On the global singularity theory of Legendre mappings

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A simple example of a wave front is an equidistant of a smooth closed hypersurface in $\mathbb{R}^n$.

A generic wave front in the plane can have only cusps (singularities of type $A_2$) and transversal intersections of two smooth branches (singularities of type $A_1^2$; nonsingular points of a front are called singularities of type $A_1$).
A generic wave front in three-dimensional space can have singularities of types $A_2$ (cusp- 
didal edges), $A_1^2$ (transversal intersections of two smooth branches), $A_3$ (swallowtails), $A_2A_1$ 
(transversal intersections of cuspidal edges and smooth branches), and $A_3^2$ (transversal inter-
sections of three smooth branches).
1 Legendre mappings

Let $E$ be a smooth $(2n - 1)$-dimensional manifold. A contact element on $E$ is a pair of the form $(\pi, x)$, where $x \in E$ and $\pi \subset T_x E$ is a tangent hyperplane at the point $x$.

Consider a smooth field of contact elements on $E$. Locally such a field is given by a smooth 1-form $\omega$ as the field of its zeros. A field of contact elements is called a contact structure if $\omega \wedge (d\omega)^{n-1} \neq 0$. Integral $(n - 1)$-dimensional manifolds of a contact structure are called Legendre manifolds.

We fix a contact structure on $E$. A smooth fiber bundle $\rho : E \to V$ over a smooth $n$-dimensional manifold $V$ is called a Legendre bundle if all its fibers are Legendre submanifolds.

Let $\rho$ be a Legendre bundle and $L$ be a smooth Legendre submanifold in $E$. Then the mapping $f = \rho \circ i : L \xrightarrow{i} E \xrightarrow{\rho} V$, where $i : L \hookrightarrow E$ is the identical embedding, is called Legendre mapping. The range $F = f(L)$ of the Legendre mapping $f$ is called wave front.

We consider only proper Legendre mappings.

**Example.** Let us consider the space $\mathbb{R}^{2n-1} = \{(y, t, x, p, q)\}$, where

$$t = (t_1, \ldots, t_k), \quad x = (x_1, \ldots, x_k), \quad p = (p_{k+1}, \ldots, p_{n-1}), \quad q = (q_{k+1}, \ldots, q_{n-1}).$$

The form $dy + tdx + pdq$ defines a contact structure on $\mathbb{R}^{2n-1}$. The mapping

$$\rho : \mathbb{R}^{2n-1} \to \mathbb{R}^n, \quad \rho : (y, t, x, p, q) \mapsto (y, x, q)$$

is a Legendre bundle over $\mathbb{R}^n = \{(y, x, q)\}$.

Let $S = S(t, q)$ be a family of smooth functions of $t$ smoothly depending on $q$. Then the system of equations

$$y = -S(t, q) + t \frac{\partial S(t, q)}{\partial t}, \quad x = -\frac{\partial S(t, q)}{\partial t}, \quad p = \frac{\partial S(t, q)}{\partial q}$$

defines a smooth Legendre submanifold $\mathbb{R}^{n-1} = \{(t, q)\} \xrightarrow{i} \mathbb{R}^{2n-1}$. The mapping

$$\mathbb{R}^{n-1} \to \mathbb{R}^n, \quad (t, q) \mapsto \left(-S(t, q) + t \frac{\partial S(t, q)}{\partial t}, -\frac{\partial S(t, q)}{\partial t}, q\right)$$

is a Legendre mapping $f : \mathbb{R}^{n-1} \xrightarrow{i} \mathbb{R}^{2n-1} \xrightarrow{\rho} \mathbb{R}^n$. 

3
2 Legendre singularities

Legendre mappings

\[ f_1 : L_1 \overset{i_1}{\rightarrow} E_1 \overset{\rho_1}{\rightarrow} V_1, \quad f_2 : L_2 \overset{i_2}{\rightarrow} E_2 \overset{\rho_2}{\rightarrow} V_2 \]

are called equivalent if there exist diffeomorphisms \( \Phi : E_1 \rightarrow E_2, \varphi : V_1 \rightarrow V_2 \), and \( \varepsilon : L_1 \rightarrow L_2 \) such that \( \Phi \) transforms the contact structure on \( E_1 \) to the contact structure on \( E_2 \) and the diagram

\[
\begin{array}{c}
L_1 \overset{i_1}{\rightarrow} E_1 \overset{\rho_1}{\rightarrow} V_1 \\
\downarrow \varepsilon \quad \downarrow \Phi \\
L_2 \overset{i_2}{\rightarrow} E_2 \overset{\rho_2}{\rightarrow} V_2
\end{array}
\]

is commutative.

The equivalence class of a Legendre mapping germ is called singularity.

We equip the space \( W \) of all embeddings \( i : L \rightarrow E \) with Whitney’s fine \( C^\infty \)-topology. A Legendre mapping

\[ f = \rho \circ i : L \overset{i}{\rightarrow} E \overset{\rho}{\rightarrow} V \]

is called stable if every Legendre mapping close to \( f \) is equivalent to \( f \).

Arnold’s Theorem. Stable mappings are dense in the space of Legendre mappings to a smooth manifold of the dimension \( n \leq 6 \). Their singularities are equivalent to singularities of the mapping

\[ \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n, \ (t, q) \mapsto \left( -S(t, q) + t \frac{\partial S(t, q)}{\partial t}, -\frac{\partial S(t, q)}{\partial t}, q \right) . \]

at the origin, where \( S = S(t, q) \) is a function of one of the following types (\( \mu \) is integer):

- \( A_\mu : S = t_1^{\mu+1} + q_{\mu-1}t_1^{\mu-1} + \ldots + q_{2}t_1^2, \ 1 \leq \mu \leq n; \)
- \( D^\pm_\mu : S = t_1^2t_2^2 + t_2^{\mu-1} + q_{\mu-1}t_2^{\mu-2} + \ldots + q_{3}t_2^2, \ 4 \leq \mu \leq n; \)
- \( E_6 : S = t_1^3 + t_2^3 + q_3t_1t_2^2 + q_4t_1t_2 + q_3t_2^2, \ (\mu = 6 \leq n). \)

The number \( \mu \) is called the codimension of the singularity. If \( \mu \) is odd, then singularities of types \( D^+_\mu \) and \( D^-_\mu \) are equivalent (they are denoted by \( D_\mu \)).

Singularities of types \( A_\mu, D^\pm_\mu, E_6 \) are stable for Legendre mappings to a smooth manifold of any dimension \( n \).
3 Multisingularities of Legendre mappings

Let $\mathbb{S}$ be the free Abelian multiplicative semigroup whose generators are the symbols

$$A_\mu (\mu = 1, 2, \ldots), \quad D_{2+2k}^-, D_{2+2k}^+, D_{3+2k} (k = 1, 2, \ldots), \quad E_6.$$ 

The identity element of this semigroup is denoted by $1$.

We consider an arbitrary non-identity element $A = X_1 \ldots X_p \in \mathbb{S}$, where $X_j, j = 1, \ldots, p$, is any generator of $\mathbb{S}$.

**Definition.** A Legendre mapping $f : L \to V$ has a multisingularity of type $A$ at a point $y \in V$ if:

1) $f^{-1}(y)$ consists of $p$ distinct points;

2) there exists an order $x_1, \ldots, x_p$ of points from $f^{-1}(y)$ such that $f$ has at these points singularities of types $X_1, \ldots, X_p$, respectively.

We say also that the mapping $f$ has a multisingularity of type $1$ at each point of the complement to the front $\mathcal{F} = f(L)$ in $V$. The front $\mathcal{F}$ has a singularity of type $A \in \mathbb{S}$ at a point $y \in V$ if $f$ has a multisingularity of type $A$ at $y$.

The sum of the codimensions of singularities of types $X_1, \ldots, X_p$ is called the codimension of a multisingularity of type $A$ and is denoted $\text{codim} A$. Multisingularities of generic Legendre mappings to a space of the dimension $n \leq 6$ are multisingularities of types $A \in \mathbb{S}$ such that $\text{codim} A \leq n$. 

4 The adjacency indices of Legendre multisingularities

Let us consider a stable Legendre mapping $f : L \rightarrow V$. The set $A_f$ of points $y \in V$ such that $f$ has a multisingularity of type $A$ at $y$ is a smooth submanifold in $V$. Its codimension is equal to codim $A$.

Let $f$ has a multisingularity of type $B \in S$ at $y \in V$, where codim $B = c$. We take a neighborhood $U$ of the origin 0 in $\mathbb{R}^c$ and consider a smooth embedding $h : U \rightarrow V$ such that $h(0) = y$ and the submanifold $h(U) \subset V$ is transversal to the manifold $B_f$ at $y$.

Let $D_\varepsilon \subset \mathbb{R}^c$ be the open $c$-dimensional ball of radius $\varepsilon > 0$ centered at 0. Then there exists a positive $\varepsilon_0 = \varepsilon_0(f, y, h)$ such that for any $A \in S$ and $\varepsilon < \varepsilon_0$ all connected components of the intersection $h(S_\varepsilon) \cap A_f$ are contractible and the number of these components depends only on $A$ and $B$. This number is denoted by $J_A(B)$ and, in the case $A \neq B$, is called the adjacency index of multisingularities of type $B$ to multisingularities of type $A$.

Example.

$$J_{A_1}(A_2) = J_{A_3}(A_3) = 2, \quad J_{A_1}(A_3) = 4, \quad J_{A_2^1}(A_3) = 1.$$
5 The adjacency indices of multisingularities of Legendre mappings to spaces of low dimension

Proposition. For any $A, B, C \in S$,
\[ J_A(B \cdot C) = \sum_{(X, Y) \in S \times S : X \cdot Y = A} J_X(B)J_Y(C). \]

Theorem. Let $A = A_{\mu_1}^{k_1} \ldots A_{\mu_p}^{k_p} \in S$, where $\mu_1, \ldots, \mu_p$ are positive integers different in pairs. Then for any positive integer $\mu$
\[ J_A(A_{\mu}) = \sum_{0 \leq k_0 \leq N, k_0 \equiv N \pmod{2}} \frac{(k_0 + k_1 + \ldots + k_p)!}{k_0! \cdot k_1! \cdot \ldots \cdot k_p!}, \quad \text{where} \quad N = \mu + 1 - \sum_{i=1}^{p} \mu_i (\mu_i + 1). \]

Corollary. The nonzero adjacency indices $J_A(A_{\mu})$, $4 \leq \mu \leq 6$, are given in the following tables:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$J_A(A_4)$</th>
<th>$J_A(A_5)$</th>
<th>$J_A(A_6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$1$</td>
<td>$A_1$</td>
<td>$A_2$</td>
</tr>
<tr>
<td>$J_A(A_4)$</td>
<td>$3$</td>
<td>$6$</td>
<td>$4$</td>
</tr>
</tbody>
</table>
Theorem. The nonzero adjacency indices $J_A(D^\mu)$, $\mu \leq 6$, are given in the following tables:

<table>
<thead>
<tr>
<th>$J_A(D^+_4)$</th>
<th>$1$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_2^2$</th>
<th>$A_3$;</th>
<th>$A$</th>
<th>$1$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_2^2$</th>
<th>$A_3$</th>
<th>$A_3^2$</th>
<th>$A_1^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$3$</td>
<td>$4$</td>
<td>$2$</td>
<td>$1$</td>
<td>$2$;</td>
<td>$A$</td>
<td>$7$</td>
<td>$14$</td>
<td>$6$</td>
<td>$9$</td>
<td>$6$</td>
<td>$2$;</td>
<td></td>
</tr>
<tr>
<td>$J_A(D^-_5)$</td>
<td>$6$</td>
<td>$15$</td>
<td>$9$</td>
<td>$12$</td>
<td>$8$</td>
<td>$6$</td>
<td>$3$</td>
<td>$1$</td>
<td>$1$</td>
<td>$2$</td>
<td>$2$</td>
<td>$1$;</td>
<td></td>
</tr>
</tbody>
</table>

| $J_A(D^+_5)$ | $6$ | $16$ | $10$ | $14$ | $10$ | $4$ | $3$ | $1$ | $2$ | $6$ | $2$ | $2$ | $A_2A_2^2$ |
| $J_A(D^-_5)$ | $2$ | $2$ | $2$ | $A_3A_2$; |

| $J_A(D^+_6)$ | $11$ | $34$ | $20$ | $40$ | $20$ | $26$ | $18$ | $1$ | $3$ | $6$ | $18$ | $4$ | $4$ | $A_4^1$ |
| $J_A(D^-_6)$ | $2$ | $2$ | $4$ | $2$ | $2$ | $A_3A_2^2$ |

Theorem. The nonzero adjacency indices $J_A(E_6)$ are given in the following tables:

| $J_A(E_6)$ | $1$ | $A_1$ | $A_2$ | $A_2^2$ | $A_3$ | $A_2A_1$ | $A_3^2$ | $D_4^+$ | $D_4^-$ | $D_4$ | $A_4$ | $A_3A_1$ | $A_3^2$ | $A_2A_1^2$ |
| $A$ | $5$ | $15$ | $11$ | $17$ | $10$ | $16$ | $6$ | $1$ | $1$ | $6$ | $6$ | $2$ | $5$ |
| $J_A(E_6)$ | $5$ | $15$ | $11$ | $17$ | $10$ | $16$ | $6$ | $1$ | $1$ | $6$ | $6$ | $2$ | $5$ |

| $J_A(E_6)$ | $1$ | $2$ | $2$ | $A_2A_1$; | $J_A(E_6)$ | $2$ | $2$ | $1$; |
6 The coexistence of Legendre multisyngularities

Theorem. Let \( f \) be a stable Legendre mapping to a smooth manifold of the dimension \( n \leq 6 \). Assume that the front of this mapping is compact. Then:

1) For any \( \mathcal{A} \in \mathbb{S} \setminus \{1\} \) such that codim \( \mathcal{A} < n \) and codim \( \mathcal{A} \equiv n - 1 \mod 2 \), the Euler characteristic \( \chi(\mathcal{A}_f) \) of the manifold \( \mathcal{A}_f \) is calculated by the formula

\[
\chi(\mathcal{A}_f) = \sum \limits_{\mathcal{B}} K_\mathcal{A}(\mathcal{B}) \chi(\mathcal{B}_f),
\]

where the summation is taken over all \( \mathcal{B} \in \mathbb{S} \) such that codim \( \mathcal{B} \in [\text{codim} \mathcal{A} + 1, n] \), codim \( \mathcal{B} \equiv n \mod 2 \); the coefficients \( K_\mathcal{A}(\mathcal{B}) \) are rational numbers depending only on \( \mathcal{A} \) and \( \mathcal{B} \).

2) If \( n \) is odd, then (1) implies the following relations:

\[
\begin{align*}
\chi(A_2) &= \chi(A_3) + \chi(A_2 A_1) - \frac{5}{3} \chi(D_5) - \chi(D_4^+ A_1) - 3 \chi(D_4^- A_1) - 2 \chi(A_5) - 2 \chi(A_4 A_1) - \frac{3}{2} \chi(A_3 A_2) - 2 \chi(A_3 A_2^2) - 2 \chi(A_2 A_2^2) - 2 \chi(A_2 A_2^2), \\
\chi(A_1^2) &= \frac{1}{2} \chi(A_3) + \chi(A_2 A_1) + 3 \chi(A_1^2) - \frac{7}{2} \chi(D_5) - \frac{5}{2} \chi(D_4^+ A_1) - \frac{19}{2} \chi(D_4^- A_1) - 2 \chi(A_5) - 2 \chi(A_4 A_1) - \frac{3}{2} \chi(A_3 A_2) - 2 \chi(A_3 A_2^2) - 7 \chi(A_2 A_2^2) - 9 \chi(A_2 A_2^2) - 20 \chi(A_1^2), \\
\chi(D_4^+) &= \frac{1}{2} \chi(D_5) + \chi(D_4^+ A_1), \\
\chi(D_4^-) &= \frac{1}{2} \chi(D_5) + \chi(D_4^- A_1), \\
\chi(A_4) &= \chi(D_5) + \chi(A_5) + \chi(A_4 A_1), \\
\chi(A_3 A_1) &= \chi(D_5) + \chi(D_4^+ A_1) + 3 \chi(D_4^- A_1) + \chi(A_5) + \chi(A_4 A_1) + \chi(A_3 A_2) + 2 \chi(A_3 A_1^2), \\
\chi(A_3 A_2) &= \frac{1}{2} \chi(A_5) + \chi(A_3 A_2) + \chi(A_3 A_2^2), \\
\chi(A_2 A_2^2) &= \frac{1}{2} \chi(D_5) + \chi(A_4 A_1) + \frac{1}{2} \chi(A_3 A_2) + \chi(A_3 A_1^3) + 2 \chi(A_2 A_1) + 3 \chi(A_2 A_1^2), \\
\chi(A_1^4) &= \chi(D_4^- A_1) + \frac{1}{2} \chi(A_3 A_1^2) + \chi(A_2 A_2^3) + 5 \chi(A_1^5).
\end{align*}
\]
3) If \( n \) is even, then (1) implies the following relations:

\[
\begin{align*}
\chi(A_1) &= \chi(A_2) + 2\chi(A_1^2) - 5\chi(D_0^-) - 2\chi(A_1) - \frac{5}{2}\chi(A_3A_1) - 2\chi(A_2^2) - 4\chi(A_2A_1^2) - 8\chi(A_4) + 13\chi(E_6) + 7\chi(D_6^-) + 27\chi(D_0^+); \\
&\quad + 2\chi(A_3^3) + 5\chi(D_1^+A_1^2) + 51\chi(D_4^-A_1^2) + 9\chi(A_6) + 14\chi(A_5A_1) + \frac{5}{2}\chi(A_4A_2) + 21\chi(A_1^2A_1^2) + 10\chi(A_3^2) + \frac{31}{2}\chi(A_2A_2A_1) + 31\chi(A_2A_2^2) + 12\chi(A_2^3) + 24\chi(A_2^2A_1^2) + 48\chi(A_2A_1^3) + 96\chi(A_1^4); \\
\chi(A_2A_1) &= \chi(A_1) + \chi(A_3A_1) + 2\chi(A_2^2) + 2\chi(A_2A_2^2) - 5\chi(E_6) - 3\chi(D_0^+) - 7\chi(D_6^-); \\
&\quad - \frac{1}{4}\chi(D_5A_1) - \chi(D_1^+A_2) - 8\chi(D_4^-A_2) - 2\chi(D_1^+A_1^2) - 6\chi(D_4^-A_1^2) - 4\chi(A_6) - 5\chi(A_5A_1) - 5\chi(A_4A_2) - 6\chi(A_3A_2^2) - 4\chi(A_3^2) - \frac{11}{2}\chi(A_3A_2A_1) - 6\chi(A_3A_1^2) - 6\chi(A_2^3) - 8\chi(A_2A_1^2) - 8\chi(A_2A_1^2); \\
\chi(A_1^2) &= \chi(D_5^+) + \frac{1}{2}\chi(A_3A_1) + \chi(A_2A_1^2) + 4\chi(A_1^4) - \frac{5}{2}\chi(E_6) - \chi(D_0^+) - 5\chi(D_6^-); \\
&\quad - 5\chi(A_5A_1) - \frac{1}{2}\chi(D_4^-A_2) - \frac{11}{2}\chi(D_4^-A_1^2) - 16\chi(D_4^-A_1^2) - 8\chi(A_6) - \frac{5}{2}\chi(A_4A_2) - 5\chi(A_4A_1^2) - 2\chi(A_3^2) - \frac{5}{2}\chi(A_3A_2A_1) - \frac{11}{2}\chi(A_3A_1^2) - 2\chi(A_2^3) - 6\chi(A_2A_1^2) - 16\chi(A_2A_1) - 40\chi(A_1^4); \\
\chi(D_5) &= \chi(E_6) + \chi(D_0^-) + \chi(D_6^-) + \chi(D_5A_1), \\
\chi(D_0^-A_1) &= \chi(D_0^-) + \frac{1}{2}\chi(D_5A_1) + \chi(D_1^+A_2) + 2\chi(D_1^+A_1^2), \\
\chi(D_4^-A_1) &= \chi(D_4^-) + \frac{1}{2}\chi(D_5A_1) + \chi(D_1^+A_2) + 2\chi(D_1^+A_1^2), \\
\chi(A_6) &= \chi(E_6) + 2\chi(D_0^-) + \chi(A_6) + \chi(A_5A_1), \\
\chi(A_4A_1) &= \chi(E_6) + \chi(D_5A_1) + \chi(A_6) + \chi(A_5A_1) + \chi(A_4A_2) + 2\chi(A_4A_1^2), \\
\chi(A_3A_2) &= \chi(D_5^+) + \chi(D_0^-) + \chi(D_1^+A_2) + 3\chi(D_1^+A_2) \\
&\quad + \chi(A_6) + \chi(A_4A_2) + 2\chi(A_3^2) + \chi(A_3A_2A_1), \\
\chi(A_3A_1^2) &= \chi(D_6^-) + \chi(D_5A_1) + \chi(D_1^+A_1^2) + 3\chi(D_1^+A_1^2) \\
&\quad + \chi(A_5A_1) + \chi(A_4A_1^2) + \chi(A_3A_2A_1) + 3\chi(A_3A_1), \\
\chi(A_2A_1^2) &= \frac{1}{2}\chi(E_6) + \frac{1}{2}\chi(A_5A_1) + \chi(A_4A_2) + \chi(A_3A_2A_1) + 3\chi(A_3^2) + 2\chi(A_2A_1^2), \\
\chi(A_2A_1^2) &= \frac{1}{2}\chi(D_5A_1) + \chi(D_4^-A_2) \\
&\quad + \chi(A_4A_1^2) + \frac{1}{2}\chi(A_3A_2A_1) + \chi(A_3A_1^2) + 2\chi(A_2A_1^2) + 4\chi(A_2A_1^2), \\
\chi(A_1^2) &= \chi(D_1^+A_1^2) + \frac{1}{2}\chi(A_3A_1^2) + \chi(A_2A_1^2) + 6\chi(A_1^4). \\
\end{align*}
\]
Corollary. The numbers of isolated singularities of types $D_5$, $A_5$, $A_3A_2$, and $A_3A_1^2$ on any compact generic front in a 5-dimensional space are even. The numbers of isolated singularities of types $D_5A_1$, $A_3A_2A_1$, and $A_3A_1^3$ on any compact generic front in a 6-dimensional space are even; the numbers of singularities of types $E_6$ and $A_5A_1$ have the same parity.