Linear systems associated to unicuspidal rational plane curves

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joint work with Daniel Daigle

Bruce 60 and Wall 75
Workshop on Singularity Theory, its Applications and Future Prospects

Liverpool, 18–22 June 2012
1. First open problems

2. Classification of rational curves

3. \( \mathbb{P}^1 \)-rulings on rational surfaces

4. Rational unicuspidal plane curves
$C$ irreducible proj. curve, $n : \tilde{C} \to C$ normalization $g(C) := p_a(\tilde{C})$

$p \in C$, $\delta(C, p) := \dim_C(n_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C)_p$

1. $p$ non-singular of $C \iff \delta(C, p) = 0$
2. $p$ node of $C \iff \delta(C, p) = 1$

$$\delta(C) := \sum_{p \in C} \delta(C, p), \quad p_a(C) = g(C) + \delta(C)$$

$C$ rational and cuspidal if $\tilde{C} \cong \mathbb{P}^1$ ($\iff g(C) = 0$), and all singular points are locally irreducible.
First open problems
Classification of rational curves
$\mathbb{P}^1$-rulings on rational surfaces
Rational unicuspidal plane curves

\[ C \subset \mathbb{P}^2 \text{ rational, } \deg(C) = d, \ \operatorname{Sing}(C) = \{p_i\}^r_{i=1} \]
\[ \delta(C) = p_a(C) = (d - 1)(d - 2)/2 \]

- If $C$ general $\Rightarrow$ $C$ has $\delta(C) = (d - 1)(d - 2)/2$ nodes.

Problem (Sakai)

\[ C \subset \mathbb{P}^2 \text{ rational cuspidal } \# \operatorname{Sing}(C) \leq 4. \ (\# \operatorname{Sing}(C) \leq 8 \ [\text{Tono}]) \]

Many examples with $\# \operatorname{Sing}(C) = 1, 2$ or $3$ but $\exists$ ! with $4$ cusps:
\[ d = 5, [2_3], [2], [2], [2], \ (\text{Namba’s book}) \]
Parametrization $\left(t^3 - 1, t^5 + 2t^2, t\right)$
Conjecture (Nagata-Coolidge)

*For every rational cuspidal plane curve there exists a birational map of \( \mathbb{P}^2 \), that is Cremona transformation, such that the proper transform of the curve is a line.*

For all known examples the conjecture has been verified, (if \( \# \text{Sing}(C) > 4 \) [Palka, 2012] true).

A general rational curve of degree 6 has \((6 - 1)(6 - 2)/2 = 10\) nodes and cannot be transformed into a line by a Cremona transformation of \( \mathbb{P}^2 \) (Coolidge)
Problem (Main goal)

*Characterize those collections of local embedded topological types \{ T_i \}_{i=1}^{\nu} (without fixing the positions of the points p_i) which can be realized by a projective plane curve C of degree d.*

Same problem if C rational and cuspidal.

<table>
<thead>
<tr>
<th>d</th>
<th># Sing(C)</th>
<th>type of cusp</th>
<th>−N(t)</th>
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<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>[4]</td>
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</tr>
<tr>
<td>5</td>
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</tr>
<tr>
<td>5</td>
<td>2</td>
<td>[3, 2], [2_2]</td>
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<tr>
<td>5</td>
<td>2</td>
<td>[3], [2_3]</td>
<td>2t</td>
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<tr>
<td>5</td>
<td>2</td>
<td>[2_2], [2_4]</td>
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<tr>
<td>5</td>
<td>3</td>
<td>[3], [2_2], [2]</td>
<td>2t</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>[2_2], [2_2], [2_2]</td>
<td>6t</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>[2_3], [2], [2], [2]</td>
<td>8t</td>
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</table>
ClassIFICATION RATIONAL AND CUSPIDAL PLANE CURVE DEGREE 6

<table>
<thead>
<tr>
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<th>$# \text{Sing}(C)$</th>
<th>type of cusp</th>
<th>$-N(t)$</th>
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<td>6</td>
<td>1</td>
<td>[5]</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>[4, 24]</td>
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</tr>
<tr>
<td>6</td>
<td>1</td>
<td>[3, 2]</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>[3], [2]</td>
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<tr>
<td>6</td>
<td>2</td>
<td>[3, 2], [3]</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>[4, 23], [2]</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>[4, 22], [22]</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>[4], [24]</td>
<td>$t + t^2$</td>
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Some rational cuspidal plane curves degree 7

<table>
<thead>
<tr>
<th>reference</th>
<th>$d$</th>
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<th>type of cusp</th>
<th>$-N(t)$</th>
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<tbody>
<tr>
<td>$x^6z + y^7$</td>
<td>7</td>
<td>1</td>
<td>$[6]$</td>
<td>$0$</td>
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<tr>
<td>$x^5z^2 + y^7$</td>
<td>7</td>
<td>2</td>
<td>$[5, 2_2], [2_3]$</td>
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<tr>
<td>$x^4z^3 + y^7$</td>
<td>7</td>
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<td>$[4, 3], [3_2]$</td>
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<tr>
<td>Fenske</td>
<td>7</td>
<td>2</td>
<td>$[5], [2_5]$</td>
<td>$2t + 2t^2 + 2t^3$</td>
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<tr>
<td>Fenske</td>
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<td>2</td>
<td>$[4, 2_3], [3_2]$</td>
<td>$t + 2t^2 + t^3$</td>
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<tr>
<td>Fenske</td>
<td>7</td>
<td>2</td>
<td>$[4, 2_2], [3_2, 2]$</td>
<td>$2t + 2t^2 + 2t^3$</td>
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<tr>
<td>Fenske</td>
<td>7</td>
<td>2</td>
<td>$[4], [3_3]$</td>
<td>$2t + 2t^2 + 2t^3$</td>
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<tr>
<td>Flenner-Zai.</td>
<td>7</td>
<td>3</td>
<td>$[4, 2_2], [3_2], [2]$</td>
<td>$3t + 4t^2 + 3t^3$</td>
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<tr>
<td>Flenner-Zai.</td>
<td>7</td>
<td>3</td>
<td>$[5], [2_4], [2]$</td>
<td>$3t + 3t^3$</td>
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<tr>
<td>Flenner-Zai.</td>
<td>7</td>
<td>3</td>
<td>$[5], [2_3], [2_2]$</td>
<td>$3t + 3t^3$</td>
</tr>
</tbody>
</table>
Fibonacci sequence: \[ \varphi_0 = 0, \varphi_1 = 1, \varphi_{j+2} = \varphi_{j+1} + \varphi_j. \]

**Theorem (FLMN)**

There exists a rational unicuspidal plane curve of degree \( d \) with only one Puiseux pair \((a, b)\) if and only if \((d, a, b)\) appear in the following list

1. \((a, b) = (d - 1, d)\);
2. \((a, b) = (d/2, 2d - 1)\), where \( d \) is even;
3. \((a, b) = (\varphi_{j-2}^2, \varphi_j^2)\) and \( d = \varphi_{j-1}^2 + 1 = \varphi_{j-2}\varphi_j \), where \( j \) is odd and \( \geq 5 \);
4. \((a, b) = (\varphi_{j-2}, \varphi_{j+2})\) and \( d = \varphi_j \), where \( j \) is odd and \( \geq 5 \);
5. \((a, b) = (\varphi_4, \varphi_8 + 1) = (3, 22)\) and \( d = \varphi_6 = 8 \);
6. \((a, b) = (2\varphi_4, 2\varphi_8 + 1) = (6, 43)\) and \( d = 2\varphi_6 = 16 \).
First open problems
Classification of rational curves
\( \mathbb{P}^2 \)-rulings on rational surfaces
Rational unicuspidal plane curves

\( X \) non-singular alg. surface \( \Rightarrow \exists \) alg. proj. nonsing. surface \( S \):

1. \( X \) open dense in \( S \),
2. \( S \setminus X \) codim 1 in \( S \);
3. \( S \setminus X \) has only normal crossing.

\( D \) reduced effective div. supported on \( S \setminus X \), \( K_S \) canonical div. \( S \).

[\textit{Iitaka}] \textbf{Logarithmic Kodaira dimension} \( \overline{\kappa}(X) \) of \( X \)

\[
\overline{\kappa}(X) := \sup_{n>0,|n(D+K_S)|\neq \emptyset} \dim \Phi_{|n(D+K_S)|}(S)
\]

\( \overline{\kappa}(X) \) independent of \( S \) and \( \overline{\kappa}(X) \in \{-\infty, 0, 1, 2\} \)
\[ \pi_{\text{emb}} : S_{\text{emb}} \rightarrow \mathbb{P}^2 \text{ minimal embedded resolution of } C. \]

\[ D_{\text{emb}} := \pi_{\text{emb}}^{-1}(C) \text{ normal crossing and } \tilde{C}_{\text{emb}} \text{ strict transform of } C. \]

\[ \bar{\kappa}(\mathbb{P}^2 \setminus C) := \bar{\kappa}(S_{\text{emb}} \setminus D_{\text{emb}}). \]

1. \textbf{(Yoshihara) if } C \text{ rational unicuspidal, } \bar{\kappa}(\mathbb{P}^2 \setminus C) = -\infty \text{ iff } \tilde{C}^2_{\text{emb}} > -2. \]

\[ C \subset \mathbb{P}^2, S_{\text{min}} \rightarrow \mathbb{P}^2 \text{ minimal resolution of } C, \]

\[ C_{\text{min}} \text{ strict transform of } C \text{ en } S_{\text{min}}, \]

\[ C \text{ non negative (resp. positive) if } C^2_{\text{min}} \geq 0 \text{ (resp. } C^2_{\text{min}} > 0). \]

\[ \mathbb{P}^2 = S_0 \xleftarrow{\pi_1} S_1 \xleftarrow{\pi_2} \cdots S_{\text{min}} \xleftarrow{\pi_{n+1}} \cdots S_{N-1} \xleftarrow{\pi_N} S_{\text{emb}} \]
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$C \subset \mathbb{P}^2$ rational cuspidal curve

1. (Wakabayashi) if $\# \operatorname{Sing}(C) = 2 \Rightarrow \bar{\kappa}(\mathbb{P}^2 \setminus C) \geq 0$, 
2. (Wakabayashi) if $\# \operatorname{Sing}(C) \geq 3 \Rightarrow \bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$, 
3. (Tsunoda) there exist no rational cuspidal curve with $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 0$,

<table>
<thead>
<tr>
<th>$\bar{\kappa}(\mathbb{P}^2 \setminus C)$</th>
<th>$# \operatorname{Sing}(C)$</th>
<th>Classificación</th>
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<tr>
<td>$-\infty$</td>
<td>1</td>
<td>Kashiwara</td>
</tr>
<tr>
<td>0</td>
<td>≠</td>
<td>Tsunoda</td>
</tr>
<tr>
<td>1</td>
<td>1 and 2</td>
<td>Tsunoda-Tono</td>
</tr>
<tr>
<td>2</td>
<td>1, 2, 3 and $\exists \ 4 \leq 8$</td>
<td></td>
</tr>
</tbody>
</table>

Alejandro Melle Hernández joint work with Daniel Daigle
Bruc Linear systems associated to unicuspidal rational plane curves
$S$ rational nonsingular projective surface.\newline
$\Lambda$ pencil on $S$ is $\mathbb{P}^1$-ruling if base-point-free and general member $\cong \mathbb{P}^1$,\newline
$\Sigma \subset S$ section of $\Lambda$ if $\Sigma \cdot D = 1$ for $D \in \Lambda$\newline
$C \subset S$ satisfies $C \cong \mathbb{P}^1$ and $C^2 = 0$ then $|C|$ on $S$ is $\mathbb{P}^1$-ruling.
Gizatullin’s Theorem. \( \Lambda \mathbb{P}^1 \)-ruling on \( S \). Then \( \Lambda \) has section \( \Sigma \):

1. if \( D \in \Lambda \), support of \( D \) is a tree of rational non-singular curves.
2. there exist birational morphism \( \rho : S \to F_k \) with \( F_k \) a Nagata-Hirzebruch surface

- the exceptional locus of \( \rho \) is the union of the irreducible curves \( C \subset S \) which are \( \Lambda \)-vertical and disjoint from \( \Sigma \)
- the linear system \( \mathbb{L} = \rho_* (\Lambda) \) base-point-free pencil on \( F \) each of whose elements is a projective line, and the curve \( \Delta = \rho(\Sigma) \) is a section of \( \mathbb{L} \)
- if \( \Sigma^2 \leq 0 \) then \( \Sigma^2 = -k \) and \((F, \mathbb{L}, \Delta) = (F_k, \mathbb{L}_k, \Delta_k)\).
C rational unicuspidal plane curve deg d, i.e Sing(C) = \{p\}
For \((\ell, j) \in \mathbb{N}^2\) such that \(\ell > 0\) and \(j \leq \ell d\),

\[ X_{\ell,j}(C) := \{D \text{ effec div in } \mathbb{P}^2 : \deg(D) = \ell \quad i_p(C, D) \geq j\} \]

Then

1. \(X_{\ell,j}(C)\) is a linear system \(\forall \ell, j\), \(\dim X_{\ell,j}(C) \geq 1\) si \(\ell \geq d\), and is nonempty whenever \(\ell \geq d\).
2. \(\dim X_{d,j}(C) = \#[j, d^2] \cap \Gamma(C, p)\).
3. If \(j\) tal que \((d - 1)(d - 2) \leq j \leq d^2\),

\[ \dim X_{d,j}(C) = d^2 - j + 1 \]

4. \(\Lambda_C := X_{d,d^2}(C)\) pencil,
5. \(N_C := X_{d,d^2-1}(C)\) net,
6. \(C \in \Lambda_C \subset N_C\).
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Theorem (Daigle, —)

$C \subset \mathbb{P}^2$ rational unicuspidal,

1. $\Lambda_C$ unique pencil on $\mathbb{P}^2$ s.t. $C \in \Lambda_C$ and $Bs(\Lambda_C) = \{P\}$.
2. $C$ no-negative iff $\Lambda_C$ rational pencil
3. $C$ positive iff $N_C$ rational net.

All known unicuspidal rational curves are non-negative.
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\[ C \subset \mathbb{P}^2 \text{ rational and cuspidal} \]

1. \( C \) has at most 4 singular points.
2. \( C \) can be transformed into a line by a Cremona transf.
3. \( C \) verifies compatibility conditions.
4. \( C \) unicuspidal then \( C \) non-negative.
5. If \( \kappa(\mathbb{P}^2 \setminus C) = 2 \) then \( \mathbb{P}^2 \setminus C \) is rigid rígida, that is all deformations are non obstructed (Flenner-Zaidenberg)
6. Classification of rational cuspidal curves

Alejandro Melle Hernández joint work with Daniel Daigle
First open problems
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\(C \subset \mathbb{P}^2\) rational cuspidal \(\deg C = d, \text{Sing}(C) = \{p_1\}_{i=1}^r\)

\((C, p_i) \subset (\mathbb{C}^2, p_i)\) \(\Delta_i(t)\) carateristic polynomial of monodormomy

\[
\deg(\Delta_i(t)) = 2\delta(C, p_i), \Delta_i(1) = 1, \Delta'_i(1) = \delta(C, p_i)
\]

\[
\Delta(t) := \prod_{i=1}^r \Delta_i(t) = 1 + (1 - t)\delta(C) + (1 - t)^2 \tilde{Q}(t)
\]

\[
\tilde{Q}(t) = \sum_{l \mid d} b_l t^l + \sum_{l=0}^{d-3} c_l t^{(d-3-l)d}
\]

Problem (Compatibility conditions, FLMN)

\(\text{If } C \text{ exists then: }\bullet \ c_l \leq \frac{(l+1)(l+2)}{2}, \quad \forall l = 0, \ldots, d - 3\)

\(\bullet \ N(t) = \sum_{l=0}^{d-3} \left(c_l - \frac{(l+1)(l+2)}{2}\right) t^{d-3-l} \text{ has negative coeff.}\)
Theorem (FLMN)

Compatibility conditions are verified for all known curves
Muchas gracias por su atención