On 105.28: Peter Giblin writes: In his interesting Note Clive Johnson establishes conditions for a sequence of the form $x_{m+1} = \dfrac{ax_m + b}{cx_m + d}$, $m = 0, 1, 2, \ldots$, where $ad \neq bc$ and x_0 is a given real initial term, to be periodic with a given period n. (Traditionally if $cx_m + d = 0$ then x_{m+1} is declared to be ∞ and the next term x_{m+2} is a/c. That is the real line is 'compactified with a single point at infinity'.) In what follows I shall assume the sequence is real, that is a, b, c, d and x_0 are real numbers, but this is not strictly necessary. Summarising his statements:

> The above sequence has period $n > 1$, that is $x_n = x_0$ for all (real) choices of initial term x_0 and this does not hold for smaller values of n, if, and only if, the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has the property that M_n is a scalar matrix (the identity matrix multiplied by a real number), and this is not true for any smaller value of n. Furthermore this holds if, and only if, the eigenvalues λ_1, λ_2 of M satisfy $(\lambda_1/\lambda_2)^n = 1$ and λ_1/λ_2 is a primitive nth root of 1. (The (necessarily complex) eigenvalues of M are the roots of
>
> $$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$
>
> and λ_1^n, λ_2^n are the eigenvalues of M^n.)

As the author remarks, it is hard to find explicit references to these results, though they are in some sense 'well known'! It is very useful to have a reference for them, especially as the detailed implications are laid bare. I have two supplementary remarks, the first of which I owe to a conversation with Bronisław Wajnryb of the Rzeszów University of Technology in Poland some years ago.

Let $\lambda_1/\lambda_2 = r$ say so that $a + d = \lambda_1 + \lambda_2 = \lambda_2(1 + r)$ and $ad - bc = \lambda_1\lambda_2 = r\lambda_2^2$. Hence

$$ad - bc = r\left(\frac{a + d}{1 + r}\right)^2 \text{ so that } \frac{(a + d)^2}{ad - bc} = \frac{1}{r} + 2 + r = 2\big(\mathrm{Re}(r) + 1\big),$$

where $\mathrm{Re}(r)$ is the real part of r, namely $\cos\big(2\pi k/n\big)$ for some k coprime to n. The point of this is that it shows that all primitive nth roots of 1 with distinct real parts give *quadratic* conditions on the coefficients a, b, c, d determining the sequence. We can take $c = 1$ without loss of generality and then each quadratic condition is of the form $(a + d)^2 = 2(ad - b)\big(\mathrm{Re}(r) + 1\big)$ which is a surface in a, b, d space given as a graph, with b a quadratic function of a, c, d. It is not hard to check that for $n > 2$ this surface is a paraboloid lying in the half-space $b \leqslant 0$ (for $n = 2$ it is the plane $a + d = 0$) and that all the surfaces have as limiting points the constant sequences where $ad - bc = 0$, that is $b = ad$ here. The result is particularly striking in the case $n = 5$ where there are explicit formulas for $\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$ and $\cos\frac{4\pi}{5} = -\frac{1 + \sqrt{5}}{4}$ which are the distinct real parts of two primitive fifth roots of 1. These give an explicit factorization of the condition for a Möbius sequence of period 5, namely

$$\left(2a^2 + (3 + \sqrt{5})bc + (1 - \sqrt{5})ad + 2d^2\right)\left(2a^2 + (3 - \sqrt{5})bc + (1 + \sqrt{5})ad + 2d^2\right) = 0.$$

Attractive plots can be made of the nested paraboloids arising from this construction (taking $c = 1$ as above) and these have been incorporated into a number of projects by Nuffield Research Placement sixth formers, for example Matthew Temple, whose work is written up in [1].

The second remark is that it is easily proved by induction that $2\cos nx$ is a *monic* polynomial in $2\cos x$ with integer coefficients, that is with leading coefficient 1. For example, writing C for $2\cos x$, we have

$$2\cos 3x = C^3 - 3C$$

and $$2\cos 8x = C^8 - 8C^6 + 20C^4 - 16C^2 + 2.$$

It follows from this that if $2\cos x$ is rational, say p/q in lowest terms with $q > 1$, then x cannot be of the form $2k\pi/n$. Since $\cos 2k\pi = 1$ the latter leads to the equation

$$2q^N = p^N + a_1 q p^{N-1} + a_2 q^2 p^{N-2} + \dots ,$$

for some N and integers $a_1, a_2, \dots$. This is impossible as q divides every term except p^N. From this we deduce for example that the Möbius sequence with $a = 1, b = 1, c = -2, d = 1$ cannot give an n-cycle for any n, since the ratio of eigenvalues comes to $\frac{1}{3}\left(-1 + 2\sqrt{2}i\right)$ with real part $-\frac{1}{3}$, that is $2\cos x = -\frac{2}{3}$ in the notation above. So the ratio of eigenvalues cannot be an nth root of 1 for any n. This Möbius sequence is 'chaotic'.

References

1. Matthew Temple, *Oscillating Möbius sequences*, Nuffield Bursary Project 2012, https://www.liv.ac.uk/~pjgiblin/papers/Matthew.pdf

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