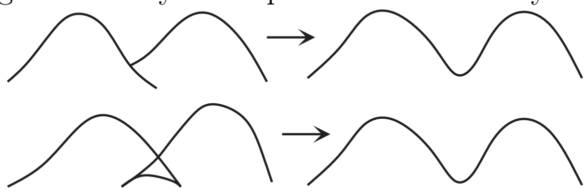
Singularity Theory and Computer Vision, Peter Giblin

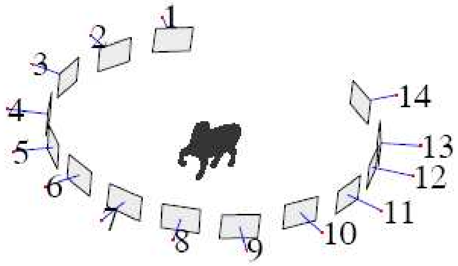
If you drive past two connected hills, their outline changes, as in the upper diagram. A 'T-junction' changes to a smooth curve. The same effect can be observed (with more trouble) walking past a two-humped camel. If the hills, or the camel, were transparent, then the outlines would look more like the lower diagram, with two sharp points (cusps) on the outline, and a crossing, all disappearing in the second view. This behaviour is called *instability*, where some significant or dramatic change occurs when a parameter (your viewing position) changes smoothly. This particular instability is called a *swallowtail*



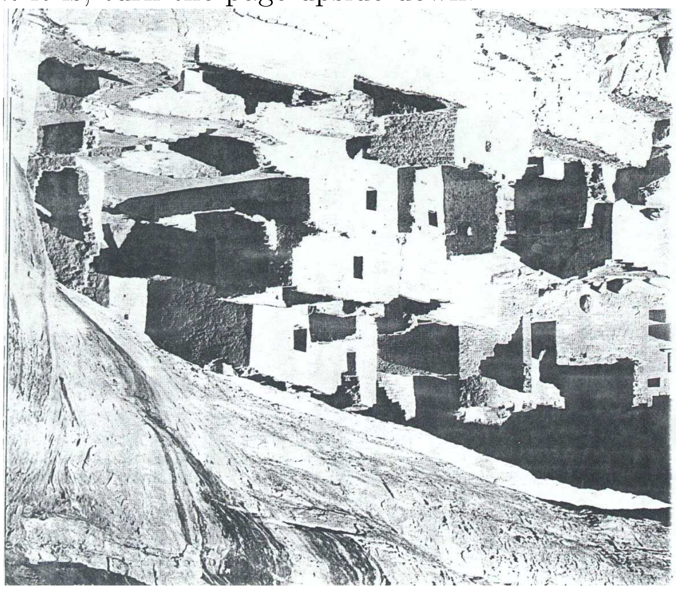
singularity, and changing the viewing position is said to 'unfold' the singularity. It is the business of singularity theory to classify such instabilities and, since they occur in very many contexts, it is the business of people such as myself to apply the profound results of singularity theory where they might be useful. My own research over several decades has worked out applications in optics, symmetry, and various aspects of 'computer vision'. This is the science of deriving information about the world from 2-dimensional images. Of course we, in common with most animals, do this all the time, using our eyes to obtain information about the world. Understanding how this works is the subject of *psychophysics* and that is not my field; I can only say that the combination of eye and brain to decode information is so extraordinary that trying to imitate it by artificial means is (at present) completely hopeless. Instead we have to use mathematical tools, together with fast computers, to do a half-way decent job. The mathematics of singularity theory (a combination of geometry, algebra and calculus of vector valued functions) plays a prominent role here.

Nearly all my research has been done in collaboration with others in Europe or the United States, some of them mathematicians, some computer scientists and some engineers. I'll write about two particular projects in which I have been involved. The first is about *reconstructing the* 3*-dimensional world from* 2*-dimensional information*, while the second is more like *reducing shapes to their essentials, exploiting various kinds of symmetry.*

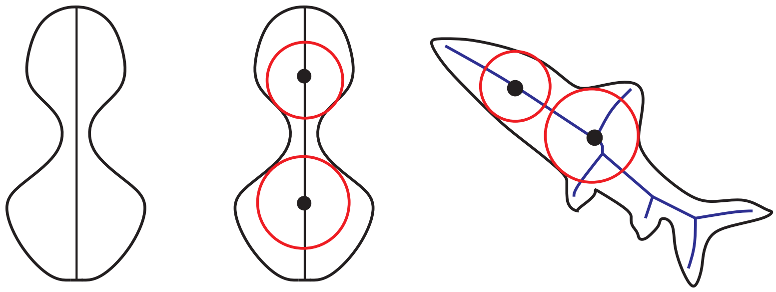
Reconstruction and classification Reconstructing scenes from the real 3D world from images—either a video film or, more challengingly, from a sequence of snapshots—is an area of very active interest. Earlier work with a Cambridge engineer, Roberto Cipolla, was written up in a book *Visual Motion of Curves and Surfaces*, and you can find a lot about this and the fascinating later developments on Roberto's webpage. It is rather amazing that a real 3D model of a scene can be made without knowing much about where the camera was positioned when taking the snapshots. Here we only need to know that the motion is roughly circular.



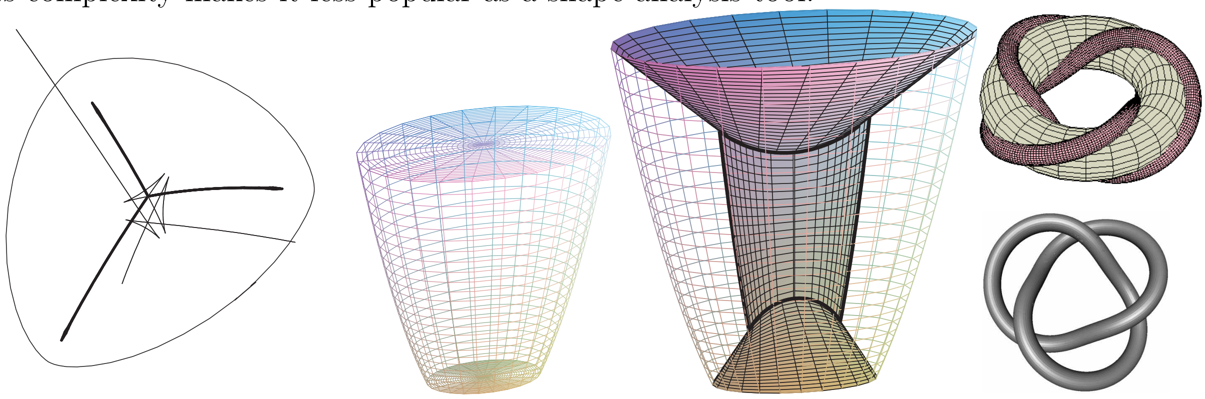
More recently I have worked on using *curves* in images to help with reconstruction (this involves some pretty heavy algebra, completely unsuitable for writing about here!); and also a thorough analysis of how illumination affects our views of surfaces as we move past them. This work is done jointly with James Damon from the University of North Carolina and a former postdoc employed on a European Community funded project, Gareth Haslinger. It actually requires the full power of recent methods of classification (due partly to Damon, and partly to three mathematicians all of whom have connexions with Liverpool, Bill Bruce, Andrew du Plessis and Terry Wall). To emphasize how important shadows are to our understanding of scenes take a look at the picture below. If you can't quite make out what it is, turn the page upside-down!



Symmetry The symmetric shape on the left in the figure below has a straight 'axis of symmetry'. There is a nice way to generalize this idea. Circles centred on the axis of symmetry will be tangent to the shape in two places, as in the middle figure. So for a more complicated shape let's draw circles tangent twice to the shape (and lying entirely inside) and trace their centres, as in the blue lines on the right. These form the *medial axis* of the shape.



The instabilities—singularities—here occur at the centres of three-times-tangent circles, where the medial axis branches in a Y shape, and at the endpoints where the medial axis just stops and the circle becomes *very tangent* to the shape: the two contact places coincide. Imagine the circle near the head of the fish sliding north-west and shrinking until it fits snugly in the top left corner of the shape: the centre will be at the extreme end of the medial axis. The medial axis forms a sort of 'skeleton' of the shape, encoding information about it in a concise way—In fact it is very widely used to compare and classify shapes. (There are lots of references to 'medial axis' and the related 'symmetry set' on the internet.) Symmetry sets also trace centres of circles but allow the circles to extend outside the shape. An example is shown on the left in the next figure: the shape is the outer 'rounded triangle', the medial axis is the thick Y-curve and the symmetry set is the complicated cuspy thin line. For example the end of the spike at the north-west is the centre of a circle 'very tangent' to the shape at the south-east and completely *enclosing* the shape. Although the symmetry set carries 'more information' than the medial axis its complexity makes it less popular as a shape-analysis tool.



The middle two figures show a sort of 'elliptical bin' with a top and bottom, and, somewhat larger, its medial axis. Here we use centres of spheres, tangent twice, to replace centres of circles in the examples above. Medial axes and more general 'skeletal structures' are being investigated intensively as they provide an excellent way of encoding 3D shapes. On the right is shown a 'torus knot', first wrapped around a torus and then detached from it. A particularly interesting challenge is to describe in detail the medial axis of 'knot complements', that is all 3-space with a knot (such as the one shown) removed. Knot complements arise in other contexts, such as 'hyperbolic 3-manifolds' which are related to the famous *Poincaré Conjecture*.