g_{02} = 0, g_{03} \neq 0$ and M is not parabolic at $(0, 0, 0)$. Analogously to (17) and (18) the singularity is versally unfolded if and only if the following matrix evaluated at the origin

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \frac{\partial \phi_1}{\partial \varepsilon} & \frac{\partial \phi_2}{\partial \varepsilon} & \frac{\partial \phi_3}{\partial \varepsilon} \\ \frac{\partial^2 \phi_1}{\partial \varepsilon^2} & \frac{\partial^2 \phi_2}{\partial \varepsilon^2} & \frac{\partial^2 \phi_3}{\partial \varepsilon^2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ g_{11} & 0 & 0 \\ 2g_{11} + 2g_{12} & 6g_{03} & 0 \end{pmatrix}$$

has nonzero determinant, which is clearly the case unless $g_{11} = 0$. If this occurs (Case 2b), then to be versally unfolded as the more degenerate singularity

$$\widehat{B}_3^* : \mathcal{F} = -q_3 + \varepsilon(u^2 \pm v^2 \pm \varepsilon^2 + (\tau \pm q_1^2)\varepsilon + q_2)$$

we would require the first two rows of the matrix to be independent. But $g_{11} = 0$ prevents this from happening, so $\widehat{B}_3^*$ does not occur in our geometrical context and instead this case results in the singularity $\widehat{B}_3^{**}$.

8. Conclusions

In this article we have applied the techniques of modern singularity theory to study the curves and surfaces occurring in the classical theory of thermodynamics applied to the mixing of fluids, as derived by D.J. Korteweg and expounded by Sengers [25]. In fact to some extent we are responding to a suggestion on page 85 of this book: 'It would be interesting to compare Korteweg's method of continuous deformation of surfaces with the methodology of catastrophe theory'. We have presented a complete list of 'normal forms' which describe the many different 'criminant surfaces' and 'binodal curves' which arise in this context, with proofs in selected cases. The cases selected, and the figures derived from them, are of particular significance in Korteweg's work. One drawback of the methods of singularity theory used here is that certain geometrical information is lost in passing to a normal form; in particular the number of inflexions on the binodal curves cannot be predicted. For this we need to address 'dual' properties of the surfaces and curves and this is the subject of ongoing work.

Data availability

No data was used for the research described in the article.

Acknowledgements

References

[1] Thomas F. Banchoff, Terence Gaffney, Clint McCrory, Cusps of Gauss Mappings, Chapman and Hall/CRC Research Notes in Mathematics, 1982; web version by Daniel Dreibelbis at https://www.emis.de/monographs/CGM/index.html.

[2] J.W. Bruce, P.J. Giblin, F. Tari, Families of surfaces: height functions, Gauss maps and duals, in: W.L. Marar (Ed.), Real and Complex Singularities, in: Pitman Research Notes in Mathematics, vol. 333, 1995, pp. 148–178.

[3] Herbert B. Callen, Thermodynamics and an Introduction to Thermostatistics, second edition, John Wiley and Sons, 1985.

[4] K. Denbigh, The Principles of Chemical Equilibrium, 4th ed., Cambridge University Press, 1981.

[5] Wojciech Domitrz, Miriam Manoel, Pedro de M. Rios, The Wigner caustic on shell and singularities of odd functions, J. Geom. Phys. 71 (2013) 8–72.

[6] Wojciech Domitrz, Michał Zwierzyński, The geometry of the Wigner caustic and a decomposition of a curve into parallel arcs, Anal. Math. Phys. 12 (7) (2022).

[7] D. Dreibelbis, W. Olsen, Structure and transitions of line bitangencies in a family of surface pairs, J. Geom. 113 (2022) 36.

[8] Peter Giblin, Graham Reeve, Equidistants for families of surfaces, J. Singul. 21 (2020) 97–118.

[9] Peter J. Giblin, Graham Reeve, Centre symmetry sets of families of plane curves, Demonstr. Math. 48 (2015) 167–192.

[10] P.J. Giblin, V.M. Zakalyukin, Singularities of centre symmetry sets, Proc. Lond. Math. Soc. 90 (2005) 132–166.

[11] Peter J. Giblin, Vladimir M. Zakalyukin, Recognition of centre symmetry set singularities, Geom. Dedic. 130 (2007) 43–58.

[12] D.J. Korteweg, Uber Faltenpunkte (On plait points), Sitzungsber. Akad. Wiss. Wien, Math-Naturwiss. Kl., Abt. 2A (1889) 1154–1191.

[13] D.J. Korteweg, Vraagstukken CXXXVII, CXXXVIII, CXXXIX (Problems), in: Wiskundige opgaven met oplossingen van het Genootschap E.O.A.K.A.T.B., vol. 4 (Mathematical exercises with solutions by the society E.O.A.K.T.B.), Nabu Press, Charleston SC, United States, 1890, pp. 331–338, The motto represented by the acronym signifies: Untiring Labour Shall Overcome Everything, reprinted: publication date 16 Apr 2012.

[14] D.J. Korteweg, Sur les points de plissement, Arch. Néerl. 24 (1891) 57–98, French translation of [12].

[15] D.J. Korteweg, La théorie générale des plis, Arch. Néerl. 24 (1891) 295–368.

[16] D.J. Korteweg, Plaitpoints and corresponding plaits in the neighborhood of the sides of the c-surface of Van der Waals, Proc. K. Akad. 5 (1903) 445–465.

[17] Gaetano Mangiapia, Roberto Sartorio, Radu P. Lungu, Plait point in ternary system with a miscibility gap: analysis of the procedures involved in its determination and development of novel procedures, Fluid Phase Equilib. 409 (2016) 49–58.

[18] https://math24.net/van-der-waals-equation.html.

[19] Paul H.E. Meijer, Theory of coexisting states: calculation of binodals, Phys. A, Stat. Mech. Appl. 237 (1997) 31–44.

[20] William Edward Olsen, Transitions in Line Bitangency Submanifolds for a One-Parameter Family of Immersion Pairs, University of North Florida Master's Thesis, 2014, https://digitalcommons.unf.edu/etd/521/.

[21] Graham Reeve, Singularities of Systems of Chords in Affine Space, PhD thesis, University of Liverpool, 2012.

[22] Graham M. Reeve, Vladimir M. Zakalyukin, Singularities of the Minkowski set and affine equidistants for a curve and a surface, Topol. Appl. 159 (2), 555–561.

[23] Graham M. Reeve, Vladimir M. Zakalyukin, Singularities of the affine chord envelope for two surfaces in four-space, Proc. Steklov Inst. Math. 277 (2012) 221–233.

[24] George Ruppeiner, Alex Seftas, Thermodynamic curvature of the binary van der Waals fluid, Entropy 22 (2020) 1208, 10 pages.

[25] Joanna Levelt Sengers, How Fluids Unmix, Koninklijke Nederlandse Academie van Wetenschappen, Amsterdam, ISBN 90-6984-357-9, 2002.

[26] R. Uribe-Vargas, A projective invariant for swallowtails and godrons, and global theorems on the flecnodal curve, Mosc. Math. J. 6 (2006) 731–768.

[27] A.N. Varchenko, Evolutions of convex hulls and phase transitions in thermodynamics, J. Sov. Math. 52 (1990) 3305–3325.