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Liverpool
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2 $\delta_1$-minimal surfaces

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Let us consider the plane curve \((Y, 0)\) with the \(E_6\) singularity parameterised as the image of the map germ \(\gamma(v) = (v^3, v^4)\). We want to deform it into a new curve \(\tilde{Y}\) which has only cusps \(A_2\) (i.e., simple cusps) and nodes \(A_1\) (i.e., transverse double points).
Introduction

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Consider the mini-versal deformation:

\[
\Gamma(v; a, b, c) = (v^3 + av, v^4 + bv^2 + cv).
\]

Up to coordinate changes, any nearby deformation of \(\gamma\) is obtained from \(\Gamma\) by choosing appropriate coefficients \(a, b, c\).
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We find three different types of generic deformations, according to the numbers of cusps or nodes:
The first deformation has 3 nodes and corresponds to the case that:
\[
\left(16a^3 - 48a^2b + 36ab^2 + 27c^2\right) \left(32a^3 - 48a^2b + 24ab^2 - 4b^3 + 27c^2\right) (a-b) \neq 0.
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In fact, the non-vanishing of the 3 factors above prevents the appearance of either: a cusp, a self-tangency or a triple point, respectively.
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The 3 factors can be computed easily by means of resultants:

\[
\text{In[1]:= } p = y^3 + ay; q = y^4 + by^2 + cy; pu = p /. y \rightarrow u; qu = q /. y \rightarrow u;
\]

\[
ecl = \text{Factor}[(p-\text{pu})/(y-u)]; \text{ec2} = \text{Factor}[(q-\text{qu})/(y-u)];
\]

\[
\text{lambda = Resultant[ecl, ec2, u]; Factor[Resultant[lambda, D[lambda, y, y], y]]}
\]

\[
\text{Out[1]= } (a-b)^6 \left(16a^3 - 48a^2 b + 36ab^2 + 27c^2\right) \left(32a^3 - 48a^2 b + 24ab^2 - 4b^3 + 27c^2\right)^2
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The second deformation has 2 nodes and 1 cusp and occurs when:

\[
16a^3 - 48a^2b + 36ab^2 + 27c^2 = 0, \quad (c, 2a - 3b) \neq (0, 0),
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\text{In[1]:= } p = y^3 + ay; \quad q = y^4 + by^2 + cy; \quad pu = p/.y \to u; \quad qu = q/.y \to u;
\quad e1 = \text{Factor}\left[(p-pu)/(y-u)\right]; \quad e2 = \text{Factor}\left[(q-qu)/(y-u)\right];
\quad lambda = \text{Resultant}[e1, e2, u]; \quad \text{Factor}[\text{Resultant}[lambda, D[lambda, y], y]]
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Finally, the third deformation has 1 node and 2 cusps and is given by:
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(c, 2a-3b) = (0,0),
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The image of \( f \) is an irreducible surface \((X, 0)\) with 1-dimensional singular set \( \Sigma \).
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**GOAL:** Characterize those surfaces $(X, 0)$ which are the total space of an unfolding of a plane curve $(Y, 0)$ with only cusps and nodes and generalize the above formulas.
\( \delta_1 \)-minimal surfaces

Let \((X,0)\) be an irreducible surface in \((\mathbb{C}^3,0)\) with 1-dimensional singular set \(\Sigma\). Given a generic plane \(0 \in H \subset \mathbb{C}^3\), we denote by \(Y = X \cap H\) the transverse slice of \(X\). The delta invariant of \(Y\) does not depend on \(H\) and we denote it by \(\delta_1(X,0) = \delta(Y,0)\), the \textit{transverse delta invariant} of \((X,0)\).
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**Theorem**

We have \(\delta_1(X,0) \geq m_0(\Sigma,0)\), with equality iff \((X,0)\) admits a corank 1 parameterisation \(f : (\mathbb{C}^2,0) \rightarrow (\mathbb{C}^3,0)\) such that the only singularities outside the origin are semicubic cuspidal edges and transverse double points.
Definition

If $\delta_1(X, 0) = m_0(\Sigma, 0)$, then we say that $(X, 0)$ is $\delta_1$-minimal.
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- $(X, 0)$ is the total space of an unfolding of a plane curve with only cusps and nodes.

\[\begin{align*}
\kappa &= \text{the number of cusps of } Y_t, \text{ for } t \neq 0. \\
\nu &= \text{the number of nodes of } Y_t, \text{ for } t \neq 0.
\end{align*}\]

Obviously, we have $\kappa + \nu = \delta_1(X, 0)$. 

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Unfolding plane curves with cusps and nodes
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Given a \( \delta_1 \)-minimal surface \((X, 0)\), we denote:

- \( \kappa \) = the number of cusps of \( Y_t \), for \( t \neq 0 \).
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Obviously, we have \( \kappa + \nu = \delta_1(X, 0) \).
If \((X, 0)\) is \(\delta_1\)-minimal and \(f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)\) is the corank 1 parameterisation, after linear change of coordinates in \(\mathbb{C}^3\) and reparametrisation, we can assume

\[
f(u, v) = (u, p(u, v), q(u, v)),
\]

for some functions \(p, q \in m_2\). Then, the generic plane \(H\) is given by \(x = 0\) and \(\gamma_t(v) = (p(t, v), q(t, v))\) is the parameterisation of the deformation \(Y_t\).
If \((X, 0)\) is \(\delta_1\)-minimal and \(f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)\) is the corank 1 parameterisation, after linear change of coordinates in \(\mathbb{C}^3\) and reparametrisation, we can assume

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**Proposition**

We have \(0 \leq \kappa \leq m_0(X, 0) - 1\). Moreover, they are equivalent:
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for some functions \(p, q \in \mathfrak{m}_2\). Then, the generic plane \(H\) is given by \(x = 0\) and \(\gamma_t(v) = (p(t, v), q(t, v))\) is the parameterisation of the deformation \(Y_t\).

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- \(\kappa = 0\).
- \(f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)\) is finitely determined with respect to the \(\mathcal{A}\)-equivalence.
- For each \(t \neq 0\), \(\gamma_t\) is stable with respect to the \(\mathcal{A}\)-equivalence.
Consider the projectivized cotangent bundle $PT^*\mathbb{C}^{n+1}$ with the canonical contact structure and denote the projection by $\pi : PT^*\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. 
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By definition, a hypersurface singularity $(X, 0)$ in $(\mathbb{C}^{n+1}, 0)$ is called a front if it is the image of a composition of $\pi$ with a Legendrian embedding $L$:

$$(\mathbb{C}^n, 0) \xrightarrow{L} (PT^*\mathbb{C}^{n+1}, ([\nu_0]; 0)) \xrightarrow{\pi} (\mathbb{C}^{n+1}, 0).$$

If we do not assume that $L$ is an embedding, but we just suppose it is Legendrian, then $(X, 0)$ is said to be a frontal (in Zakalyukin’s terminology). By abuse of language, we also use the word frontal for the map germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$. 
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This means that there is a holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ and a holomorphic vector field $\nu : (\mathbb{C}^n, 0) \to \mathbb{C}^{n+1} \setminus \{0\}$ such that

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Proposition

Let \((X, 0)\) be a hypersurface in \((\mathbb{C}^{n+1}, 0)\) parameterised by a corank 1 map germ \(f(u, v) = (u, p(u, v), q(u, v))\). Then \((X, 0)\) is a frontal iff either \(p_v|q_v\) or \(q_v|p_v\).
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Example

Any irreducible plane curve \((Y, 0)\) is a frontal.

The swallowtail \(f(u, v) = (u, v^3 + uv, v^4 + 2^3uv^2)\) is a frontal.

The cross-cap \(f(u, v) = (u, v^2, uv)\) is not a frontal.
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We recall that a map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is said to be $\mathcal{A}$-stable if any unfolding is trivial. The same is true for multigerms.
**Proposition**

Let \((X, 0)\) be a hypersurface in \((\mathbb{C}^{n+1}, 0)\) parameterised by a corank 1 map germ \(f(u, v) = (u, p(u, v), q(u, v))\). Then \((X, 0)\) is a frontal iff either \(p_v | q_v\) or \(q_v | p_v\).

**Example**

- Any irreducible plane curve \((Y, 0)\) is a frontal.
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By the Mather-Gaffney geometric criterion, a map germ \(f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)\) with \(n < p\) is \(\mathcal{A}\)-finite if and only if there is a proper representative \(f : U \rightarrow V\) such that \(f^{-1}(0) = \{0\}\) and the multigerm at any point \(y \in V \setminus \{0\}\) is \(\mathcal{A}\)-stable.
By analogy with these definitions we have:

**Definition**

We say that a frontal $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is $\mathcal{F}$-stable if any frontal unfolding of $f$ is trivial. The same definition is also valid for multigerms.
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The \( \mathcal{F} \)-stable singularities of a frontal surface \( (X, 0) \) are either generic fronts (described by [Arnold, Uspehi Mat. Nauk '75] and [Zakalyukin, Funct. Anal. Appl. '76]) or the folded umbrella ([Ishikawa, Asian J. Math. '05]).
As a consequence, a frontal surface \((X, 0)\) is \(\mathcal{F}\)-finite iff it the only singularities outside the origin are transverse double points or semicubic cuspidal edges.
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Recall that if \((X, 0)\) is \(\delta_1\)-minimal then \(0 \leq \kappa \leq m_0(X, 0) - 1\), where \(\kappa\) is the number of cusps.

**Proposition**

*They are equivalent:*

\[(X, 0)\] is \(\delta_1\)-minimal with \(\kappa = m_0(X, 0) - 1\). 
\[(X, 0)\] is a corank 1 \(F\)-finite frontal surface. 
\[(X, 0)\] is the total space of a \(F\)-stabilization of a frontal curve.
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Local Euler obstruction

The local Euler obstruction was first introduced by [MacPherson, Ann Math. '74] in the construction of characteristic classes of singular algebraic varieties. Here we prefer to use the approach of [Lê-Teissier, Ann. Math. '81] in terms of polar multiplicities:

\[
\text{Eu}(V, 0) = \sum_{i=0}^{d-1} (-1)^i m_i(V, 0),
\]

where \( (V, 0) \) is a \( d \)-dimensional complex analytic set germ and \( m_i(V, 0) \) denotes the \( i \)th-polar multiplicity. In particular, for a surface \( (X, 0) \),

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Moreover, if \((X, 0)\) has 1-dimensional singular set \(\Sigma\), then we can use a formula due to [Brasselet-Lê-Seade, Topology '00]:

$$\text{Eu}(X, 0) = \chi(Y_t) - m + \sum_{i=1}^{m} \text{Eu}(X, x_i),$$

for \(t \neq 0\), where \(Y_t = X \cap H_t\), \(Y_t \cap \Sigma = \{x_1, \ldots, x_m\}\).
Theorem

Let \((X, 0)\) be a \(\delta_1\)-minimal surface with \(\kappa = \# \) of cusps. Then,

\[
\text{Eu}(X, 0) = 1 + \kappa.
\]

In particular, \(1 \leq \text{Eu}(X, 0) \leq m_0(X, 0)\).

J.J. Nuño-Ballesteros

Unfolding plane curves with cusps and nodes
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Let \((X, 0) \subset (\mathbb{C}^3, 0)\) be an irreducible surface with 1-dimensional singular locus. Then,

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1. \((X, 0)\) is the image of a corank 1 \(\mathcal{A}\)-finite map germ iff it is \(\delta_1\)-minimal and \(\text{Eu}(X, 0) = 1\).
2. \((X, 0)\) is the image of a corank 1 \(\mathcal{F}\)-finite frontal iff it is \(\delta_1\)-minimal and \(\text{Eu}(X, 0) = m_0(X, 0)\).
Theorem

Let \((X, 0)\) be a \(\delta_1\)-minimal surface with \(\kappa = \#\) of cusps. Then,

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It was showed in [Jorge-Pérez & Saia, Int. J. Math. '06] that if \((X, 0)\) is the image of a corank 1 \(\mathcal{A}\)-finite map germ, then \(\text{Eu}(X, 0) = 1.\)
For a curve \((Y, 0) \subset (\mathbb{C}^N, 0)\), we introduced in [Nuño & Tomazella, BLMS '08]:

\[
m_1(Y, 0) := \mu(\ell|_{(Y, 0)}),
\]

where \(\ell : \mathbb{C}^N \to \mathbb{C}\) is a generic linear form and \(\mu(\ell|_{(Y, 0)})\) is the Milnor number in the sense of [Mond & van Straten, JLMS '01]. Then, we showed:

\[
m_1(Y, 0) = \mu(Y, 0) + m_0(Y, 0) - 1,
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**Proposition**

Let \((X, 0)\) be the total space of an unfolding of a curve \((Y, 0)\). Then for \(t \neq 0\),

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**Corollary**

They are equivalent:

- \(m_1(X, 0) = 0.\)
- The unfolding is \(m_1\)-constant.
- \((X, 0)\) is a frontal (in the hypersurface case \(N = 2\)).