Inflections and vertices of germs of singular plane curves

Farid Tari

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The problem(s)

- An -versal deformation \((t^2, t^3 + ut)\) of the cusp curve.
- The group \(A\) does not preserve the geometry of curves.

How to study deformations of germs of plane curves taking into consideration singularities, inflections and vertices? (Question valid for singular varieties \(\mathbb{R}^n\). So far there is no general theory for this.)

How many inflections and vertices are concentrated at a singular point of a plane curve?
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Work on single germs


None of the approaches has built in them a way to study deformations of the curve as well as its flat (and round) geometry.

Observation: We are dealing with parametrised curves. There is some work on germs of curves given by equations in [Diatta and Giblin, 2007] and [Capitano and Diatta, 2009].
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In [M. Salarinoghabi and -, Flat and round singularity theory, preprint 2015] we propose the following approach (similar to the one used to study bifurcations of vector fields and implicit differential equations).

Two families $\gamma_s$ and $\eta_u$ of a plane curve singularity are FRS-fibre topologically equivalent if there exists a germ of a homeomorphism $k: (\mathbb{R}^m, (S_1, 0)) \to (\mathbb{R}^m, (S_2, 0))$, where $S_1$ and $S_2$ are stratifications of the parameter spaces, such that

(i) $\gamma_s$ is diffeomorphic to $\eta_{k(s)}$ in each stratum of $S_1$;

(ii) $\gamma_s$ and $\eta_{k(s)}$ have the same number of inflections and vertices in each stratum;

(iii) the relative position of the inflections, vertices, singularities and double points on $\gamma_s$ and $\eta_{k(s)}$ is the same in each stratum.

Basically, FRS-equivalence means the instantaneous configurations of the curves $\gamma_s$ and $\eta_{k(s)}$ (as well as their vertices, inflections and evolutes) are the same.
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- Smooth curves with vertices of finite order
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The smooth case: at a vertex of finite order

Away from inflections, we have the theory of Lagrangian singularities and caustics (generating family is the (big) family of distance squared functions).

$R$-versal deformations $\approx R$-versal deformation of $D_{a_0}(t_0)$.

Birth of an inward and an outward vertex at a second order vertex ($\kappa'(t_0) = \kappa''(t_0) = 0$, $\kappa'''(t_0) \neq 0$).

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At an inflections of finite order

\[ \kappa(t_0) = \kappa'(t_0) = \ldots = \kappa(k-1)(t_0) = 0; \]

so here we have an accumulation of inflections and vertices.

• For the inflections, we use the height functions.

• For the vertices and the evolute, we consider the inverse of the stereographic projection \( \phi: \mathbb{R}^2 \to \mathbb{S}^2 \) to understand what happens at infinity.

The contact of \( \gamma(t) = (t, \beta(t)) \) with circles in \( \mathbb{R}^2 \) is the same as that \( \phi(\gamma) \) with circles in \( \mathbb{S}^2 \), and this is measured by the singularities of the members of the family \( D: \mathcal{J} \times \mathbb{S}^2 \to \mathbb{R} \), with

\[
D(t, v) = v_1 t + v_2 \beta(t) - v_3 1 + t^2 + \beta(t)^2,
\]

where \( \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{S}^2 \).

We have \( H(0, 1) \) has and \( A_k \)-singularity \( \iff D(0, 1, 0) \) has an \( A_k \)-singularity.
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\[
\mathcal{D}(t, v) = \frac{v_1 t + v_2 \beta(t) - v_3}{1 + t^2 + \beta(t)^2}, \quad v = (v_1, v_1, v_3) \in S^2.
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At an inflections of finite order

\[ \gamma_s = (t, \beta_s(t)) \] be a family of curves with \( \gamma_0 = \gamma \).

We have a big family of height functions \( \tilde{H}(t, v, s) \) and a big family \( \tilde{D}(t, v, s) \).

Then, \( \tilde{H} \) is an \( \mathbb{R}^+ \)-versal \( \iff \) \( \tilde{D} \) is \( \mathbb{R}^+ \)-versal.

At an inflection of finite order:

\[ \text{FR-versal deformations} = \mathbb{R}^+ \text{-versal deformation of } \mathcal{H}_n(t_0) \]
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At an inflection of finite order:

**FR-versal deformations = $\mathcal{R}^+\text{-versal deformation of } H_{n(t_0)}$.**
FR-deformation of a second order inflection

Change from outward to inward vertex through infinity at a second order inflection.

model: \((t, t^4 + ut^2)\).

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**FR-deformation of a second order inflection**

Change from outward to inward vertex through infinity at a second order inflection. **FR-model:** \((t, t^4 + ut^2)\).
Singular curves: the cusp case

A-model: $(t^2, t^3)$

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$\gamma(t) = \gamma_0(t) = (\alpha(t), \beta(t))$, singular at $t = 0$.

For $t_0 \in J$, we take $\gamma(t_0)$ to be the origin and write

$\gamma(t) = (\alpha'(t_0)(t - t_0) + \frac{1}{2!}\alpha''(t_0)(t - t_0)^2 + ..., \beta'(t_0)(t - t_0) + \frac{1}{2!}\beta''(t_0)(t - t_0)^2 + ...)$

This gives the Taylor map (of order $k$)

$\phi_{\gamma}(t_0) = (\alpha'(t_0), \frac{1}{2!}\alpha''(t_0), ..., \beta'(t_0), \frac{1}{2!}\beta''(t_0), ...)$

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Singular curves: the cusp case

\( A \)-model : \((t^2, t^3)\)
Singular curves: the cusp case

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This gives the Taylor map (of order \( k \)) \( j^k \phi_{\gamma} : J \to J^k(1, 2) \), with

\[
j^k \phi_{\gamma}(t_0) = (\alpha'(t_0), \frac{1}{2!}\alpha''(t_0), \ldots; \beta'(t_0), \frac{1}{2!}\beta''(t_0), \ldots).
\]
The cusp case

Denote by \((a_1, a_2, \ldots; b_1, b_2, \ldots)\) the coordinates in \(J_{k(1,2)}\) (identified with \(\mathbb{R}^k \times \mathbb{R}^k\)).

We have the following stratification \(S\) of \(J_{k(1,2)}\) at a cusp:

**Cusps (C):**
\[a_1 = b_1 = 0\]

**Inflections (I):**
\[a_1 b_2 - a_2 b_1 = 0\]

**Vertices (V):**
\[-2(a_1 b_2 - a_2 b_1)(a_1 a_2 + b_1 b_2) + (a_1^2 + b_1^2)(a_1 b_3 - a_3 b_1) = 0\]

We have \(C \subset I\), \(C \subset V\), \(V = V_1 \cup V_2\), \(V_1\) tangent to \(I\) along \(C\) and \(V_2\) transverse to \(C\).

(A product stratification and a 2-dimensional transverse slice is given by intersecting with the plane \((a_1, 0, \ldots, 0; b_1, 0, \ldots, 0)\).)
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**Vertices (V):**

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We have \(C \subset I, \ C \subset V\)
Denote by \((a_1, a_2, \ldots; b_1, b_2, \ldots)\) the coordinates in \(J^k(1, 2)\) (identified with \(\mathbb{R}^k \times \mathbb{R}^k\)).

We have the following stratification \(S\) of \(J^k(1, 2)\) at a cusp:

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The cusp case

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(A product stratification and a 2-dimensional transverse slice is given by intersecting with the plane \((a_1, 0, \ldots, 0; b_1, 0, \ldots, 0)\).)
The cusp case

Definition

We say that a $1$-parameter family of curves $\gamma_s$, with $\gamma_0 = \gamma$ a cusp curve, is FRS-generic if the family of Taylor maps $j_k \Phi : (\mathbb{R} \times \mathbb{R}, (0, 0)) \to J_k(1, 2)$, given by $j_k \Phi(t, s) = j_k \phi_{\gamma_s}(t)$, is transverse to the cusp stratum $C$, and hence to the stratification $S$. 

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Inflections and vertices of germs of singular plane curves
The cusp case

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Inflections and vertices of germs of singular plane curves
The cusp case

A 1-parameter family of curves $\gamma_s$, with $\gamma_0 = \gamma$, a cusp curve is \textit{FRS}-generic if and only if it is an \textit{Ae}-deformation of the cusp singularity of $\gamma$. Therefore, for the cusp singularity, \textit{FRS}-generic = \textit{Ae}-versal.
A 1-parameter family of curves $\gamma_s$, with $\gamma_0 = \gamma$, a cusp curve is $FRS$-generic if and only if it is an $A_e$-deformation of the cusp singularity of $\gamma$. 
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The cusp case
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The image of $\gamma_s$ by the Taylor map and its position with respect to the stratification of $J^k(1, 2)$. 
The cusp case: \textit{FRS}-bifurcations

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Inflections and vertices of germs of singular plane curves
The cusp case: \textit{FRS}-bifurcations

\textit{FRS}-bifurcations of the cusp. \textit{FRS}-model: \((t^2, t^3 + ut)\).
The evolute of a cusp

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Inflections and vertices of germs of singular plane curves
The evolute of a cusp

*FRS*-generic bifurcations of a cusp curve and of its evolute.

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Inflections and vertices of germs of singular plane curves
The ramphoid cusp

We have to work a bit harder here (especially for the multi-local strata). We need to take into consideration the $A_h$-model of the ramphoid cusp:

\[(t_2, t_4 + t_5), (t_2, t_5 + t_6), (t_2, t_5)\].

We consider a ramphoid cusp $A_h$ equivalent to $(t_2, t_4 + t_5)$ and obtain a stratification of the jet space. We found that a model of an FRS-versal deformation is $(t_2, t_4 + t_5 + t_6 + u t_3 + v t_4)$. 

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Inflections and vertices of germs of singular plane curves
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FRS-Bifurcations of the ramphoid cusp

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Inflections and vertices of germs of singular plane curves
FRS-Bifurcations of the ramphoid cusp
Bifurcations of the evolute
Bifurcations of the evolute

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Inflections and vertices of germs of singular plane curves
An application of *FRS*-deformations to profiles

The two generic types of geometric deformations of the beaks singularity of the profile (black) and of its evolute (red) [M. Hasegawa and M. Salarinoghabi, in preparation].

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Inflections and vertices of germs of singular plane curves
An application of *FRS*-deformations to profiles

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Counting inflections and vertices

\[ \gamma(t) = (x(t), y(t)) = \left( t^m, \sum_{i=m+1}^{k} a_i t^i + O(t^{k+1}) \right). \]

Number of inflections at \( t = 0 \):
\[ I = \text{ord} \left( x' y'' - x'' y' \right). \]

Number of vertices at \( t = 0 \):
\[ V = \text{ord} \left( (x'^2 + y'^2) (x''' y' - x'' y') + 3(x' x'' + y' y'') (x'' y' - x' y'') \right). \]

We have formulae for \( I \) and \( V \).
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[With Fabio Scalco Dias (UF Itajuba)]

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We have formulae for $I$ and $V$. 

Farid Tari Inflections and vertices of germs of singular plane curves
Example: the cusp

We have $V = 3$ and $I = 2$. The evolute of the cusp is the union of $\ell = 2$ lines and a smooth curve, which we call the proper evolute of the cusp curve.

Observe that $I = \ell$. 

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Inflections and vertices of germs of singular plane curves
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Example: the cusp

We have $V = 3$ and $I = 2$.
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Counting vertices and inflections
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Inflections and vertices of germs of singular plane curves
Counting vertices and inflections

Theorem

\[ I = \ell \]

\( l \) lines

Proper evolute

\( \mu = \text{Milnor number } D_{c_0} \)

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Inflections and vertices of germs of singular plane curves
Counting vertices and inflections

Theorem

\[ I = \ell \]
\[ V = I + \mu - 2 \]
Realisations

Q: Is there a deformation $\gamma_s$ of $\gamma_0$ such that $\gamma_s$ has $I$ inflections and $V$ vertices?

- **Cusp ($A_2$):** $V = 3$, $I = 2$
- **Ramphoid cusp ($A_4$):** $V = 5$, $I = 3$
- **$E_6$-singularity:** $\gamma(t) = (t^3, t^4)$, $V = 7$ and $I = 4$.

The curve $\gamma_\alpha(t, s) = (t^3 - 0.018s, t^4 - 0.01s^2)$, for $s$ near zero, has $V = 7$ vertices and $I = 4$ inflections.

In general ???
Q: Is there a deformation $\gamma_s$ of $\gamma_0$ such that $\gamma_s$ has $I$ inflections and $V$ vertices?

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Inflections and vertices of germs of singular plane curves
Realisations

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- $E_6$-singularity: $\gamma(t) = (t^3, t^4)$, $V = 7$ and $I = 4$.

The curve $\gamma_\alpha = (t^3 - 0.018st, t^4 - 0.01st^2)$, for $s$ near zero, has $V = 7$ vertices and $I = 4$ inflections.
Realisations

Q: Is there a deformation $\gamma_s$ of $\gamma_0$ such that $\gamma_s$ has $l$ inflections and $V$ vertices?

True for

- Cusp ($A_2$): $V = 3, l = 2$
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- $E_6$-singularity: $\gamma(t) = (t^3, t^4)$, $V = 7$ and $l = 4$.

The curve $\gamma_\alpha = (t^3 - 0.018st, t^4 - 0.01st^2)$, for $s$ near zero, has $V = 7$ vertices and $l = 4$ inflections.

- In general ???
Thank you and Happy Birthday Victor!