Rigidity of bi-Lipschitz equivalence of weighted homogeneous function-germs in the plane

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Aim is to discuss the classification of (real or complex) analytic functions in a neighborhood of an isolated singular point, that we can assume to be 0.

Notation:

\[ f : \mathbb{K}^n, 0 \to \mathbb{K}, 0, \mathbb{K} = \mathbb{R} \text{ ou } \mathbb{C}, \]

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Theorem: (A.Fernandes, —, 2012, [2])

If two weighted homogeneous (but not homogeneous) function-germs \((\mathbb{C}^2, 0) \to (\mathbb{C}, 0)\) are strongly bi-Lipschitz equivalent then they are analytically equivalent.
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Summary

• Analytic equivalence
• Bi-Lipschitz equivalence
• Henry and Parusinski’s example
• The main result and idea of the proof
• Open problems
Analytic equivalence

\[ f \sim g \quad \text{if} \quad \exists \text{ a germ of analytic diffeomorphism } h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0) \]

such that \( g = f \circ h \).

\[ \mathcal{R} = \{ h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0), \text{ germs of analytic diffeomorphisms} \} \]

is the group of right equivalences.
If \( f \) and \( g \) are analytically equivalent then \( h(\{g = c\}) = \{f = c\} \).

**Figure: Analytic Equivalence**
The classification of function germs with respect to analytic equivalence has moduli. The classical example is due to H. Whitney.

(H. Whitney, 1965)

\[ f_t(x, y) = xy(x - y)(x - ty), \quad 0 < |t| < 1. \]

For each \( t \), \( X_t = f_t^{-1}(0) \) is the set of 4 lines through the origin in \( \mathbb{K}^2 \).

The classical invariant is the cross ratio

\[ j = \frac{CA}{CB} / \frac{DA}{DB} \]
\[ j = \frac{AB}{AC} / \frac{AC}{AD} \]
Thom Levine Theorem ([3])

Let $U$ be a domain in $\mathbb{K}$, $W$ a neighborhood of 0 in $\mathbb{K}^n$ and $F : W \times U \to \mathbb{K}$, such that $F(0, t) = 0$, $F$ analytic. Let $f_t(x) = F(x, t)$, $\forall t \in U$, $\forall x \in W$.

The following conditions are equivalent

(i) There exists a family of analytic diffeomorphisms $H : W \times U \to W$, $h_t(0) = 0$, $\forall t$ $h_0(x) = x$, such that

$$f_t \circ h_t = f_0$$

(ii) There exists a family of analytic vector fields $v : W \times U \to \mathbb{K}^n$, $v(0, t) = 0$ $\forall t \in U$, such that

$$\frac{\partial f_t}{\partial t}(x) = df_t(x)(v(x, t)), \forall t \in U, \forall x \in W$$
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Write

\[ v(x, t) = \sum_1^n v_i(x, t) \frac{\partial}{\partial x_i}, \quad v_i(0, t) = 0, \quad V(x, t) = \frac{\partial}{\partial t} - v(x, t) \]

then

\[ (ii) \iff V(x, t) \text{ is tangent to the levels of } F \]
The conditions corresponding to (i) and (ii) for bi-Lipschitz equivalence are not equivalent. Clearly

\[(ii) \implies (i)\]

But, it is only known that the derivative of a bi-Lipschitz homeomorphism is bounded and exists almost everywhere.
Definition

The family $F : W \times U \to \mathbb{C}$, $F(0, t) = 0$, $f_t(x) = F(x, t)$ is strongly bi-Lipschitz trivial when there exists a continuous family of Lipschitz vector fields $v_t : W \to \mathbb{C}^n$, $v(0, t) = 0$ such that

$$\frac{\partial f_t}{\partial t}(x) = df_t(x)(v(x, t)), \quad \forall t \in U, \forall x \in W$$

Remark

If $f_t$ is strongly bi-Lipschitz trivial, then for all $t \neq t' \in U$,

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\[ f_t(x, y) = x^3 + y^6 - 3t^2xy^4 ; 0 < |t| < \frac{1}{2} \]

Parusinski and Henry proved that given \( t \neq s \), there is no \( \phi: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \) germ of bi-Lipschitz homeomorphism such that \( f_t = f_s \circ \phi \), i.e. \( f_t \) is not bi-Lipschitz equivalent to \( f_s \).

This shows in particular that the bi-Lipschitz classification of function germs has modality.

Remark

The bi-Lipschitz classification of analytic sets has no modality (Mostowskii, 1985).

Risler and Trotman asked in 1997:

\[ f^{-1}(0) \sim_{bi-Lipschitz} g^{-1}(0) \implies f \sim_{bi-Lipschitz} g ? \]

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\( f^{-1}(0) \sim_{bi-Lipschitz} g^{-1}(0) \implies f \sim_{bi-Lipschitz} g ? \)
The strategy used by them was to introduce a new invariant based on the observation that the bi-Lipschitz homeomorphism does not move much certains regions around the relative polar curves $\frac{\partial F}{\partial x} = 0$.

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In general, a bi-Lipschitz homeomorphism is not obtained by integration of a Lipschitz vector field. The motivation of Parusinski-Henry invariant comes from the following result.

**Theorem, Henry-Parusinski**

There is no bi-Lipschitz vector field

\[ V(x, y, t) = \frac{\partial}{\partial t} + v_1(x, y, t) \frac{\partial}{\partial x} + v_2(x, y, t) \frac{\partial}{\partial y}, \quad v_1(0, 0, t) = v_2(0, 0, t) = 0, \]

defined in a neighborhood of \((0, 0, t_0)\) and tangent to the levels of

\[ f(x, y, t) = x^3 - 3t^2xy^4 + y^6. \]
Proof:

Let us suppose $V$ does exist. Then $\frac{\partial F}{\partial v} = 0$, that is,

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial t} + v_1(x, y, t) \frac{\partial F}{\partial x} + v_2(x, y, t) \frac{\partial F}{\partial y} = 0$$

Now, let $\Gamma(x, y, t)$ be the family of polar curves of $F$:

$$\Gamma(x, y, t) = \{(x, y, t) \mid \frac{\partial F}{\partial x} = 3(x^2 - t^2 y^4) = 0\}$$

$\Gamma$ consists of 2 branches, $x = \pm ty^2$. 
Evaluating $v_2(x, y, t)$ along the two branches of the polar, we get

$$v_2(\pm ty^2, y, t) = \frac{\pm t^2 y}{1 \mp 2t^3}$$

Comparing $v_2$ on the two branches of the polar curve $\Gamma$, we get

$$(I) : \quad v_2(ty^2, y, t) - v_2(-ty^2, y, t) = \frac{t^2 y}{1 - 2t^3} - \frac{-t^2 y}{1 + 2t^3} \sim y$$

On the other hand, since $v_2$ is bi-Lipschitz, then

$$(II) : \quad |v_2(ty^2, y, t) - v_2(-ty^2, y, t)| \leq C|ty^2|$$

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Definition

Let \( w = (w_1, \ldots, w_n) \) be a \( n \)-tuple of positive integers. We say that a polynomial function \( f(x_1, \ldots, x_n) \) is \( w \)-homogeneous of degree \( d \) if

\[
f(s^{w_1} x_1, \ldots, s^{w_n} x_n) = s^d f(x_1, \ldots, x_n),
\]

for every \( s \in \mathbb{C}^* \).

We denote by \( H^d_w(n, 1) \) the space of \( w \)-homogeneous polynomials in \( n \)-variables of degree \( d \).
Let $f(x, y, t)$ be a polynomial function such that for every $t \in U$, the function $f_t(x, y) = F(x, y, t)$ is $w$-homogeneous ($w_1 > w_2$) with isolated singularity in $(0, 0) \in \mathbb{C}^2$. If $\{f_t : t \in U\}$ as a family of function germs in $(0, 0) \in \mathbb{C}^2$, is strongly bi-Lipschitz trivial, then $f_{t_1}$ is analytically equivalent to $f_{t_2}$ $\forall t_1, t_2 \in U$. 
Proof.

From the hypothesis, there exists a Lipschitz vector field

\[ V(x, y, t) = \frac{\partial}{\partial t} + v_1(x, y, t) \frac{\partial}{\partial x} + v_2(x, y, t) \frac{\partial}{\partial y}, \quad v_i(0, 0, t) = 0, \quad i = 1, 2. \]

This vector field is tangent to the level sets of \( F \), that is,

\[ \frac{\partial F}{\partial V} = \frac{\partial F}{\partial t} + v_1(x, y, t) \frac{\partial F}{\partial x} + v_2(x, y, t) \frac{\partial F}{\partial y} = 0 \]

Let

\[ \Gamma_t = \{(x, y, t) : \frac{\partial F}{\partial x} = 0\} \]

be the family of polar curves of the family \( F \).
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be the family of polar curves of the family \( F \).
The polar is a family of algebraic curves and may have multiple components. The proof of the theorem in this general case is the most difficult part of the theorem.

We assume $\Gamma_t$ is reduced and $y = 0$ is not a factor of $\frac{\partial F}{\partial x} = 0$.

In this case, let $a_1(t), \ldots, a_k(t)$ be the roots of

$$\frac{\partial F}{\partial x}(x, 1, t) = 0,$$

(as the degree of $\frac{\partial F}{\partial x}(x, 1, t) = 0$ does not depend on $t$, the functions $a_i(t)$ are continuous.)
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(as the degree of $\frac{\partial F}{\partial x}(x, 1, t) = 0$ does not depend on $t$, the functions $a_i(t)$ are continuous.)
Since $w_1 > w_2$, the parametrization of the branch $\gamma_i$ of $\Gamma$ is

$$\gamma_i(s) = (a_i(t)s^{w_1}, s^{w_2}, t), \quad i = 1, \ldots, k.$$
We define the functions $k_1(t), \ldots, k_r(t)$ as

$$k_i(t) = \frac{\partial F}{\partial t} (a_i(t), 1, t) \frac{\partial F}{\partial y} (a_i(t), 1, t)$$

and prove the following proposition:

**Proposition**

With the above notation $k_i(t) = k_j(t)$, $\forall i, j = 1, \ldots, r$
Proof of the proposition

Notice that on the polar set, we have

\[ v_2(x, y, t) = -\frac{\partial F}{\partial t}(x, y, t) - \frac{\partial F}{\partial y}(x, y, t) \]

Comparing \( v_2 \) on two branches of the polar we get

\[ \left| \frac{\partial F}{\partial t}(a_i(t)s^{w_1}, s^{w_2}, t) - \frac{\partial F}{\partial t}(a_j(t)s^{w_1}, s^{w_2}, t) \right| = |k_i(t) - k_j(t)||s|^{w_2} \]
On the other hand, since $v_2(x, y, t)$ is Lipschitz, then

$$|v_2(a_i(t)s^{w_1}, s^{w_2}, t) - v_2(a_j(t)s^{w_1}, s^{w_2}, t)| \leq C|s|^{w_1}$$

But $w_1 > w_2$, then we must have $k_i(t) = k_j(t) = k(t)$, and this proves the proposition.
Proof of the theorem (when the family of polar curves is reduced)

Now, for fixed \( t \), the function

\[
\frac{\partial F}{\partial t}(x, y, t) - k(t)y \frac{\partial F}{\partial y}(x, y, t)
\]

is analytic and identically zero on each branch of the polar.

If \( \frac{\partial F}{\partial x} \) is a reduced family of plane curves (without repeated branches) then there exists \( b(x, y, t) \), analytic function such that

\[
\frac{\partial F}{\partial t}(x, y, t) = k(t)y \frac{\partial F}{\partial y} + b(x, y, t) \frac{\partial F}{\partial x}.
\]

We now use the weighted homogeneity of \( F \) to get that the family is analytically trivial.
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We now use the weighted homogeneity of $F$ to get that the family is analytically trivial.
Problems

We first recall the following result and example:

**Theorem**

A. Fernandes,—, 2004, [1]: Let $f : \mathbb{K}^n, 0 \to \mathbb{K}, 0$ be the germ of a weighted homogeneous polynomial function of type $(w_1, \ldots, w_n : d)$, $w_n \leq \ldots \leq w_1$ with isolated singularity. Let $f_t(x) = f(x) + t\theta(x, t)$, $t \in [0, 1]$, be a deformation of $f$. If $\text{fil}(\theta) \geq d + w_1 - w_n$, then $f_t$ is strongly bi-Lipschitz trivial.

**Example**

The family

$$f_t(x, y) = x^3 - 3t^2xy^{3n-2} + y^{3n}$$

is not strongly bi-Lipschitz trivial. Moreover, Parusinski and Henry’s invariant does not distinguish the elements of the family $f_t$. 
Problems

(1) Prove that the rigidity theorem also holds for deformations

\[ f_t(x, y) = f(x, y) + t\theta(x, y), \text{ with } d < \text{fil}(\theta) < d + w_1 - w_n. \]

(2) Investigate the bi-Lipschitz invariance of higher order terms of the asymptotic expansion of \( f \) over the branches of the polar curves.
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with \( d < \text{fil}(\theta) < d + w_1 - w_n \).

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asymptotic expansion of \( f \) over the branches of the polar curves.

