Projections of hypersurfaces with boundary in $\mathbb{R}^4$ to planes

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Liverpool- June 2012

Dedicated to Bruce and Wall.

Joint work with Luciana F. Martins - UNESP - Brazil
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$M$ hypersurface with boundary in $\mathbb{R}^4$.

We study singularities of these projections, that measure the contact of $M$ with planes.

At interior points we classify it by using the Mather’s group $\mathcal{A}$ (2003).

For boundary points we need to use the subgroup $\mathcal{B}$ (Damon’s geometric group) of $\mathcal{A}$ which preserves a plane in the source.

J. W. Bruce and P. J. Giblin investigated the singularities of projections of surfaces with boundary in $\mathbb{R}^3$ to a plane.
The projection

The family of planes in $\mathbb{R}^4$ close to the plane $u$ generated by $a = (1, 0, 0, 0)$ and $b = (0, 0, 0, 1)$ may be given taking $(1, \beta_1, \gamma_1, 0)$ and $(0, \beta_2, \gamma_2, 1)$ as generators of those planes, for $\beta_1, \beta_2, \gamma_1, \gamma_2$ close to 0. So, we get that the projection is locally the map

$$\Pi_{\beta, \gamma}(X, Y, Z, W) = (X + \beta_1 Y + \gamma_1 Z, W + \beta_2 Y + \gamma_2 Z),$$

where $\beta$ and $\gamma$ denote the pairs $(\beta_1, \beta_2)$ and $(\gamma_1, \gamma_2)$. Note that $\Pi_{0,0} = \Pi_u$.

Let $i(x, y, z) = (X, Y, Z, W)$ a germ of immersion, then we classify simple germs of maps $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 and $B_e$-codimension $\leq 4$. 
The method of classification of map germs is well known for the group $\mathcal{A}$, but also works for the geometric group $\mathcal{B}$.

We used the **Transversal package** that was written by Neil Kirk (see http://old.lms.ac.uk/jcm//3/lms1999-002/). The program was written in Maple and we adapted such package to work in the case of hypersurface with boundary that is for the group $\mathcal{B}$. 
Theorem

(Bruce, du Plessis, Wall, 1987)

A germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is $(2r + 1)$-$A_1$-determined if

$$m_n^{r+1} \mathcal{E}(n, p) \subset LA_1.f + m_n^{2r+2} \mathcal{E}(n, p).$$
Proposition

(Complete transversal, Bruce, Kirk, du Plessis, 1997) Let $g$ be a $k$-jet in $J^k(n, p)$, and let $T$ be a vector subspace of the set $H^{k+1}(n, p)$ of homogeneous jets of degree $k + 1$, such that

$$H^{k+1}(n, p) \subset T + L(J^{k+1}A_1)(g).$$

Then any $(k + 1)$-jet $j^{k+1}f$ with $j^k g = j^k f$ is $J^{k+1}A_1$-equivalent to $g + t$ for some $t \in T$. (The vector subspace $T$ is called the complete $(k + 1)$-transversal of $g$.)

Lemma

(Mather’s Lemma, 1969) Let $G$ be a Lie group acting smoothly on a finite dimensional manifold $X$. Let $V$ be a connected submanifold of $X$. Then $V$ is contained in a single orbit of $G$ if and only if

1. for each $x \in V$, $T_x V \subset T_x G(x) = L_G(x);$
2. $\dim T_x G(x)$ is constant for all $x \in V$. 
The codimension of the orbit of $f$ is given by
\[
\dim_\mathbb{R}(m_3.E(3, 2)/L.A(f))
\]
and the codimension of the extended orbit ($A_e$-codimension) is
\[
\dim_\mathbb{R}(E(3, 2)/L_e.A(f)).
\]
The classification of germs is carried out inductively on the jet level until a sufficient jet is found. To do this we have three main results: finite determinacy (FD); Complete Transversal Theorem (CT) and Mather’s Lemma (ML).

- $j^k f$ is f.d? If yes then stop. (FD result by using the unipotent group $A_1$.)

- If not we add some $k + 1$-degree monomials given by CT (that also uses $A_1$).

- Finally Mather’s Lemma says it these monomials are really necessary for the $A$ group.

After that we return to the FD result to see if the $j^{k+1} f$ is f.d. and so on.
Then to work with the Transversal package we use two simultaneous windows of maple:
• one to work with $A_1$ (to apply FD and CT results)
• the other one to work with the group $A$ (to apply ML).

This method also works for the group $B$ then we adapted the Transversal Program to use the group $B$. 
The group $B$ is the subgroup of $A$ consisting of pairs of germs of diffeomorphisms $(h, k)$ in $\text{Diff}(\mathbb{R}^3) \times \text{Diff}(\mathbb{R}^3)$ with $h$ preserving the manifold as well as its boundary $(\mathbb{R}^2, 0)$ (that is, $h$ takes the variety $V = \{(x, y, z) : z \geq 0\}$ into itself).

Therefore, if $(h, k) \in B$ we can write
\[ h(x, y, z) = (h_1(x, y, z), h_2(x, y, z), zh_3(x, y, z)) \] with $h_3(0, 0, 0) > 0$, for germs of smooth functions $h_i, \ i = 1, 2, 3$. 
The $B$ (resp. $B_1$, i.e. the subgroup of $B$ whose elements have with 1-jets at 0 the identity) tangent space of $f \in m(x, y, z).\mathcal{E}(3, 2)$ is given by

\[
LB.f = m(x, y, z).\{f_x, f_y\} + \mathcal{E}_3\{zf_z\} + f^*m(u, v).\{e_1, e_2\},
\]
\[
LB_1.f = m^2(x, y, z).\{f_x, f_y\} + m(x, y, z).\{zf_z\} + f^*m^2(u, v).\{e_1, e_2\}
\]

where $f_x, f_y$ denotes the partial derivative with respect to $x, y$, \{e_1, e_2\} is the standard basis vectors of $\mathbb{R}^2$ considered as elements of $\mathcal{E}(3, 2)$, and $f^*(m_2)$ is the pull-back of the maximal ideal in $\mathcal{E}_2$.

The extended tangent space to the $B$-orbit of $f$ at the germ $f$ is given by

\[
L_eB(f) = \mathcal{E}_3.\{f_x, f_y\} + \mathcal{E}_3\{zf_z\} + f^*m(u, v).\{e_1, e_2\},
\]
The 1-jets

\[ j^1 f = (a_1 x + a_2 y + a_3 z, b_1 x + b_2 y + b_3 z). \]

Cases \( a_1 b_2 - a_2 b_1 \neq 0 \) and \( a_1 b_2 - a_2 b_1 = 0 \), give us the orbits in \( J^1(3, 2) \):

\[ (x, y), (x, z), (x, 0), (z, 0)^{\text{ns}}, (0, 0). \]

\( (0, 0) \) leads to germs of corank 2
\( (x, y) \) is 1-\( \mathcal{B} \)-determined and is stable (\( \mathcal{B}_e \)-cod 0).
Higher jets

1. \( j^1 f \sim (x, z) \) then \( f \sim (x, z + g(x, y)) \), \( \deg(g) \geq 2 \).

**Proposition**

The map germ \( f(x, y, z) = (x, z + g(x, y)) \) is \( r-B \)-determined (resp. simple) if and only if the map-germ \( h(x, y) = (x, g(x, y)) \) is \( r-A \)-determined (resp. simple). We also have \( \mathcal{B}_e \text{-cod}(f) = \mathcal{A}_e \text{-cod}(h) \).

**Proof.**

Observe that

\[
LB.f = \mathcal{E}_3.\{(z, 0), (0, z)\} + LA.h.
\]
Table 1: \(A\)-simple germs of map-germs \(\mathbb{R}^2, 0 \rightarrow \mathbb{R}^2, 0, \epsilon = 1\), given by Rieger.

<table>
<thead>
<tr>
<th>Type</th>
<th>Normal form</th>
<th>(A_e)-codimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((x, y))</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>((x, \epsilon y^2))</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>((x, xy + y^3))</td>
<td>0</td>
</tr>
<tr>
<td>4(_k)</td>
<td>((x, y^3 + (\pm 1)^{k-1}x^ky), \ k \geq 2)</td>
<td>(k - 1)</td>
</tr>
<tr>
<td>5</td>
<td>((x, xy + \epsilon y^4))</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>((x, xy + y^5 \pm y^7))</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>((x, xy + y^5))</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>((x, xy + \epsilon y^6 + y^9))</td>
<td>4</td>
</tr>
<tr>
<td>11(_{2k+1})</td>
<td>((x, xy^2 + \epsilon y^4 + y^{2k+1}), \ k \geq 2)</td>
<td>(k)</td>
</tr>
<tr>
<td>12</td>
<td>((x, xy^2 + y^5 + \epsilon y^6))</td>
<td>3</td>
</tr>
<tr>
<td>13</td>
<td>((x, xy^2 + y^5 \pm y^9))</td>
<td>4</td>
</tr>
<tr>
<td>16</td>
<td>((x, x^2y + \epsilon y^4 \pm y^5))</td>
<td>3</td>
</tr>
<tr>
<td>17</td>
<td>((x, x^2y + \epsilon y^4))</td>
<td>4</td>
</tr>
</tbody>
</table>
2. \( j^1 f \sim (x, 0) \). A complete 2-transversal is given by

\[
(x, b_1 xy + b_2 xz + b_3 yz + b_4 y^2 + b_5 z^2)
\]

for some \( b_1, b_2, b_3, b_4, b_5 \in \mathbb{R} \). We obtain the following orbits in \( J^2(3, 2) \):

Case (a)
\( b_4 \neq 0 \) \( \Rightarrow \) \((x, y^2)^{(ns)}, (x, y^2 + xz \pm z^2), (x, y^2 + xz), (x, y^2 \pm z^2)\).

Case (b)
\( b_4 = 0, b_1 \neq 0 \) \( \Rightarrow \) \((x, xy)^{(ns)}, (x, xy + yz), (x, xy + z^2)\).

Case (c)
\( b_4 = 0, b_1 = 0 \) \( \Rightarrow \) \((x, yz)^{(ns)}, (x, xz + z^2)^{(ns)}, (x, xz)^{(ns)}, (x, z^2)^{(ns)}, (x, 0)^{(ns)}\).
Case (a): \( j^2 f = (x, y^2) + j^2(0, g(x, y, z)) \).

**Proposition**

The map-germ \( f(x, y, z) = (x, y^2 + g(x, z)) \) is \( r-\mathcal{B} \)-determined (resp. simple) if and only if the map-germ \( h(x, z) = (x, g(x, z)) \) is \( r-\mathcal{B} \)-determined (resp. simple). We have \( \mathcal{B}_e\text{-cod}(f) = \mathcal{B}_e\text{-cod}(h) \).

**Proof.**

Observe that

\[
LB.f = E_3 \cdot \{(y, 0), (0, xy), (0, y^2), (0, yz)\} + LB.h.
\]
Table 2: Corank 1 $\mathcal{B}$-simple singularities of map-germs $g : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0, \epsilon = 1$, from Bruce and Giblin.

<table>
<thead>
<tr>
<th>Normal form</th>
<th>$\mathcal{B}_\epsilon$-codimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x, \epsilon z^2 + (\pm 1)^k x^{k-1} z), \ k \geq 2$</td>
<td>$k - 2$</td>
</tr>
<tr>
<td>$(x, xz + \epsilon z^3)$</td>
<td>1</td>
</tr>
<tr>
<td>$(x, xz + \epsilon z^4 \pm z^6)$,</td>
<td>2</td>
</tr>
<tr>
<td>$(x, xz + \epsilon z^4)$</td>
<td>3</td>
</tr>
</tbody>
</table>

Then for us the germs are $(x, y^2 + g(x, y))$ with $\epsilon = \pm 1$. We follow the other cases with $2 < cod \leq 4$. We show that they are all no simple germs.
Case (b): \( j^2 f \sim (x, xy + yz) \) or \( j^2 f \sim (x, xy + z^2) \)

The simple cases we summarize in the next table.

**Table 3: Corank 1 \( \mathcal{B} \)-singularities of map-germs \( f : \mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0 \) of \( \mathcal{B}_e \)-codimension \( \leq 4 \)**

<table>
<thead>
<tr>
<th>Normal form</th>
<th>( \mathcal{B}_e )-codimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x, yz + xy + \epsilon y^3))</td>
<td>1</td>
</tr>
<tr>
<td>((x, yz + xy + y^4 + \epsilon y^6))</td>
<td>2</td>
</tr>
<tr>
<td>((x, yz + xy + y^4))</td>
<td>3</td>
</tr>
<tr>
<td>((x, xy + z^2 + y^3 + \epsilon y^k z), k \geq 2)</td>
<td>(k)</td>
</tr>
</tbody>
</table>
As a consequence of the analysis, we have the following results.

**Theorem**

The map-germ $f : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ with 1-jet equivalent to $(z, 0)$ or 2-jet equivalent to $(x, y^2), (x, xy), (x, yz), (x, xz + z^2), (x, xz), (x, z^2), (x, 0)$ or 3-jet $(x, xy + z^2 + \epsilon y^2 z), (x, xy + z^2)$ or 4-jet $(x, y^2 + xz), (x, xy + yz)$ are no simple germs.

**Theorem**

The map-germ $f : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank at most 1 and $B_e$-codimension $\leq 1$ are simple germs.

We study geometrically the germs given by the last theorem.
The geometry of codimension $\leq 1$ singularities

Table 4: $\mathcal{B}$-simple map germs $(\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^2, 0)$ of codimension $\leq 1$ ($\varepsilon$ and $\bar{\varepsilon}$ are independently $\pm 1$).

<table>
<thead>
<tr>
<th>No.</th>
<th>Normal form</th>
<th>$\mathcal{B}_e$-cod</th>
<th>Versal unfolding</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$(x, y)$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>$(x, z + \varepsilon y^2)$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>$(x, z + xy + y^3)$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>$(x, z + \varepsilon x^2 y + y^3)$</td>
<td>1</td>
<td>$(x, z + y^3 + \varepsilon x^2 y + \lambda y)$</td>
</tr>
<tr>
<td>V</td>
<td>$(x, z + xy + \varepsilon y^4)$</td>
<td>1</td>
<td>$(x, z + xy + \varepsilon y^4 + \lambda y^2)$</td>
</tr>
<tr>
<td>VI</td>
<td>$(x, y^2 + \varepsilon z^2 + xz)$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>VII</td>
<td>$(x, y^2 + \varepsilon z^2 + \bar{\varepsilon} x^2 z)$</td>
<td>1</td>
<td>$(x, y^2 + \varepsilon z^2 + \bar{\varepsilon} x^2 z + \lambda z)$</td>
</tr>
<tr>
<td>VIII</td>
<td>$(x, y^2 + xz + \varepsilon z^3)$</td>
<td>1</td>
<td>$(x, y^2 + xz + \varepsilon z^3 + \lambda (z + z^2))$</td>
</tr>
<tr>
<td>IX</td>
<td>$(x, yz + xy + \varepsilon y^3)$</td>
<td>1</td>
<td>$(x, yz + xy + \varepsilon y^3 + \lambda (y^2 + z))$</td>
</tr>
</tbody>
</table>

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\[ \Pi_{\beta, \gamma}(X, Y, Z, W) = (X + \beta_1 Y + \gamma_1 Z, W + \beta_2 Y + \gamma_2 Z), \]

where \( \beta \) and \( \gamma \) denote the pairs \((\beta_1, \beta_2)\) and \((\gamma_1, \gamma_2)\). \( \Pi_{0,0} = \Pi_u \).

To realize a versal unfolding with 1-parameter we take
\( \beta_1 = \gamma_1 = \beta_2 = 0 \) and call \( \gamma_2 = \lambda \), where
\[ i : (x, y, z) \rightarrow (X, Y, Z, W) \]
is an imersion at \((0, 0, 0)\).

I. \( i(x, y, z) = (x, 0, z, y) \)
II. \( i(x, y, z) = (x, y, 0, z + \varepsilon y^2) \)
III. \( i(x, y, z) = (x, y, 0, z + xy + y^3) \)
IV. \( i(x, y, z) = (x, y, 0, z + y^3 + \varepsilon x^2 y) \)
V. \( i(x, y, z) = (x, y^2, y, z + xy + \varepsilon y^4) \)
VI. \( i(x, y, z) = (x, y, z, y^2 + \varepsilon z^2 + xz) \)
VII. \( i(x, y, z) = (x, y, z, y^2 + \varepsilon z^2 + \varepsilon x^2 z) \)
VIII. \( i(x, y, z) = (x, y, z + z^2, y^2 + xz + \varepsilon z^3) \)
IX. \( i(x, y, z) = (x, y, y^2 + z, yz + xy + \varepsilon y^3) \)
Geometrical information in Cases I-IX.

We consider the following geometric sets to recognize the cases:
- the fibre $f^{-1}(0)$;
- kernel $K$ of $df(0)$;
- $\Sigma$, the critical set of the map $f$;
- singularities of the restriction of $f$ to the boundary
- critical loci (image of $\Sigma$) and the image of the boundary, for $f$ and for members of the versal family unfolding $f$.

Parts of $\Sigma$ or its image that is ‘virtual’, in the sense of corresponding to the part $z < 0$ in the source $\mathbb{R}^3$, appear dashed in the figures. The boundary (plane-xy) and its image are drawn with the gray color.
The following geometric facts recognize the cases. Cases I to V are submersions.

I. The unique case where $K$ is a line transverse to the boundary that is, on $M$, the direction of projection is transverse to $\partial M$.

II. The fibre $f^{-1}(0)$ is a curve tangent to the boundary, for $\varepsilon = -1$ and $f^{-1}(0) = 0$, for $\varepsilon = 1$. The singularity of the restriction of $f$ to the boundary is the fold $(x, y^2)$.

III. The set $f^{-1}(0)$ is a half curve tangent to the boundary. The singularity of the restriction of $f$ to the boundary is the cusp $(x, xy + y^3)$.
Figure: Submersions. (a) Case I. (b) Cases II and V, for $\varepsilon = -1$. (c) Cases II and V, for $\varepsilon = 1$. (d) Cases III and IV.
The cod 1 submersions IV and V have a cod 1 singularity for the restriction of $f$ to the boundary.

IV. The singularity of the restriction of $f$ to the boundary is the lips/beaks $(x, \varepsilon x^2 y + y^3)$, which is a codimension 1 singularity.

V. The singularity of the restriction of $f$ to the boundary is the swallowtail $(x, xy + y^4)$, which is a codimension 1 singularity. Besides the fibre $f^{-1}(0)$ is a curve tangent to the boundary, for $\varepsilon = -1$ and $f^{-1}(0) = 0$, for $\varepsilon = 1$.

Figure: Left: The image of the boundary of Case V. Right: Bifurcation diagrams of the singularities for the restriction of $f$ to the boundary, cases IV.
VI-IX. These are those germs with $K = \ker df(0)$ being the plane-\(yz\).

For each no submersion germ \(f\) of codimension 1 we give a bifurcation diagram to show, for germs close to \(f\) in the \(\mathcal{B}_e\)-versal unfolding, which types of boundary singularities occur.

VI. The image do \(\Sigma\) and the fibre \(f^{-1}(0)\) distinguish cases \(\varepsilon = 1\) and \(\varepsilon = -1:\ f^{-1}(0) = 0\) or it is a set transverse to the boundary, respectively.
VII. The singular set $\Sigma$ (or the image of $\Sigma$) distinguishes this case of all other cases. $f^{-1}(0)$ distinguishes the cases $\epsilon$.

**Figure:** Case VII for $\epsilon \bar{\epsilon} = 1$. 

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Figure: Case VII for $\varepsilon \varepsilon = -1$. 
VIII. $f|_\Sigma$, unlike all other cases, has a cusp singularity.
IX. This is the unique no submersion whose image of the boundary is the hole plane. $f^{-1}(0)$ distinguishes the cases $\varepsilon$. 

\[ \lambda > 0 \quad \lambda = 0 \quad \lambda < 0 \]

$\varepsilon = 1$

$\varepsilon = -1$
Some references:


Thank you very much for your attention!

Happy birthday, Bruce and Wall!!