Curves of Constant Width & Centre Symmetry Sets

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Contents

1 Introduction

This project explores the interesting area of curves of constant width and centre symmetry sets. Initially one might think that the only curve of constant width would be a circle, but this is in fact not the case as we shall find out. The centre symmetry set of a curve is almost a visual representation of the nature of that curve's symmetry.

Section 2 outlines the fundamental definitions and concepts, introducing the idea of defining a curve in terms of its support function, which is central to this project.

In section 3, we look at some examples of curves of constant width and their centre symmetry sets. With many graphical displays, it is hoped that this will make the concepts outlined in section 2 seem less abstract, improving the understanding of the reader.

In section 4, we explore the possibility that parametric examples (some of which are taken from section 3) could also be expressed algebraically.

Section 5 fills in the gaps for some proofs taken from various literature (all of which are listed in the Bibliography) and the main theorem prooved here is that of Barbier.

Section 6 looks at Morse's lemma and Morse functions which can be used for finding parallel tangents to curves (which is very useful for curves of constant width and centre symmetry sets as we shall see).

Section 7 asks whether it would be possible to start with 2 pieces of curve (say X and Y) and find another piece of curve Z such that both X and Z have Y as their centre symmetry set.

Finally, section 8 features all the Maple programmes I have used in the making of graphics and lengthly calculations for this project (with annotations).

2 Basic Ideas

My main references for this section are [CGI] and [R].

2.1 Support Function

Consider a curve γ to be smooth, closed and convex such that the orgin is inside γ. If we take an arbitrary tangent (namely the support line $l(t)$ below) and draw a perpendicular line from this tangent to the origin, then the length of this perpendicular is called the support function $h(t)$, where t is the angle between the $x-$axis and the perpendicular.

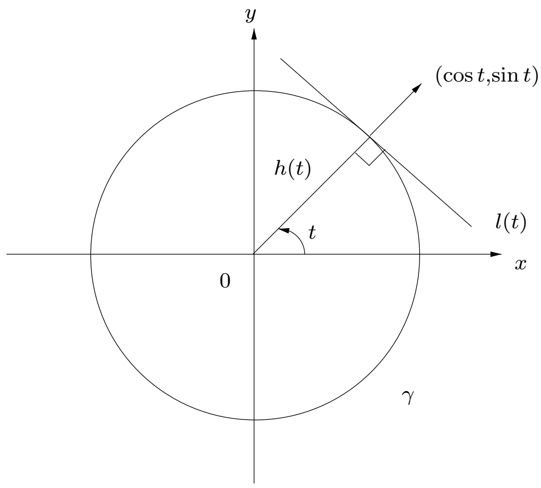


Figure 1: Curve γ with support function $h(t)$ and support line $l(t)$.

If we are given the support function to γ, then we can also find the equation of γ itself and use the fact that the curve will be, by definition, the envelope of its tangents. We can say that the equation for $l(t)$ (a tangent to γ) is,

$$x \cos t + y \sin t = h(t) \tag{1}$$

and so for our family of tangents, we have that

$$F(t, x, y) = x \cos t + y \sin t - h(t).$$

We would like to find the envelope of this family (equal to the parametrisation of our curve γ), i.e. we want to solve $F = \frac{\partial F}{\partial t} = 0$ where,

$$\frac{\partial F}{\partial t}(t, x, y) = -x \sin t + y \cos t - h'(t).$$

After some calculation, we find the parametrisation of γ to be given by,

$$x(t) = h(t)\cos t - h'(t)\sin t \tag{2}$$

$$y(t) = h(t)\sin t + h'(t)\cos t \tag{3}$$

where $'$ denotes a derivative with respect to t (as it will do throughout unless otherwise stated).

Theorem 2.1 *Our curve γ is singular $\iff$ $h(t) + h''(t) = 0$.*

Proof. Well, we have in equations (2) and (3) our parametrisation for γ and from these we can derive,

$$x'(t) = h'(t)\cos t - h(t)\sin t - h''(t)\sin t - h'(t)\cos t$$

$$y'(t) = h'(t)\sin t + h(t)\cos t + h''(t)\cos t - h'(t)\sin t.$$

The curve γ is then singular if $x'(t) = y'(t) = 0$, i.e. where,

$$0 = \big(h(t) + h''(t)\big)\sin t$$

$$0 = \big(h(t) + h''(t)\big)\cos t$$

and since $\sin t$ and $\cos t$ never simultaneously equal 0, we have that our condition for γ to be singular is,

$$h(t) + h''(t) = 0. \tag{4}$$

∎

2.2 A Curve of Constant Width

The width of a closed, convex curve in a specified direction is determined by the distance between 2 parallel tangents and if the distance between all parallel tangents is equal, then we have a curve of constant width (CCW). An obvious example of such a curve would be a circle, however there also exist non-circular CCW, taking forms such as those of 20p and 50p coins.

We can see from Figure 2 that the width of our curve γ in the direction t is equal to $h(t) + h(t + \pi)$. Therefore, our condition for γ to be a CCW is that,

$$h(t) + h(t + \pi) = k \tag{5}$$

where k equals some constant.

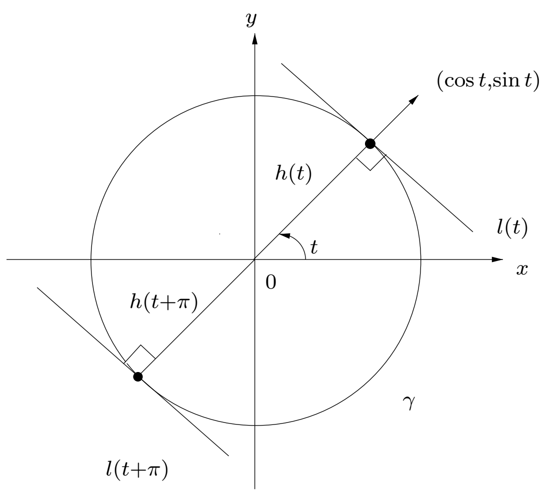


Figure 2: A CCW γ defined by support function $h(t)$.

Theorem 2.2 *The chord joining the contact points of the parallel tangents $l(t)$ and $l(t+\pi)$ will be a common normal to both $l(t)$ and $l(t+\pi)$.*

Proof. We can use our equation for $l(t)$ (equation (1)) by substituting $t = t+\pi$ into $l(t)$ to give us,

$$x(t+\pi)\cos(t+\pi) + y(t+\pi)\sin(t+\pi) = h(t+\pi).$$

Now use the fact that $\sin(t+\pi) = -\sin t$ and $\cos(t+\pi) = -\cos t$ which give us our equation for $l(t+\pi)$ to be,

$$-x(t+\pi)\cos t - y(t+\pi)\sin t = h(t+\pi).$$

However, we would like all the terms in this equation to be expressed in terms of t. From equations (2) and (3) we have $x(t)$ and $y(t)$ which, using our rules concerning sines and cosines from above, imply that,

$$\begin{aligned} x(t+\pi) &= -h(t+\pi)\cos t + h'(t+\pi)\sin t & (6) \\ y(t+\pi) &= -h(t+\pi)\sin t - h'(t+\pi)\cos t. & (7) \end{aligned}$$

Well, we can then calculate,

$$x(t+\pi) - x(t) = -\left[h(t+\pi) + h(t)\right]\cos t + \left[h'(t+\pi) + h'(t)\right]\sin t$$

$$y(t+\pi) - y(t) = -\left[h(t+\pi) + h(t)\right]\cos t - \left[h'(t+\pi) + h'(t)\right]\sin t$$

which can then be simplified using equation (5) to,

$$x(t) - x(t+\pi) = k\cos t$$

$$y(t) - y(t+\pi) = k\sin t.$$

Therefore the direction of our chord joining parallel tangents is,

$$(x(t) - x(t+\pi), y(t) - y(t+\pi)) = k\,(\cos t, \sin t)$$

and, with reference to Figure 2 we can see that this is a vector, equal in length and direction to $h(t) + h(t+\pi)$ (i.e. itself or any parallel vetor) and by definition, the support function is orthogonal to the support line. Therefore parallel tangents to a CCW share a common normal (the chord $h(t) + h(t+\pi)$ or any parallel). ∎

With this in mind, we would now like to find a parametrisation of the envelope of parallel tangent chords to a CCW.

2.3 Centre Symmetry Set & Evolute of γ

For now, we will not assume that our curve γ is a CCW (it is however smooth, closed and convex). Let us find find the envelope of chords joining parallel tangents and, from above, we know that these chords will join the points $(x(t), y(t))$ to $(x(t+\pi), y(t+\pi))$ on γ (see equations (2), (3), (6) and (7)). So the equation of such a chord (not the vector direction as calculated previously) is given by,

$$\frac{Y - y(t)}{X - x(t)} = \frac{y(t+\pi) - y(t)}{x(t+\pi) - x(t)}$$

where (X, Y) are coordinates of a chord in $\mathbb{R}^2$. From this we can obtain our family of chords by taking all terms over to one side,

$$F(t, X, Y) = [Y - y(t)][x(t+\pi) - x(t)] - [X - x(t)][y(t+\pi) - y(t)].$$

For the envelope of these chords, we need to substitute our known values into $F = \frac{\partial F}{\partial t} = 0$. From $F = 0$ we obtain, after some simplification,

$$Y[(h' + h'_\pi)s - (h + h_\pi)c] + X[(h_\pi + h)s + (h'_\pi + h')c] - hh'_\pi + h'h_\pi = 0.$$

Finding $\frac{\partial F}{\partial t}$ and setting it equal to 0 gives,

$$Y[(h + h'' + h_\pi + h''_\pi)s] + X[(h + h'' + h_\pi + h''_\pi)c] - hh''_\pi + h''h_\pi = 0$$

and now we can find our envelope of chords. There are some very large expressions involved in this calculation but, with some help from Maple (see Appendix 1), we find that,

$$X = \frac{-h'h_\pi \sin t + h''h'_\pi \sin t + hh'_\pi \sin t - h'h''_\pi \sin t + hh''_\pi \cos t - h''h_\pi \cos t}{h + h'' + h_\pi + h''_\pi}$$

$$Y = \frac{-h'h_\pi \cos t + h''h'_\pi \cos t + hh'_\pi \cos t - h'h''_\pi \cos t + h''h_\pi \sin t - hh''_\pi \sin t}{h + h'' + h_\pi + h''_\pi}$$

(denominator is non-zero, by (4)) yet there is more that we can say about this, consider first the centre symmetry set.

Definition
The envelope of chords joining parallel tangent points is called the centre symmetry set (CSS).

Proposition 2.1 *(X, Y) is a parametrisation of γ's CSS.*

However, what if γ is a CCW? Well, then we can use the afore mentioned property of a CCW (equation (5)) and its derivatives in order to express our (X, Y) in a simpler form, i.e. the following results allow us to find a parametrisation for the CSS on a CCW:

$$\begin{aligned} h(t+\pi) &= k - h(t) && (8) \\ h'(t+\pi) &= -h'(t) && (9) \\ h''(t+\pi) &= -h''(t). && (10) \end{aligned}$$

Again, these calculations are very complicated, so using Maple (see Appendix 2) we find that the CSS for a CCW can be parametrised by,

$$(X^*, Y^*) = (-h'(t) \sin t - h''(t) \cos t, h'(t) \cos t - h''(t) \sin t)$$

but this is a a special case.

When γ is a CCW, our chords are common normals to the tangents at points $(x(t), y(t))$ and $(x(t+\pi), y(t+\pi))$ on γ. From this we can conclude that the envelope of chords joining parallel tangent points is equal to the envelope of γ's normals. Hence (X^*, Y^*) is a parametrisation of γ's evolute.

(Note that for the CSS, we only need values of t between 0 and π since taking values of t between π and 2π will cover the common normals twice, i.e. if we observe the envelope of chords for $t \in [0, 2\pi)$, then we will see a double cover of the CSS. Therefore the evolute of γ is equal to the double cover of the CSS).

Proposition 2.2 *(X^*, Y^*) is the parametrisation of both the CSS and the evolute of a CCW.*

It seems only appropriate that we should also find the evolute of γ when it is not necessarily a CCW. The family of normals to γ can be represented by,

$$G(t, A, B) = (A - x(t)) \sin t - (B - y(t)) \cos t$$

and differentiating this with respect to t gives us,

$$\frac{\partial G}{\partial t}(t, A, B) = \big(h'' \sin t + B - h' \cos t\big) \sin t + \big(A + h' \sin t + h'' \cos t\big) \cos t.$$

Solving $G = \frac{\partial G}{\partial t} = 0$ gives us,

$$(A, B) = (-h'(t) \sin t - h''(t) \cos t, h'(t) \cos t - h''(t) \sin t)$$

where (A, B) is a parametrisation of γ's evolute and notice that this is identical to (X^*, Y^*) above, where γ was of constant width!

Proposition 2.3 *The evolute of a smooth, convex curve with support function $h(t)$ can be parametrised by (X^*, Y^*).*

Theorem 2.3 *Points (X^*, Y^*) are singular $\iff$ $h'(t) + h'''(t) = 0$.*

Proof. By defintion these points are singular if $\frac{d}{dt}(X^*) = \frac{d}{dt}(Y^*) = 0$ where,

$$\frac{d}{dt}(X^*) = -h''(t) \sin t - h'(t) \cos t - h'''(t) \cos t + h''(t) \sin t$$

$$\frac{d}{dt}(Y^*) = -h'''(t) \sin t - h''(t) \cos t - h''(t) \cos t - h'(t) \sin t$$

and its easy to see that by cancelling terms and taking out a common factor that,

$$\left(\frac{d}{dt}(X^*), \frac{d}{dt}(Y^*)\right) = -\big(h'(t) + h'''(t)\big)(\cos t, \sin t).$$

Well, since $\sin t$ and $\cos t$ never equal 0 simultaneously, we have that the condition for singular points on the CSS and evolute to our CCW or the evolute of any smooth, convex curve with support function $h(t)$ is that,

$$h'(t) + h'''(t) = 0.$$

∎

3 Examples of Curves of Constant Width & their CSSs

3.1 Example 1

In considering an example of a curve of constant width, we said that we require the support function $h(t)$ to satisfy the condition $h(t) + h(t + \pi) = k$ where k is a constant, an example of which is,

$$h(t) = a \cos^2 \left(\frac{3t}{2}\right) + b.$$

This can also be expressed in the form,

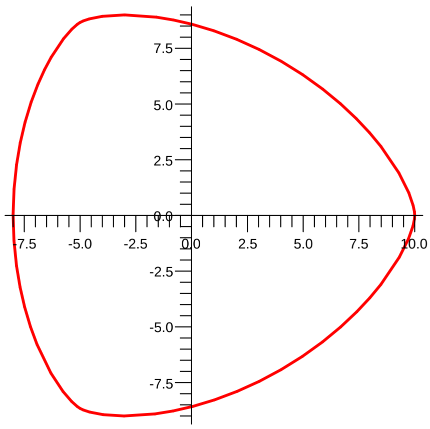
$$h(t) = \frac{a}{2} \cos 3t + \frac{a}{2} + b$$

using the trigonometric identity $\cos 2A = 2\cos^2 A - 1$. Substituting this into our parametrisation of γ gives us a CCW with parametric equations,

$$x = \left(\frac{a}{2} \cos 3t + \frac{a}{2} + b\right) \cos t + \frac{3a}{2} \sin 3t \sin t$$

$$y = \left(\frac{a}{2} \cos 3t + \frac{a}{2} + b\right) \sin t - \frac{3a}{2} \sin 3t \cos t.$$

Let us now consider values of a and b so that we might plot γ, say $a = 2, b = 8$.



Notice here that the width of the curve is equal to 18 (the curve is symmetrical about the x-axis and we can see that the intersection points on the axis are a distance of 18 apart) and this is equal to twice the sum of constants in our support function $h(t)$ (that is $2\left(\frac{a}{2} + b\right) = 2(1+8)$ for our chosen values). We observe that it has 6 vertices.

Theorem 3.1 *The CCW with support function of the form $h(t) = (\sum P_i \cos Q_i t) + R$, where $P > 0$ and R are constants and Q is some odd integer, is symmetrical about the horizontal axis.*

Proof. For $h(t)$ as defined, our CCW would have parameters,

$$x(t) = \left[\left(\sum P_i \cos Q_i t\right) + R\right] \cos t - \left(-\sum P_i Q_i \sin Q_i t\right) \sin t$$

$$y(t) = \left[\left(\sum P_i \cos Q_i t\right) + R\right] \sin t + \left(-\sum P_i Q_i \sin Q_i t\right) \cos t$$

here substituting our $h(t)$ into equations (2) and (3) and expanding our brackets it follows that,

$$x(t) = \sum P_i \cos Q_i t \cos t + R \cos t + \sum P_i Q_i \sin Q_i t \sin t$$

$$y(t) = \sum P_i \cos Q_i t \sin t + R \sin t - \sum P_i Q_i \sin Q_i t \cos t$$

are the points on our CCW above the horizontal axis (assume $t > 0$). Now let $t = -t$ (such that we are observing points below the horizontal axis) in these equations, then

$$x(-t) = \sum P_i \cos Q_i t \cos t + R \cos t + \sum P_i Q_i \sin Q_i t \sin t$$

$$y(-t) = -\sum P_i \cos Q_i t \sin t - R \sin t + \sum P_i Q_i \sin Q_i t \cos t$$

using $\cos(-t) = \cos t$ and $\sin(-t) = \sin t$ we can concude that $x(-t) = x(t)$ whereas $y(-t) = y(t)$ so our CCW is symmetrical about the horizontal axis. ∎

The corresponding envelope of chords joining points of contact on parallel tangents (the CSS) can be seen in Figure 3 (the CSS being the new curve which appears to emerge). This time I have removed the axes to make the CSS as clear as possible (see Appendix 3). We also observe that it has 3 cusps.

Theorem 3.2 *The CSS to the curve with support function $h(t) = \cos 3t + 9$ is a standard deltoid.*

Proof. Consider the parmetrisation of a standard deltoid,

$$(C, D) = (6 \cos t + 3 \cos 2t, 6 \sin t - 3 \sin 2t)$$

for values of $t \in [0, 2\pi]$ and the parametrisation of our CSS,

$$(E, F) = (3 \sin 3t \sin t + 9 \cos 3t \cos t, -3 \sin 3t \cos t + 9 \cos 3t \sin t)$$

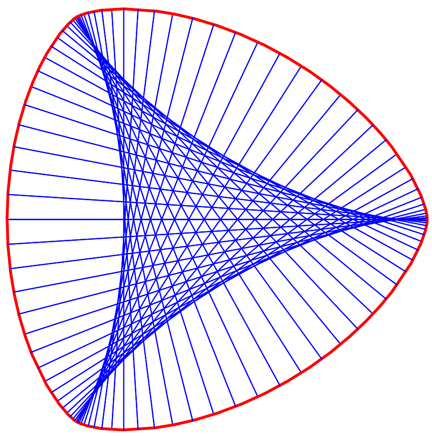


Figure 3: CCW & its CSS defined by $h(t) = \cos 3t + 9$.

for $0 \leq t \leq \pi$ (calculated using (X^*, Y^*) from previously). Since we are taking t over half the range in the case of our CSS, consider the change of variable $u = 2t$ (such that $0 \leq u \leq 2\pi$). Substituting $t = \frac{u}{2}$ into (E, F) gives,

$$\left(3\sin\left(\frac{3u}{2}\right)\sin\left(\frac{u}{2}\right) + 9\cos\left(\frac{3u}{2}\right)\cos\left(\frac{u}{2}\right), -3\sin\left(\frac{3u}{2}\right)\cos\left(\frac{u}{2}\right) + 9\cos\left(\frac{3u}{2}\right)\sin\left(\frac{u}{2}\right)\right).$$

Now let us use the formulae,

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{11}$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{12}$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{13}$$

in (E, F). Using (11) and (12) in E and (13) in F we can simplify (E, F) down to the following,

$$\left(\frac{3}{2}(\cos u - \cos 2u) + \frac{9}{2}(\cos u + \cos 2u), -\frac{3}{2}(\sin 2u + \sin u) + \frac{9}{2}(\sin 2u - \sin u)\right)$$

and so we can see that,

$$(E, F) = (6\cos u + 3\cos 2u, 3\sin 2u - 6\sin u).$$

This is the same as our deltoid parametrisation (C, D) (except that $F = -D$, but this is not a problem, since our CSS (and deltoid) are both symmetrical in the horizontal axis.)

3.2 Example 2

Another example of a support function satisfying our condition for a curve of CCW would be,

$$h(t) = a\cos^2\left(\frac{5t}{2}\right) + b.$$

This can also be expressed as,

$$h(t) = \frac{a}{2}\cos 5t + \frac{a}{2} + b$$

and so perhaps it becomes clear that any support function of the form $h(t) = P\cos Qt + R$ where Q is an odd integer ≥ 3 will give a non-circular CCW, but it is natural to ask why this is the case.

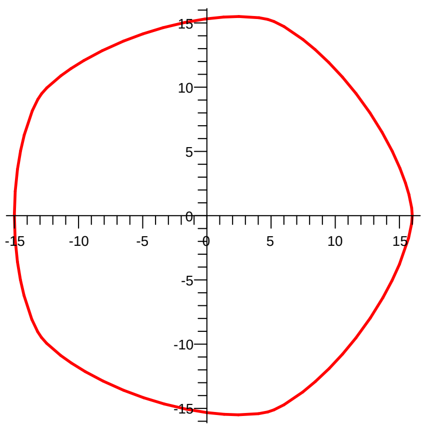
Well, Q has to be odd because, if Q were even then $h(t) + h(t+\pi) \neq k$, where k is a constant, i.e. we would not have a CCW. Also, Q must be ≥ 3 (not 1) since if Q were 1, then γ would be a circle.

Substituting $h(t)$ into our parametrisation of γ (equations (2) and (3)), gives us the parametric equations,

$$x = \left(\frac{a}{2}\cos 5t + \frac{a}{2} + b\right)\cos t + \frac{5a}{2}\sin 5t \sin t$$

$$y = \left(\frac{a}{2}\cos 5t + \frac{a}{2} + b\right)\sin t - \frac{5a}{2}\sin 5t \cos t.$$

Consider values of a and b so that we might plot γ, say for $a = 1$ and $b = 15$,



noticing that the width is 31 and it has 10 vertices. The corresponding CSS can be seen in Figure 4 and it has 5 cusps. So it would appear that a smooth curve γ with support function $h(t) = P\cos Qt + R$ has width $2R$, $2Q$ vertices and the CSS has Q cusps.

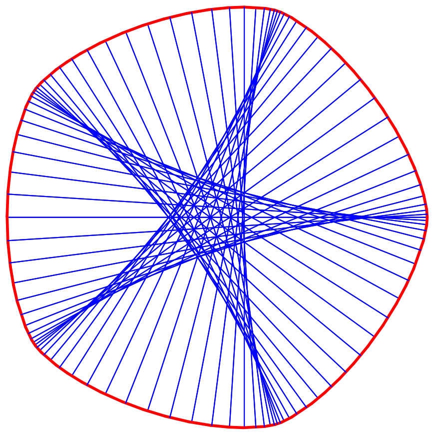


Figure 4: CCW & its CSS defined by $h(t) = \frac{1}{2}\cos 5t + \frac{31}{2}$.

Theorem 3.3 *A CCW with support function of the form $h(t) = (\sum P_i \cos Q_i t) + R$, where $P > 0$ and R are constants and Q is some odd integer, has width $w = 2R$*

Proof. The width of our CCW is, by definition,

$$w = h(t) + h(t + \pi)$$

from previously. Well substituting in our $h(t)$ and $h(t + \pi)$ this becomes,

$$w = \left(\sum P_i \cos Q_i t\right) + R + \left(\sum P_i \cos Q_i (t + \pi)\right) + R$$

but we know that $\cos(t + \pi) = -\cos t$ so in fact,

$$w = \left(\sum P \cos Q_i t\right) + R - \left(\sum P_i \cos Q_i t\right) + R = 2R$$

∎

Theorem 3.4 *A smooth CCW with support function of the form $h(t) = P\cos Qt + R$, where $P > 0$ and R are constants and Q is an odd integer ≥ 3, has $2Q$ vertices.*

Proof. If our curve has $2Q$ vertices, then this is equivalent to $\kappa'(t)$ having $2Q$ zeros where, from the proof of Theorem 5.2, we have that,

$$\rho(t) = h(t) + h''(t)$$

which implies,

$$\kappa(t) = \frac{1}{h(t) + h''(t)}.$$

Note that the denominator here does not equal 0, by Theorem 2.3 (so $\kappa \neq 0$). So, we want to find the zeros of $\kappa'(t)$, but we observe that, dropping t from our notation,

$$\frac{d}{dt}\left(\frac{1}{\kappa}\right) = -\frac{1}{\kappa^2}\kappa'$$

or equivalently,

$$\rho' = -\frac{1}{\kappa^2}\kappa'$$

($\kappa \neq 0$ so we have divide by it). Rearranging this to make κ' the subject,

$$\kappa' = -\kappa^2 \rho'$$

and therefore κ' has zeros $\iff$ ρ' has zeros. Substituting h and its derivatives into $\rho = h + h''$ gives,

$$\rho = P\cos Qt + R - PQ^2\cos Qt$$

differentiating with respect to t throughtout provides,

$$\rho' = -PQ\sin Qt + PQ^3 \sin Qt$$

and factorising we find that,

$$\rho' = PQ(Q+1)(Q-1)\sin Qt.$$

Therefore, zeros of ρ' occur at $PQ(Q+1)(Q-1) = 0$ or $\sin Qt = 0$, but the former is not the case since $Q \neq 1 or -1$ (since $Q \geq 3$) and $P \neq 0$ (since $P > 0$).

Thus, our CCW has vertices when $\sin Qt = 0$, where $0 \leq t < 2\pi$ (and so $0 \leq Qt < 2Q\pi$). Therefore, since $\sin t$ has 2 zeros in the interval $[0, 2\pi)$, it follows that $\sin Qt$ has $2Q$ zeros in the interval $[0, 2Q\pi)$. Hence, our CCW has $2Q$ vertices. ∎

3.3 Example 3

Our final example is particularly interesting, consider the CCW with support function,

$$h(t) = a\cos 3t + b\cos 5t + c\sin 3t + d.$$

Substituting $h(t)$ into equations (2) and (3) gives us the parameters for γ to be,

$$x = (a\cos 3t + b\cos 5t + c\sin 3t + d)\cos t - (-3a\sin 3t - 5b\sin 5t + 3c\cos 3t)\sin t$$

$$y = (a\cos 3t + b\cos 5t + c\sin 3t + d)\sin t + (-3a\sin 3t - 5b\sin 5t + 3c\cos 3t)\cos t$$

whilst our CSS can be parametrised by,

$$X^* = -(-3a\sin 3t - 5b\sin 5t + 3c\cos 3t)\sin t - (-9a\cos 3t - 25b\cos 5t - 9c\sin 3t)\cos t$$

$$Y^* = (-3a\sin 3t - 5b\sin 5t + 3c\cos 3t)\cos t - (-9a\cos 3t - 25b\cos 5t - 9c\sin 3t)\sin t.$$

Now, let us consider the graph of γ and its CSS for the values $a = 1, b = 1, c = 10$ and $d = 110$ as shown in Figure 5.

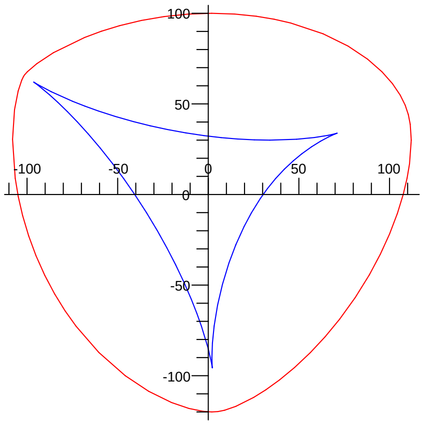


Figure 5: CCW & its CSS defined by $h(t) = \cos 3t + \cos 5t + 10\sin 3t + 110$.

Theorem 3.5 *Our CCW with support function* $h(t) = \cos 3t + \cos 5t + 10\sin 3t + 110$ *is asymmetrical.*

Proof. To prove this, we must first examine the curvature of our CCW (as defined in the proof to Theorem 3.4),

$$\kappa = \frac{1}{-8\cos 3t - 24\cos 5t - 80\sin 3t + 100}$$

and, using Maple (see Appendix 4), we can draw the graph of κ (vertical axis) against t (see Figure 6).

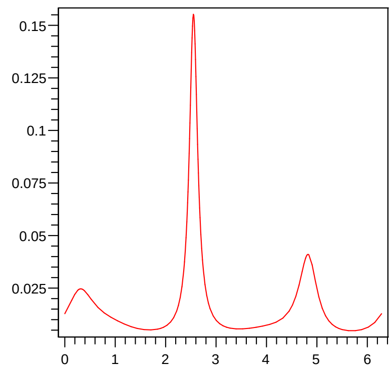


Figure 6: Curvature of CCW defined by $h(t) = \frac{1}{2}\cos 5t + \frac{31}{2}$.

Then we use the fact that vertices are maxima of curvature, i.e. $\kappa = \kappa' = 0$ so our values of t for which our CCW has vertices will be the values of t for which we have peaks on our graph. Using Maple (see Appendix 4) we find these to be,

$$t_1 = 0.3168141150, t_2 = 2.552446218, t_3 = 4.824267340$$

and, by substituting these into our (x, y) we find that our corresponding vertices on our CCW are the points,

$$\begin{aligned} v_1 &= (109.6820353, 46.48287336) && (14) \\ v_2 &= (-101.6689180, 65.60666933) && (15) \\ v_3 &= (4.934800624, -119.8416015) && (16) \end{aligned}$$

respectively. One of the features of an asymmetrical curve (no reflectional or rotational symmetry) with 3 vertices (ordinary) is that the triangle joining them will be scalene, that is the 3 sides will be of different length. We can calculate these lengths using Pythagoras' theorem and we find them to be,

$$l_1 = 213.9051518, l_2 = 196.5599505, l_3 = 212.2143846$$

and therefore our CCW is asymmetrical. ∎

4 Algebraic Expressions

In our section on support functions, we found a parametric expression for our curve γ (we called this $(x(t), y(t))$) and also for the CSS for γ of constant width (which we called (X^*, Y^*)). Now we would like to determine whether the CCW and CSS are algebraic curves.

4.1 Example 4

Take for example the curve we constructed from the support function in Example 1. Our parameters of γ in that case were,

$$x = \left(\frac{a}{2}\cos 3t + \frac{a}{2} + b\right)\cos t + \frac{3a}{2}\sin 3t \sin t$$

$$y = \left(\frac{a}{2}\cos 3t + \frac{a}{2} + b\right)\sin t - \frac{3a}{2}\sin 3t \cos t,$$

so in order to find an algebraic expression, we need to eliminate our parameter t from these equations. This process quickly becomes very complicated, but it can be done with the help of Maple (see Appendix 5). First of all, simplifying our (x, y) above using Maple's knowledge of trigonometrical identites gives,

$$x = -4a\cos^4 t + 6a\cos^2 t + \frac{a}{2}\cos t + b\cos t - \frac{3a}{2}$$

$$y = -\frac{1}{2}\sin t\left(8a\cos^3 t - a - 2b\right)$$

and we notice that here x is expressed completely in terms of $\cos t$ but that we have a $\sin t$ term in y. In order to eliminate this $\sin t$ (we want x, y both expressed in terms of the same variable), we square y, thus creating a $\sin^2 t$ term which can be replaced by $1 - \cos^2 t$ and the result is,

$$y^2 = \frac{1}{4}\left(1 - \cos^2 t\right)\left(64a^2\cos^6 t - 32ab\cos^3 t - 16a^2\cos^3 t + a^2 + 4ab + 4b^2\right).$$

So we have 2 equations both expressed in terms of the same variable $\cos t$, let this equal a new variable C, consequently,

$$x = -4aC^4 + 6aC^2 + \frac{a}{2}C + bC - \frac{3a}{2}$$

$$y^2 = \frac{1}{4}(1 - C^2)\left(64a^2C^6 - 32abC^3 - 16a^2C^3 + a^2 + 4ab + 4b^2\right).$$

If we now eliminate C from these equations (using Maple, see Appendix 5) and use the same values of a and b as in Example 1, we find that our algebraic equation for our curve γ (of constant width) is,

$x^8+16x^7+19x^6+4(x^6y^2+x^2y^6)-5544x^5-16x^5y^2-41283(x^4+y^4)-519x^4y^2+6y^4x^4+266382x^3-80x^3y^4+11088x^3y^2+7950960x^2+441x^2y^4+16632y^4x-82566x^2y^2-48y^6x-799146y^2x+7950960y^2-45y^6+y^8=373248000$

and we notice that this curve is of degree 8.

Using a similar approach, we find that the algebraic expression for the corresponding CSS, previously parametrised by (X^*, Y^*) where,

$$X^* = \frac{3a}{2}\sin 3t \sin t + \frac{9a}{2}\cos 3t \cos t$$

$$Y^* = -\frac{3a}{2}\sin 3t \cos t + \frac{9a}{2}\cos 3t \sin t$$

is given by,

$$X^{*4} - 24X^{*3} + 162X^{*2} + 2X^{*2}Y^{*2} + 72X^*Y^{*2} + 162Y^{*2} + Y^{*4} = 2187$$

and we notice that this curve is of degree 4.

In fact, through the observation of other examples, we conjecture that generally, for any support function of the form $h(t) = P\cos Qt + R$, where P and R are some constants and Q is some odd integer ≥ 3, that its curve γ will have degree $2\beta + 2$ and the corresponding CSS has degree $\beta + 1$ (half that of the curve).

4.2 Example 5

An interesting question might be: are there other support functions, besides those of the form $h(t) = P\cos 3t + R$, which produce a CCW with 6 vertices (and hence a CSS with 3 cusps)? Well, let us consider the curve with support function,

$$h(t) = a\cos 3t + b\cos 5t + c$$

where b is small and our curve can be parametrised by (x, y) such that,

$$x = (a\cos 3t + b\cos 5t + c)\cos t - (-3a\sin 3t - 5b\sin 5t)\sin t$$

$$y = (a\cos 3t + b\cos 5t + c)\sin t + (-3a\sin 3t - 5b\sin 5t)\cos t$$

and the corresponding CSS can be parametrised by (X^*, Y^*) such that,

$$X^* = -(-3a\sin 3t - 5b\sin 5t)\sin t - (-9a\cos 3t - 25b\cos 5t)\cos 5t$$

$$Y^* = (-3a\sin 3t - 5b\sin 5t)\cos t - (-9a\cos 3t - 25b\cos 5t)\sin 5t.$$

Using a similar method to that of Example 4 we find that, by eliminating t from our equation and by letting $a = 1, b = 0.01$ and $c = 10$ we find that our curve is algebraic with equation,

$$\begin{aligned}
&-1173632.509x - 0.0401768067y^6x + 0.0232928689y^6x^2 + 0.02858369159x^4y^4 \\
&+ 3.117481181x^4y^2 - 58.37739244x^3y^2 - 0.03305188292y^2x^6 + 1.696502092x^3y^4 \\
&- 0.5258350304x^5y^2 + 116.4626578y^4x + 25.86689693y^4x^2 + 4752.134265y^2x \\
&- 1471.699139x^2y^2 - 1.786474297y^6 - 0.6566123013x^7 + 0.0004083423248y^2x^8 \\
&+ 0.000013309184y^2x^9 + 0.0004118065498xy^8 + 0.01697225998x^8 - 78.37006461x^5 \\
&+51371.41114x^2+1411.175166x^4+22533.25936x^3-15.10876437x^6+0.00664677896y^8 \\
&- 151.4573255y^4 + 0.0000001536y^{10}x^2 + 0.000000384y^4x^8 + 0.000000512y^6x^6 \\
&+ 0.000000384x^4y^8 + 0.000237166864y^6x^4 + 0.000101015864y^8x^2 \\
&+ 0.000022987264y^4x^7 + 0.000019245568y^6x^5 + 0.00000769664y^8x^3 \\
&+ 0.00000114304y^{10}x + 0.0004329471456y^4x^6 + 0.004011831741y^2x^7 \\
&+ 0.001954836976y^4x^5 + 0.0000001536x^{10}y^2 + 0.001595414176y^6x^3 \\
&+ 99874.04907y^2 + 0.000029166984y^{10} + 0.003066641175x^9 + 0.0001408326026x^{10} \\
&+ 0.000003013888x^{11} + 0.0000000256y^{12} + 0.0000000256x^{12} = 7237792.1
\end{aligned}$$

and we can see that this is a curve of degree 12 whilst our CSS has equation,

$$\begin{aligned}
&130.5789041X^* - 0.0352734375X^{*4} + 0.0004X^{*6} - 107.3467125X^* \\
&+ 0.02493X^{*5} + 128.8374885Y^{*2} + 64.8856125X^*Y^{*2} + 4.445203125X^{*2}Y^{*2} \\
&+ 0.9272765625X^{*4} + 0.06705X^*Y^{*4} + 0.0963Y^{*2}X^{*3} + 0.0012\,Y^{*2}X^{*4} \\
&+ 0.0004Y^{*6} + 0.0012Y^{*4}X^{*2} - 12.74613750X^{*3} = 1772.281877
\end{aligned}$$

and it is clear that this has degree 6 (half that of our curve). Now consider the graph of these algebraic curves as shown in Figure 7.

We can see that this curve has 6 vertices and the CSS has 3 cusps, just as in the case of $h(t) = P\cos 3t + R$. This despite the fact that we had a curve of degree 12 and CSS of degree 6. We conclude that there does not appear to be a consistent link between the degree of our CCW and the number of vertices exhibited by that curve or, the degree of our CSS and the number of associated cusps.

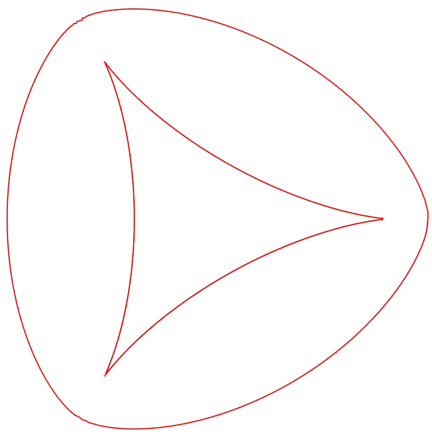


Figure 7: Algebraic plots of CCW & its CSS defined by $h(t) = \cos 3t + 0.01 \cos 5t + 10$.

4.3 More General Case?

Let us suppose that we have a support function of the form,

$$h(t) = a\cos 3t + b\sin 3t + c\cos t + d\sin t + e. \tag{17}$$

This would appear to be a more general case of example 1 (where effectively we had $h(t) = M\cos 3t + N$, where M and N are constants), however we will now show that it is not. Consider our curve γ constructed as in our section on the support function where we said that for the support line $l(t)$ we had,

$$x\cos t + y\sin t = h(t).$$

What if we now translated the curve γ by a vector, say (u, v) as shown?

Well, our equation for $l(t)$ would become,

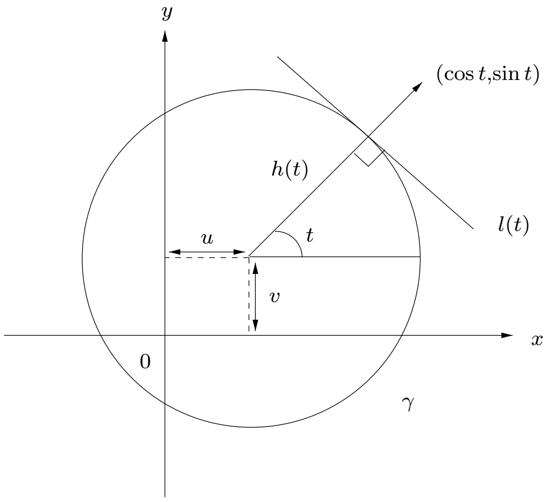
$$(x\cos t + y\sin t) + (u\cos t + v\sin t) = h(t)$$

or equivalently,

$$(x\cos t + y\sin t) = h(t) - (u\cos t + v\sin t)\,.$$

So translating our curve γ by a vector (u, v) has the effect of replacing any support function $h(t)$ by $h(t) - u\cos t - v\sin t$. Yet we are still looking at the same curve γ (we've just "moved" it) and therefore it has the same CSS. Thus, we can conclude that adding (or subtracting) $\sin t$ and $\cos t$ terms to our original $h(t) = M\cos 3t + N$ (see Example 1) will not affect the curve or its CSS. Hence we can reduce our "more general" case in equation (17) to,

$$h(t) = a\cos 3t + b\sin 3t + e. \tag{18}$$



Now let us see what happens if we rotate the curve γ. We would like to show at this point that, given a and b as follows, then there exists $A > 0$ and α as follows, such that the equation below is satisfied,

$$a \cos 3t + b \sin 3t = A \cos 3(t + \alpha)$$

(since we are trying to show that (18) can be simplified to $h(t) = M \cos 3t + N$ as claimed at the start of this section). Well, let us expand the RHS of the previous equation,

$$a \cos 3t + b \sin 3t = A [\cos 3t \sin 3\alpha - \sin 3t \cos 3\alpha]$$

and, by comparing the coefficients of $\cos 3t$ and $\sin 3t$ on either side of this equation, we find that,

$$a = A \sin 3\alpha \tag{19}$$

$$b = -A \cos 3\alpha \tag{20}$$

By squaring these 2 equations and then adding them together, we can find our expression for A,

$$a^2 + b^2 = A^2$$

which implies that $A = \sqrt{a^2 + b^2}$. If we now substiute this value for A back into our simultaneous equations above then we have,

$$a = \sqrt{a^2 + b^2} \sin 3\alpha \tag{21}$$

$$b = -\sqrt{a^2 + b^2} \cos 3\alpha \tag{22}$$

and these enable us to define our angle α in terms of a and b,

$$\sin 3\alpha = \frac{a}{\sqrt{a^2 + b^2}}$$

$$\cos 3\alpha = \frac{-b}{\sqrt{a^2 + b^2}}$$

(we always define an angle in terms of its sine and cosine). In conclusion we have shown that, given a and b there exists $A > 0$ and α such that $a \cos 3t + b \sin 3t = A \cos 3(t + \alpha)$ so equation (18) can be simplified to,

$$h(t) = A \cos 3t + e. \tag{23}$$

for rotated axes.

5 Barbier's Theorem

My main reference for this section is [F].

Theorem 5.1 (Barbier) *For a closed convex curve of constant width $w \geq 0$, its perimeter is equal to πw.*

Proof. Let our curve γ be parametrised by $(x, y) = (h(t)\cos t - h'(t)\sin t, h(t)\sin t + h'(t)\cos t)$ where $h(t)$ represents our support function and $'$ represents differentiation with respect to t. Consider the arc length s defined by,

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.dt$$

for our case where, from now on we shall write h in place of $h(t)$, h' instead of $h'(t)$ and so on. We can easily find the following terms,

$$\left(\frac{dx}{dt}\right)^2 = \left(h^2 + 2hh'' + h''^2\right)\sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = \left(h^2 + 2hh'' + h''^2\right)\cos^2 t$$

which when substituted into our equation for s gives,

$$s = \int_0^{2\pi} \sqrt{\left(h^2 + 2hh'' + h''^2\right)}.dt = \int_0^{2\pi} \sqrt{(h + h'')^2}.dt = \int_0^{2\pi} \left(h + h''\right).dt$$

noting that arc length must be positive. At this point we can use the fact that, if our curve is of constant width w, then the following poperties hold;

$$h(t) + h(t + \pi) = w \tag{24}$$

$$h'(t) + h'(t + \pi) = 0 \tag{25}$$

$$h''(t) + h''(t + \pi) = 0. \tag{26}$$

In order to use these, we split the interval of our integral for s into $(0, \pi)$ and $(\pi, 2\pi)$ as follows,

$$s = \int_0^{\pi} \left(h + h''\right).dt + \int_{\pi}^{2\pi} \left(h + h''\right).dt$$

and looking at our second integral, we would like to use a change of variable, letting $u = t - \pi$. Together with the corresponding change of limits this gives us,

$$s = \int_0^{\pi} \left(h(t) + h''(t)\right).dt + \int_0^{\pi} \left(h(u + \pi) + h''(u + \pi)\right).du$$

where $'$ here denotes derivatives of the respective variables t and u. Now we want to use our properties above (for variable u rather than t) and rearrange such that the following conditions hold;

$$h(u+\pi) = w - h(u) \tag{27}$$
$$h''(u+\pi) = -h''(u). \tag{28}$$

Substituting these into our integral gives us,

$$s = \int_0^\pi \big(h(t) + h''(t)\big)\,.dt + \int_0^\pi \big(w - h(u) - h''(u)\big)\,.du$$

and remember that t and u are just variables so we can combine the 2 integrals, perhaps letting $t = u = z$ for example, so we have that

$$s = \int_0^\pi \big(h(z) + h''(z) + w - h(z) - h''(z)\big)\,.dz = \pi w.$$

∎

Another interesting theorem relating to a CCW is Theorem 5.2.

Theorem 5.2 *The radii of curvature at opposite points $(x(t), y(t))$ and $(x(t+\pi), y(t+\pi))$ have a constant sum equal to w (w as before).*

Proof. As before, let our plane curve γ be a CCW parametrised by,

$$(x(t), y(t)) = (h(t)\cos t - h'(t)\sin t, h(t)\sin t + h'(t)\cos t)$$

where $h(t)$ represents our support function and $'$ represents differentiation with respect to t. It follows that,

$$(x(t+\pi), y(t+\pi)) = (-h(t+\pi)\cos t + h'(t+\pi)\sin t, -h(t+\pi)\sin t - h'(t+\pi)\cos t)$$

and we would like to find the curvature at these points. We can define the curvature κ at a point on the γ, with arc length s, by the formula,

$$\kappa = \frac{dt}{ds}.$$

Then, since the radius of curvature ρ is the inverse of κ we have that,

$$\rho = \frac{ds}{dt} = \left|\frac{d\gamma}{dt}\right|.$$

So for the radius of curvature at the point $(x(t), y(t))$, we have

$$\rho(t) = \sqrt{(h' \cos t - h \sin t - h'' \sin t - h' \cos t)^2 + (h' \sin t + h \cos t + h'' \cos t - h' \sin t)^2}.$$

where $h = h(t)$. We can see that terms will cancel here leaving,

$$\rho(t) = \sqrt{(-h \sin t - h'' \sin t)^2 + (h \cos t + h'' \cos t)^2}$$

which we can then factorise as follows,

$$\rho(t) = \sqrt{[-(h + h'')(\sin t)]^2 + [(h + h'')(\cos t)]^2}$$

and this simplifies to,

$$\rho(t) = \sqrt{(h + h'')^2(\sin^2 t + \cos^2 t)} = h + h''.$$

Now, in a similar way, let us consider the radius of curvature at the point $(x(t + \pi), y(t + \pi))$, which after cancellation is seen to be,

$$\rho(t + \pi) = \sqrt{(h(t + \pi) \sin t + h''(t + \pi) \sin t)^2 + (-h(t + \pi) \cos t - h''(t + \pi) \cos t)^2}.$$

Gathering terms together in the same way as we did previously we find that,

$$\rho(t + \pi) = \sqrt{[(h_\pi + h''_\pi) \sin t]^2 + [-(h_\pi + h''_\pi) \cos t]^2} = h_\pi + h''_\pi.$$

Therefore, the sum of the radii of curvature at opposite points have sum,

$$\rho(t) + \rho(t + \pi) = h(t) + h''(t) + h(t + \pi) + h''(t + \pi)$$

and using our properties for a CCW (i.e. $h(t)+h(t+\pi) = w$ and $h''(t)+h''(t+\pi) = 0$), this becomes,

$$\rho(t) + \rho(t + \pi) = w.$$

∎

6 Morse's Lemma

My main reference for this section is [BG].

Say we have a closed, non-convex curve Γ, then we can pick out parallel tangents with the function,

$$h(t_1, t_2) = T(t_1).N(t_2)$$

where $T(t_i)$ is the tangent at $\Gamma(t_i)$ and $N(t_i)$ is the normal at $\Gamma(t_i)$ for $i = 1$ or 2. We use the fact that $h(t_1, t_2) = 0$ if the tangent at $\Gamma(t_1)$ is perpendicular to the normal at $\Gamma(t_2)$ (and therefore the respective tangents are parallel). Clearly, a trivial solution would arise if $t_1 = t_2$ since

$$h(t_1, t_2) = T(t_1).N(t_1) = 0$$

by definition. We want to find the parameter of points on Γ where the tangents are parallel (either t_1 or t_2).

First consider the Implicit Function Theorem (taken from MATH443) and the Jacobian of $h = h(t_1, t_2)$ at the origin (which means $t_1 = t_2$) .

Theorem 6.1 (Implicit Function Theorem) *Let $\mathbb{R}^m, v \to \mathbb{R}^q, c$ $(m \geq q)$ be a smooth map (defined near $v \in \mathbb{R}^m$ such that $f(v) = c$) and suppose v is a regular point of f ($J(f)$ has rank q at v). There are q linearly independent columns of $J(f)$ at v (non-zero $q \times q$ minor).*

Then, for $mf^{-1}(c)$ close to v, we can express the variables corresponding to the q independent columns as smooth functions of those remaining.

Let us examine points near $(t_1, t_2) = (0, 0)$ using the Jacobian matrix,

$$J(h) = \left(\frac{\partial h}{\partial t_1}, \frac{\partial h}{\partial t_2}\right)$$

which gives us that,

$$J(0, 0) = (0, 0)$$

so J is singular, thus we cannot use the Implicit Function Theorem (IFT) and we resort to the Morse lemma. Instead of using the Jacobian matrix, we use the Hessian matrix for our function $h(t_1, t_2) = T(t_1).N(t_2)$.

Definition

A Morse function $f(t_1, t_2)$ mapped by $f : \mathbb{R}^2 \to \mathbb{R}$ is one for which $\frac{\partial f}{\partial t_1} = \frac{\partial f}{\partial t_2} = 0$, i.e. f is singular at (t_1, t_2) and the Hessian matrix,

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial t_1^2} & \frac{\partial^2 f}{\partial t_1 \partial t_2} \\ \frac{\partial^2 f}{\partial t_1 \partial t_2} & \frac{\partial^2 f}{\partial t_2^2} \end{pmatrix}$$

is non-singular at (t_1, t_2), i.e.

$$|H(f)| = \begin{vmatrix} \frac{\partial^2 f}{\partial t_1^2} & \frac{\partial^2 f}{\partial t_1 \partial t_2} \\ \frac{\partial^2 f}{\partial t_1 \partial t_2} & \frac{\partial^2 f}{\partial t_2^2} \end{vmatrix} \neq 0.$$

Lemma 6.1 (Morse) *If $f(t_1, t_2)$ is a Morse function at $(t_1, t_2) = (0, 0)$ then, $f^{-1}(0)$ is locally diffeomorphic to,*

(M1) $\{(t_1, t_2) : t_1^2 - t_2^2 = 0\}$ *OR*

(M2) $\{(t_1, t_2) : t_1^2 + t_2^2 = 0\}$.

Since the map is $f : \mathbb{R}^2, (0, 0) \to \mathbb{R}, 0$, then we can see that $(M1)$ and $(M2)$ do not allow complex solutions. Thus $(M1)$ corresponds to the graph of $(t_1+t_2)(t_1-t_2) = 0$ which implies that,

$$t_1 = t_2 \text{ or } t_1 = -t_2$$

and so the graph would look like Figure 8,

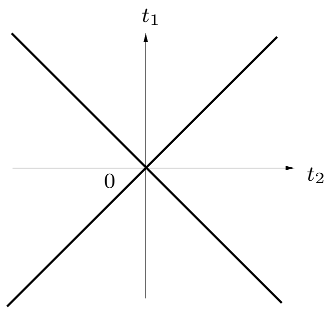


Figure 8: Graph for (M1)

whilst the only real solution of $(M2)$ is

$$t_1 = t_2 = 0$$

since t_1^2 and t_2^2 must both be ≥ 0, therefore its graph is as shown in Figure 9.

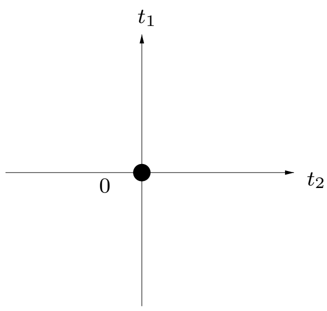


Figure 9: Graph for (M2)

At $(t_1, t_2) = (0, 0)$ we have that $t_1 = t_2$, so $h(t_1, t_2) = 0$ implying parallel tangents (from previous). So if $|H(h)| \neq 0$ and $\frac{\partial h}{\partial t_1} = \frac{\partial h}{\partial t_2} = 0$ at (t_1, t_2) then we have that $h = h(t_1, t_2)$ is a Morse function and, if $h(t_1, t_2)$ is a Morse function, we can also use Morse's lemma to describe $h^{-1}(0)$.

6.1 When is $h(t_1, t_2)$ a Morse function?

For this we need to make the following calculations (assume Γ to be a unit speed curve for simplicity),

$$\frac{\partial h}{\partial t_1} = \kappa_1 N_1 . N_2 \tag{29}$$

$$\frac{\partial h}{\partial t_2} = -T_1 . \kappa_2 T_2 \tag{30}$$

$$\frac{\partial^2 h}{\partial t_1^2} = \left(\kappa_1' N_1 - \kappa_1^2 T_1\right) . N_2 \tag{31}$$

$$\frac{\partial^2 h}{\partial t_2^2} = -T_1 . \left(\kappa_2' T_2 + \kappa_2^2 N_1\right) \tag{32}$$

$$\frac{\partial^2 h}{\partial t_1 \partial t_2} = -\left(\kappa_1 N_1\right) . \kappa_2 T_1 \tag{33}$$

where κ_i' means the derivative of $\kappa(t_i)$ with respect to t_i for $i = 1$ or 2. Let us consider the example of $(t_1, t_2) = (0, 0)$ so that we might use Morse's Lemma.

$$\begin{aligned}
\frac{\partial h}{\partial t_1} &= \kappa(0)N(0).N(0) = \kappa(0) && (34)\\
\frac{\partial h}{\partial t_2} &= -T(0).\kappa(0)T(0) = -\kappa(0) && (35)\\
\frac{\partial^2 h}{\partial t_1^2} &= \left(\kappa'(0)N(0) - \kappa(0)^2 T(0)\right).N(0) = \kappa'(0) && (36)\\
\frac{\partial^2 h}{\partial t_2^2} &= -T(0).\left(\kappa'(0)T(0) + \kappa(0)^2 N(0)\right) = -\kappa'(0) && (37)\\
\frac{\partial^2 h}{\partial t_1 \partial t_2} &= -\left(\kappa(0)N(0)\right).\kappa(0)T(0) = 0 && (38)
\end{aligned}$$

and so since for a Morse function, we require $\frac{\partial h}{\partial t_1} = \frac{\partial h}{\partial t_2} = 0$, we can see that $\kappa(0) = -\kappa(0) = 0$, i.e. we have an inflexion at the origin. Our corresponding Hessian matrix would be,

$$H(0,0) = \begin{pmatrix} \kappa'(0) & 0 \\ 0 & -\kappa'(0) \end{pmatrix}$$

and we know that a Morse function requires this to be non-singular, i.e. we must have that $-\kappa'(0)^2 \neq 0$. In fact, if this is the case, then $\kappa'(0) \neq 0$ and so we have that $-\kappa'(0)^2 < 0$. This means that our condition for a Morse function when $(t_1, t_2) = (0, 0)$ is that,

$$\kappa(0) = 0 \ \&\ \kappa'(0) \neq 0$$

i.e. an ordinary inflexion at the origin. The parallel tangents of a curve with an ordinary inflexion at $(t_1, t_2) = (0, 0)$ would not only occur for our trivial solution $(t_1 = t_2)$, but also for $t_1 \neq t_2$ when near the origin (see Figure 10).

The set $h^{-1}(0)$ can be reprsented by Figure 11, i.e. $h^{-1}(0)$ has a local solution comprising a pair of 2 intersecting curves, one of which is the diagonal $t_1 = t_2$ (which is clearly smooth). We have found that $h^{-1}(0)$ is locally a pair of smooth transverse curves, using the Morse lemma; hence the true parallel tangent set is the other smooth branch of $h^{-1}(0)$ besides the diagonal.

Proposition 6.1 *In a neighbourhood of an ordinary inflexion, the parallel tangent set is a smooth curve.*

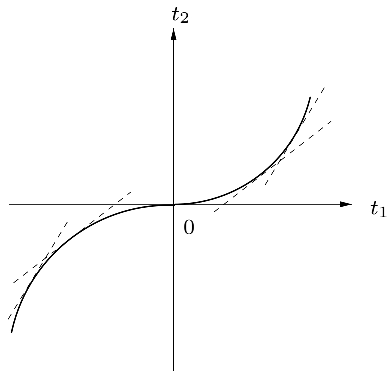


Figure 10: Parallel tangents of an ordinary inflexion.

This situation is illustrated in Figure 11.

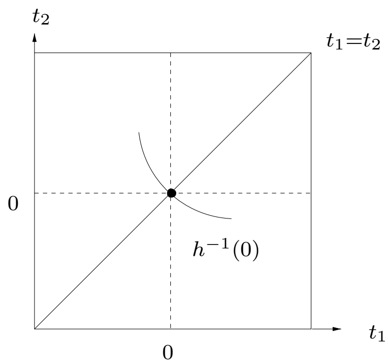


Figure 11: Local solutions for $h\,(t_1, t_2) = 0$.

7 Reconstruction of a curve from its CSS

We would like to find out if, given a curve, say A parametrised by $A(s) = (h(s), s)$ and another piece of curve, say B parametrised by $B(u) = (u, q + g(u))$, is it possible to find another piece of curve, say C parametrised by $C(t) = (t, p + f(t))$ such that B and C have A as their CSS?

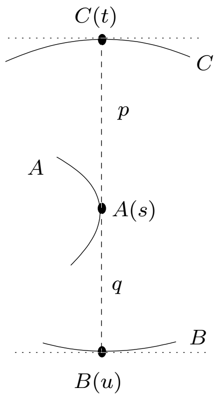


Figure 12: Curve A is the CSS to curves B and C here.

Here the dotted lines at the top and bottom of the diagram represent parallel tangents at $B(u)$ and $C(t)$ whilst the dashed line joining them is tangential to A. Also, the constants p and q are equal to the distances between the tangent points $C(t)$ and $B(u)$ respectively (represented by black circles on C and B) and the tangent point on A (represented by black circle on A). We have some conditions for our problem.

Firstly, we want $B(u)$ and $C(t)$ to lie on the tangent to $A(s)$ and secondly, we want $g'(u) = f'(s)$, i.e. we want the tangents to B and C to be parallel ($'$ here denotes differentiation with respect to the specified variable, as it will throughout). We would expect t and u to be functions of s since the points $B(u)$ and $C(t)$ lie on the tangent to the curve A, parametrised by the variable s.

Using our first condition we have,

$$(u, q + g(u)) = (h(s), s) + \mu \left(h'(s), 1\right) \tag{39}$$

$$(t, p + f(t)) = (h(s), s) + \lambda \left(h'(s), 1\right) \tag{40}$$

and we want all of the fuctions here (g, f and h) to be functions of s (variable of the CSS to curve B) so that, knowing only B and its CSS A, we can find C.

Well, we know that h is a function of s and, implicitly, we have that u is a function of s ($B(u)$ lies on the tangent to $A(s)$ since A is the CSS to B) and therefore $g = g(u)$ is a function of s. Now we want to find t, the variable of curve C, as a function of s, and thus $f = f\left(t\left(s\right)\right)$. So, rearranging equations (39) and (40) we find,

$$(u, q + g(u)) - (h(s), s) = \mu \left(h'(s), 1\right)$$

$$(t, p + f(t)) - (h(s), s) = \lambda \left(h'(s), 1\right)$$

and now, if we think of these in terms of (x, y) coordinates and we divide the x coordinates by the y coordinates then we obtain the following,

$$\frac{u - h(s)}{q + g(u) - s} = h'(s) \tag{41}$$

$$\frac{t - h(s)}{p + f(t) - s} = h'(s) \tag{42}$$

noticing that we have eliminated μ and λ. If we now consider equation (41) to be a family of chords (joining $B(u)$ to $A(s)$), say $G(s, u)$ where,

$$G(s, u) = u - h(s) - h'(s)\left(q + g(u) - s\right).$$

Since our model is affinely invariant, we can take an affine transformation, i.e. we can set up our model in such a way that we can take a point on A to be the origin (such that $A(s) = (h(s), s) = (0,0)$) and let the tangent to A at that point, parametrised by $(h'(s), 1)$, be the vertical axis (represented by the dashed line in Figure 12).

We can then let this vertical tangent intersect with curve B at the point where its parameter u also equals 0 and such that the tangent to B at this point, parametrised by $(1, g'(u))$, is a horizontal tangent i.e. we analyse our given curves A and B at the points with parameter value 0.

So now, let us consider the Jacobian matrix,

$$J = \left(\frac{\partial G}{\partial s}, \frac{\partial G}{\partial u}\right)$$

close to $(s, u) = (0, 0)$. We have that,

$$\frac{\partial G}{\partial u} = 1 - h'(s)g'(u)$$

and we have set up our model in such a way that $h(0) = h'(0) = 0$ (and $g'(0) = 0$) so then,

$$\frac{\partial G}{\partial u}(0, 0) = 1 \neq 0$$

and, by the Implicit Function Theorem (see Theorem 6.1), we can write u as a function of s and therefore $g = g(u(s))$. Now it only remains to prove that we can find t and therefore $f(t)$ in terms of s. We can rearrange equation (42) to make $f(t)$ the subject, the result of which is,

$$f(t) = \frac{t - h(s)}{h'(s)} - p + s.$$

We want to make use of our second condition (that $\frac{dg}{du} = \frac{df}{dt}$), so let us differentiate our equation throughout with respect to s giving,

$$\frac{df}{dt}t' = \frac{h'(s)\,(t' - h'(s)) - (t - h(s))\,h''(s)}{h'(s)^2} + 1$$

where $'$ here denotes $\frac{d}{ds}$ throughout. Now multiply everything by $h'(s)^2$ to get rid of our fraction and also leads to cancellations producing,

$$\frac{df}{dt}t'h'(s)^2 = h'(s)t' - (t - h(s))\,h''(s). \tag{43}$$

In a parallel fashion, we obtain, from equation (41),

$$\frac{dg}{du}u'h'(s)^2 = h'(s)u' - (u - h(s))\,h''(s)$$

and using our second condition, this becomes

$$\frac{df}{dt}u'h'(s)^2 = h'(s)u' - (u - h(s))\,h''(s). \tag{44}$$

Now, dividing (43) by (44) and cross-multiplying we have,

$$t'\left[h'(s)u' - (u - h(s))\,h''(s)\right] = u'\left[h'(s)t' - (t - h(s))\,h''(s)\right]$$

then, multiplying out we see that, after cancellations,

$$-t'uh''(s) + t'h(s)h''(s) = -u'th''(s) + u'h(s)h''(s).$$

Divide through by $h''(s)$ and take out common factors to find that,

$$(h(s) - u)\,\frac{dt}{ds} + u't = u'h(s) \tag{45}$$

i.e. here we have an equation for t, where t is the only unknown (doesn't involve $f(t)$) in terms of h and u which are both functions of the variable s, so we know that we can find t (and so $f = f(t)$) in terms of s.

Proposition 7.1 *Given a curve A and another piece of curve B, we can find another piece of curve C, such that B and C have A as their CSS.*

7.1 Example 6

Let our function $h(s) = s^2$, so that our CSS curve A is parametrised by,

$$A(s) = \left(s^2, s\right)$$

and similarly, let our function $g(u) = \alpha u^2$ so that our given piece of curve B is parametrised by,

$$B(u) = \left(u, q + \alpha u^2\right).$$

We don't know an explicit value for t but we do know that it can be expressed in terms of s, so express t as a power series in s up to degree 5,

$$t = t_1 s + t_2 s^2 + t_3 s^3 + t_4 s^4 + t_5 s^5$$

and similarly, let us describe f as a power series in t as follows,

$$f = f_2 t^2 + f_3 t^3 + f_4 t^4 + f_5 t^5$$

where t_i and f_i for $1 \leq i \leq 5$ are coefficients to be found. However, we have an equation for t in (45), but for this we need explicit expressions for u and $\frac{du}{ds}$. So, by rearranging equation (41), we have,

$$u - h(s) - h'(s)\left(q + g(u) - s\right) = 0$$

and, by substituting our values from $A(s)$ above into this, we find that s^2 terms cancel to give,

$$u - 2sq - 2s\alpha u^2 + s^2 = 0$$

i.e. we have a quadratic equation in u, so using the quadratic formula we have that,

$$u = \frac{1 \pm \sqrt{1 - 8s^2\alpha\left(2q - s\right)}}{4s\alpha}$$

and we want to choose the sign ($+$ or $-$) which gives a finite value for $u(0)$ when we express our square root term as a power series in s. It is clear that we must take u_-, since if we took u_+ our first term in the power series for $u(s)$ would be $\frac{1}{2s\alpha}$ which would be infinite when $s = 0$. Now, using Maple (see Appendix 6) we find u and $\frac{du}{ds}$ up to degree 5,

$$u = 2qs - s^2 + 8\alpha q^2 s^3 - 8\alpha q s^4 + \frac{2\alpha^2 + 64\alpha^3 q^3}{\alpha} s^5$$

and

$$\frac{du}{ds} = 2q - 2s + 24\alpha q^2 s^2 - 32\alpha q s^3 + \frac{10\alpha^2 + 320\alpha^3 q^3}{\alpha} s^4$$

which we can then substitute into $g(u)$ in order to apply the parallel tangents condition, that is $\frac{df}{dt} - \frac{dg}{du} = 0$. Using Maple (see Appendix 6), we can write this as a power series,

$$\left(2f_2t_1 - 4\alpha q\right)s + \left(2f_2t_2 + 2\alpha + 3f_3t_1^2\right)s^2 + \left(-16\alpha^2q^2 + 2f_2t_3 + 6f_3t_1t_2 + 4f_4t1^3\right)s^3$$
$$+ \left(2f_2t_4 + 16\alpha^2q + 3f_3\left(2t_1t_3 + t_2^2\right) + 12f_4t_1^2t2 + 5f_5t_1^4\right)s^4$$
$$+ \left(2f_2t_5 + 3f_3\left(2t_1t_4 + 2t_2t_3\right) - 4\alpha^2 - 128\alpha^3q^3 + 4f_4\left(t1\left(2t_1t_3 + t_2^2\right) + t_3t_1^2 + 2t_2^2t_1\right)\right.$$
$$\left. + 20f_5t_1^3t_2\right)s^5$$

then we wish to utilise equation (42), expressing it as a series in s. First we rearrange it to the form

$$(t - h(s)) - h'(s)\left(p + f(t) - s\right) = 0$$

into which we can then substitute our chosen value of $h(s)$ and our power series expansions of t and $f(t)$ giving us, in ascending powers of s, the following series.

$$\left(t_1 - 2p\right)s + \left(t_2 + 1\right)s^2 + \left(-2f_2t_1^2 + t_3\right)s^3 + \left(t_4 - 4f_2t_1t_2 - 2f_3t_1^3\right)s^4 +$$
$$\left(t_5 - 2f_2\left(2t_1t_3 + t_2^2\right) - 2f_4t_1^4 - 6f_3t_1^2t_2\right)s^5$$

We say that these 2 series are satisfied for any s, so we equate the coefficients of powers of s to 0 in order to find our t_i and f_i coefficients, for $1 \le i \le 5$. These calculations (see Appendix 6) heed results from which we can say that,

$$t(s) = 2ps - s^2 + 8\alpha qps^3 - \left(\frac{16}{3}\alpha q + \frac{8}{3}\alpha p\right)s^4 + 2\alpha\left(32p\alpha q^2 + 1\right)s^5$$

$$f\left(t(s)\right) = 4\alpha qps^2 - \frac{4}{3}\alpha\left(2q + p\right)s^3 + \alpha\left(32p\alpha q^2 + 1\right)s^4 - \frac{16}{15}\alpha^2 q\left(23p + 22q\right)s^5.$$

So now, by assigning numerical values to α, p and q, we can use Maple (see Appendix 7) to draw curves A, B and C and their tangents (see Figure 13).

Here we have let $\alpha = 1, p = 2$ and $q = -2$ and the "middle"curves represent A, the "lower"curves represent B whilst the "upper"curves represent C (very much as depicted in Figure 12). The thinner lines here represent tangents to our curves, we can see that the tangents to A join the parallel tangents of B and C (and so A is the CSS of B and C). This is emphasised in Figure 14 where we have drawn the tangents for 3 different values of s ($s = 0, s = 0.05$ and $s = -0.05$) together.

In future work, we intend to look at surfaces of constant width, together with CSSs in 3 dimensions. Also, we would like to see just how far we can take our reconstruction

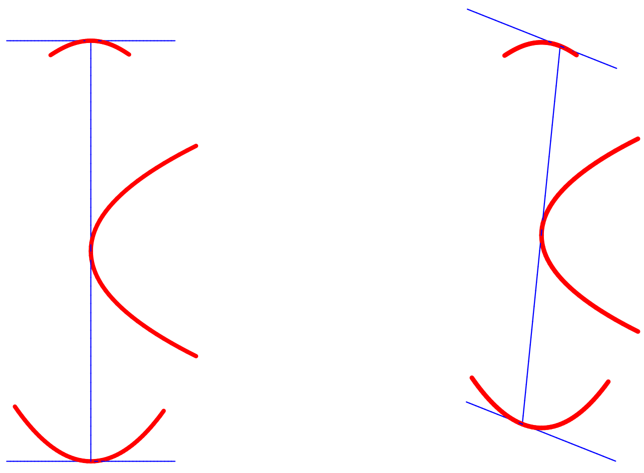


Figure 13: Tangents for $s = 0$ and $s = 0.05$ respectively.

section, i.e. given the CSS and perhaps one or 2 points on the curve, is it possible to reconstruct the whole curve?

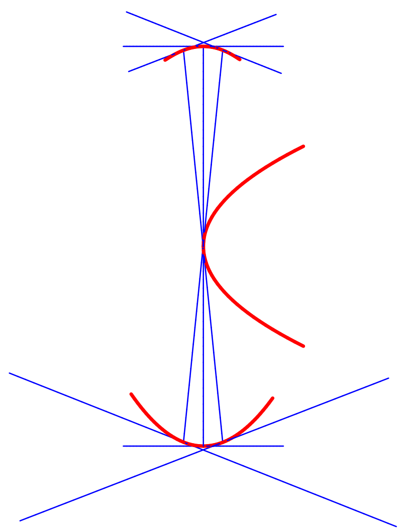


Figure 14: Tangents for $s = 0, s = 0.05$ and $s = -0.05$.

8 Maple Appendices

8.1 Appendix 1

This is the Maple programme for finding the parametrisation of the CSS of a curve, not necessarily of constant width.

```
with(plots):with(plottools):
with(linalg):with(PDEtools):
```

Consider the support function defined by $h(t)$ and $h(t+\pi)$ to be defined by $h1(t)$.

```
declare(h(t),h1(t),prime=t):
dh:=diff(h(t),t);dh1:=diff(h1(t),t);d2h:=diff(dh(t),t);
```

Note, whenever using h or $h1$, we must refer to them as $h(t)$ and $h1(t)$ respectively, otherwise Maple will not differentiate them properly. Consider our curve parametrised by $(x(t), y(t))$ as follows.

```
x:=h(t)*cos(t)-dh*sin(t);y:=h(t)*sin(t)+dh*cos(t);
```

Now consider $(x(t+\pi), y(t+\pi))$ to be defined by $(x1(t), y1(t))$.

```
x1:=-h1(t)*cos(t)+dh1*sin(t);y1:=-h1(t)*sin(t)-dh1*cos(t);
```

Our family of chords can be defined by F.

```
F:=(Y-y)*(x1-x)-(X-x)*(y1-y);
```

For the envelope of chords we need $\frac{dF}{dt}$.

```
dF:=diff(F,t);
```

The envelope of chords is given by the solution to the equations $F = \frac{dF}{dt} = 0$.

```
XY:=solve({F=0,dF=0},{X,Y});
```

We can say that X is the 2nd of the 2 arguments produced (it could be the 1st).

```
X:=op(2,XY[2]);
```

and Y is the 1st of the 2 arguments produced (it could be the 2nd).

```
Y:=op(2,XY[1]);
```

Here (X, Y) are parameters of the CSS.

8.2 Appendix 2

This picks up where Appendix 1 left off and finds the parametrisation of the CSS when we have a CCW. If our curve is of constant width, then we should have certain conditions satisfied;

(1) that $h(t) + h(t+\pi) = k$ where k = constant,

```
X1:=subs(h1=k-h,X);Y1:=subs(h1=k-h,Y);
```

(2) that $h'(t) + h'(t+\pi) = 0$,

```
X2:=subs(diff(h1(t),t)=-dh,X1);Y2:=subs(diff(h1(t),t)=-dh,Y1);
```

(3) that $h''(t) + h''(t+\pi) = 0$ and

```
X3:=subs(diff(diff(h1(t),t),t)=-d2h,X2);
Y3:=subs(diff(diff(h1(t),t),t)=-d2h,Y2);
```

(4) that since k = constant, then $k'(t) = 0$.

```
X4:=subs(diff(k(t),t)=0,X3);Y4:=subs(diff(k(t),t)=0,Y3);
```
but we can simplify these.
```
X5:=simplify(X4);Y5:=simplify(Y4);
```
Here $(X5, Y5)$ represent our conditions for a CCW, i.e. if we have a CCW, then the parameters of the envelope of its chords joining parallel tangents (its CSS) will take the form $(X5, Y5)$.

8.3 Appendix 3

We want to draw a CCW and the envelope of its chords joining points on the curve at $(x(t), y(t))$ to points $(x(t+\pi), y(t+\pi))$.
```
with(plots):with(plottools):
```
Define a fixed number m.
```
m:=3;
```
Define $p = p(t)$ and its derivative with respect to t. This is called our support function.
```
p:=a*(cos(m*t/2))^2+b;
dp:=diff(p,t);
```
Define our closed, convex curve C in terms of the following parametric equations.
```
x:=p*cos(t)-dp*sin(t);y:=p*sin(t)+dp*cos(t);
```
Define (x, y) for $t = (t+\pi)$, this will be used later.
```
x2:=subs(t=t+Pi,x);y2:=subs(t=t+Pi,y);
```
Now consider values of (a, b) in order to plot our curve C.
```
a:=2;b:=8;
```
Define the graph of C.
```
C:=plot([x,y,t=0..2*Pi],thickness=2,scaling=constrained):
```
We want the envelope of chords joining points at t to points at $t+\pi$, i.e. we want to draw many chords. These chords join points the $(x(t), y(t))$ to points $(x(t+\pi), y(t+\pi))$. We use a "for"loop.
```
for_i_from_0_to_100_do
t:=(i/100)*(2*Pi):
chord[i]:=line([x,y],[x2,y2]):
end_do:
display(C,seq(chord[i],i=0..100),color=blue,axes=NONE);
```
Here we have a CCW (for m) and the envelope of its chords joining points at t and $t+\pi$ (this is the CSS).

8.4 Appendix 4

We propose that our CCW as shown in Figure 5 is assymetrical. This is the Maple programme with which we check this by considering the curvature of our CCW. First define the curvature.

```
kappa:=1/(h+d2h);
```

Then substitute the values for our specific support function.

```
kappa1:=subs(a=a1,b=b1,c=c1,d=d1,kappa);
```

Define the derviative of κ with respect to t.

```
dkappa:=diff(kappa1,t);
```

Plot the graph of curvature κ against t

```
plot([t,kappa1,t=0..2*Pi]);
```

Let t_1, t_2 and t_3 be our maxima of curvature.

```
t1:=fsolve(dkappa=0,t=0..1);
t2:=fsolve(dkappa=0,t=2..3);
t3:=fsolve(dkappa=0,t=4..5);
```

Now consider the corresponding points on our curve (for these values of t).

```
xx1:=evalf(subs(t=t1,x1));yy1:=evalf(subs(t=t1,y1));
xx2:=evalf(subs(t=t2,x1));yy2:=evalf(subs(t=t2,y1));
xx3:=evalf(subs(t=t3,x1));yy3:=evalf(subs(t=t3,y1));
```

Now consider the lengths of the sides of a triangle formed by joining these points.

```
len1:=sqrt((xx2-xx3)^2+(yy2-yy3)^2);
len2:=sqrt((xx3-xx1)^2+(yy3-yy1)^2);
len3:=sqrt((xx1-xx2)^2+(yy1-yy2)^2);
```

8.5 Appendix 5

We would like to derive algebraic equations for a CCW and its CSS.

```
with(linalg):with(plots):with(plottools):
```

Define our support function and its derivatives. We would like to consider $h = a\left(\cos\frac{3t}{2}\right)^2 + b$ but using the formula $\cos 2A = 2cos^2A - 1$ we can simplify this as follows.

```
h:=a/2*cos(3*t)+a/2+b;dh:=diff(h,t);d2h:=diff(dh,t);
```

Use these to obtain the parameters of our curve.

```
x:=h*cos(t)-dh*sin(t);y:=h*sin(t)+dh*cos(t);
```

Use Maple's knowledge of trigonometric identities to simplify our expressions.

```
x1:=simplify(x);simplify(y);
```

Notice that our equation for x is purely in terms of $\cos t$ whilst we have an unwanted $\sin t$ in our expression for y. If we square our equation for y then we will have a $\sin^2 t$ term which we can then replace by $1 - \cos^2 t$.

```
y1:=simplify(y^2);
```

Now we can see that for $x1$ and $y1$ (which $= y^2$) we have 2 polynomials all in terms of $\cos t$ (once we've used $\sin^2 t = 1 - cos^2t$ in $y1$).

```
x2:=subs(cos(t)=C,x1);
```

```
y2:=subs(cos(t)=C,(sin(t))^2=1-C^2,y1);
```

So now we have 2 equations in terms of the 3 unknowns ($x2, y2$ and C) so we want to eliminate C (the variable we don't want). Use YY here since we are looking at y^2, not y.

```
elim1:=eliminate({X=x2,YY=y2},C);
```

The 1st component above gives us a value of C and the 2nd component is the remaining equation, now that C has been eliminated. So we want this 2nd component, use the following command (note that the 1 in the brackets here gets rid of the brackets in the output of the command above for us).

```
elim2:=op(1,op(2,elim1))/a;
```

Now we would like to replace YY by Y^2.

```
elim3:=subs(YY=Y^2,elim2);
```

This is our algebraic equation for a CCW! We would like to study the graph of this, so let us consider our equation for the following values of a and b.

```
a1:=2;b1:=8;
```

Substitute these values into our elimination.

```
eq1:=subs(a=a1,b=b1,elim3);
```

Simplify it.

```
eq2:=(eq1)/(-16);
```

Notice that this equation is of degree 8 (perhaps since we had $3t$ in our $h(t)$ and this is $2 \times 3 + 2$). Plot the algebraic graph.

```
implicitplot(eq1=0,X=-10..10,Y=-10..10,grid=[100,100]);
```

But, is this the same as the graph defined by our parametric equations? Define non-algebraic parameters,

```
x1:=subs(a=a1,b=b1,x);y1:=subs(a=a1,b=b1,y);
```

and plot them.

```
plot1:=plot([x1,y1,t=0..2*Pi],scaling=constrained):
display(plot1);
```

The graphs are indeed the same and it would seem that squaring y has not introduced any extra components! Now we would like to obtain an algebraic expression for the corresponding CSS (we do this in a parallel fashion to the way we did it for our CCW). Remember we know the parametrisation of the CSS for a CCW.

```
cx:=-dh*sin(t)-d2h*cos(t);cy:=dh*cos(t)-d2h*sin(t);
```

We can simplify these.

```
cx1:=simplify(cx);cy1:=simplify(cy^2);
cx2:=subs(cos(t)=C,cx1);cy2:=subs(cos(t)=C,(sin(t))^6=(1-C^2)^3,cy1);
```

Notice that we have 2 equations whose RHSs are expressed completely in terms of C^2. So consider them in terms of a new variable, say $CC = C^2$.

```
cxx2:=subs(C^2=CC,C^4=CC^2,cx2);cyy2:=subs(C^2=CC,cy2);
elim4:=eliminate({CX=cxx2,CYY=cyy2},CC);
elim5:=op(1,op(2,elim4))/a;
```

```
elim6:=subs(CYY=CY^2,elim5);
```

This is our algebraic expression for the CSS. Notice that this equation is of degree 4 (notably half that of the curve itself). Also since the parametrisation of the CSS did not depend on b (since parametrisation depends on dh and $d2h$, not h), it is right that our equation does not either. Consider this for values of a and b (where $a = 2, b = 8$ as before).

```
eq2:=subs(a=a1,b=b1,elim6);
eq3:=(eq2)/(-16);
implicitplot(eq2=0,CX=-9..9,CY=-8..8,grid=[200,200]);
cx2:=subs(a=a1,b=b1,cx);cy2:=subs(a=a1,b=b1,cy);
plot2:=plot([cx2,cy2,t=0..Pi],scaling=constrained):
display(plot2);
```

The graphs are indeed the same and it would seem that squaring cy has not introduced any extra components!

8.6 Appendix 6

This is the Maple programme for the reconstruction section. Firstly, we define t as a power series in s.

```
t:=t1*s+t2*s^2+t3*s^3+t4*s^4+t5*s^5;
```

Then we define $h(s)$ and its derivative, for example let,

```
h:=s^2;dh:=diff(h,s);
```

Defining $g(u)$ (and its derivative) allows us to change it later. For example let,

```
g:=alpha*u^2;dg:=diff(g,u);
```

Define $f(t)$ as a power series in t,

```
f:=f2*t^2+f3*t^3+f4*t^4+f5*t^5;
```

We know that $u(s)$ is the solution of the following equation,

```
u_sol:=solve(u-h-dh*(q+(subs(s=u,g))-s)=0,u);
```

We need to choose the solution which is 0 at $s = 0$, this is u_- (as apposed to u_+).

```
u_val:=u_sol[2];
```

We can now express $u(s)$ as a power series in s.

```
u_val_5:=series(u_val,s,7);
```

Find the derivative of $u(s)$, i.e. $u'(s)$,

```
du:=diff(u_val_5,s);
```

We know that $t'(0) = t_1$ (this is one of our B.C.s along with $t(0) = 0$). We have 3 main conditions (of which we only need 2 for now).

Condition (1) says $f'(t) = g'(u)$ i.e. our 2 pieces of curve have parallel tangents.

```
eq1:=series(2*f2*t+3*f3*t^2+4*f4*t^3+5*f5*t^4-subs(u=u_val_5,dg),s,6);
```

Condition (2) says $(t - h) - h'(p + f - s) = 0$ i.e. points on our piece of curve we wish to derive, are points on tangent line to CSS curve.

```
eq2:=series((t-h)-dh*(p+f-s),s,6);
```
We can see that by equating coefficients of powers of s to 0 in $eq3$, we can find t_1 and t_2.
```
t1_val:=solve(coeff(eq2,s)=0,t1);t2_val:=solve(coeff(eq2,s^2)=0,t2);
```
Substitute these known values into $eq1$ and $eq2$.
```
eq1a:=subs(t1=t1_val,t2=t2_val,eq1);
eq2a:=subs(t1=t1_val,t2=t2_val,eq2);
```
Now do the same to solve for other t coefficients and f coefficients too.
```
f2_val:=solve(coeff(eq1a,s)=0,f2);
```
Repeat the process.
```
eq1b:=subs(f2=f2_val,eq1a);eq2b:=subs(f2=f2_val,eq2a);
```
We can see that we can find f_3 from $eq1b$ and t_3 from $eq2b$ (since they won't be in terms of other, as yet, unknown coefficients.
```
f3_val:=solve(coeff(eq1b,s^2)=0,f3);
t3_val:=solve(coeff(eq2b,s^3)=0,t3);
eq1c:=subs(f3=f3_val,t3=t3_val,eq1b);
eq2c:=subs(f3=f3_val,t3=t3_val,eq2b);
```
Notice that the coefficient of s^2 in $eq1c$ does equal 0 (for some reason Maple doesn't realise)! If this had not been the case then we would have had that p, q were not arbitrary.
```
f4_val:=solve(coeff(eq1c,s^3)=0,f4);
t4_val:=solve(coeff(eq2c,s^4)=0,t4);
eq1d:=subs(f4=f4_val,t4=t4_val,eq1c);
eq2d:=subs(f4=f4_val,t4=t4_val,eq2c);
```
Again, notice that the coefficient of s^4 in $eq2d$ does equal 0!
```
f5_val:=solve(coeff(eq1d,s^4)=0,f5);
t5_val:=solve(coeff(eq2d,s^5)=0,t5);
```
So now we have all of our f and t coefficients explicitly, sub these into their respective equations. Do t first, since $f = f(t)$.
```
t_val:=subs(t1=t1_val,t2=t2_val,t3=t3_val,t4=t4_val,t5=t5_val,t);
f_val:=subs(f2=f2_val,f3=f3_val,f4=f4_val,f5=f5_val,t=t_val,f);
```
Try simplifying this.
```
f_val2:=simplify(f_val);
```
The way to get a nicer expression for f is to find the series, simplify and find the series again.
```
f_val2a:=series(simplify(series(f_val2,s,6)),s,6);
```

8.7 Appendix 7

This is the Maple programme for constructing our 3 pieces of curve A, B and C such that B and C have A as their CSS. We need to use certain packages when plotting graphs which we have to call.

```
with(plots):with(plottools):
```

Consider a piece of curve A, parametrised by $(h(s), s)$, this will represent our CSS.

```
h:=s^2;
```

Define the graph of A

```
plot1:=plot([h,s,s=-1..1],thickness=3,scaling=constrained):
```

and then to display the graph, we use the following command.

```
display(plot1);
```

Now we wish to consider a piece of curve B, parametrised by $(u, q + g(u))$.

```
g:=alpha*u^2;
```

From previous calculations, we have an expression for u.

```
u:=2*q*s-s^2+8*alpha*q^2*s^3-8*alpha*q*s^4-1/4*
(-8*alpha^2-256*alpha^3*q^3)*s^5/alpha;
```

For B we want $q + g(u)$ as the "y"coordinate.

```
By:=q+g;
```

Give some values to our parameters.

```
q1:=-2;alpha1:=1;
```

Substitute these into our expressions for u, g and By.

```
u1:=subs(q=q1,alpha=alpha1,u);
g1:=subs(q=q1,alpha=alpha1,g);
u2:=subs(q=q1,alpha=alpha1,g=g1,By);
```

Then define the graph of B.

```
plot2:=plot([u1,u2,s=-0.2..0.2],thickness=3,scaling=constrained):
```

Now we seek a 3rd piece of curve, say C, parametrised by $(t, p + f(t))$. Give a numerical value to parameter p.

```
p1:=2;
```

We found $t(s)$ to be as follows.

```
t:=2*p*s-s^2+8*alpha*q*p*s^3-(16/3)*alpha*q*s^4-
(8/3)*alpha*p*s^4+2*alpha*32*p*alpha*q^2*s^5+2*alpha*s^5;
```

And we found $f(t(s))$.

```
f:=4*alpha*q*p*s^2-4/3*alpha*(2*q+p)*s^3+alpha
*(32*p*alpha*q^2+1)*s^4-16/15*alpha^2*q*(22*q+23*p)*s^5;
```

For curve C we want $p + f(t)$ as the "y"coordinate.

```
Cy:=p+f;
```

Substitute our parameter values into these functions.

```
t1:=subs(p=p1,alpha=alpha1,q=q1,t);
f1:=subs(p=p1,alpha=alpha1,q=q1,f);
t2:=subs(p=p1,alpha=alpha1,q=q1,f=f1,Cy);
```

Define the graph of C.

```
plot3:=plot([t1,t2,s=-0.1..0.1],axes=NONE,
thickness=3,scaling=constrained):
```

Now we want to draw a line which is tangent to A. Let us begin by drawing a single tangent to a point on A, say where $s = 0$, call this point s_0.

```
s0:=0;
```

We can see that the tangent must pass through (A_{s0_h}, A_{s0_s}). This is our point of contact on A.

```
A_s0_h:=subs(s=s0,h);A_s0_s:=subs(s=s0,s);
```

We need the derivative of the path (w.r.t. s) of $(h(s), s)$, call these dh and ds respectively.

```
dh:=diff(h,s);ds:=diff(s,s);
```

So when $s = s_0$, the tangent vector must be (T_{s0_h}, T_{s0_s}).

```
T_s0_h:=subs(s=s0,dh);T_s0_s:=subs(s=s0,ds);
```

A parametrisation of the tangent line is (TL_{s0_h}, TL_{s0_s}).

```
TL_s0_h:=A_s0_h+(s-s0)*T_s0_h;TL_s0_s:=A_s0_s+(s-s0)*T_s0_s;
```

Define the plot of the tangent to A.

```
tan0:=plot([TL_s0_h,TL_s0_s,s=-2..2],axes=NONE,
color=blue,thickness=1,scaling=constrained):
```

We want to draw the tangent to the point of contact on B. We can see that this tangent must pass through $(B_{s0_{u1}}, B_{s0_{u2}})$.

```
B_s0_u1:=subs(s=s0,u1);B_s0_u2:=subs(s=s0,u2);
```

We need the derivative of the path (w.r.t. s) of (u_1, u_2), call these du_1 and du_2 respectively.

```
du1:=diff(u1,s);du2:=diff(u2,s);
```

So when $s = s_0$, the tangent vector must be $(T_{s0_{u1}}, T_{s0_{u2}})$.

```
T_s0_u1:=subs(s=s0,du1);T_s0_u2:=subs(s=s0,du2);
```

A parametrisation of the tangent line to B is $(TL_{s0_{u1}}, TL_{s0_{u2}})$.

```
TL_s0_u1:=B_s0_u1+(s-s0)*T_s0_u1;TL_s0_u2:=B_s0_u2+(s-s0)*T_s0_u2;
```

Define the tangent to B.

```
tan1:=plot([TL_s0_u1,TL_s0_u2,s=-0.2..0.2],color=blue,
thickness=1,scaling=constrained):
```

We find the parallel tangent to this on C by multiplying by -1 (by the way we have set up our model).

```
tan2:=plot([-1*TL_s0_u1,-1*TL_s0_u2,s=-0.2..0.2],
color=blue,thickness=1,scaling=constrained):
```

To view the 3 curves and their respective tangents, we use the following command.

```
display(plot1,plot2,plot3,tan0,tan1,tan2);
```

Now consider another value of s, say $s_1 = 0.05$ and find the tangents to A, B and C in a parallel fashion to the way we did for s_0.

9 Bibliography

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