Lectures on supersymmetry and superstrings

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Abstract: These are informal notes on some mathematical aspects of supersymmetry, based on two 'Mathematics and Theoretical Physics Meetings' in the Autumn Term 2009.[1] First super vector spaces, Lie superalgebras and supercommutative associative superalgebras are briefly introduced together with superfunctions and superspaces. After a review of the Poincaré Lie algebra we discuss Poincaré Lie superalgebras and introduce Minkowski superspaces. As a minimal example of a supersymmetric model we discuss the free scalar superfield in two-dimensional $N = (1, 1)$ supersymmetry (this is more or less copied from [1]). We briefly discuss the concept of chiral supersymmetry in two dimensions. None of the material is original. We give references for further reading. Some 'bonus material' on (super-)conformal field theories, (super-)string theories and symmetric spaces is included. These are topics which will play a role in future meetings, or which came up briefly during discussions.

1 General superalgebra

Definition: A super vector space V is a vector space with a decomposition

$$V = V_0 \oplus V_1$$

and a map (parity map)

$$\tilde{\cdot} : (V_0 \cup V_1) - \{0\} \to \mathbb{Z}_2$$

which assigns even or odd parity to every homogeneous element. Notation:

$$a \in V_0 \to \tilde{a} = 0 \;, \quad \text{'even'} \;, \quad b \in V_1 \to \tilde{b} = 1 \quad \text{'odd'} \;.$$

Definition: A Lie superalgebra $\mathbf{g}$ (also called super Lie algebra) is a super vector space

$$\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{g}_1$$

together with a bracket respecting the grading:

$$[\mathbf{g}_i \;, \mathbf{g}_j] \subset \mathbf{g}_{i+j}$$

(addition of index mod 2), which is required to be $\mathbb{Z}_2$ graded anti-symmetric:

$$[a, b] = -(-1)^{\tilde{a}\tilde{b}}[b, a]$$

(multiplication of parity mod 2), and to satisfy a $\mathbb{Z}_2$ graded Jacobi identity:

$$[a, [b, c]] = [[a, b], c] + (-1)^{\tilde{a}\tilde{b}}[b, [a, c]] \;.$$

Remark: If one imposes that the bracket has the standard properties (antisymmetric, Jacobi identity) and is compatible with the grading, then one obtains a $\mathbb{Z}_2$ graded Lie algebra (= 'normal' Lie algebra with a $\mathbb{Z}_2$ grading).

[1] See `www.liv.ac.uk/mohaupt/p+m_meetings.html`

But sometimes the term '$\mathbb{Z}_2$ graded Lie algebra' is used synonymously to super Lie algebra.

Example: If V is a super vector space, then $\mathrm{End}(V)$ is naturally a super vector space, and in fact a Lie superalgebra (with the graded commutator of maps as Lie superbracket.)

Definition: An associative superalgebra A is a super vector space which is an associative ring such that the multiplication is compatible with the grading:

$$A_i \cdot A_j \subset A_{i+j}$$

Definition: An associative superalgbra A is called supercommutative if homogeneous elements satisfy

$$a \cdot b = (-1)^{\tilde{a}\tilde{b}} b \cdot a \ .$$

Defintion of superfunctions and of superspace $\mathbb{R}^{m|n}$.

Let $U \subset \mathbb{R}^m$ be an open set. Then the smooth functions on U with values in the Grassmann algebra $G_n(\mathbb{R})$ with n generators form the ring

$$O_{m,n}(U) \simeq C^\infty(U) \otimes G_n(\mathbb{R}) \ .$$

If we denote the generators of $G_n(\mathbb{R})$ by θ_i, $i = 1, \ldots, n$, where

$$\theta_i \theta_j = -\theta_j \theta_i$$

then such a function admits an expansion

$$f(x,\theta) = \sum_{\epsilon_1=0}^{1} \cdots \sum_{\epsilon_n=0}^{1} f_{\epsilon_1 \cdots \epsilon_n}(x) \theta_1^{\epsilon_1} \cdots \theta_n^{\epsilon_n} =: \sum_{\epsilon} f_\epsilon(x) \theta_1^{\epsilon_1} \cdots \theta_n^{\epsilon_n} \qquad (1)$$

where $x \in U$, and $f_\epsilon \in C^\infty(U)$. If we vary U over the topology, we obtain a presheaf of rings $(\mathbb{R}^m, O_{m,n})$, which can be shown to be complete. The resulting ringed space

$$\mathbb{R}^{m|n} = (\mathbb{R}^m, O_{m,n})$$

is the standard superspace of even dimension m and odd dimension n. Interpretation: $x^1, \ldots, x^m$ are 'even coordinates' and $\theta^1, \ldots, \theta^n$ are 'odd coordinates'.

$\mathbb{R}^{m|n}$ serves as the local model for supermanifolds. **Definition:** a supermanifold of even dimension m and odd dimension n is a ringed space which 'locally looks like a domain in $\mathbb{R}^{m|n}$.' (Define supercharts, etc. Underlying topological space should be Hausdorff and second-countable.)

Literature: The above is partially based on [2].

2 The Poincaré Lie algebra

As a vector space, the Poincaré Lie algebra has the form

$$\mathrm{Poinc}(V) = iso(V) = V \oplus so(V)$$

where V is a real vector space equipped with a non-degenerate bilinear form and $so(V)$ the Lie algebra of infinitesimal orthogonal transformations. If we denote

V equipped with the non-degenerate bilinear form of signature (p, q) by $\mathbb{R}^{p,q}$ then we can write

$$\mathrm{Poinc}(p, q) = iso(p, q) = \mathbb{R}^{p,q} \oplus so(p, q)$$

Strictly speaking the Poincaré Lie algebra corresponds to the case where $p = 1$ or $q = 1$, i.e. one direction is 'time-like', the others 'space-like'. But sometimes one considers general (p, q). The case $p = 0$ or $q = 0$ is the Euclidean Lie algebra (isometries of affine space with positive/negative definite scalar product).

The basis elements or generators are denoted P_m (translations) and L_{mn} (rotations and pseudo-rotations/Lorentz boosts). Schematically, the Lie algebra structure is as follows:

1. The rotations form a subalgebra isomorphic to $so(p, q)$:
$$[L, L] \subset L$$
2. The translations form an abelian subalgebra
$$[P, P] = 0$$
3. The translations transform in the fundamental representation of $so(p, q)$:
$$[L, P] \subset P$$

Due to the third property the above sum is semi-direct, rather than direct (as a sum of Lie algebras). The algebra is not semi-simple, because the translations form an invariant abelian subalgebra. This complicates the representation theory (use method of induced representations, Wigner, McKay).

3 Poincaré Lie superalgebras

There is a natural way to extend the Poincaré Lie algebra (minimally) into a Lie superalgebra. As a vector space, the even part is $iso(p, q)$, while the odd part is taken to be 'the' spinor S representation of $so(p, q)$. (By 'the' spinor representation we refer to the irreducible representation of the Clifford algebra $\mathrm{Cliff}(p, q)$ associated with the bilinear form on $\mathbb{R}^{p,q}$, which is unique up to equivalence.)

$$\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{g}_1 = (\mathbb{R}^{p,q} \oplus so(p, q)) \oplus S$$

The basis elements or generators of S are denoted Q_α, and are called supertransformations, supersymmetry transformations or supertranslations. The Lie superalgebra structure looks schematically as follows (it is understood that $iso(p, q)$ is an even subalgebra):

1. Translations and supertransformations commute:
$$[P, Q] = 0$$
2. Generators of supertransformations transform in the spinor representation of $so(p, q)$:
$$[L, Q] \subset Q$$

3. Supertransformations 'anticommute into translations':

$$\{Q, Q\} \subset P$$

Here $\{\cdot, \cdot\}$ denotes the anticommutator: $\{Q_\alpha, Q_\beta\} = Q_\alpha Q_\beta + Q_\beta Q_\alpha$.

The interesting piece we need to spell out in more detail is the 'supertranslation subalgebra' generated by P_m and Q_α. Schematically it has the form $Q = \sqrt{P}$, i.e. 'supertransformations are square roots of translations'.[2] For concreteness, let us give the anticommutation relations explicitly for $p = 1, q = 3$ (4d Minkowski space):

$$\{Q_\alpha, Q_\beta\} = (\gamma^m C)_{\alpha\beta} P_m \tag{2}$$

Here we have taken the spinor representation to be real (Majorana representation). We could equivalently have taken it to be complex but chiral (Weyl representation). γ^m are the generators of the Clifford algebra

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}\mathbb{I}$$

where η^{mn} is the matrix of the bilinear form on $\mathbb{R}^{p,q}$. The Clifford generators are not symmetric with respect to the bilinear form, but there is a more or less unique element C of the Clifford algebra (called the charge conjugation operator), such that $\gamma^m C$ are symmetric. The fact that the 'sturcture constants' of supertranslations are determined by the Clifford algebra follows from the fact that the bracket of two supercharges must be 'spin equivariant': the left hand side and right hand side (2) must transform in the same respresenation of the Lorentz algebra. The structure of the supertranslation algebra (2) is universal, though details depend on the dimension and signature of $\mathbb{R}^{p,q}$.

One physical motivation for supersymmetry is 'the unification of forces and matter': representations of the Poincaré Lie superalgebra necessarily combine irreducible representations of the Poincaré Lie algebra with different spin (one of the Casmir operators). In the minimal case considered above any representation of the Poincaré Lie superalgebra combines two representations of the Poincaré Lie algebra, where the spin differs by $\frac{1}{2}$. Since states with integer spin represent 'forces', while states with half-integer spin represent 'matter', forces and matter are combined into irreducible representations.

There are two types of non-mininal super extensions of the Poincaré Lie algebra.

1. One can take several copies of S as the odd part:

$$\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{g}_1 = (\mathbb{R}^{p,q} \oplus so(p,q)) \oplus (S \oplus S \oplus \cdots)$$

The supersymmetry generators Q^I_α, $I = 1, \ldots, N$ anti-commute for $I \neq J$ and close into translations for $I = J$. If there are N copies of S, this is called the N-fold extended supersymmetry algebra. Physical considerations restrict the size of N. If N is too large, then any representation contains states where no physically meaningful interactions can be defined. For $(p, q) = (1, 3)$ the bound is $N \leq 8$, for $(p, q) = (1, 10)$ it is $N \leq 1$, i.e. only 'simple supersymmetry'.

[2]Not so weird: the Dirac operator is a square root of the Laplace/D'Alembert operator, and one can get a lot of geometry out of this [3].

2. For $N > 1$ the algebras also admit non-trivial central extensions. More recently it has become clear in the context of supergravity and superstring theory that one should also consider polyvector extensions (aka BPS charges). Polyvector charges are invariant under supertransformations but transform as antisymmetric tensors under Lorentz transformations. Polyvector-extended Poincaré Lie superalgebras have been classified for arbirary (p, q).

Literature: This is partially based on [4] and [5], where Poincaré Lie superalgebras with arbitrary (p, q) and their polyvector extensions were classfied.

4 Superspace and superfields

Our above discussion of superspaces and supermanifolds did not take into account that in supersymmetric theories we require that the Poincaré Lie superalgebra acts on superspace and superfields. The general concept of superspace and supermanifolds is useful by itself, but now we need to incorporate Poincaré supertransformations to connect our discussion to the construction of supersymmetric theories.

First note that Minkowski space $\mathbb{R}^{p,q}$ is a homogeneous space (in fact a symmetric space[3]) for the Poincaré group:

$$\mathbb{R}^{p,q} = \frac{ISO(p, q)}{SO(p, q)}$$

(We use the same symbol $\mathbb{R}^{p,q}$ for the affine space and the vector space). Translations act on this space simply by

$$x \to x + a$$

and hence on functions by

$$f(x) \to f(x + a)$$

Historically, superspace was introduced as the homogeneous space of the $(N = 1)$ Poincaré super Lie group:

$$\mathbb{R}^{p,q|n} = \frac{SuperISO(p, q)}{SO(p, q)} .$$

Here n is equal to the dimension of S, the spinor representation. Using superspace, supertransformation can be realized as 'translations of the odd coordinates', i.e. they are really 'supertranslations':

$$\theta^\alpha \to \theta^\alpha + \epsilon^\alpha$$

Here θ^α is an anticommuting coordinate in the sense introduced earlier, and ϵ^α the parameter of the supertransformation. This implies that ϵ^α should also have odd paritiy, i.e. it is an anticommuting parameter. Moreover, given that $so(p, q)$ acts on S by the spinor representation, both θ^α and ϵ^α are 'anticommuting spinors'.

[3] See appendix B.

The space of anticommuting spinors is sometimes denoted ΠS to distinguish it from the space of 'commuting spinors' S. Here Π is the 'parity functor' which changes the parity of the elements of a super vector space.

$$\Pi V_0 = V_1' \ , \quad \Pi V_1 = V_0'$$

Remark: we can take any normal (i.e. 'purely even') vector space $V = V_0$ and 'make it purely odd'

$$\Pi V = \Pi(V_0) = V' = V_1'$$

and we can take any purely odd vector space and obtain a 'normal' vector space by 'forgetting the grading'.

To have a consistent operation of the algebra on superspace, a translation of the odd coordinates must be accompanied by a translation of the even coordinates:

$$x^m \to x^m + \langle \epsilon | \gamma^m \theta \rangle$$

Here $\langle \cdot | \cdot \rangle$ is a suitably normalized bilinear form on ΠS, the space of anticommuting spinors. In physics spinor variables are taken to be odd to implement the Pauli principle.

5 An example

The following example is taken from [1] Section 4.1. Conventions and notation differ somewhat from those used above.

As an example, we consider supersymmetry in two-dimensional Minkowski space $\mathbb{R}^{1,1}$. The two-dimensional case has special features, which are not present in higher dimensions and deserve a comment. The Lorentz group $SO(1,1)$ is abelian and its irreducible representations are one-dimensional. For the quantum theory, the relevant group is its double cover $Spin(1,1)$. Its representations are labeled by the half integer helicity $\lambda \in \frac{1}{2}\mathbb{Z}$.

We take the metric to be

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The generators of the Clifford algebra can be taken to be purely imaginary (pseudo-Majorana representation, we could also take them purely real):

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ .$$

These matrices obey the Clifford algebra

$$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta} \ .$$

The coordinates on Minkowski space $\mathbb{R}^{1,1}$ are denoted σ^α, $\alpha = 0, 1$. To obtain the corresponding superspace $\mathbb{R}^{1,1|2}$ we introduce two anticommuting coordinates θ^A, $A = 1, 2$. These must form a spinor under the Lorentz algebra. Since we work with a (pseudo-)Majorana representation, θ^A should form the components of a Majorana spinor θ. The Lorentz invariant symmetric bilinear form on the 'anticommuting' spinor representation is

$$\overline{\psi}\chi = \rho^0_{AB}\psi_A\chi_B = -\rho^0_{AB}\chi_B\psi_A = \rho^0_{BA}\chi_A\psi_B = \overline{\chi}\psi$$

Note that the anticommutativity of odd spinors is compensated by the antisymmetry of the Clifford generator ρ^0. (The Clifford generator is needed to obtain a Lorentz invariant bilinear form.) The quanity

$$\overline{\theta} = \theta^T \rho^0$$

is called the Majorana conjugate of θ. (The Majorana condition, which is a reality condition for spinors, can be expressed as equality of Dirac conjugate $\theta^+ \rho^0$ and Majorana conjugate.)

A general superfield takes the form

$$Y(\sigma, \theta) = X(\sigma) + \overline{\theta}\psi(\sigma) + \frac{1}{2}\overline{\theta}\theta B(\sigma)$$

Compared to (1) two points deserve comment. First, we used the bilinear form to write the terms in the expansion in Lorentz covariant form. The transformation properties of the component fields are manifest: $X(\sigma)$ is a scalar field, $\psi(\sigma)$ is a Majorana spinor field, $B(\sigma)$ is a scalar field. Second in (1), the component fields were real or complex valued. But in physics we need to implement the Pauli principle, which requires that fields with half integer spins (fermions) are odd (anti-commute) while fields with integer spin (bosons) are even (commute). Therefore, the field $\psi(\sigma)$ is taken to be an anticommuting spinor field. This requires to tensor the algebra of functions with an additional odd parameter space. If we allow all component fields to be valued in this additional parameter space, then the above superfield can be characterized as the general even function (odd powers of θ have odd component functions, even powers of θ have even component functions.)

Supertransformations act as translations on the odd coordinates, thus the generator should involve derivatives with respect to θ. Since only polynomials of finite order in odd coordinates can occur, the only rule of differentiation needed is

$$\frac{\partial}{\partial \theta^A} \theta^B = \delta_A^B$$

Anticommutativity implies that the Leibnitz rule must take a graded form:

$$\frac{\partial}{\partial \theta^A}(\theta^B \theta^C) = \frac{\partial}{\partial \theta^A}(\theta^B)\theta^C - \theta^B \frac{\partial}{\partial \theta^A}\theta^C \; .$$

Thus the derivative $\frac{\partial}{\partial \theta^A}$ carries odd parity itself. (Derivatives are derivations (on the algebra of germs of functions). For derivatives wrt odd variables one defines 'superderivations' by replacing the Leibnitz rule by a graded version.)

Supertransformations act on super Minkowski space by the differential operators

$$Q_A = \frac{\partial}{\partial \overline{\theta}^A} + i(\rho^\alpha \theta)_A \partial_\alpha$$

(Here $\overline{\theta}^A$ are the components of the 'Majorana conjugate' $\overline{\theta} = \theta^T \rho^0$.) We will see that the second term is needed when we verify that this differential operator satisfies the anticommutation relations of the Poincaré Lie superalgebra (we only need to check the subalgebra of supertranslations).

Now take ϵ to be the parameter of an infinitesimal supersymmetry transformation and compute the action on odd and even coordinates:

$$\begin{aligned} \delta\theta^A &:= [\overline{\epsilon}Q, \theta^A] = \epsilon^A \\ \delta\sigma^\alpha &:= [\overline{\epsilon}Q, \sigma^\alpha] = i\overline{\epsilon}\rho^\alpha\theta \end{aligned}$$

Since

$$[\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] = -2i\bar{\epsilon}_1 \rho^\alpha \epsilon_2 \partial_\alpha = a^\alpha \partial_\alpha$$

is a 'normal' translation with parameter $a^\alpha = -2i\bar{\epsilon}_1 \rho^\alpha \epsilon_2$, we see that these differential operators realize the supertranslation algebra

$$\{Q_A, Q_B\} = 2(\rho^\alpha \rho^0)_{AB} P_\alpha$$

where P_α is the (Hermitian) generator of translations, $P_\alpha = -i\partial_\alpha$.

The action on superfields is

$$\begin{aligned} \delta Y &= [\bar{\epsilon} Q, Y] = (\bar{\epsilon} Q Y) \\ [\delta_1, \delta_2] Y &= -a^\alpha \partial_\alpha Y \end{aligned}$$

By expansion into 'component fields', one obtains

$$\begin{aligned} \delta X &= \bar{\epsilon} \psi \\ \delta \psi &= -i\rho^\alpha \epsilon \partial_\alpha X + B\epsilon \\ \delta B &= -i\bar{\epsilon} \rho^\alpha \partial_\alpha \psi \end{aligned}$$

Interpretation: under supersymmetry, the scalar field X transforms into the spinor field ψ, the spinor field ψ transforms into the derivative of the scalar field plus an 'auxiliary scalar field' B, the auxiliary scalar field B transforms into the derivative of the spinor field. In applications, X and ψ are dynamical field, while B is a dummy field which allows to represent the algebra without imposing field equations. Indeed, if we impose $B = 0$, then the above algebra 'closes' if we require that ψ satisfied the massless Dirac equation $\rho^\alpha \partial_\alpha \psi = 0$.

To construct an action functional for this superfield, we need two further concepts.

- Integration for odd coordinates is defined by

  $$\int d^2\theta (a + \theta^1 b_1 + \theta^2 b_2 + \theta^1 \theta^2 c) = c$$

  This 'Berezin integral' is a linear functional on the Grassmann algebra, which simply picks the coefficient of the top element. We do not need to interprete this in terms of an underlying measure theory, but rather consider it as a purely algebraic operation.

- To write down actions we need derivatives of superfields, but the odd partial derivative does not transform in a simple way under supersymmetry. One would like to have a 'covariant derivative' which has a simple transformation behaviour. This is not the case for the odd partial derivative $\frac{\partial}{\partial \theta^A}$, but it is straightforward to check that

  $$D_A = \frac{\partial}{\partial \bar{\theta}^A} - i(\rho^\alpha \theta)_A \partial_\alpha$$

  anticommutes with supersymmetry transformations

  $$\{D_A, Q_B\} = 0 \ .$$

Also note that the product of two superfields is again a superfield. Thus all operations needed to construct actions are at our disposal.

To obtain action functionals which are *invariant* under supersymmetry one uses the result that for any superfield F the action

$$S = \int d^2\sigma d^2\theta F$$

is invariant supersymmetry, up to boundary terms resulting from integration by parts. Since $\overline{D}YDY = \overline{D}_A Y D_A Y$ is a superfield,

$$S = \int d^2\sigma d^2\theta \overline{D}YDY$$

is an invariant action. This is in fact the supersymmetric generalization of the standard action of a free massless scalar field. When decomposing the integrand into component fields and integrating over odd variables one finds (dropping a multiplicative constant)

$$S = \int d^2\sigma(\partial_\alpha X \partial^\alpha X - i\overline{\psi}\rho^\alpha\partial_\alpha\psi - B^2)$$

The first term is the standard kinetic term for massless scalar. The equation of motion is the massless wave equation. The second term is the standard kinetic term for a massless spinor field. The equation of motion is the massless Dirac equation. The third term is purely algebraic, and therefore the field B is non-dynamical and can be eliminated by its equation of motion $B = 0$. However, if we eliminate B the action is only supersymmetry invariant modulo the equations of motion (on shell supersymmetry), while as long as we keep B we do not need to impose the equations of motion (supersymmetry is realized off shell).

6 Chirality and chiral supersymmetry in two dimensions

If we write the equations of motion

$$\begin{aligned} \partial_\alpha\partial^\alpha X &= 0 \\ \rho^\alpha\partial_\alpha\psi &= 0 \end{aligned}$$

in light cone coordinates $\sigma^\pm = \sigma^0 \pm \sigma^1$,

$$\begin{aligned} \partial_+\partial_- X &= 0 \\ \partial_\pm\psi_\mp &= 0 \end{aligned}$$

then we realize that the general solution is given by decoupled left- and right-moving waves:

$$\begin{aligned} X &= X_L(\sigma^+) + X_R(\sigma^-) \\ \psi &= \begin{pmatrix} \psi_-(\sigma_-) \\ \psi_+(\sigma_+) \end{pmatrix} \end{aligned}$$

The decoupling of left- and rightmoving degrees of freedom is a particular property of massless two-dimensional fields. One interesting consequence is the existence of 'chiral' supersymmetry algebras, which act differently on the left- and right-moving degrees of freedom.

In our previous example, the supersymmetry charges formed a Majorana spinor Q_A and acted symmetrically on the left- and rightmoving fields. In two-dimensions (with Minkowski signature), a Majorana spinor can be decomposed into two Majorana-Weyl spinors $Q_\pm$ whic act according to

$$\begin{array}{lll} Q_+ & : & X_L(\sigma^+) \to \psi_+(\sigma^+) \to \cdots \\ Q_- & : & X_R(\sigma^-) \to \psi_-(\sigma^-) \to \cdots \end{array} \tag{3}$$

We can therefore eliminate the Majorana spinor ψ_- (by imposing a constraint), and obtain a theory where the dynamical fields are $X = X_L + X_R$ and ψ_+. This theory is invariant under Q_+ transformation which map $X_L \to \psi_+$ and leave X_R invariant. Such models are called 'heterotic', and the corresponding supersymmetry algebra is called the $N = (1,0)$ supersymmetry algebra. The standard '$N = 1$' of our previous example, which was left-right symmetry is then called $N = (1,1)$. More generally one can construct algebras and models with $N = (p,q)$ supersymmetry, where p and q count left and right-moving supercharges.

7 Non-linear sigma models

The supersymmetric model considered above has a bilinear action and, hence, linear equations of motion. Such models are 'free', they do not describe interactions. There are various way of deforming a free field theory into an interacting one. One way of generalizing the scalar part of the action is

$$S[X] = \int d^2\sigma \partial_\alpha X^i \partial^\alpha X^j g_{ij}(X(\sigma)) \; .$$

This is called a non-linear sigma model (for irrelevant historical reasons). The scalar fields X^i can be interpreted as the components of a map X from Minkowski space into a (pseudo-)Riemannian manifold N with metric g. As a further generalization, we can replace Minkowski space by a general (pseudo-)Riemannian space time M with metric h. The resulting Polyakov action is

$$S[X,h] = \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j g_{ij}(X(\sigma))$$

The scalar fields X^i are the components of a map

$$X \; : (M,h) \to (N,g)$$

between (pseudo)-Riemannian spaces. In geometrical terms the Polyakov action simply is

$$S[X,h] = \langle dX, dX \rangle \; ,$$

the norm of the vector valued differential

$$dX \in \Gamma(M, T^*M \otimes X^*TN) \; .$$

At $p \in M$ the local form is

$$dX(\sigma) = \partial_\alpha X^i(\sigma) d\sigma_p^\alpha e_i(X(\sigma)) \ ,$$

where e_i is a local frame on N.

The Polyakov action is the 'energy functional' for maps, and the stationary points are harmonic maps

$$\mathrm{tr}_h DdX = 0 \ .$$

In components this reads

$$\Delta_h X^i + \Gamma[g]^i_{jk} h^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^J = 0$$

where Δ_h is the Laplace or D'Alambertian on (M, h) and where $\Gamma[g]$ are the Christoffel symbols of (N, g).

The non-linear sigma model can be extended to a supersymmetric models. The spinors ψ are vector-valued sections of the spinor bundle over M,

$$\psi \in \Gamma(M, \Pi S \otimes X^* TN) \ .$$

Supersymmetry implies the presence of various new terms in the action, including a 'four-fermion term' of the form

$$R_{ijkl} \bar{\psi}^i \psi^j \bar{\psi}^k \psi^l$$

where R_{ijkl} is the Riemann tensor of (N, g).

Supersymmetry might impose restrictions on the geometry of the target space N. While any (pseudo-)Riemannian manifold N can be used to define a model with two-dimensional $N = (1, 1)$ supersymmetry, $N = (2, 2)$ supersymmetry requires that the target space is Kähler. If one uses the $N = 2$ version of superspace, then the action is

$$\int d^2\sigma d^4\theta K(\Phi, \bar{\Phi})$$

where the function K is real analytic in the (chiral) $N = 2$ superfield Φ. It turns out the K (when evaluated on the scalar part of the superfield) is a Kähler potential for the metric g of the scalar sigma model. Thus the Kähler geometry of N follows constructively from the $N = 2$ superspace formulation (see Mirian's lectures for more details).

If one enlarges the supersymmetry algebra even further, the geometry of N becomes even more restricted. For example $N = (4, 4)$ implies that N must be hyper-Kähler. Left-right asymmetric supersymmetry algebras lead to geometries which are less familiar. For example $N = (4, 0)$ leads to hyper-Kähler with torsion (HKT). Some of these geometries anticipated the concept of 'generalized complex structures' introduced by Hitchin. For sufficiently high N only symmetric spaces are possible.

Non-linear sigma models and their supersymmetric extensions can also be considered on higher-dimensional space-times M. In four dimension $N = 1$ requires Kähler geometry, and $N = 2$ requires, depending on the representation used, affine special Kähler or hyper-Kähler geometry (or others for more exotic representations). In four-dimensions the presence of a non-trivial space-time metric h modifies the possible target space geometries. For $N = 1$ it must be Kähler-Hodge, for $N = 2$ projective special Kähler or Quaternion-Kähler (or others, depending on the representation). For $N \geq 4$ only (specific) symmetric spaces are possible.

8 Conformal field theories

The Polyakov action is invariant under conformal transformations of the metric h on M (also called Weyl transformations)

$$h_{\alpha\beta} \to e^{2\Lambda(\sigma)} h_{\alpha\beta} \; .$$

This implies that if we fix a metric on h we can still compensate conformal diffeomorphisms by a Weyl transformation. In the quantum version of the theory this symmetry is 'anomalous', which means that it cannot be implemented 'verbatim'. Concretely, in the quantum theory we encounter a central extension c of the Lie algebra of conformal diffeomorphisms.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m-1)(m+1)$$

This is the famous Virasoro algebra. In conformal field theories, the Hilbert space of states carries a unitary representation of this algebra (this requires that $c > 0$ on irreducible representations, and thus one cannot implement the unextended Lie algebra of conformal diffeomorphisms as a symmetry). This is viewed as the quantum version of 'conformal invariance'. Conformal invariance of a non-linear sigma model 'at the quantum level' imposes restrictions on the geometry of the target space N. This can be expressed as the vanishing of a so-called β-function:

$$\bar{\beta}[g(X)]_{ij} = -\frac{1}{4\pi} \left(R_{ij} + \frac{l^2}{2} R_{iabc} R_j{}^{abc} + \mathcal{O}(l^2) \right) \stackrel{!}{=} 0$$

Here R_{ij} and R_{ijkl} are the Ricci tensor and the Riemann tensor of (N, g). The above equation is an expansion in derivatives, which is controlled by a parameter l which has the dimension of a length. You should think of l as controlling the curvature of (N, g), with $l \to 0$ corresponding vanishing curvature. Clearly, an expansion in derivatives only makes sense in the limit of small curvature.

To lowest order the condition on (N, g) is Ricci-flatness, $R_{ij} = 0$. Corrections to this condition can be obtained iteratively, but the important point is that N must *admit* a Ricci-flat metric. This condition is topological in nature and therefore not modified by the corrections.

A conformally invariant sigma model can be extended to a superconformal field theory (SCFT). Supersymmetry and conformal symmetry (Virasoro algebra) get linked together and as a result one obtains infinite dimenesional Lie superalgebras (various N-extended super-Virasoro algebras). As we have seen $N = (2, 2)$ supersymmetry requires that the target space N is Kähler. If we require $N = (2, 2)$ superconformal symmetry then we must impose that N admits a Ricci-flat metric, and thus N must be Calabi-Yau.

9 Conformal field theories and strings

The Polyakov action can be re-interpreted as describing the motion of strings through a space-time (N, g) rather than a field theory on a space-time (M, g) with values in a target space (N, g). Thus M is re-interpreted as a generalized worldline or 'world sheet'. We will use the symbol Σ instead of M.

In string theory the interactions between strings are encoded in the topology of the surface Σ. Intial and final states of an interaction are represented by punctures, and to obtain the probability amplitude for a given process one must sum over all topologies which can contribute.

Different string theories are distinguished by the classes of surfaces one admits. The most restricted choice are closed oriented surfaces (closed oriented strings). One can admit boundaries subject to Neumann or Dirichlet boundary conditions (open strings) and/or admit non-orientable surfaces (non-oriented strings).

Consistency of string theory requires to impose that conformal symmetry is realised exactly at the quantum level (critical string) or to add an additional degree of freedom with particular properties, the Liouville mode (non-critical string theory). We only consider the first option. In contrast to 'normal' conformal field theories one can achieve conformal invariance (despite the anomaly), because the metric h on the world sheet (M, h) is not fixed but treated as a dynamical variable. As it turns out both h and the fields X (interpreted as the coordinates of the string) contribute to the anomaly with opposite signs, achieving cancellation if the dimension of N is fixed to be 26.

10 Superconformal field theories and Superstrings

The so-called RNS formulation (Ramond, Neveu, Schwarz) of superstrings is obtained by extending the Polyakov action to a supersymmetric action. Since the world sheet metric h acquires a superpartner which contributes to the conformal anomaly, the 'critical' dimension of the space-time N, where the conformal vanishes, is now 10 rather than 26.

Since we now also have spinor fields on the world sheet, further choices arise because these fields can be either periodic or anti-periodic around each non-contactible loop on Σ. Such choices correspond to picking a spin structure on Σ. If one takes the space-time to be ten-dimensional Minkowski space, and requires that the resulting theory is also supersymmetric in the *ten-dimensional* sense, then only five choices are consistent.[4]

1. Type IIA. This theory has closed, oriented strings and $N = (1, 1)$ world sheet supersymmetry. The spin structures are chosen such that the ground states form a representation of the non-chiral maximal ten-dimensional supersymmetry algebra.

2. Type IIB. This theory has closed, oriented strings and $N = (1, 1)$ world sheet supersymmetry. The spin structures are chosen such that the ground states form a representation of the chiral maximal ten-dimensional supersymmetry algebra.

3. Type I. This theory has closed and open non-oriented strings. The world sheet supersymmetry is the diagonal part of the $N = (1, 1)$ algebra. (The boundary conditions couple the left- and right-moving sectors and reduce

[4]Let us note that in ten dimensions there are only three relevant (discarding the option of massless fields with spin larger than two) Poincaré Lie superalgebras, the minimal one (supercharges form a Majorana-Weyl spinor), and two maximal ones, one of which is non-chiral (supercharges form two Majorana-Weyl spinors of opposite chirality) and one chiral (supercharges form two Majorana-Weyl spinors of same chirality).

the symmetry.) The ground states fall into representations of the minimal ten-dimensional supersymmetry algebra.

4. Type HE. This theory has closed, oriented strings and $N = (1, 0)$ supersymmetry. The additional fields required in the right-moving sector to cancel the conformal anomaly must realise a 'current algebra' based on an even selfdual Euclidean lattice. (This requirement follows from 'modular invariance' i.e. the invariance of the world sheet under large diffeomorphisms.) There are only two such lattices (up to isometries), and for the HE theory one chooses the root lattice of $E_8 \otimes E_8$. The letter H stands for 'heterotic'. This is one of the two heterotic string theories.

5. Type HO. As HE, but based on the lattice generated by the roots of $SO(16)$ together with the weights of either of the Majorana Weyl spinor representations.

These five distinct ten-dimensional supersymmetric string theories have been argued to be related by 'dualities' and are believed to be limits of a single underlying 'M-theory'. (Consider this a working program rather than a theory or fact.)

A Literature

In addition to the already cited, the following references might be useful. Manin [6] has developed a mathematical approach to gauge theories which is based on complex geometry and supermanifolds. Huybrechts [7] makes a few useful remarks in Chapter 3B. Moreover, this book on complex geometry covers precisely those aspects which are relevant in the context of supersymmetric theories. De Witt's book is one of the classical works on supermanifolds. Volume 1 of [8] contains a lot of introductory material.

B Symmetric Spaces

$\mathbb{R}^{p,q} = \frac{ISO(p,q)}{SO(p,q)}$ is a symmetric space, both in the Riemannian sense (Riemann tensor is parallel, each point is a fixed point of an involutive isometry), and in the Lie sense (a homogeneous space G/H, where H is an open subgroup of fixed points of involutive automorphisms of G). Since $ISO(p, q)$ is not semi-simple, the Killing form is degenerate according to Cartan's criterion. Therefore the (flat) metric on $\mathbb{R}^{p,q}$ is not induced by a non-degenerate metric on $ISO(p, q)$. Instead one has to procede as follows. The group $ISO(p, q)$ can be obtained from semi-simple groups by a contraction (singular limit). There are two main (but equivalent) versions of such contraction: an explicit, coordinate-based version (Wigner-Inönö contraction) and a coordinate-free version (Saletan contraction). Both a described in the last chapter of Gilmore's book [9].

The semi-simple group from which $ISO(p, q)$ is obtained by contraction is not unique. Two possible choices are $SO(p + 1, q)$ and $SO(p, q + 1)$. The contraction can be performed at the level of the associated symmetric spaces:

$$\frac{SO(p+1,q)}{SO(p,q)} \longrightarrow \frac{ISO(p,q)}{SO(p,q)} = \mathbb{R}^{p,q} \longleftarrow \frac{SO(p,q+1)}{SO(p,q)}$$

In the positive signature case this corresponds to obtaining flat space as a limit of a sphere, and of a hyperboloid, respectively:

$$S^n = \frac{SO(n+1)}{SO(n)} \longrightarrow \mathbb{R}^n = \frac{ISO(n)}{SO(n)} \longleftarrow H^n = \frac{SO(n,1)}{SO(n)}$$

S^n and H^n are the standard spaces of positive and negative curvature respectively. Flat space is obtained by sending the curvature to zero.

In the Lorentzian signature case, Minkowski space is obtained as a limit of de Sitter space and of Anti de Sitter space, respectively.

$$dS^n = \frac{SO(1,n)}{SO(1,n-1)} \longrightarrow \mathbb{R}^{1,n-1} = \frac{ISO(1,n-1)}{SO(1,n-1)} \longleftarrow AdS^n = \frac{SO(2,n-1)}{SO(1,n-1)}$$

De Sitter and anti de Sitter space are the maximally symmetric solutions of the Einstein equations with positive and negative cosmological constant, respectively. Therefore the zero curvature limit corresponds to sending the cosmological constant to zero.

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