Affine Normal Curvature of Hypersurfaces from the Point of View of Singularity Theory

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Abstract

We consider smoothly embedded hypersurfaces $M \subset \mathbb{R}^{n+1}$ under the action of the special affine group $\mathrm{SL}(n+1, \mathbb{R}) \ltimes \mathbb{R}^{n+1}$. We construct a differential invariant, called affine normal curvature, which assigns to a point and a tangent direction a number. We prove some of its nice properties which connect it with affine principal directions, affine umbilics, and affine mean curvature.

Contents

1 Introduction

In this paper we consider the equi-affine differential geometry of smoothly embedded hypersurfaces in $\mathbb{R}^{n+1}$, i.e. differential invariants of hypersurfaces embedded in $\mathbb{R}^{n+1}$ under the action of the Lie group $\mathrm{ASL}(n+1,\mathbb{R}) := \mathrm{SL}(n+1,\mathbb{R}) \ltimes \mathbb{R}^{n+1}$, i.e. volume preserving linear transformations composed with translations.

Given a smoothly embedded hypersurface $M \subset \mathbb{R}^{n+1}$ we formulate an interpretation of affine normal curvature using singularity theory. We find an explicit formula for the affine normal curvature $\mu : \mathbb{P}TM \to \mathbb{R} \cup \{\infty\}$. We then prove that it has many of the desirable properties of the Euclidean normal curvature.

In §2 we give a review of affine differential geometry from the point of view of connexions and metrics. This follows the approach of Nomizu and Sasaki [8].

In §3 we give a motivation for the affine normal curvature. The motivation is to find an affine analogue to the Euclidean normal curvature which posses many of the same nice properties.

In §4 we establish a link between the affine differential geometry of plane curves and the singularity types of families of functions defined over these curves. This link is then generalised to hypersurfaces.

In §5 we list our main results. We recall the definition of the family of affine distance functions defined over a hypersurface, and we generalise the findings of §4 to this case. We give a definition for affine normal curvature, and then derive an expression for it. We then go on to prove some of its properties. In particular we show that it has an extrema in a direction if, and only if, that direction is affine principal. We show that it is constant on a linear subspace if, and only if, that subspace is an eigenspace of the affine shape operator. We show that in the case where our surface has a positive definite affine metric the integral of the affine normal curvature is related to the affine mean curvature of the surface.

2 Affine Differential Geometry

Here we give a brief review of the affine differential geometry of hypersurfaces, see [8] for further details.

Let M be a smoothly embedded hypersurface in $\mathbb{R}^{n+1}$ with a transverse vector field ξ. Let $\mathfrak{X}(M)$ denote the $C^\infty(M,\mathbb{R})$-module of smooth vector fields on M. Let D denote the standard covariant derivative on $\mathbb{R}^{n+1}$. Given $\mathbf{v},\mathbf{w} \in \mathfrak{X}(M)$ we can decompose $D_\mathbf{v}\mathbf{w}$ into the sum of a tangential component and a transverse component parallel to ξ. This gives the equation of Gauss:

$$D_\mathbf{v}\mathbf{w} = \nabla_\mathbf{v}\mathbf{w} + h(\mathbf{v},\mathbf{w})\xi \tag{1}$$

where ∇ is a torsion free connexion on M and h is symmetric bilinear form; both of which are dependent on the choice of ξ. Notice that the non-degeneracy of h is independent of the choice of ξ and depends only on the hypersurface (see [8]). We shall always assume that h is non-degenerate.

We can decompose $D_\mathbf{v}\xi$ into a tangential component and a transverse component parallel to ξ. This gives the equation of Weingarten:

$$D_\mathbf{v}\xi = -S\mathbf{v} + \tau(\mathbf{v})\xi \tag{2}$$

where $(-S)$ is a tensor of type $(1,1)$ called the affine shape operator and τ is a one-form called the transverse connexion form.

Let Ω be a volume form on $\mathbb{R}^{n+1}$, i.e. a non-degenerate skew-symmetric $(n+1)$-form. We can define a volume form on M, denoted by ω, as follows: given $\mathbf{v}_i \in \mathfrak{X}(M)$ for $1 \leq i \leq n$ we set

$$\omega(\mathbf{v}_1,\ldots,\mathbf{v}_n) := \Omega(\mathbf{v}_1,\ldots,\mathbf{v}_n,\xi)\ .$$

The metric h also defines a volume element on M. Given $\mathbf{v}_i \in \mathfrak{X}(M)$ for $1 \leq i \leq n$, let $H := (h_{i,j})$ be the $n \times n$ matrix where $h_{i,j} := h(\mathbf{v}_i, \mathbf{v}_j)$. Then we define

$$\nu(\mathbf{v}_1, \ldots, \mathbf{v}_n) := |\det(H)|^{1/2} .$$

Theorem 2.1 (Nomizu and Sasaki [8]) *There is, up to sign, a unique transverse vector field ξ, for which the following two conditions are met:*

1. $\nabla_{\mathbf{v}}\omega = 0$ *for all* $\mathbf{v} \in \mathfrak{X}(M)$,
2. $\omega(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \nu(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ *for all* $\mathbf{v}_i \in \mathfrak{X}(M)$.

This vector field is called the affine normal vector field and shall be denote by $\mathbf{A}$. It is also called the Blaschke normal field.
A short calculation shows that $\nabla_{\mathbf{v}}\omega = \tau(\mathbf{v})\omega$. It follows that $\nabla_{\mathbf{v}}\omega = 0$ for all $\mathbf{v} \in \mathfrak{X}(M)$ if, and only if, $\tau(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathfrak{X}(M)$. Thus $D_{\mathbf{v}}\mathbf{A}$ is tangent to M for all $\mathbf{v} \in \mathfrak{X}(M)$.

3 Motivation

In this paper we construct an affine analogue of Euclidean normal curvature. The original motivation comes from surfaces in three-space. Let $M \subset \mathbb{R}^3$ be a smoothly embedded surface. Let $p \in M$ and $\mathbf{v} \in T_pM$. Denote by P the plane passing through p spanned by $\mathbf{v}$ and the the Euclidean unit normal vector $\mathbf{N}$. The intersection of the plane P with the surface gives a plane curve contained in P. The Euclidean plane curve curvature at p of this plane curve is defined to be the Euclidean normal curvature of M at p in the direction $\mathbf{v}$. We denote this by $\kappa_p(\mathbf{v})$.

Here we list some of the nice properties of Euclidean normal curvature. It has an extremum in a direction if, and only if, the direction is principal. The value of κ_p in one of these directions is the corresponding principal curvature. Euclidean normal curvature is constant if, and only if, p is an umbilic. Euclidean normal curvature is zero in a direction if, and only if, $D^2_{\mathbf{v}}\mathbf{N} \in T_pM$. If $H_p(M)$ denotes the mean curvature of M at p then

$$\frac{1}{\pi}\int_{\mathbb{RP}^1} \kappa_p(\mathbf{v})\ d\mathbf{v} = H_p(M) .$$

A candidate for an affine analogue would be to consider the plane passing through p which is spanned by $\mathbf{A}$ and $\mathbf{v}$. This again gives a plane curve, and one might take the affine normal curvature of M at p in the direction $\mathbf{v}$ to be the affine plane curve curvature at p of this plane curve (see [7] for a study of affine plane curves and [5] for affine curves in $\mathbb{R}^n$). We discover that this construction does not have any of the desired analogous properties of Euclidean normal curvature, and so it would seem that it is not the correct one.

4 Singularity Theory

The affine plane curve curvature of a smooth plane curve without inflexions can be written in terms of the singularities of the family of affine distance functions. Let $I \subseteq \mathbb{R}$ be an open interval and let $\gamma : I \to \mathbb{R}^2$ be an embedding such $\gamma(I)$ is without inflexions. We assume that γ is parametrised by affine arc-length (i.e. $[\gamma', \gamma''] = 1$ for all $s \in I$), see [5] and [7]. With such a parameter the affine plane curve curvature is given by $\mu = [\gamma'', \gamma''']$. The family of affine distance functions is given by $\Delta : \mathbb{R}^2 \times I \to \mathbb{R}$ where $\Delta(\mathbf{x}, s) := [\gamma', \mathbf{x} - \gamma]$.

The basic method when applying singularity theory to the differential geometry of plane curves is to look for certain $\mathbf{x} \in \mathbb{R}^2$ for which $\Delta : \{\mathbf{x}\} \times I \to \mathbb{R}$ has certain singularity types. Since $\Delta : \{\mathbf{x}\} \times I \to \mathbb{R}$ is a map from the line to the line we expect $\Delta : \{\mathbf{x}\} \times I \to \mathbb{R}$ to have an A_k singularity for some $k \in \mathbb{N}$, i.e. to be $\mathcal{R}$-equivalent to $\pm s^{k+1}$. The reader is referred to [1] and [2] for further details.

We can show that $\Delta_s(\mathbf{x}, s_0) = 0$ if, and only if, $\mathbf{x} = \gamma(s_0) + \lambda\gamma''(s_0)$ for some $\lambda \in \mathbb{R}$. In this case $\Delta : \{\mathbf{x}\} \times I \to \mathbb{R}$ is said to have an $A_{\geq 1}$ singularity at $s = s_0$. Furthermore, assuming that $\mu(s_0) \neq 0$ we can show that $\Delta_s(\mathbf{x}, s_0) = \Delta_{ss}(\mathbf{x}, s_0) = 0$ if, and only if, $\mathbf{x} = \gamma + \mu(s_0)^{-1}\gamma''(s_0)$. In this case $\Delta : \{\mathbf{x}\} \times I \to \mathbb{R}$ is said to have an $A_{\geq 2}$ singularity at $s = s_0$. The points $\mathbf{x} \in \mathbb{R}^2$ for which $\Delta : \{\mathbf{x}\} \times I \to \mathbb{R}$ has an $A_{\geq 2}$ for some $s \in I$ are also called affine focal points. Let $\rho := \gamma(s_0) + \mu(s_0)^{-1}\gamma''(s_0)$ then $\Delta(\rho, s_0) = 1/\mu(s_0)$, and so $\mu(s_0) = \Delta(\rho, s_0)^{-1}$.

In order to compute the affine plane curve curvature we simply look for affine focal points, and then the reciprocal of the affine distance from the affine focal point to the base curve point is the affine curvature.

For a detailed study of the use of singularity theory to investigate the Euclidean differential geometry of curves see [2]. In the following section we generalise this idea of taking planer sections and looking for focal points to smoothly embedded hypersurfaces in $\mathbb{R}^{n+1}$.

5 Main Results

Given a hypersurface $M \subset \mathbb{R}^{n+1}$ we can define the family of affine distance functions, as in [4], by $\Delta : \mathbb{R}^{n+1} \times M \to \mathbb{R}$ as follows: given an ambient point $\mathbf{x} \in \mathbb{R}^{n+1}$ and a surface point $p \in M$ we define the affine distance from $\mathbf{x}$ to p implicitly by

$$p - \mathbf{x} = \mathbf{z}(\mathbf{x}, p) + \Delta(\mathbf{x}, p)\mathbf{A}(p) \tag{3}$$

where $\mathbf{z} \in T_pM$. Let P denote the plane passing through p which is spanned by the affine normal $\mathbf{A}$ and some non-zero $\mathbf{v} \in T_pM$. Let C denote the intersection $M \cap P$ which, close to p will be a regular curve. Consider the restriction of $\Delta : \mathbb{R}^{n+1} \times M \to \mathbb{R}$ where $\mathbf{x} \in P$ and $p \in C$. This gives a family of functions $\widetilde{\Delta} : P \times C \to \mathbb{R}$. Locally this is a two parameter family of functions from $\mathbb{R}$ to $\mathbb{R}$, i.e. $\widetilde{\Delta} : \mathbb{R}^2 \times \mathbb{R} \rightarrowtail \mathbb{R}$.

Remark 5.1 *The key point here is that the restriction* $\widetilde{\Delta} : P \times C \to \mathbb{R}$ *does not coincide with the family of affine distance functions which arises from considering the cross-sectional curve* C *as a plane curve in* P *and using the two-dimensional distance functions.*

Definition 5.2 *The affine normal curvature of* M *at* p *in the direction* $\mathbf{v}$*, denoted by* $\mu_p(\mathbf{v})$*, is given by* $\widetilde{\Delta}(\mathbf{x}, p)^{-1}$ *where* $\mathbf{x} \in P$ *is such that* $\widetilde{\Delta} : \{\mathbf{x}\} \times C \to \mathbb{R}$ *has a degenerate singularity at* $p \in C$*.*

Let S be as in Equation (2) and h as in Equation (1) then we have

Proposition 5.3 *Assuming that* $h(\mathbf{v}, \mathbf{v}) \neq 0$ *we have the following:*

$$\mu_p(\mathbf{v}) = -\frac{h(S\mathbf{v}, \mathbf{v})}{h(\mathbf{v}, \mathbf{v})} \ . \tag{4}$$

From this point on we shall assume that $h(\mathbf{v},\mathbf{v}) \neq 0$. At points where h fails to be positive or negative definite we simply say that μ_p is not defined in directions for which $h(\mathbf{v},\mathbf{v}) = 0$. Before we prove Proposition 5.3 it will be convenient to give some definitions and formulate some lemmas.

Definition 5.4 *The affine principal directions of M at p are the eigendirections of the affine shape operator $(-S)_p$. The corresponding eigenvalues are called affine principal curvatures.*

Definition 5.5 (Cecil [4]) *A point $\mathbf{x} \in \mathbb{R}^{n+1}$ is called an affine focal point if it is in the envelope of the family of affine normal lines to M, i.e. the unoriented lines spanned by the affine normals. Thus a point $\mathbf{x} = p + \lambda\mathbf{A}$ is an affine focal point of M at p if, and only if, $1/\lambda$ is an affine principal curvature of M at p.*

Lemma 5.6 *A member of the family of affine distance functions $\Delta : \{\mathbf{x}\} \times M \to \mathbb{R}$ has a critical point at $p \in M$ if, and only if, $p - \mathbf{x}$ is parallel to $\mathbf{A}_p$. If Δ has a critical point at $p \in M$ then it is degenerate if, and only if, $\mathbf{x}$ is an affine focal point.*

Proof [Lemma 5.6] The family of affine distance functions are such that $p - \mathbf{x} = \mathbf{z} + \Delta\mathbf{A}$ where $\mathbf{z} \in T_pM$. Differentiating by $\mathbf{v} \in T_pM$ we have

$$\mathbf{v} = \nabla_{\mathbf{v}}\mathbf{z} + h(\mathbf{v},\mathbf{z})\mathbf{A} + \mathbf{v}\Delta\mathbf{A} - \Delta S\mathbf{v} \ ,$$

where $\mathbf{v}\Delta$ denotes the directional derivative of the function Δ by the vector $\mathbf{v}$. Comparing tangential and transverse components we have

$$\nabla_{\mathbf{v}}\mathbf{z} = (\Delta S + E)\mathbf{v} \ , \qquad (5)$$

$$h(\mathbf{v},\mathbf{z}) = -\mathbf{v}\Delta \ , \qquad (6)$$

where E denotes the identity operator. Since h is non-degenerate Equation (6) tells us that Δ has a critical point if, and only if, $\mathbf{z} = \mathbf{0}$. Next we consider the Hessian $(\mathbf{v},\mathbf{w}) \mapsto \mathbf{w}(\mathbf{v}\Delta)$. From Equation (6) we see that

$$\mathbf{w}(\mathbf{v}\Delta) = -\mathbf{w}(h(\mathbf{v},\mathbf{z})) \ .$$

Using standard properties of connexions we have

$$\mathbf{w}(h(\mathbf{v},\mathbf{z})) = (\nabla_{\mathbf{w}} h)(\mathbf{v},\mathbf{z}) + h(\nabla_{\mathbf{w}}\mathbf{v},\mathbf{z}) + h(\mathbf{v},\nabla_{\mathbf{w}}\mathbf{z}) \ .$$

If $\Delta : \{\mathbf{x}\} \times M \to \mathbb{R}$ has a critical point then $\mathbf{z} = \mathbf{0}$ and so the Hessian is given by

$$(\mathbf{v},\mathbf{w}) \mapsto -h(\mathbf{v},\nabla_{\mathbf{w}}\mathbf{z}) \ .$$

From Equation (5) we see that $h(\mathbf{v},\nabla_{\mathbf{w}}\mathbf{z}) = h(\mathbf{v},(\Delta S + E)\mathbf{w})$. Since h is non-degenerate it follows that the Hessian is degenerate if, and only if, $\det(\Delta S + E) = 0$. It follows that $\mathbf{x}$ is an affine focal point. □

Proof [Proposition 5.3] Let $\gamma : I \to M$ be a smooth embedding such that $\gamma(I) = C$. Let $\gamma(0) = p$ and let $\mathbf{v} := \dot{\gamma}(0)$. To find the $A_{\geq 2}$ points of the family $\widetilde{\Delta} : P \times C \to \mathbb{R}$ we need to solve $(\mathbf{v}\widetilde{\Delta})(0) = (\mathbf{v}(\mathbf{v}\widetilde{\Delta}))(0) = 0$.

From Lemma 5.6 we see that $\mathbf{x} \in P$ is such that $(\mathbf{v}\widetilde{\Delta})(0) = (\mathbf{v}(\mathbf{v}\widetilde{\Delta}))(0) = 0$ if, and only if, $p - \mathbf{x}$ is parallel to $\mathbf{A}$ and $h(\mathbf{v},(\widetilde{\Delta}S + E)\mathbf{v}) = 0$. The affine distance of this focal point to the base point is given by $\widetilde{\Delta}$, and so by definition the affine normal curvature of M at p in the direction $\mathbf{v}$ is given by solving for $1/\widetilde{\Delta}$. This gives $\widetilde{\Delta}^{-1} = -h(S\mathbf{v},\mathbf{v})/h(\mathbf{v},\mathbf{v})$; as required. □

Remark 5.7 Since the affine normal curvature is a quotient of symmetric quadratic forms we see, for non-zero $\lambda \in \mathbb{R}$ and $\mathbf{v} \in T_pM$, that $\mu_p(\lambda\mathbf{v}) = \mu_p(\mathbf{v})$ and so we may consider μ_p as a function from the projectivised tangent space, i.e. $\mu_p : \mathbb{P}T_pM \to \mathbb{R} \cup \{\infty\}$.

Proposition 5.8 *We have $\mu_p(\mathbf{v}) = 0$ if, and only if, $D^2_{\mathbf{v}}\mathbf{A} \in T_pM$.*

Proof From Equation (2) we have $D_{\mathbf{v}}\mathbf{A} = -S\mathbf{v}$. It follows from Equation (1) that

$$D^2_{\mathbf{v}}\mathbf{A} = -(\nabla_{\mathbf{v}}(S\mathbf{v}) + h(\mathbf{v},S\mathbf{v})\mathbf{A}) \ .$$

Hence $D^2_{\mathbf{v}}\mathbf{A} \in T_pM$ if, and only if, $h(\mathbf{v},S\mathbf{v}) = 0$, i.e. if, and only if, $h(S\mathbf{v},\mathbf{v}) = 0$.

□

Proposition 5.9 *The function* $\mathbf{v} \mapsto \mu_p(\mathbf{v})$ *has an extrema if, and only if,* $\mathbf{v}$ *is an affine principal direction of* M *at* p*. Furthermore, the value of* $\mu_p(\mathbf{v})$ *in such a direction is the corresponding affine principal curvature.*

Proof We wish to show that $(d\mu)(\mathbf{v}) = 0$ if, and only if, $\mathbf{v}$ is an affine principal direction. That is to say $(\mathbf{w}\mu)(\mathbf{v}) = 0$ for all $\mathbf{w}$ if, and only if, $\mathbf{v}$ is an affine principal direction. Consider the derivative of Equation (4) by $\mathbf{w}$, this gives

$$-h(\mathbf{v},\mathbf{v})^2 \cdot (\mathbf{w}\mu)(\mathbf{v}) = h(\mathbf{v},\mathbf{v}) \cdot (h(S\mathbf{w},\mathbf{v}) + h(S\mathbf{v},\mathbf{w})) - 2h(S\mathbf{v},\mathbf{v}) \cdot h(\mathbf{v},\mathbf{w}) \ . \quad (7)$$

The Ricci equation (see [8]) says that $h(S\mathbf{w},\mathbf{v}) = h(\mathbf{w},S\mathbf{v})$ for all $\mathbf{v}$ and $\mathbf{w}$. Using this, and the fact that h is symmetric, Equation (7) becomes

$$-h(\mathbf{v},\mathbf{v})^2 \cdot (\mathbf{w}\mu)(\mathbf{v}) = 2h(\mathbf{v},\mathbf{v}) \cdot h(S\mathbf{v},\mathbf{w}) - 2h(S\mathbf{v},\mathbf{v}) \cdot h(\mathbf{v},\mathbf{w}) \ . \quad (8)$$

Let us first assume that $\mathbf{v}$ is an affine principal direction, i.e. $(-S)\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$. In this case the RHS of Equation (8) vanishes. Since $h(\mathbf{v},\mathbf{v}) \neq 0$ it clearly follows that $(\mathbf{w}\mu)(\mathbf{v}) = 0$ for all $\mathbf{w}$.
Next, let us assume that $(\mathbf{w}\mu)(\mathbf{v}) = 0$ for all $\mathbf{w}$. Equation (8) becomes

$$h(\mathbf{v},\mathbf{v}) \cdot h(S\mathbf{v},\mathbf{w}) - h(S\mathbf{v},\mathbf{v}) \cdot h(\mathbf{v},\mathbf{w}) = 0 \ .$$

If E denotes the identity operator then we can rearrange this as follows:

$$h((h(\mathbf{v},\mathbf{v}) \cdot S - h(S\mathbf{v},\mathbf{v}) \cdot E)\mathbf{v},\mathbf{w}) = 0 \ .$$

Since h is non-degenerate this is true for all $\mathbf{w}$ if, and only if,

$$(h(\mathbf{v},\mathbf{v}) \cdot S - h(S\mathbf{v},\mathbf{v}) \cdot E)\mathbf{v} = \mathbf{0} \ .$$

Finally, rearranging this, we see that

$$(-S)\mathbf{v} = -\frac{h(S\mathbf{v},\mathbf{v})}{h(\mathbf{v},\mathbf{v})}\mathbf{v} \ ,$$

i.e. $\mathbf{v}$ is an affine principal direction with affine principal curvature $\mu_p(\mathbf{v})$. $\square$

Proposition 5.10 *Let V be an m-dimensional subspace of T_pM. The function μ_p is constant, with value λ, on V if, and only if, every $\mathbf{v} \in V$ is an affine principal direction with affine principal curvature λ.*

Proof Consider $\mu_p(\mathbf{v}) = h((-S)\mathbf{v}, \mathbf{v})/h(\mathbf{v}, \mathbf{v})$. First assume that every $\mathbf{v} \in V$ is an affine principal direction with affine principal curvature λ, i.e. $(-S)\mathbf{v} = \lambda\mathbf{v}$ for all $\mathbf{v} \in V$. It follows that $\mu_p(\mathbf{v}) = \lambda$ for all $\mathbf{v} \in V$, and so μ_p is constant, with value λ, on V. Next assume that the function μ_p is constant, with value λ, on V, i.e. $h((-S)\mathbf{v}, \mathbf{v}) = \lambda h(\mathbf{v}, \mathbf{v})$ for all $\mathbf{v} \in V$. It follows that $h((S + \lambda E)\mathbf{v}, \mathbf{v}) = 0$ for all $\mathbf{v} \in V$. Since h is non-degenerate we must have $(S + \lambda E)\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$, i.e. every $\mathbf{v} \in V$ is an affine principal direction with affine principal curvature λ. □

Definition 5.11 *A point p is called an affine umbilic if the affine shape operator is a multiple of the identity, i.e. every direction is an affine principal direction.*

Corollary 5.12 *The affine normal curvature μ_p is a constant function if, and only if, p is an affine umbilic.*

Next we consider the integral of μ_p. Since μ_p can be thought of as a function on the projectivised tangent space we shall integrate around the unit sphere.

Lemma 5.13 *We can parametrise the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ by $\mathbf{X} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ where, assuming that $\theta_{n+1} := 0$, the i-th component, for $1 \le i \le n+1$, of $\mathbf{X}$ is*

$$\mathbf{X}_i(\theta_1, \ldots, \theta_n) = \cos\theta_i \prod_{j=1}^{i-1} \sin\theta_j \ .$$

Proof We prove this by induction. Consider $\mathbb{S}^1$ lying in the xy-plane of $\mathbb{R}^3$. We can rotate about the x-axis to give $\mathbb{S}^2$ lying in $\mathbb{R}^3$. In general consider $\mathbb{S}^k \subset \mathbb{R}^{k+2}$, where $\mathbb{S}^k$ is contained in the coordinate hyperplane given by $x_{k+2} = 0$. Let E_k denote the $k \times k$ identity matrix, and let

$$R := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} .$$

Consider the rotation given by the application of the $(k+2)\times(k+2)$-matrix

$$\left(\begin{array}{c|c} E_k & 0 \\ \hline 0 & R \end{array}\right) .$$

This gives $\mathbb{S}^{k+1}$ lying in $\mathbb{R}^{k+2}$. Induction now proves the result. □

Lemma 5.14 *Consider $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. The integral of monomials of the form $x_i x_j$ for $i \neq j$ restricted to $\mathbb{S}^n$ is zero.*

Proof Let dv_k be the volume form on the k-sphere, and consider the integral

$$\int_{\mathbb{S}^k} x_i x_j \; dv_k \ .$$

We take a change of coordinates: consider x_i and x_j as functions of $\{\theta_1, \ldots, \theta_k\}$. Let $x_i := \mathbf{X}_i(\theta_1, \ldots, \theta_k)$, where $\mathbf{X}_i$ is as in Lemma 5.13. We must also consider a term coming from the Jacobian of $\mathbf{X}$. Let

$$\left| \frac{\partial(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1})}{\partial(\theta_1, \ldots, \theta_k)} \right|$$

denote the determinant of the $k \times k$ Jacobian matrix of the mapping

$$(\theta_1, \ldots, \theta_k) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1}) \ .$$

Furthermore, let $\mathcal{J}$ be given by

$$\mathcal{J} := \sqrt{\sum_{i=1}^{k} \left| \frac{\partial(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1})}{\partial(\theta_1, \ldots, \theta_k)} \right|^2} \ .$$

From standard results in calculus we see that

$$\int_{\mathbb{S}^k} x_i x_j \; dv_k = \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \mathbf{X}_i \mathbf{X}_j \mathcal{J} \; d\theta_1 \wedge \ldots \wedge d\theta_k \ .$$

We can show, again by iteration, that

$$\mathcal{J} = \prod_{i=1}^{k-1} |\sin\theta_i|^{k-i} \ .$$

Next we look at $\mathbf{X}_i$ and $\mathbf{X}_j$ and from Lemma 5.13. Notice that by the symmetry of the sphere we can permute any of the $\mathbf{X}_i$, i.e. if $(\mathbf{X}_1, \ldots, \mathbf{X}_{k+1})$ parametrises the k-sphere then so too does

$$(\mathbf{X}_1, \ldots, \mathbf{X}_{i-1}, \mathbf{X}_j, \mathbf{X}_{i+1}, \ldots, \mathbf{X}_{j-1}, \mathbf{X}_i, \mathbf{X}_{j+1}, \ldots, \mathbf{X}_{k+1}) \ .$$

Thus, we need only consider the simplest case, i.e. $\mathbf{X}_i = \cos\theta_1$ and $\mathbf{X}_j = \cos\theta_2 \sin\theta_1$. Furthermore, notice that $\mathcal{J}$ is unaffected by such a permutation of the components of the parametrisation. We then consider

$$\int_0^{2\pi}\int_0^{\pi} \cos\theta_1 \cos\theta_2 \sin\theta_1 |\sin\theta_1|^{k-1} |\sin\theta_2|^{k-2} \ d\theta_1 \wedge d\theta_2 \ .$$

A straightforward calculation shows this to be zero, and so the result follows.

□

Corollary 5.15 *Assume that h is positive definite, then the integral of $\mu_p(\mathbf{v})$ over $\mathbb{S}^{n-1}$ is given by*

$$\sum_{i=1}^{n} \int_{\mathbb{S}^{n-1}} s_{i,i} x_i^2 \ dv_{n-1} \ ,$$

where the affine shape operator $(-S)$ at p is given by $(-S) := (s_{i,j})$.

Proof Consider an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, i.e. $T_pM = \langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle$ and $h(\mathbf{v}_i, \mathbf{v}_j) = \varepsilon_i \delta_{i,j}$ where $\varepsilon_i = \pm 1$ and $\delta_{i,j}$ is the Kronecker-δ. In this case h is positive definite, and so $\varepsilon = +1$ for all $1 \leq i \leq n$. Let the affine shape operator be such that $(-S)\mathbf{v}_i := s_{1,i}\mathbf{v}_1 + \cdots + s_{n,i}\mathbf{v}_n$, for $s_{i,j} \in \mathbb{R}$. Given $x_i \in \mathbb{R}$ consider $\mathbf{v} := x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$. A direct computation shows that

$$\mu_p(\mathbf{v}) = \sum_{i=1}^{n} \frac{x_i(x_1 s_{i,1} + \cdots + x_n s_{i,n})}{x_1^2 + \cdots + x_n^2} \ .$$

Expanding this expression we find that:

$$\mu_p(\mathbf{v}) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{(s_{i,j}+s_{j,i})x_i x_j}{x_1^2+\cdots+x_n^2} \ .$$

Let us consider the integral of $\mu_p(\mathbf{v})$. We want to restrict $\mathbf{v}$ to $\mathbb{S}^{n-1}$, this is done by restricting the x_i so that $x_1^2+\cdots+x_n^2=1$. In this case we have

$$\mu_p(\mathbf{v}) = \frac{1}{2}\sum_{i,j=1}^{n}(s_{i,j}+s_{j,i})x_i x_j \ .$$

The result now follows from Lemma 5.14. □

Consider a point where p is positive definite. We now show that the integral of μ_p around all of the directions is related directly to the affine mean curvature of the surface, i.e. the mean average of the affine principal curvatures.

Proposition 5.16 *Assume that h is positive definite, then*

$$\int_{\mathbb{S}^{n-1}} \mu_p(\mathbf{v})\ dv_{n-1} = \lambda(s_{1,1}+\cdots+s_{n,n})$$

for some constant λ.

Proof The integrals over the sphere are independent of the choice of parameterisation. Using the symmetry of the sphere and the same permutation of parametrisation components as in Lemma 5.14 we see that the result must follow. □

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