Liverpool Lectures on String Theory
The Free Bosonic String
Summary Notes
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1 The classical relativistic particle

1.1 Relativistic kinematics and dynamics

D-dimensional Minkowski space(-time): $\mathbb{M} = \mathbb{M}^D = \mathbb{R}^{1,D-1}$.

Points = Events: $x = (x^\mu) = (x^0, x^i) = (t, \vec{x})$,
where $\mu = 0, \ldots, D-1$, $i = 1, \ldots, D-1$ and we set $c = 1$.

Minkowski metric ('mostly plus'):

$$\eta = (\eta_{\mu\nu}) = \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_{D-1} \end{pmatrix} . \tag{1}$$

Scalar product (relativistic distance between events):

$$\eta_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu = -t^2 + \vec{x}^2 \begin{cases} < 0 & \text{for timelike distance,} \\ = 0 & \text{for lightlike distance,} \\ > 0 & \text{for spacelike distance.} \end{cases} \tag{2}$$

World line of a particle:

$$x(t) = (x^\mu(t)) = (t, \vec{x}(t)) . \tag{3}$$

where t = time measured in some inertial frame.

Velocity :

$$\vec{v} = \frac{d\vec{x}}{dt} . \tag{4}$$

Note:

$$\vec{v} \cdot \vec{v} \begin{cases} < 1 (= c^2) & \text{for massive particles (timelike worldline) ,} \\ = 1 (= c^2) & \text{for massless particles (lightlike worldline) .} \end{cases} \tag{5}$$

Proper time:

$$\tau(t_1, t_2) = \int_{t_1}^{t_2} dt \sqrt{1 - \vec{v}^2} . \tag{6}$$

To find $\tau(t)$, take upper limit to be variable:

$$\tau(t_1, t) = \int_{t_1}^{t} dt' \sqrt{1 - \vec{v}^2} . \tag{7}$$

Implies:

$$\frac{d\tau}{dt} = \sqrt{1 - \vec{v}^2} = \sqrt{1 - \frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt}} . \tag{8}$$

NB: this makes sense for massive particles, only, because we need $\vec{v}^2 < 1$. (See below for massless particles).

Relativistic velocity:

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{dt} \frac{dt}{d\tau} \right) = \frac{(1, \vec{v})}{\sqrt{1 - \vec{v}^2}} . \tag{9}$$

Note that relativistic velocity is normalised:

$$\dot{x}^\mu \dot{x}_\mu = -1 . \tag{10}$$

Relativistic momentum (kinetic momentum):

$$p^\mu = m\dot{x}^\mu = (p^0, \vec{p}) = \left(\frac{m}{\sqrt{1-\vec{v}^2}}, \frac{m\vec{v}}{\sqrt{1-\vec{v}^2}} \right) . \tag{11}$$

NB: m is the mass (=total energy measured in the rest frame).
For $c = 1$ note that $E = p^0$ is the total energy, and:

$$p^\mu p_\mu = -m^2 = -E^2 + \vec{p}^2 . \tag{12}$$

In the rest frame we have $E = m (= mc^2)$.

Relativistic version of Newton's second axiom:

$$\frac{d}{dt}\vec{p} = \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}} \right) = \vec{F} , \tag{13}$$

where $\vec{F}$ is the force acting on the particle.

NB: This is the correct relativistic version of Newton's second axiom, but it is not (yet) manifestly covariant.

1.2 Action principle for the massive free particle: non-covariant version

Action and Lagrangian:

$$S[\vec{x}] = \int dt L(\vec{x}(t), \vec{v}(t)) . \tag{14}$$

Action and Lagrangian for a free massive relativistic particle:

$$S = -m \int dt \sqrt{1-\vec{v}^2} = -\text{mass times proper time} . \tag{15}$$

NB: we have set $\hbar = 1$, action is dimensionless. The action of a particle is proportional to its proper time. The factor m is needed to make the action dimensionless.

Action principle/Variational principle: the equations of motion are found by imposing that the action is stationary with respect to variations $x \to x + \delta x$ of the path. Initial and final point of the path are kept fixed: $\delta x(t_i) = 0$, where t_1 = time at initial position and t_2 = time at final position.

Substitute variation of path into action and perform Taylor expansion:

$$S[\vec{x} + \delta\vec{x}, \vec{v} + \delta\vec{v}] = S[\vec{x}, \vec{v}] + \delta S + \cdots \tag{16}$$

When performing the variation, we can use the chain rule:

$$\delta\sqrt{1-\vec{v}^2} = \frac{\partial}{\partial v^i}\sqrt{1-\vec{v}^2}\delta v^i \tag{17}$$

(To prove this, you peform a Taylor expansion of the square root).
$\vec{v}, \delta\vec{v}$ are not independent quanties:

$$\delta\vec{v} = \delta\frac{d\vec{x}}{dt} = \frac{d}{dt}\delta\vec{x} \ . \tag{18}$$

To find δS, we need to collect all terms proportional to δx. Derivatives acting on δx are removed through integration by parts. Result:

$$\delta S = -\int_{t_1}^{t_2} \left(\frac{d}{dt} \frac{m v_i}{\sqrt{1-\vec{v}^2}} \right) \delta x^i dt + \left. \frac{m v_i}{\sqrt{1-\vec{v}^2}} \delta x^i \right|_{t_1}^{t_2} \ . \tag{19}$$

Now use that the ends of the path are kept fixed (thus boundary terms vanish) and that the action principle requires $\delta S = 0$ for all choices of δx to obtain the equation of motion for a free massive particle:

$$\frac{d}{dt} \frac{m\vec{v}}{\sqrt{1-\vec{v}^2}} = \frac{d}{dt}\vec{p} = 0 \ . \tag{20}$$

1.3 Canonical momenta and Hamiltonian

Canonical momentum:

$$\pi^i := \frac{\partial L}{\partial v_i} \ . \tag{21}$$

For $L = -m\sqrt{1-\vec{v}^2}$:

$$\pi^i = p^i \ . \tag{22}$$

Canonical momentum = kinetic momentum (not true in general. Example: charged particle in a magnetic field).

Hamiltonian:

$$H(\vec{x}, \vec{\pi}) = \vec{\pi} \cdot \vec{v} - L(\vec{x}, \vec{v}(\vec{x}, \vec{\pi})) \ . \tag{23}$$

Lagrangian $\rightarrow$ Hamiltonian: go from variables $(\vec{x}, \vec{v})$ (coordinates and velocities) to new variables $(\vec{x}, \vec{\pi})$ (coordinates and canonical momenta) by a Legendre transform:

$$H(\vec{x}, \vec{\pi}) = \frac{\partial L}{\partial \vec{v}} \cdot \vec{v} - L(\vec{x}, \vec{v}(\vec{x}, \vec{\pi})) \ . \tag{24}$$

For $L = -m\sqrt{1-\vec{v}^2}$:

$$H = \vec{\pi} \cdot \vec{v} - L = \vec{p} \cdot \vec{v} - L = \frac{m}{\sqrt{1-\vec{v}^2}} = p^0 = E \ . \tag{25}$$

Thus: Hamiltonian = Total energy (not true in general. See below.)

Our treatment of the relativistic particle provokes the following questions:

- How can we include massless particles?

- How can we make Lorentz covariance manifest?
- How can we show that 'the physics' does not depend on how we parametrise the world line?

We will answer these questions in reverse order.

1.4 Length, proper time and covariance on the world line

Mathematical description of a curve (for concreteness, a curve in Minkowski space $\mathbb{M}$):

$$C \ : \mathbb{R} \ni \sigma \longrightarrow x^\mu(\sigma) \in \mathbb{M} \ . \tag{26}$$

σ is an arbitrary curve parameter.

Tangent vectors of the curve:

$$x'^{\,\mu} := \frac{dx^\mu}{d\sigma} \ . \tag{27}$$

Scalar products of tangent vectors, using Minkowski metric:

$$x'^{\,2} = \eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \begin{cases} > 0 & \text{for spacelike curves,} \\ = 0 & \text{for lightlike like curves (also called null curves),} \\ < 0 & \text{for timelike curves.} \end{cases} \tag{28}$$

Length of a spacelike curve:

$$L(\sigma_1, \sigma_2) = \int_{\sigma_1}^{\sigma_2} \sqrt{\eta \mu \nu \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} d\sigma \ . \tag{29}$$

The length is independent of the parametrisation, i.e., it does not change if one 'reparametrises' the curve:

$$\sigma \rightarrow \tilde{\sigma}(\sigma) \ , \quad \text{where} \frac{d\tilde{\sigma}}{d\sigma} \neq 0 \ . \tag{30}$$

Note that $\tilde{\sigma}$ is an invertible, but otherwise arbitrary function of σ. Often one imposes

$$\frac{d\tilde{\sigma}}{d\sigma} > 0 \ , \tag{31}$$

which means that the orientation (direction) of the curve is preserved.

For timelike curves one can define analogously

$$\tau(\sigma_1, \sigma_2) = \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} \tag{32}$$

and this is nothing but the proper time between the two events located at σ_1 and σ_2.

To find τ (proper time) as a function of σ (arbitrary curve parameter), take the upper limit to be variable:

$$\tau(\sigma) = \int_{\sigma_1}^{\sigma} d\sigma' \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma'} \frac{dx^\nu}{d\sigma'}} \ . \tag{33}$$

Differentiate:

$$\frac{d\tau}{d\sigma} = \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} \ . \tag{34}$$

Compare tangent vector with relativistic velocity:

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\sigma} \frac{d\sigma}{d\tau} = \frac{\frac{dx^\mu}{d\sigma}}{\sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}}} \ . \tag{35}$$

This implies (as required if τ is really proper time):

$$\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1 \ . \tag{36}$$

Relativistic velocity = normalised tangent vector.

Use proper time τ instead of σ as curve parameter:

$$\int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} = \int_{\tau_1}^{\tau_2} d\tau = \tau_2 - \tau_1 \ . \tag{37}$$

This confirms that τ is indeed the proper time.

NB: Similarly, the expression for the length of a spacelike curve takes a simple form when using the length itself as the curve parameter.

1.5 Covariant action for the massive particle

Use new, covariant expression for proper time to rewrite the action:

$$S[x^\mu, x'^\mu] = -m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} \ . \tag{38}$$

This action does now depend on x^μ (and the corresponding relativistic velocity), rather than on $\vec{x}$ (and the velocity). It is now covariant in the following sense:

- The action is invariant under reparametrisations $\sigma \to \tilde{\sigma}(\sigma)$ of the world-line. (Reparametrisations are by definition invertible, hence $\frac{d\tilde{\sigma}}{d\sigma} \neq 0$.)
- The action is manifestly invariant under Poincaré transformations,

  $$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu \ , \tag{39}$$

  where

  $$(\Lambda^\mu{}_\nu) \in O(1, D-1) \quad \text{and} \quad (a^\mu) \in \mathbb{M} \tag{40}$$

  are constant. (This is manifest, because the action is constructed out of Lorentz vectors).

Compute equations of motion through variation $x^\mu \to x^\mu + \delta x^\mu$:

$$\frac{\delta S}{\delta x^\mu} = 0 \Leftrightarrow \frac{d}{d\sigma}\left(\frac{m\, x'^{\,\mu}}{\sqrt{-\, x'^{\,\mu}\, x'_{\,\mu}}}\right) = 0 \ . \tag{41}$$

To get the physical interpretation, replace arbitrary curve parameter σ by proper time τ:

$$\frac{d}{d\tau}\left(m\frac{dx^\mu}{d\tau}\right) = m\ddot{x}^\mu = 0 \ . \tag{42}$$

We claim that this is the same as (13) with $\vec{F} = \vec{0}$.

More generally, the covariant version of (13) is

$$\frac{d}{d\tau}\left(m\frac{dx^\mu}{d\tau}\right) = f^\mu \ , \tag{43}$$

where

$$(f^\mu) = \frac{(\vec{v}\cdot\vec{F}, \vec{F})}{\sqrt{1-\vec{v}^2}} \tag{44}$$

is the relativistic force. To see that this is equivalent to (13), note

$$\frac{dp^0}{dt} = \vec{v}\cdot\frac{d\vec{p}}{dt} \tag{45}$$

(The change of energy per unit of time is related to the acting force $\vec{F}$ by $\vec{v}\cdot\vec{F}$. I.p., if the force and velocity are orthogonal, like for charged particle in a homogenous magnetic field, the energy is conserved.)

Then (13) implies

$$\frac{dp^0}{dt} = \vec{v}\cdot\vec{F} \ . \tag{46}$$

Multiplying by $\frac{d\tau}{dt}$ and using the chain rule we get the manifestly covariant version (43) of (13). The additional equation for p^0 is redundant required for having manifest covariance.

For a free massive particle the equation of motion is

$$m\ddot{x}^\mu = 0 \ . \tag{47}$$

Solution = straight (world)line:

$$x^\mu(\tau) = x^\mu(0) + \dot{x}^\mu(0)\tau \ . \tag{48}$$

Actions with interaction

- If the force f^μ comes from a potential, $f_\mu = \partial_\mu V(x)$, then the equation of motion (43) follows from the action

$$S = -m \int \sqrt{-\dot{x}^2} d\tau - \int V(x(\tau)) d\tau \ . \tag{49}$$

  For simplicity, we took the curve parameter to be proper time. In the second term, the potential V is evaluated along the worldline of the particle.

- If f^μ is the Lorenz force acting on a particle with charge q, $f^\mu = F^{\mu\nu}\dot{x}_\nu$, then the action is

$$S = -m \int \sqrt{-\dot{x}^2} d\tau - q \int A_\mu dx^\mu \ . \tag{50}$$

  In the second term, the (relativistic) vector potential A_μ is integrated along the world line of the particle

$$\int A_\mu dx^\mu = \int A_\mu(x(\tau)) \frac{dx^\mu}{d\tau} d\tau \ . \tag{51}$$

  The resulting equation of motion is

$$\frac{d}{d\tau}\left(m \frac{dx^\mu}{d\tau} \right) = q F^{\mu\nu} \dot{x}_\nu \ , \tag{52}$$

  where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. Equation (52) is the manifestly covariant version of

$$\frac{d\vec{p}}{dt} = q \left(\vec{E} + \vec{v} \times \vec{B} \right) \ . \tag{53}$$

- The coupling to gravity can be obtained by replacing the Minkowsk metric $\eta_{\mu\nu}$ by a general (pseudo-)Riemannian metric $g_{\mu\nu}(x)$:

$$S = -m \int d\tau \sqrt{-g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu} \ . \tag{54}$$

  The resulting equation of motion is the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0 \ . \tag{55}$$

1.6 Canonical momenta and Hamiltonian for the covariant action

Action and Lagrangian:

$$S = \int L d\sigma = -m \int d\sigma \sqrt{-x'^{\,2}} \ . \tag{56}$$

Canonical momenta:

$$\pi^{\mu} = \frac{\partial L}{\partial\, x'_{\mu}} = m \frac{x'^{\,\mu}}{\sqrt{-\, x'^{\,2}}} = m \dot{x}^{\mu}\ . \tag{57}$$

Canonical momenta are dependent:

$$\pi^{\mu}\pi_{\mu} = -m^2 \tag{58}$$

Going from a general curve parameter σ to proper time τ we see that

$$\pi^{\mu} = p^{\mu} \tag{59}$$

and the constraint is recognised as the mass shell condition $p^2 = -m^2$.

Hamiltonian:

$$H = \pi^{\mu}\dot{x}_{\mu} - L = 0 \tag{60}$$

Hamiltonian $\neq$ Total energy. Rather Hamiltonian $= 0$. Typical for covariant Hamiltonians. Reason: canonical momenta not independent. The 'Hamiltonia constraint' $H = 0$ defines a subspace of the phase space. Solutions to the equations of motion live in this subspace. $H = 0$ does not mean that the dynamics is trivial, it just reflects that the momenta are dependent. To stress this one sometimes writes $H \approx 0$ (read: H is weakly zero). One can also write: $H = \lambda(p^2 + m^2)$, where λ is an arbitrary constant. In this form it is clear that the Hamiltonian is not identically zero (strongly zero, implying trivial physics), but that it vanishes whenever the particle satisfies the mass shell condition $p^2 = -m^2$.

We still need to find a way to include massless particles.

1.7 Covariant action for massless and massive particles

Introduce invariant line element $e d\sigma$, where $e = e(\sigma)$ transforms under reparametrisations such that $e d\sigma$ is invariant:

$$\left.\begin{array}{l} \tilde{e}(\tilde{\sigma}) = e(\sigma)\frac{d\sigma}{d\tilde{\sigma}} \\ d\tilde{\sigma} = d\sigma\frac{d\tilde{\sigma}}{d\sigma} \end{array}\right\} \Rightarrow e d\sigma = \tilde{e} d\tilde{\sigma}\ . \tag{61}$$

Action:

$$S[x,e] = \frac{1}{2}\int e d\sigma \left(\frac{1}{e^2}\left(\frac{dx^{\mu}}{d\sigma}\right)^2 - m^2 \right)\ . \tag{62}$$

We allow $m^2 \geq 0$. (In fact, we could allow m^2 to become negative, $m^2 < 0$.)

Symmetries of the action:

- $S[x,e]$ is invariant under reparametrisations $\sigma \to \tilde{\sigma}$.
- $S[x,e]$ is invariant under Poincaré transformations $x^{\mu} \to \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu}$.

The action depends on the fields $x = (x^\mu)$ and e. The corresponding equations of motion are found by variations $x \to x + \delta x$ and $\to e + \delta e$, respectively.

Equations of motion:

$$\frac{d}{d\sigma}\left(\frac{x'^{\,\mu}}{e}\right) = 0 \, , \tag{63}$$

$$x'^{\,2} + e^2 m^2 = 0 \, . \tag{64}$$

The equation of motion for e is algebraic. e is an auxiliary field, not a dynamical field. If $m^2 \neq 0$, we can solve for e:

$$e^2 = -\frac{x'^{\,2}}{m^2} \tag{65}$$

For massive particles we have $m^2 > 0$ and $x'^{\,2} < 0$, so that e^2 is positive. We take the positive root:

$$e = \frac{\sqrt{-\,x'^{\,2}}}{m} \, . \tag{66}$$

Substituting the solution for e into (62) we recover the action (38) of a massive particle. However, the new action allows us to deal with massless particles as well.

According to (64), e controlls the norm of the tangent vector, which can be changed by reparametrisation. Instead of eliminating e by its equation of motion, we can bring it to a prescribed 'gauge' (=parametrisation).

- For $m^2 > 0$, impose the gauge

  $$e = \frac{1}{m} \tag{67}$$

  The equations of motion become:

  $$\ddot{x}^\mu = 0 \tag{68}$$

  $$\dot{x}^2 = -1 \tag{69}$$

  Comment: in this gauge, σ becomes proper time τ.

- For $m^2 = 0$, impose the gauge

  $$e = 1 \, . \tag{70}$$

  The equations of motion become:

  $$\ddot{x}^\mu = 0 \, , \tag{71}$$

  $$\dot{x}^2 = 0 \, . \tag{72}$$

  Comment: for $m^2 = 0$, (64) tells us that the worldline is lightlike, as expected for a massless particle. In this case there is no proper time, but one can choose a so-called affine parametrisation. $e = 1$ (or any other constant value) corresponds to picking an affine parameter.

Note that in both cases the dynamical equation of motion $\ddot{x}^\mu = 0$ must be supplemented by a constraint,

$$\phi = \left\{ \begin{array}{c} \dot{x}^2 + 1 \\ \dot{x}^2 \end{array} \right\} = 0 \tag{73}$$

to capture the full information.

2 The classical relativistic string

2.1 The Nambu-Goto action

Replace particle by one-dimensional string. Worldline becomes a surface, called the worldsheet Σ.

$$X : \Sigma \ni P \longrightarrow X(P) \in \mathbb{M} \,. \tag{74}$$

Coordinates on $\mathbb{M}$ are $X = (X^\mu)$, where $\mu = 0, 1, \ldots, D-1$.
Coordinates on Σ are $\sigma = (\sigma^0, \sigma^1) = (\sigma^\alpha)$. The worldsheet has one spacelike direction ('along the string') and one timelike direction ('point on the string moving forward in time'). Take σ^0 to be time-like, σ^1 to be space-like:

$$\dot{X}^2 \leq 0 \,, \quad (X')^2 > 0 \,. \tag{75}$$

We use the following notation:

$$\begin{aligned} \dot{X} &= (\partial_0 X^\mu) = \left(\frac{\partial X^\mu}{\partial \sigma^0} \right) \,, \\ X' &= (\partial_1 X^\mu) = \left(\frac{\partial X^\mu}{\partial \sigma^1} \right) \,. \end{aligned} \tag{76}$$

Range of worldsheet coordinates:

- Spacelike coordinate:
$$\sigma^1 \in [0, \pi] \,. \tag{77}$$
- Timelike coordinate
$$\sigma^0 \in (-\infty, \infty) \,, \tag{78}$$
for propagation of a non-interacting string for infinite time, or
$$\sigma^0 \in [\sigma^0_{(1)}, \sigma^0_{(2)}] \,. \tag{79}$$
for propagation of a non-interacting string starting for a finite intervall of time).

Nambu-Goto action $\sim$ area of world sheet:

$$S_{\rm NG}[X] = -T A(\Sigma) = -T \int_\Sigma d^2 A \tag{80}$$

T has dimension $(\text{length})^{-2}$ or energy/length = string tension ($\hbar = c = 1$).

Invariant area element on Σ (induced from $\mathbb{M}$):

$$d^2A = d^2\sigma \sqrt{|\det(g_{\alpha\beta})|} \,, \tag{81}$$

where

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{82}$$

is the induced metric (pull back). Note $\det(g_{\alpha\beta}) < 0$.

Action is invariant under reparametrizations of Σ,

$$\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^0, \sigma^1) \,, \quad \text{where} \quad \det\left(\frac{\partial\tilde{\sigma}^\alpha}{\partial\sigma^\beta}\right) \neq 0 \,. \tag{83}$$

Action is also invariant under Poincaré transformations of $\mathbb{M}$.

Action, more explicitly:

$$S_{\text{NG}} = \int d^2\sigma \mathcal{L} = -T \int d^2\sigma \sqrt{(\dot{X}X')^2 - \dot{X}^2(X')^2} \,. \tag{84}$$

$\mathcal{L}$ is the Lagrangian (density). We compute

$$\begin{aligned} \Pi^\mu = P_0^\mu = \frac{\partial\mathcal{L}}{\partial\dot{X}_\mu} &= T\frac{(X')^2\dot{X}^\mu - (\dot{X}X')X'^\mu}{\sqrt{(\dot{X}X')^2 - \dot{X}^2(X')^2}} \,, \\ P_1^\mu = \frac{\partial\mathcal{L}}{\partial X'_\mu} &= T\frac{\dot{X}^2 X'^\mu - (\dot{X}X')\dot{X}^\mu}{\sqrt{(\dot{X}X')^2 - \dot{X}^2(X')^2}} \,. \end{aligned} \tag{85}$$

Π^μ is the canonical momentum, P_α^μ are the momentum densities on Σ.

The equations of motion are found by variations $X \to X + \delta X$ of the worldsheet, where the initial and final configuration are kept fixed, $\delta X(\sigma^0 = \sigma^0_{(1)}, \sigma^0_{(2)}) = 0$. Since the ends of the string are not fixed for other values of σ^0, variation yields a boundary term, which we need to require to vanish:

$$\int d\sigma^0 \left[\frac{\partial\mathcal{L}}{\partial X'_\mu}\delta X_\mu\right]_{\sigma^1=0}^{\sigma^1=\pi} \stackrel{!}{=} 0 \,. \tag{86}$$

This boundary term must vanish $\Rightarrow$ admissible boundary conditions.

1. Periodic boundary conditions:

$$X(\sigma^1) = X(\sigma^1 + \pi) \tag{87}$$

   Closed strings, the world sheet does not have (time-like) boundaries.

2. Neumann boundary conditions:

$$\left.\frac{\partial\mathcal{L}}{\partial X'_\mu}\right|_{\sigma^1=0,\pi} = 0 \tag{88}$$

(also called: free boundary conditions, natural boundary conditions).
Open strings, ends can move freely. (The ends of an open string always move with the speed of line and thus their worldlines are light-like.)
The momentum at the end of the string is conserved. (P_1^μ is the momentum density along the space-like direction of Σ, i.e., along the string at a given 'time'.)

3. Dirichlet boundary conditions:

$$X^i(\sigma^1 = 0) = x_0 \ , \quad X^i(\sigma^1 = \pi) = x_1 \ . \tag{89}$$

Open strings with ends kept fixed along the i-direction (spacelike).
(Dirichlet boundary conditions in the time direction make only sense in imaginary time, in the Euclidean version of the theory, where they describe instantons).

Consider Neumann boundary conditions along time and p spacelike conditions and Dirichlet boundary conditions along $D - p$ directions. Then the ends of the string are fixed on p-dimensional spacelike surfaces, called Dirichlet p-branes. Momentum is not conserved at the ends of the string in the Dirichlet directions (obvious, since translation invariance is broken) $\Rightarrow$ p-branes are dynamical objects. Interpretation: strings in a solitonic background ($\neq$ vacuum).

Equations of motion (with either choice of boundary condition):

$$\partial_0 \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} + \partial_1 \frac{\partial \mathcal{L}}{\partial X'_\mu} = 0 \tag{90}$$

or

$$\partial_\alpha P^\alpha_\mu = 0 \tag{91}$$

Canonical momenta are not independent. Two constraints:

$$\begin{aligned} \Pi^\mu X'_\mu &= 0 \\ \Pi^2 + T^2 (X')^2 &= 0 \end{aligned} \tag{92}$$

Canonical Hamiltonian (density):

$$\mathcal{H}_{\text{can}} = \dot{X}\Pi - \mathcal{L} = 0 \ . \tag{93}$$

2.2 The Polyakov action

2.2.1 Action, symmetries, equations of motion

Intrinsic metric $h_{\alpha\beta}(\sigma)$ on the world-sheet Σ, with signature $(-)(+)$.
Polyakov action:

$$S_{\text{P}}[X,h] = -\frac{T}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \ , \tag{94}$$

where $h = -\det(h_{\alpha\beta}) = |\det(h_{\alpha\beta})|$.

Local symmetries with respect to Σ:

1. Reparametrizations $\sigma \to \tilde{\sigma}(\sigma)$, which act by

$$\begin{aligned} \tilde{X}^\mu(\tilde{\sigma}) &= X^\mu(\sigma) , \\ \tilde{h}_{\alpha\beta}(\tilde{\sigma}) &= \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta} h_{\gamma\delta}(\sigma) . \end{aligned} \quad (95)$$

2. Weyl transformations:

$$h_{\alpha\beta}(\sigma) \to e^{2\Lambda(\sigma)} h_{\alpha\beta}(\sigma) . \quad (96)$$

Remarks:

1. A Weyl transformation is not a diffeomorphism, but the multiplication of the metric by a positive function. Mathematicians usually call this a conformal transformation, because it changes the metric but preserves the conformal structure of $(\Sigma, h_{\alpha\beta})$.
2. The invariance under Weyl transformation is special for strings, it does not occure for particles, membranes and higher-dimensional p-branes.
3. Combining Weyl with reparametrization invariance, one has three local transformations which can be used to gauge-fix the metric $h_{\alpha\beta}$ completely. Thus $h_{\alpha\beta}$ does not introduce new local degrees of freedom: it is an auxiliary ('dummy') field.

Global symmetries with respect to $\mathbb{M}$: Poincaré transformations.

Equations of motion from variations $X \to X + \delta X$ and $h_{\alpha\beta} \to h_{\alpha\beta} + \delta h_{\alpha\beta}$.

$$\frac{1}{\sqrt{h}}\partial_\alpha \left(\sqrt{h} h^{\alpha\beta}\partial_\beta X^\mu\right) = 0 , \quad (97)$$

$$\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu = 0 . \quad (98)$$

Boundary conditions: as before.

The X-equation (97) is the covariant two-dimensional wave equation, alternatively written as

$$\Box X^\mu = 0 \Leftrightarrow \nabla_\alpha \nabla^\alpha X^\mu = 0 \Leftrightarrow \nabla_\alpha \partial^\alpha X^\mu = 0 \quad (99)$$

The h-equation (98) is algebraic and can be used to eliminate $h_{\alpha\beta}$ in terms of the induced metric $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$. It implies

$$\det(g_{\alpha\beta}) = \frac{1}{4}\det(h_{\alpha\beta})(h^{\gamma\delta} g_{\gamma\delta})^2 . \quad (100)$$

Substituting this into the Polyakov action one gets back the Nambu-Goto action. More generally one can show that $S_{\text{P}} \geq S_{\text{NG}}$, where equality holds if $g_{\alpha\beta}$ and $h_{\alpha\beta}$ are related by a Weyl transformation (i.e., they are conformally equivalent).

2.2.2 Relation to two-dimension field theory

Alternative point of view: Σ is a two-dimensional 'space-time', populated by D scalar fields $X = (X^\mu)$, which take values in $\mathbb{M}^D$. The Polyakov action has the form of a standard for a two-dimension scalar field theory. More precisely, it defines a two-dimensional sigma-model with 'target space' $\mathbb{M}$.

More generally: Mechanics of p-brane (p-dimensional extended object) = Field theory on a $p+1$ dimensional space-time (the worldvolume of the brane).

The energy-momentum tensor of the two-dimensional field theory is:

$$T_{\alpha\beta} := -\frac{1}{T}\frac{1}{\sqrt{h}}\frac{\delta S_P}{\delta h^{\alpha\beta}} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu \qquad (101)$$

(Defining $T_{\alpha\beta}$ by the Noether procedure gives a tensor which might differ from this by a total derivative.)

The h-equation of motion in terms of $T_{\alpha\beta}$:

$$T_{\alpha\beta} = 0\ . \qquad (102)$$

(Remember that the h-equation is a constraint.)

This equation can also be interpreted as the two-dimensional Einstein equation, since the variation of two-dimensional Einstein-Hilbert action vanishes identically. The Polyakov action describes a two-dimensional sigma model which is coupled to two-dimensional gravity.

$T_{\alpha\beta}$ is conserved:

$$\nabla^\alpha T_{\alpha\beta} = 0 \qquad (103)$$

To show this, one needs to use the equations of motion. The equation holds only 'on shell'.

$T_{\alpha\beta}$ is traceless:

$$h^{\alpha\beta}T_{\alpha\beta} = 0 \qquad (104)$$

This follows directly from the definition of $T_{\alpha\beta}$. It holds 'off shell', independently of whether the equations of motion are satisfied.

2.2.3 Polyakov action in conformal gauge

Three local symmetries, while the intrinsic metric $h_{\alpha\beta}$ has three independent components $\Rightarrow$ can bring $h_{\alpha\beta}$ to standard form (locally):

$$h_{\alpha\beta} \rightarrow \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\ . \qquad (105)$$

The condition $h_{\alpha\beta} \stackrel{!}{=} \eta_{\alpha\beta}$ is called the conformal gauge. Strictly speaking, the gauge condition should be imposed on the equations of motion, not on the action. For the case at hand the result is correct nevertheless. Therefore, let us follow the naive procedure and start from the Polyakov action in the conformal gauge.

The action:

$$S_P = -\frac{T}{2}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \; . \tag{106}$$

Equations of motion for X (variation of gauge-fixed action gives correct gauge fixed equations of motion):

$$\Box X^\mu = -(\partial_0^2 - \partial_1^2)X^\mu = 0 \tag{107}$$

This is the two-dimensional wave equation, which is known to have the general solution:

$$X^\mu(\sigma) = X_L^\mu(\sigma^0+\sigma^1) + X_R^\mu(\sigma^0-\sigma^1) \; . \tag{108}$$

Interpretation: decoupled left- and rightmoving waves.
Boundary conditions:

$$\begin{aligned} X^\mu(\sigma^1+\pi) &= X^\mu(\sigma) && \text{periodic} \\ X'_\mu\Big|_{\sigma^1=0,\pi} &= 0 && \text{Neumann} \\ \dot{X}_\mu\Big|_{\sigma^1=0,\pi} &= 0 && \text{Dirichlet} \end{aligned} \tag{109}$$

The equations coming from the h variation must now be added by hand:

$$T_{\alpha\beta} = 0 \tag{110}$$

Note that the trace of $T_{\alpha\beta}$ is

$$\text{Trace}(T) = T_\alpha^\alpha = \eta^{\alpha\beta}T_{\alpha\beta} = -T_{00} + T_{11} \; . \tag{111}$$

The trace of $T_{\alpha\beta}$ vanishes identically (off shell), and therefore $T_{\alpha\beta} = 0$ gives only rise to two independent non-trivial constraints:

$$\begin{aligned} T_{01} = T_{10} &= \frac{1}{2}\dot{X}X' = 0 \\ T_{00} = T_{11} &= \frac{1}{4}(\dot{X}^2 + X'^2) = 0 \end{aligned} \tag{112}$$

These equations are equivalent to the constraints derived from the Nambu-Goto action.

We have two-dimensional energy-momentum conservation (from the X equation):

$$\partial^\alpha T_{\alpha\beta} = 0 \tag{113}$$

and tracelessness (without equations of motion):

$$h^{\alpha\beta}T_{\alpha\beta} = 0 \; . \tag{114}$$

Note that energy-momentum conservation is non-trivial because $T_{\alpha\beta}$ only vanishes on-shell.

2.2.4 Lightcone coordinates

Equation (108) suggests to introduce lightcone coordinates (null coordinates):

$$\sigma^{\pm} := \sigma^0 \pm \sigma^1 \; . \tag{115}$$

We write σ^a, where $a = +, -$ for lightcone coordinates and σ^α, where $\alpha = 0, 1$ for standard coordinates.

The Jacobian of the coordinate transformation and its inverse:

$$\begin{aligned} (J_\alpha{}^a) &= \frac{D(\sigma^+, \sigma^-)}{D(\sigma^0, \sigma^1)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ (J_a{}^\alpha) &= \frac{D(\sigma^0, \sigma^1)}{D(\sigma^+, \sigma^-)} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned} \tag{116}$$

Converting lower indices:

$$v_a = J_a{}^\alpha v_\alpha \; , \quad v_\alpha = J_\alpha{}^a v_a \; . \tag{117}$$

Example: the lightcone derivatives

$$\partial_\pm = \frac{1}{2} \left(\partial_0 \pm \partial_1 \right) \tag{118}$$

Converting upper indices:

$$w^a = w^\alpha J_\alpha{}^a \; , \quad w^\alpha = w^a J_a{}^\alpha \; . \tag{119}$$

Example: the lightcone differentials

$$d\sigma^\pm = d\sigma^0 \pm d\sigma^2 \; . \tag{120}$$

When converting tensors, act on each index. Example: lightcone metric

$$h_{ab} = J_a{}^\alpha J_a{}^\beta h_{\alpha\beta} \; . \tag{121}$$

Explicitly: metric in standard coordinates

$$(h_{\alpha\beta}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = (h^{\alpha\beta}) \; . \tag{122}$$

Metric in lightcone coordinates

$$(h_{ab}) = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \quad (h^{ab}) = -2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; . \tag{123}$$

The lightcone coordinates of the (symmetric traceless) energy-momentum tensor are:

$$T_{++} = \frac{1}{2}(T_{00} + T_{01}) \; , \quad T_{--} = \frac{1}{2}(T_{00} - T_{01}) \; , \quad T_{+-} = 0 = T_{-+} \; . \tag{124}$$

Note that the trace, evaluated in lightcone coordinates, is:

$$\text{Trace}(T) = \eta^{ab} T_{ab} = 2\eta^{+-} T_{+-} = -4T_{+-} \ . \tag{125}$$

Thus 'traceless' $\Leftrightarrow T_{+-} = 0$.

Action in lightcone coordinates:

$$\begin{aligned} S_P &= -\frac{T}{2} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \\ &= \frac{T}{2} \int d^2\sigma = (\dot{X}^2 - X'^2) = 2T \int d^2\sigma \partial_+ X^\mu \partial_- X_\mu \ . \end{aligned} \tag{126}$$

Equations of motion in lightcone coordinates:

$$\begin{aligned} \Box X^\mu &= -(\partial_0^2 - \partial_1^2) X^\mu = -4\partial_+ \partial_- X^\mu = 0 \\ \Leftrightarrow \partial_+ \partial_- X^\mu &= 0 \ . \end{aligned} \tag{127}$$

Then, it is obvious that the general solution takes the form:

$$X^\mu(\sigma) = X_L^\mu(\sigma^0 + \sigma^1) + X_R^\mu(\sigma^0 - \sigma^1) \ . \tag{128}$$

The constraints in lightcone coordinates

$$\begin{aligned} T_{++} = 0 \quad &\Leftrightarrow \quad \partial_+ X^\mu \partial_+ X_\mu = 0 \Leftrightarrow \dot{X}_L^2 = 0 \ , \\ T_{--} = 0 \quad &\Leftrightarrow \quad \partial_- X^\mu \partial_- X_\mu = 0 \Leftrightarrow \dot{X}_R^2 = 0 \ . \end{aligned} \tag{129}$$

We did not list $T_{+-} = 0$ as a constraint, because it holds off shell. There are two non-trivial constraints, which are equivalent to the two constraints obtained for the Nambu-Goto action.

Energy-momentum conservation in lightcone coordintes:

$$\begin{aligned} \partial_- T_{++} = 0 \qquad & T_{++} = T_{++}(\sigma^+) \\ \partial_+ T_{--} = 0 \qquad & T_{--} = T_{--}(\sigma^-) \end{aligned} \tag{130}$$

These equations are non-trivial, because $T_{\pm\pm}$ only vanish on shell.

Using this, we see that every function $f(\sigma^+)$ defines a conserved chiral current

$$\partial_-(f(\sigma^+) T_{++}) = 0 \tag{131}$$

and thus a conserved charge

$$L_f = T \int_0^\pi d\sigma^1 f(\sigma^+) T_{++} \ . \tag{132}$$

To show directly that L_f is conserved, $\frac{d}{d\sigma^0} L_f = 0$, you need to use the periodicity properties of the fields in addition to current conservation. Similarly we obtain conserved quantities from T_{--} using a function $f(\sigma^-)$. For closed strings we get two different sets of conserved quantities. For open strings T_{++} and T_{--}

are not independent, because they are related through the boundary conditons. Therefore one only gets one set of conserved charges.

The canonical momenta:

$$\Pi^\mu = \frac{\partial \mathcal{L}_P}{\partial \dot{X}_\mu} = T \dot{X}^\mu \tag{133}$$

The canonical Hamiltonian

$$\begin{aligned} H_{\text{can}} &= \int_0^\pi d\sigma^1 \left(\dot{X}\Pi - \mathcal{L}_P \right) = \frac{T}{2} \int_0^\pi d\sigma^1 \left(\dot{X}^2 + X'^2 \right) \\ &= T \int_0^\pi d\sigma^1 \left((\partial_+ X)^2 + (\partial_- X)^2 \right) \end{aligned} \tag{134}$$

is the integrated version of the constraint $T_{00} = T_{11} = 0$.

NB: The conformal gauge does not provide a complete gauge fixing. The action is still invariant under residual gauge transformations. Namely, we can combine conformal reparametrisations with a compensating Weyl transformation (see below).

2.2.5 Momentum and angular momentum of the string

Polyakov action is invariant under global Poincaré transformations of M:

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu \tag{135}$$

Momentum is the quantity which is preserved when there is translation invariance (Noether theorem). Noether trick: promote the symmetry under consideration (here translations in $\mathbb{M}$) to a local transformation: $\delta X^\mu = \delta a^\mu(\sigma)$. The action is no longer invariant, but it becomes invariant when a^μ is constant. Therefore the variation of the action with respect to the local transformation must take the form

$$\delta S = \int d^2\sigma \partial_\alpha a^\mu P^\alpha_\mu \ . \tag{136}$$

Integration by parts gives

$$\delta S = - \int d^2\sigma a^\mu \partial_\alpha P^\alpha_\mu \ . \tag{137}$$

If we impose that the equations of motion are satisfied, this must vanish for any a^μ. Hence the current P^α_μ must be conserved on shell:

$$\partial_\alpha P^\alpha_\mu = 0 \ . \tag{138}$$

Interpretation P^α_μ (μ fixed) is the conserved current on Σ associated with translations in the μ-direction of $\mathbb{M}$, in other words, the momentum density along the μ-direction. (From the point of view of the two-dimensional field theory living on Σ, translations in $\mathbb{M}$ are internal symmetries.)

While above we assumed the conformal gauge, the method works without gauge fixing.
Explicitly:

$$P_\mu^\alpha = -T\sqrt{h}h^{\alpha\beta}\partial_\beta X_\mu \overset{c.g.}{=} -T\partial^\alpha X_\mu \ , \tag{139}$$

where we only went to the conformal gauge in the last step.

To find the angular momentum density, do the same for Lorentz transformations in $\mathbb{M}$. Result:

$$J_{\mu\nu}^\alpha = X_\mu P_\nu^\alpha - X_\nu P_\mu^\alpha \tag{140}$$

While current conservation holds by construction, it can easily be checked explicitly, using the equations of motion: $\nabla_\alpha P_\mu^\alpha = 0$, $\nabla_\alpha J_{\mu\nu}^\alpha = 0$.

Conserved charges are obtained by integration of the timelike component of the current along the spacelike direction of Σ:

$$\begin{aligned} P_\mu &= \int_0^\pi d\sigma^1 P_\mu^0 = T\int_0^\pi d\sigma^1 \dot{X}_\mu \\ J_{\mu\nu} &= \int_0^\pi d\sigma^1 J_{\mu\nu}^0 = T\int_0^\pi d\sigma^1 \left(X_\mu \dot{X}_\nu - X_\nu \dot{X}_\mu \right) \end{aligned} \tag{141}$$

Again, these charges are conserved by construction, but one can also check explicitly $\partial_0 P_\mu = 0$, $\partial_0 J_{\mu\nu} = 0$.

Summary: $P_\mu, J_{\mu\nu}$ are the total momentum and angular momentum of the string, $P_\mu^\alpha, J_{\mu\nu}^\alpha$ the corresponding conserved densities on the world sheet.

2.2.6 Fourier expansion

For periodic boundary condition we write the general solution as a Fourier series:

$$X^\mu(\sigma) = x^\mu + \frac{1}{\pi T}p^\mu\sigma^0 + \frac{i}{2}\sqrt{\frac{1}{\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-} + \frac{i}{2}\sqrt{\frac{1}{\pi T}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+} \tag{142}$$

where x^μ, p^μ are real and $\alpha_n^{\mu\star} = \alpha_{-n}^\mu$, $\tilde{\alpha}_n^{\mu\star} = \tilde{\alpha}_{-n}^\mu$.

NB: no linear term in σ^1, because of boundary condition. Splitting of zero mode between left and rightmoving part arbitrary.

p^μ is the total momentum:

$$P^\mu = T\int_0^\pi d\sigma^1 \dot{X}^\mu = p^\mu \ . \tag{143}$$

The center of mass of the string moves along a straight line, like a relativistic particle:

$$x_{\text{cms}}^\mu = \frac{1}{\pi}\int_0^\pi d\sigma^1 X^\mu(\sigma) = x^\mu + p^\mu\sigma^0 = x^\mu(0) + \frac{dx^\mu}{d\tau}(0)\tau \tag{144}$$

Thus: string = relativistic particle plus left- and rightmoving harmonic oscillations.

Formulae suggest to use 'string units':

$$T = \frac{1}{\pi} \ . \tag{145}$$

(On top of $c = \hbar = 1$).

Fourier components of

$$T_{\pm\pm} = \frac{1}{2}(\partial_{\pm} X)^2 \tag{146}$$

at $\sigma^0 = 0$:

$$\begin{aligned} L_m &:= T \int_0^{\pi} d\sigma^1 \ e^{-2im\sigma^1} T_{--} = \frac{\pi T}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \ , \\ \tilde{L}_m &:= T \int_0^{\pi} d\sigma^1 \ e^{2im\sigma^1} T_{++} = \frac{\pi T}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \ , \end{aligned} \tag{147}$$

where

$$\alpha_0 = \tilde{\alpha}_0 = \frac{1}{\sqrt{4\pi T}} p \overset{\pi T = 1}{=} \frac{1}{2} p \ . \tag{148}$$

The constraints $T_{\pm\pm} = 0$ imply

$$L_m = \tilde{L}_m = 0 \ . \tag{149}$$

The L_m are conserved, take $f \sim e^{2im\sigma^1}$ in (132). If the constraint hold at $\sigma^0 = 0$ they hold for all times σ^0. The constraints are conserved under time evolution.

Canonical Hamiltonian:

$$H = \int_0^{\pi} d\sigma^1 \ \left(\dot{X}\Pi - \mathcal{L} \right) = \frac{T}{2} \int_0^{\pi} (\dot{X}^2 + (X')^2)^2 = L_0 + \tilde{L}_0 \tag{150}$$

(The world sheet Hamiltonian $L_0 + \tilde{L}_0$ generates translations of σ^0, see below.)

The 'Hamiltonian constraint' $H = 0$ allows us to express the mass of a state in terms of its Fourier modes:

$$\begin{aligned} H &= L_0 + \tilde{L}_0 = \frac{\pi T}{2} \sum_{n=-\infty}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \\ &= \frac{p^2}{4} + \pi T (N + \tilde{N}) = 0 \end{aligned} \tag{151}$$

where we defined the total occupation numbers

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \ , \quad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \ .$$

This implies the 'mass shell condition':

$$M^2 = -p^2 = 4\pi T(N + \tilde{N}) \ .$$

Since $L_0 = \tilde{L}_0$ we have the additional constraint

$$N = \tilde{N} \ ,$$

which is called level matching, because it implies that left- and rightmoving modes contribute equally to the mass.

Open string with Neumann boundary conditions:

$$X^\mu(\sigma) = x^\mu + p^\mu\sigma^0 + \frac{i}{\sqrt{\pi T}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^0} \cos(n\sigma^1) \ . \tag{152}$$

This can still be decomposed into left and rightmvoing parts $X = X_L(\sigma^+) + X_R(\sigma^-)$:

$$X^\mu_{L/R}(\sigma^\pm) = \frac{1}{2}x^\mu + \frac{1}{\pi T} p^\mu_{L/R}\sigma^\pm + \frac{i}{2\sqrt{\pi T}} \sum_{n\neq 0} \frac{1}{n} \alpha^\mu_{n\ \mathrm{(L/R)}} e^{-in\sigma^\pm} \ , \tag{153}$$

but the boundary conditions $X'^\mu(\sigma^1 = 0, \pi) = 0$ imply

$$p_L = p_R = \frac{1}{2} p \ , \quad \alpha_{n\mathrm{(L)}} = \alpha_{n\mathrm{(R)}} \ . \tag{154}$$

We see that due to the boundary conditions left and rightmoving waves are reflected at the boundaries and combine into standing waves. There are only half as many independent oscillations as for closed strings.

Combining the boundary conditions with the constraints we see that $(\dot{X}^\mu)^2(\sigma^1 = 0, \pi) = 0 \Rightarrow$ The ends of the string move with the speed of light.

Since X_L and X_R are related by $\sigma^1 \to -\sigma^1$ (world sheet parity), we can combine them into one single periodic field with doubled period, $\sigma^1 \in [-\pi, \pi]$.

Fourier decomposition gives one set of modes,

$$\begin{aligned} L_m &= 2T \int_0^\pi d\sigma^1 \left(e^{im\sigma^1} T_{++} + e^{-im\sigma^1} T_{--} \right) = \frac{T}{4} \int_{-\pi}^\pi e^{im\sigma^1} \left(\dot{X} + X' \right)^2 \\ &= \frac{1}{2} \pi T \sum_n \alpha_{m-n} \alpha_n \end{aligned} \tag{155}$$

where we defined $\alpha_0 = p$. The canonical Hamiltonian is $H = L_0$.

The L_m define the constraints: $L_m = 0$ and these constraints are conserved in time. (Note that we only get 'half as many' constraints as for the bosonic string, due to the boundary conditions.) The Hamiltonian constraint $H = L_0 = 0$ is the mass shell condition:

$$M^2 = -p^2 = 2\pi T N \ .$$

Dirichlet boundary conditions: linear term in σ^1, but no linear time in σ^0. For the osciallators, cos is replaced by sin. Thus $\dot{X} = 0$ at the ends (ends don't move), but $X' \neq 0$ at the ends, corresponding to exchange of momentum with a D-brane.

3 The quantized relativistic string

3.1 Covariant quantisation and Fock space

We set $\pi T = 0$ in the following (on top of $\hbar = 1 = c$).

Remember quantization of non-relativistic particle:

$$[x^i, p^j] = i\delta^{ij} \ ,$$

where $i = 1, 2, 3$. Here $\pi^i = p^i$ is the canonical (=kinetic) momentum.

Relativistic particle:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \ .$$

The canonical (=kinetic) momentum is now constrained by the mass shell condition $p^2 + m^2 = 0$.

Free scalar field:

$$[\phi(x), \dot{\phi}(y)]_{x^0=y^0} = i\delta^{(3)}(\vec{x} - \vec{y})$$

Note that the commutator is evaluated at equal times $x^0 = y^0$. The canonical momentum of a free scalar field $\phi(x)$ is $\Pi(y) = \frac{\partial L}{\partial(\partial_0\phi(x))} = \partial_0\phi(x) = \dot{\phi}(x)$.

Free relativistic string:

$$[X^\mu(\sigma^0, \sigma^1), \Pi^\nu(\sigma'^0, \sigma'^1)]_{\sigma^0=\sigma'^0} = i\eta^{\mu\nu}\delta(\sigma^1 - \sigma'^1) \ .$$

Here $\delta(\sigma) = \delta(\sigma + \pi)$ is the periodic δ-function, which has the Fourier series

$$\delta(\sigma) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} e^{-2\pi ik\sigma/\pi} = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} e^{-2ik\sigma} \ .$$

We are working in the conformal gauge where $\Box X^\mu = 0$, resulting in the Fourier expansions (142), (152), depending on our choice of boundary conditions. In the conformal gauge, the canonical momentum is

$$\Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 X_\mu)} = T\dot{X}^\mu = \frac{1}{\pi}\dot{X}^\mu \ .$$

Using the Fourier expansion, we can derive the commutation relations for the modes

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \ , \quad [\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu} \ , \quad [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu} \ . \quad (156)$$

Hermiticity of X^μ implies:

$$(x^\mu)^+ = x^\mu \ , \quad (p^\mu)^+ = p^\mu \ , \quad (\alpha^\mu_m)^+ = \alpha^\mu_{-m} \ .$$

We get the commutation relations of a relativistic particle plus those of an infinite number of harmonic oscillators, (with frequencies $\omega_m = m > 0$, see later) corresponding to the excitations of the string. To obtain standard creation and annihilation operators, set

$$
\begin{aligned}
a^\mu_m &= \frac{\alpha^\mu}{\sqrt{m}} \quad \text{for} \quad m > 0 \;, \\
(a^\mu_m)^+ &= \frac{\alpha^\mu}{\sqrt{-m}} \quad \text{for} \quad m < 0 \;,
\end{aligned} \tag{157}
$$

to obtain $[a^\mu_m, (a^\nu_n)^+] = \eta^{\mu\nu}\delta_{m,n}$, which generalises $[a, a^+] = 1$.

As a first step to constructing the Hilbert space of states we construct a representation space of the operators x, p, α, which we call the Fock space $\mathcal{F}$. This space is different from the physical Hilbert space $\mathcal{H}$, because we still have to implement the constraints $L_m \approx 0$ (and, for closed strings $\tilde{L}_m \approx 0$).

The ground state $|0\rangle$ of $\mathcal{F}$ is defined by two conditions: it is translation invariant (=does not carry momentum) and it is annihilated by all annihilatin operators α^μ_m, $m > 0$:

$$
p^\mu|0\rangle = 0 \;, \quad \alpha^\mu_m|0\rangle = 0 \quad \text{for} \quad m > 0 \;.
$$

Momentum eigenstates are defined by the property

$$
p^\mu|k\rangle = k^\mu|k\rangle \;,
$$

where p^μ is the momentum operator and $k = (k^\mu)$ its eigenvalue. Momentum eigenstates can be constructed out of the ground state using the operator x^μ:

$$
|k\rangle = e^{ikx}|0\rangle \;,
$$

where $kx = k_\mu x^\mu$. Exciteted string states are generated by applying creation operators. I.p.

$$
\alpha^\mu_{-m}|0\rangle
$$

is a state with excitation number $N^\mu_m = 1$ in the m-th mode along the μ-direction. A general excited state is a linear combination of states of the form

$$
\alpha^{\mu_1}_{-m_1}\alpha^{\mu_2}_{-m_2}\cdots|0\rangle \;,
$$

which have definit excitation numbers $\{N^\mu_m\}$. A basis for $\mathcal{F}$ is obtained by taking linear combinations of states of the form

$$
\alpha^{\mu_1}_{-m_1}\alpha^{\mu_2}_{-m_2}\cdots|k\rangle \;,
$$

which carry momentum $k = (k_\mu)$ and excitation numbers $\{N^\mu_m|\mu = 0, \ldots, D-1,\ m > 0\}$.

The Fock space $\mathcal{F}$ carries a natural scalar product. Let us first define it in the oscillator sector. Decompose the ground state into an oscillator groundstate $|0\rangle_{\text{osc}}$ and a momentum groundstate $|0\rangle_{\text{mom}}$:

$$
|0\rangle = |0\rangle_{\text{osc}} \otimes |0\rangle_{\text{mom}} \;.
$$

The scalar product in the oscillator sector is fixed by the properties of the creation and annihilation operators α^μ_m, once we have normalised the ground state:

$$ {}_{\rm osc}\langle 0|0\rangle_{\rm osc} = 1 \; . $$

For illustration, we compute:

$$ \left(\alpha^\mu_{-m}|0\rangle_{\rm osc}, \alpha^\nu_{-n}|0\rangle\right) = {}_{\rm osc}\langle 0|\alpha^\mu_m \alpha^\nu_{-n}|0\rangle_{\rm osc} = {}_{\rm osc}\langle 0|[\alpha^\mu_m, \alpha^\nu_{-n}]|0\rangle_{\rm osc} = m\eta^{\mu\nu}\delta_{m+n,0} \; . $$

This scalar product is manifestly Lorentz covariant, but indefinit, i.e. there are states of negative norm. This shows that $\mathcal{F}$ is not a Hilbert space. This is not a problem, as long as the physical space of states is positiv definit.

The scalar product between momentum eigenstates is

$$ \langle k|k'\rangle = \delta^D(k-k') \; . $$

This is not defined if $k = k'$. Momentum eigenstates $|k\rangle$, including the momentum ground state $0\rangle_{\rm mom}$, are not normalisable. Normalisable states are obtained by forming momentum space wave packets

$$ |\phi\rangle = \int d^D k \phi(k)|k\rangle $$

where $\phi(k)$ (the momentum space wave function) is square integrable:

$$ \langle\phi|\phi\rangle = \int d^D k \overline{\phi(k)}\phi(k) < \infty \; . $$

3.2 Implementation of constraints and normal ordering

Having constructed the Fock space $\mathcal{F}$, we still need to impose the constraints $L_m \approx 0 \approx \tilde{L}_m$. Naively we expect that the resulting space $\mathcal{F}_{\rm phys}$ of physical states is positiv definit. As we will see in due course, the situation is a bit more complicated. For now we need to decide how to implement the constraints. We decide to require that the matrix elements of the operators $L_m, \tilde{L}_m$ vanish between physical states vanish. Naively, this amounts to saying: a state $|\phi\rangle$ is a physical state, $|\phi\rangle \in \mathcal{F}_{\rm phys}$, if

$$ \langle\phi|L_m|\phi\rangle = 0 \; , $$

and, for closed strings, in addition

$$ \langle\phi|\tilde{L}_m|\phi\rangle = 0 \; , $$

for all m. But as it stands this is ambiguous, because the L_m are quadratic in the α's, which do not commute any more in the quantum theory. Thus we have an ordering ambiguity in the quantum theory, which we need to investigate.

One standard ordering prescription is normal ordering, which requires to put all annihilation operators to the right of the creation operators:

$$: \alpha^\mu_m \alpha^\nu_n := \begin{cases} \alpha^\mu_m \alpha^\nu_n & \text{if } m < 0, n > 0 \\ \alpha^\nu_n \alpha^\mu_m & \text{if } m > 0, n < 0 \\ \text{any ordering} & \text{if else.} \end{cases}$$

From the commutation relations (156) we see that normal ordering only has a non-trivial effect if $m = -n$ and $\mu = \nu$, because creation and annihilation operators corresponding to different excitations levels or different directions commmute.

Compute the normally ordered version of L_0:

$$\begin{aligned} L_0^{\rm NO} &= \frac{1}{2}\sum_{n=-\infty}^{\infty} : \alpha_{-n}\cdot\alpha_n : \\ &= \frac{1}{2}\sum_{n=-\infty}^{-1} : \alpha_{-n}\cdot\alpha_n : + \frac{1}{2}\alpha_0\cdot\alpha_0 + \frac{1}{2}\sum_{1}^{\infty} : \alpha_{-n}\cdot\alpha_n : \\ &= \frac{1}{8}p^2 + N \,, \end{aligned} \tag{158}$$

where we used that $\frac{1}{2}\alpha_0 \cdot \alpha_0 = \frac{1}{8}p^2$ for closed strings and defined the number operator

$$N = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n \,.$$

Now compare this to what we might call the classically ordered version of L_0:

$$\begin{aligned} L_0^{\rm CO} &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{-n}\cdot\alpha_n \\ &= \frac{1}{2}\sum_{n=-\infty}^{-1} \alpha_{-n}\cdot\alpha_n + \frac{1}{2}\alpha_0\cdot\alpha_0 + \frac{1}{2}\sum_{1}^{\infty} \alpha_{-n}\cdot\alpha_n \\ &= \frac{1}{8}p^2 + \frac{1}{2}\sum_{1}^{\infty}(\alpha_n\cdot\alpha_{-n} + \alpha_{-n}\cdot\alpha_n) \end{aligned} \tag{159}$$

Both version of L_0 differ by an infinite constant:

$$L_0^{\rm CO} - L_0^{\rm NO} = \frac{D}{2}\sum_{n=1}^{\infty} n \,.$$

Going from the classically ordered to the normally ordered L_0 amounts to subtracting an infinite constant. We will take the quantum version of L_0 to be the normally ordered one, $L_0 = L_0^{\rm NO}$. This is prescription usually adopted in quantum field theory. But is this the correct choice? The problem is not that the constant which we subtract is infinite (we have the freedom to define any undefined object as we like), but that there is a finite ambiguity in this procedure. In other words: why don't we take $L_0^{NO} + a$, where a is some finite constant? (Equivalently, why can't we adopt a slightly different ordering prescription for finitely many term, which shifts L_0 by a finite constant?)

There are two ways to procede from here. The first is to accept that the ordering of L_0 is ambiguous, and to formulate the constraint in the form

$$\langle\phi|L_0 - a|\phi\rangle = 0 \ , \quad \langle\phi|\tilde{L}_0 - \tilde{a}|\phi\rangle = 0 \ ,$$

where $L_0, \tilde{L}_0$ are normally ordered and $a, \tilde{a}$ are finite constants which parametrize the ambiguity. It then turns out that in order to have a postive definit space of states, one must choose $a = \tilde{a} = 1$, i.e., the ambiguity is completely fixed by physical requirements.

The other way, which gives the same result, is to find a physcial procedure for computing a. The basic insight is that the infinite constant which is subtracted corresponds to the ground state energies of infinitely many harmonic oscillators. To show this, remember that $L_0 = \frac{1}{8}p^2 + N$ (closed string). The commutation relation of N with a creation operator is

$$[N, \alpha^\mu_{-m}] = m\alpha^\mu_{-m} \ .$$

Thus

$$N\alpha^\mu_{-m}|0\rangle = m\alpha^\mu_{-m}|0\rangle$$

and

$$N\alpha^{\mu_1}_{-m_1}\alpha^{\mu_2}_{-m_2}\cdots|0\rangle = (m_1 + m_2 + \cdots)\alpha^{\mu_1}_{-m_1}\alpha^{\mu_2}_{-m_2}\cdots|0\rangle$$

where the eigenvalue $m_1 + m_2 + \cdots$ of N is the total excitation number. Thus N counts excitations (weighted with their level), hence the name. Now rewrite N in terms of canonically normalised creation and annihilation operators (157):

$$N = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n = \sum_{n=1}^{\infty} n a_n^+ \cdot a_n$$

Comparing this to the Hamilton operator of a harmonic oscillator of frequency $\omega_n = n > 0$ ($\hbar = 1$),

$$H_n = n(a_n^+ a_n + \frac{1}{2})$$

we see that N differens from $\sum_{n=1}^{\infty} H_n$ by the (divergent) sum over the ground state energies $\frac{n}{2}$. We will discuss later how this sum can be regularized and computed.[1]

[1] In QFT non-trivial effects of the ground state energy can usually be ignored, because one (i) does not couple to gravity and (ii) works in infinite volume. In finite volume, the vacuum energy depends explicitly on the volume. This is a measurable effect, the Casimir effect. Since strings have finite size, the Casimir effect cannot be ignored, and its computation leads to $a = 1$. See later.

For the time being we parametrise the normal ordering ambiguity by $a, \tilde{a}$. It is straightforward to show that the L_m, $m \neq 0$ do not suffer from a normal ordering ambiguity. The resulting definition of physical states is: $|\phi\rangle$ is a physical state, $|\phi\rangle \in \mathcal{F}_{\text{phys}}$, if

$$\begin{aligned} \langle\phi|L_m|\phi\rangle = 0 \;, \qquad & \langle\phi|\tilde{L}_m|\phi\rangle = 0 \quad \text{for} \quad m \neq 0 \;, \text{ and} \\ \langle\phi|L_0 - a|\phi\rangle = 0 \;, \qquad & \langle\phi|\tilde{L}_m - \tilde{a}|\phi\rangle = 0 \;. \end{aligned} \tag{160}$$

3.3 Analysis of the physical states

We will now start to analyse $\mathcal{F}_{\text{phys}}$. The hermiticity properties $L_m^+ = L_{-m}$ can be used to write the constraints in the form

$$\begin{aligned} L_m|\phi\rangle = 0 \;, \qquad & \tilde{L}_m|\phi\rangle = 0 \quad \text{for} \quad m > 0 \;, \\ (L_0 - a)|\phi\rangle = 0 \;, \qquad & (\tilde{L}_m - \tilde{a})|\phi\rangle = 0 \;. \end{aligned} \tag{161}$$

We start with the L_0-constraint.

$$\begin{aligned} & (L_0 - a)|\phi\rangle = 0 \\ \Rightarrow \qquad & (\frac{1}{8}p^2 + N)|\phi\rangle = a|\phi\rangle \\ \Rightarrow \qquad & \frac{1}{8}k^2 + N = a \;, \end{aligned} \tag{162}$$

where k is the momentum and N denotes, in the last line, the eigenvalue of the operator N. In the future, we will use N to denote both the operator and its eigenvalue. The meaning should be clear from the context.

Since $k^2 = -M^2$, where M is the mass, we see that this constraint expresses the mass in terms of the total leftmoving excitation number:

$$\frac{1}{8}M^2 = N - a \;.$$

In the rightmoving sector we find

$$\frac{1}{8}M^2 = \tilde{N} - \tilde{a} \;.$$

Both conditions can be recombined into the mass shell condition

$$\frac{1}{4}M^2 = N + \tilde{N} - a - \tilde{a}$$

which expresses the mass in terms of left- and rightmoving excitation numbers, and the level matching condition

$$N - a = \tilde{N} - \tilde{a} \;,$$

which states the left- and rightmoving degrees of freedom contribute symmetrically to the mass.

This formula is given in string units, $\pi T = 1$. The string tension can easily be re-installed by dimensional analysis. Traditionally, the mass formula is then expressed in terms of the Regge slope

$$\alpha' = \frac{1}{2\pi T}$$

rather than in terms of the string tension T:

$$\alpha' M^2 = 2(N + \tilde{N} - a - \tilde{a}) \; .$$

The Regge slope has dimension Length^2. $\sqrt{\alpha'}$ can be regarded as the fundamental string length. String units correspond to setting $\alpha' = \frac{1}{2}$.

For open strings we have $L_0 = \frac{1}{2}p^2 + N$, and the resulting mass shell condition is

$$\alpha' M^2 = N - a \; .$$

Let us now have a look at the lowest mass states. When listing these we anticipate that $a = \tilde{a} = 1$. Start with the open string:

N	$\alpha' M^2$	State	Spin
0	-1	$\lvert k\rangle$	Scalar
1	0	$\alpha^\mu_{-1}\lvert k\rangle$	Vector
2	1	$\alpha^\mu_{-2}\lvert k\rangle$	Vector
		$\alpha^\mu_{-1}\alpha^\nu_{-1}\lvert k\rangle$	Symmetric Tensor

We still have to impose the further constraints $L_m|\phi\rangle = 0$ for $m > 0$. For $|\phi\rangle = |k\rangle$ it is easy to see that the other conditions are satisfied automatically. This state is physical, and having minimal mass-squared it is the physical ground state of the relativistic string.

3.3.1 The scalar ground state (tachyon) of the open string

Thus the physical ground state of string is a scalar with negative mass-squared, a tachyon. Note that this state is a tachyon for any positive a. Taken as classical particles, tachyons propagate faster than the speed of light and are believed to be unphysical. The interpretation quantum field theory is more subtle. The mass-squared of a particle is given by the second derivative of the scalar potential at its minimum. If the second derivative of the potential is negative rather than positive, this means that one is looking at a local maximum and not at a minimum of the potential. This means that the true vacuum is elsewhere, at a finite value of the expectation value of the 'tachyon' (tachyon condensation). In the standard model, the Higgs field has negative mass squared when expanding the around the local maximum of the Higgs potential. However, the physical Higgs particle is found by expanding the potential around the minimum, and it has a positive mass-squared, given by the curvature of the potential at its minimum. Something similar happens for the open string. The true vacuum is believed to be the ground state of the closed string, i.e., the true vacuum does not have open string excitations.

While this is interesting, for us the relativistic string, open or closed, is only a toy model anyhow, because, as we will see, it does not contain fermions in its spectrum. The physically interesting string theories are supersymmetric string theories, where the ground state is massless rather than tachyonic. Therefore we will regard the tachyon ground state as an unphysical feature of a toy model.

3.3.2 The massless vector state (photon) of the open string

The first excited state is a massless vector, hence potentially a photon. Note that the first excited state is massless if and only if $a = 1$. For generic values of a there would be no massless states at all. This indicates that the case $a = 1$ is special (though it does not prove that this is the correct value).

In this case the L_1-constraint is non-trivial. Let us evaluate the constraint for a general linear combination $|\phi\rangle = \zeta_\mu \alpha^\mu_{-1}|k\rangle$ of level-one states:[2]

$$\begin{aligned} & L_1|\phi\rangle = 0 \\ \Rightarrow \quad & L_1 \zeta_\mu \alpha^\mu_{-1}|k\rangle = 0 \\ \Rightarrow \quad & \alpha^\nu_0 \zeta_\nu [\alpha_{1,\nu}, \alpha^\mu_{-1}]|k\rangle = 0 \\ \Rightarrow \quad & k^\mu \zeta_\mu |0\rangle = 0 \; . \end{aligned} \tag{163}$$

Thus physical level-one states satisfy

$$k^2 = 0 \; , \quad k^\mu \zeta_\mu = 0 \; .$$

This tells us that the momentum is lightlike, corresponding to a massless particle, and orthogonal (in Minkowski metric) to the vector ζ, which therefore should be interpreted as the polarisation vector.

Since one can show that $L_m \zeta_\mu \alpha^\mu_{-1}|k\rangle = 0$ holds automatically for $m > 1$, all these states are physical. Can we interprete these states as the components of a photon? In D dimensions, a photon has $D - 2$ independent components due to gauge invariance. The condition $k \cdot \zeta = 0$ reduces the number of independent polarisations from D to $D - 1$. We have to show that there are actually only $D - 2$ independent components, and we would like to see the underlying gauge invariance.

To this end, consider states of the form

$$|\psi\rangle = \lambda k_\mu \alpha^\mu_{-1}|k\rangle$$

where λ is a (real) constant.

The scalar product of such a state with itself is zero:

$$\langle\psi|\psi\rangle = \lambda^2 k_\mu k_\nu \langle k|[\alpha^\mu_1, \alpha^\nu_{-1}]|k\rangle = \lambda^2 k^2 \langle k|k\rangle = 0$$

because $k^2 = 0$.[3] The state $|\psi\rangle = \lambda k_\mu \alpha^\mu_{-1}|k\rangle$ is different from zero, $|\psi\rangle \neq 0$, yet orthogonal to itself $\langle\psi|\psi\rangle = 0$: it is a so-called null state. The existence of null

[2] Here and in the following we write $|k\rangle$ instead of $|0\rangle_{\rm osc} \otimes |k\rangle$ for simplicity.

[3] Since $\langle k|k\rangle$ is not well defined, we should actually consider wave packets. But this is only a technical complication, which does not change the result.

states shows that $\mathcal{F}_{\rm phys}$ cannot be postive definit and therefore that it is not the physical Hilbert state. (More in due course).

One can also show that $|\psi\rangle$ is orthogonal to any state with excitation level different from 1. Moreover, it is orthogonal to any physical state with excitation level 1:

$$\langle k'|\alpha_1^\mu \zeta_\mu k_\nu \alpha_{-1}^\nu |k\rangle = \zeta \cdot k \langle k'|k\rangle = 0 \; ,$$

because either $k \neq k'$ implying $\langle k|k'\rangle = 0$ or $k = k'$, implying $\zeta \cdot k = 0$.

This shows that $|\psi\rangle$ is a 'spurious state', because it drops out of any scalar product between physical states. In other words

$$(\zeta_\mu + \lambda k_\mu)\alpha_{-1}^\mu |k\rangle$$

represents the same physical state for any value of λ. The freedom of changing the value of λ is a gauge degree of freedom. The component of ζ parallel to the momentum does not contribute to any physical quantity, and thus the number of independent physical polarisations is reduced to $D-2$, as expected for a D-dimensional photon.

3.3.3 Review of the Maxwell Lagrangian

For illustration, we review how the conditions $k^2 = 0$ and $k \cdot \zeta = 0$ can be derived from electrodynamics. We start from the D-dimensional Maxwell action

$$S[A] = \int d^D x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \; ,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; .$$

By variation of A_μ we obtain one half of the Maxwell equations

$$\partial^\mu F_{\mu\nu} = 0 \; .$$

In the presence of charge matter, the action becomes

$$S[A, j] = \int d^D x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right) \; ,$$

where j^μ is the electromagnetic current. The resulting Maxwell equations are

$$\partial^\mu F_{\mu\nu} = j_\nu \; .$$

The second half of the Maxwell equations do not follow from the variation of the action. They are identities (called Bianchi identities), which are the integrability conditions guaranteeing that the field strength has a vector potential:

$$\partial_{[\nu} F_{\rho\sigma]} = 0 \Leftrightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{(locally)} \; .$$

The field strength and the action are invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \chi \Rightarrow F_{\mu\nu} \to F_{\mu\nu} \ ,$$

where χ is an arbitrary function. Note that the coupling to a current j^μ is only consistent if the current is conserved, $\partial_\mu j^\mu = 0$.

Let us restrict to free Maxwell theory, $j^\mu = 0$, and compare the properties of the photon field A_μ to those of the massless vector state of the open string. Writing out the equation of motion:

$$\partial^\mu F_{\mu\nu} = \Box A_\mu - \partial_\mu \partial^\nu A_\nu = 0 \ .$$

We can partially fix the gauge symmetry by imposing the Lorenz gauge and obtain

$$\Box A_\mu = 0 \ , \quad \text{if} \quad \partial^\nu A_\nu = 0 \ .$$

There is still a residual gauge symmetry because we can make gauge transformations where the function χ satisfies $\Box \chi = 0$, because

$$\partial^\nu A_\nu \to \partial^\nu (A_\nu + \partial_\nu \chi) = \partial^\nu A_\nu + \Box \chi \ .$$

This implies that, in the Lorenz gauge, A_μ and $A_\mu + \partial_\mu \chi$ with $\Box \chi = 0$ represent the same physical state. The number of independent physical components of A_μ is $D-2$.

Consider now a plane wave with momentum k and polarisation vector ζ:

$$A_\mu = \zeta_\mu e^{ikx} \ .$$

The general solution can be written a superposition (Fourier integral) of such plane waves. This allows us to translate the equations of motion, the Lorenz gauge, and the residual gauge symmetry into momentum space:

$$\begin{aligned} \Box A_\mu = 0 \quad &\Leftrightarrow \quad k^\mu k_\mu = 0 \\ \partial^\mu A_\mu = 0 \quad &\Leftrightarrow \quad \zeta^\mu k_\mu = 0 \\ A_\mu \simeq A_\mu + \partial_\mu \chi \quad &\Leftrightarrow \quad \zeta_\mu \simeq \zeta_\mu + \lambda k_\mu \end{aligned} \tag{164}$$

We can be more explicit about the unphysical, physical and spurious parts of A_μ, by making an explicit choice for the momentum vector:

$$k = (k^0, 0, \ldots, 0, k^0)$$

This corresponds to a massless particle (equivalently, a plane wave) propagation parallel to the $(D-1)$-axis (for $D = 4$, the 3-axis or z-axis). The polarisation vector must satisfy:

$$k \cdot \zeta = 0 \Rightarrow -k^0 \zeta^0 + k^0 \zeta D - 1 = 0 \to \zeta^0 = \zeta^{D-1} \ . \tag{165}$$

Thus physical polarisation vectors have the form

$$\zeta = (\zeta^0, \zeta^1, \ldots, \zeta^{D-2}, \zeta^{D-1}) \ .$$

Introduce the following basis for polarisation vectors:

$$\begin{aligned} k &= (k^0, 0, \ldots, 0, k^0) \ , \\ \overline{k} &= \frac{1}{2(k^0)^2}(-k^0, 0, \ldots, 0, k^0) \ , \\ e_i &= (0, \ldots, 1, \ldots, 0) \ , \quad i = 1, \ldots, D-2 \end{aligned} \tag{166}$$

This is an orthonormal basis which consists of two linearly independent lightlike vectors spanning the light cone and $D-2$ spacelike vectors which spann the directions transverse to the light cone:

$$\begin{aligned} &k \cdot k = 0 \ , \quad k \cdot \overline{k} = 1 \ , \quad \overline{k} \cdot \overline{k} = 0 \ , \\ &e_i \cdot e_j = \delta_{ij} \ , \quad k \cdot e_i = 0 \ , \quad \overline{k} \cdot e_i = 0 \ . \end{aligned} \tag{167}$$

In this basis physical polarisation vectors have the form:

$$\zeta = \zeta^i e_i + \alpha k \ ,$$

i.e. the unphysical direction in polarisation space is the one parallel to $\overline{k}$ (which is the only basis vector with a nonvanishing scalar product with k).

However, the part of ζ which is parallel to k is spurious: it has zero norm and is orthogonal to any physical polarisation vector. The $D-2$ physical degrees of freedom reside in the transverse part

$$\zeta_{\text{transv}} = \zeta^i e_i = (0, \zeta^1, \zeta^2, \ldots, \zeta^{D-2}, 0)$$

We call this the transverse part, because the spatial part

$$\vec{\zeta} = (\zeta^1, \zeta^2, \ldots, \zeta^{D-2}, 0)$$

is orthogonal to the spatial momentum

$$\vec{k} = (0, \ldots, 0, k^0) \ .$$

In $D = 4$, this is the well-known fact that a photon has two physical polarisations, which are transverse to the momentum.

Group theoretical interpretation. In (165) we have made an explicit choice for the momentum vector, which corresponds to the choice of a Lorentz frame. This choice is not unique, we can still make Lorentz transformations which are spatial rotations around the axis spanned by k. The corresponding subgroup

$$SO(D-2) \subset SO(1, D-1)$$

is known as the little group. A central result in the representation theory of the Poincaré group states that massless representations (those where $p_\mu p^\mu = -m^2 = 0$) are classified by representations of the little group. The physical components of $\zeta_\mu \alpha^\mu_{-1}|0\rangle$ and of $A_\mu = \zeta_\mu e^{ikx}$ transform in the vector representation $[D-2]$ of the little group $SO(D-2)$. This is what characterises them mathematically as a massless vector, or 'photon'.

The case $D = 4$ is special, because the group $SO(2)$ is abelian, which implies that its irreducible representations are one-dimensional. In $D = 4$ the transverse polarisation vector $\zeta_{\text{transv}} = (\zeta^1, \zeta^2)$ transforms as follows:

$$\begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} .$$

To see the decomposition into two one-dimensional representations we take complex linear combinations

$$\zeta^{\pm} = \zeta^1 \pm i\zeta^2 .$$

This corresponds to going from a basis of transverse polarisations to a basis of circular polarisations. Now

$$\begin{pmatrix} \zeta^+ \\ \zeta^- \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \begin{pmatrix} \zeta^+ \\ \zeta^- \end{pmatrix} ,$$

These states have helicity $h = \pm 1$. Mathematically, we have used the group isomorphism $SO(2) \simeq U(1)$.

In conclusion, the massless vector state of the open string has precisely the kinetic properties which characterise a photon (=a massless vector boson with a gauge invariance). Therefore the Maxwell action provides a space-time description of the massless vector state of the free string: it is the effective action of this state.

3.3.4 States of the closed string

Physical closed strings satisfy the mass shell and level matching conditions

$$\begin{aligned} \alpha' M^2 &= 2(N + \tilde{N} - a - \tilde{a}) \\ N &= \tilde{N} \end{aligned} \tag{168}$$

plus

$$L_m|\phi\rangle = 0 = \tilde{L}_m|\phi\rangle \quad \text{for} \quad m > 0 . \tag{169}$$

Evaluate the mass shell and level matching conditions, anticipating $a = \tilde{a} = 1$. The resulting states are

$N = \tilde{N}$	$\alpha' M^2$	State	Spin
0	-4	$\|k\rangle$	Scalar
1	0	$\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-n}\|k\rangle$	2nd rank tensor
2	4	$\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\tilde{\alpha}^{\rho}_{-1}\tilde{\alpha}^{\sigma}_{-1}\|k\rangle$	4th rank tensor
		$\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\tilde{\alpha}^{\rho}_{-2}\|k\rangle$	3rd rank tensor
		$\alpha^{\mu}_{-2}\tilde{\alpha}^{\rho}_{-1}\tilde{\alpha}^{\sigma}_{-1}\|k\rangle$	3rd rank tensor
		$\alpha^{\mu}_{-2}\tilde{\alpha}^{\rho}_{-2}\|k\rangle$	2nd rank tensor

The ground state is again a tachyonic scalar, which we disregard as the artifact of a toy model. The first excited state is massless (for $a = \tilde{a} = 1$) and a 2nd rank tensor. For this state the constraints (169) are non-trivial for $m = 1$ (but trivial for $m > 1$). For a general linear combination of level-two states,

$$\zeta_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|k\rangle$$

evaluation of the constraints gives

$$k^{\mu}\zeta_{\mu\nu} = 0 = \zeta_{\mu\nu}k^{\nu} \ ,$$

where $k^2 = 0$ because of the mass shell condition.

$\zeta_{\mu\nu}$ is a 2nd rank tensor under the Lorentz group $SO(1, D-1)$. This representation is reducible, it can be decomposed into its symmetric and antisymmetric part:

$$\begin{aligned} \zeta_{\mu\nu} &= s_{\mu\nu} + b_{\mu\nu} \quad \text{where} \\ s_{\mu\nu} = \zeta_{(\mu\nu)} &= \frac{1}{2}(\zeta_{\mu\nu} + \zeta_{\nu\mu}) \\ b_{\mu\nu} = \zeta_{[\mu\nu]} &= \frac{1}{2}(\zeta_{\mu\nu} - \zeta_{\nu\mu}) \end{aligned} \tag{170}$$

If we decompose the string state accordingly, the polarisation tensors must satisfy the following constraints:

$$k^{\mu}s_{\mu\nu} = 0 \ , \quad k^{\mu}b_{\mu\nu} = 0 \ .$$

Let us analyse the antisymmetric and symmetric part separately.

3.3.5 The closed string B-field and axion

The antisymmetric part

$$b_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|k\rangle \ , \quad b_{\mu\nu} = -b_{\nu\mu}$$

corresponds to an irreducible Lorentz representation. Momentum and polarisation satisfy:

$$k^2 = 0 \ , \quad k^{\mu}b_{\mu\nu} = 0 \ .$$

One can show that physical states with

$$b_{\mu\nu} = k_{\mu}a_{\nu} - k_{\nu}a_{\mu} \ , \quad \text{where} \quad k^{\mu}a_{\mu} = 0 \ ,$$

are spurious, i.e. they have zero norm and are orthogonal to all physical states. Thus we have identified a gauge symmetry

$$b_{\mu\nu} \to b_{\mu\nu} + k_{\mu}a_{\nu} - k_{\nu}a_{\mu} \ , \quad \text{where} \quad k^{\mu}a_{\mu} = 0 \ .$$

This state is a massless, 2nd rank antisymmetric tensor and has a gauge symmetry similar to the one of a photon. For a free field, the action is

$$S[B] = \int d^D x \left(-\frac{1}{2 \cdot 3!} H_{\mu\nu\rho}H^{\mu\nu\rho} \right) \ ,$$

where

$$
\begin{aligned}
H_{\mu\nu\rho} &= \frac{3}{3!}\left(\partial_\mu B_{\nu\rho} - \partial_\mu B_{\rho\nu} + \partial_\nu B_{\rho\mu} - \partial_\nu B_{\mu\rho} + \partial_\rho B_{\mu\nu} - \partial_\rho B_{\nu\mu}\right) \\
&= \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \qquad (171)
\end{aligned}
$$

is the field strength (completely antisymmetric 3rd rank tensor) and $B_{\mu\nu}$ (antisymmetric 2nd rank tensor) is the gauge field (gauge potential). The equation of motion is

$$\partial^\mu H_{\mu\nu\rho} = 0 \; ,$$

and the Bianchi identity corresponding to the existence of a gauge potential is

$$\partial_{[\mu} H_{\nu\rho\sigma]} = 0 \Leftrightarrow H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \cdots$$

The field strength $H_{\mu\nu\rho}$ and, hence, the action is invariant under gauge transformations

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \; ,$$

where the gauge parameter is now a vector field Λ_μ (and has its own gauge invariance $\Lambda_\mu \to \Lambda_\mu + \partial_\mu \chi$ in turn). Writing out the equation of motion gives:

$$\partial^\mu \left(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\right) = 0$$

This becomes the wave equation when we impose the analogon of the Lorenz gauge:

$$\Box B_{\mu\nu} = 0 \; , \quad \text{if } \partial^\mu B_{\mu\nu} = 0 \text{ and } \Box \Lambda_\mu = 0 \; , \;\; \partial^\mu \Lambda_\mu = 0 \; .$$

Solutions can be build up from plane waves. A plane wave with polarisation $b_{\mu\nu}$ and momenum k has the form

$$B_{\mu\nu}(x) = b_{\mu\nu} e^{ikx}$$

This allows us to transform the equations of motion, the Lorenz gauge and the residual gauge symmetry into momentum space:

$$
\begin{aligned}
\Box B_{\mu\nu} = 0 \quad &\Leftrightarrow \quad k^2 = 0 \\
\partial^\mu B_{\mu\nu} = 0 \quad &\Leftrightarrow \quad k^\mu b_{\mu\nu} = 0 \\
B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \; , \quad &\Leftrightarrow \quad b_{\mu\nu} \to b_{\mu\nu} + k_\mu \zeta_\nu - k_\nu \zeta_\mu \; , \\
\text{where } \partial^\mu \Lambda_\mu = 0 \quad & \quad\;\;\, \text{where } k^\mu \zeta_\mu = 0 \; . \qquad (172)
\end{aligned}
$$

This shows that the massless antisymmetric tensor state of the closed string has the properties of a massless rank 2 antisymmetric gauge field.

Tensor gauge fields and axions

In $D = 4$ a rank 2 gauge field $B_{\mu\nu}$ is equivalent to an axion, where by axion we mean a scalar which has a shift symmetry $a \to a + C$, where C is a constant. Therefore the action can only depend on a through its derivative $H_\mu := \partial_\mu a$. This might be viewed as a gauge theory with potential a and field strength H_μ.

Starting from a rank 3 field strength we can use the four-dimensional ϵ tensor to define

$$H_\mu = \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma} \ .$$

Then

$$\partial^\mu H_{\mu\nu\rho} = 0 \Leftrightarrow \partial_{[\mu}H_{\nu]} = 0 \ .$$

The equation of motion for $H_{\mu\nu\rho}$ becomes a Bianchi identity for the vector H_μ, implying that it can be obtained from a scalar field: $H_\mu = \partial_\mu a$. Moreover

$$\partial_{[\mu}H_{\nu\rho\sigma]} = 0 \Leftrightarrow \partial^\mu H_\mu = 0 \ .$$

The Bianchi identity for $H_{\mu\nu\rho}$ becomes the equation of motion $\partial^\mu\partial_\mu a$ for a massless scalar field a. This shows that in four dimensions a rank 2 gauge field can equivalently be described as an axion. The 'dual' action is:

$$S[a] = -\int d^4x - partial_\mu a \partial^\mu a \ .$$

3.3.6 The graviton and the dilaton

We now turn to the symmetric part

$$s_{\mu\nu}\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|k\rangle \ , \quad s_{\mu\nu} = s_{\nu\mu}$$

of the massless closed string state. The momentum and polarisation tensor satisfy

$$k^2 = 0 \ , \quad k^\mu s_{\mu\nu} = 0 \ .$$

One can show that states of the form

$$s_{\mu\nu} = k_\mu\zeta_\nu + k_\nu\zeta_\mu \quad \text{where} \ \ k^\mu\zeta_\mu = 0$$

are spurious. The corresponding gauge symmetry is

$$s_{\mu\nu} \to s_{\mu\nu} + k_\mu\zeta_\nu + k_\nu\zeta_\mu \quad \text{where} \ \ k^\mu\zeta_\mu = 0 \ .$$

In contrast to the antisymmetric part, a 2nd rank symmetric tensor is not irreducible under the Lorentz group. Its trace s^μ_μ is a scalar (because $\eta_{\mu\nu}$ is an invariant tensor) and therefore a symmetric tensor decomposes into two irreducible representations: the trace and the traceless part.

In order to perform the decomposition explicitly we choose a lightlike vector $\overline{k}$ which has unit scalar product with the momentum:

$$k^2 = 0 \ , \quad \overline{k}^2 = 0 \ , \quad k^\mu\overline{k}_\mu = 1 \ .$$

Then the traceless part of $s_{\mu\nu}$ is[4]

$$\psi_{\mu\nu} = s_{\mu\nu} - \frac{1}{D-2}s^\rho_\rho\left(\eta_{\mu\nu} - k_\mu\overline{k}_\nu - k_\nu\overline{k}_\mu\right)$$

[4]If you ask yourself: 'why not $\psi_{\mu\nu} = s_{\mu\nu} - \frac{1}{D}s^\rho_\rho\eta_{\mu\nu}$?' (good question!), the short answer is that this would be the 'wrong' trace. Among the components of $s_{\mu\nu}$ there are several which correspond to scalars. But only one of them is a physical degree of freedom, the dilaton, while the others are spourious (gauge) degrees of freedom.

and the remaining pure trace part of $s_{\mu\nu}$ is

$$\phi_{\mu\nu} = \frac{1}{D-2} s^\rho_\rho \left(\eta_{\mu\nu} - k_\mu \overline{k}_\nu - k_\nu \overline{k}_\mu\right)$$

Note that

$$s_{\mu\nu} = \psi_{\mu\nu} + \phi_{\mu\nu} \ , \quad \eta^{\mu\nu}\psi_{\mu\nu} = 0 \ , \quad \eta^{\mu\nu}\phi_{\mu\nu} = s^\rho_\rho \ .$$

The trace part is physical,[5]

$$k^\mu \phi_{\mu\nu} = 0$$

and not spurious (the gauge transformation does not act on the trace).

It describes a scalar, called the dilaton. The purely traceless part still contains spurious states. It is a D-dimensional graviton and its gauge symmetry

$$\phi_{\mu\nu} \to \phi_{\mu\nu} + k_\mu \zeta_\nu + k_\nu \zeta_\mu \ , \quad \text{where} \quad k^\mu \zeta_\mu = 0$$

is a linearized version of (space-time) diffeomorphism invariance.

As in the case of the photon we can be more explicit about the physical and spurious components of the graviton/dilaton field, by choosing the explicit momentum vector (165) and using the basis (167). The transversality $k^\mu \psi_{\mu\nu} = 0$ of the polarisation tensor implies:

$$k^0 \psi_{00} + k^0 \psi_{0,D-1} = 0 \Rightarrow \psi_{0,D-1} = -\psi_{00}$$

etc. Combining this with symmetry $\psi_{\mu\nu} = \psi_{\nu\mu}$ gives:

$$\begin{aligned}(\psi_{\mu\nu}) &= \begin{pmatrix} \psi_{00} & \psi_{01} & \psi_{02} & \cdots & \psi_{0,D-2} & \psi_{0,D-1} \\ \psi_{10} & \psi_{11} & \psi_{12} & \cdots & \psi_{1,D-2} & \psi_{1,D-1} \\ \vdots & & & & & \vdots \\ \psi_{D-2,0} & \psi_{D-2,1} & \psi_{D-2,2} & \cdots & \psi_{D-2,D-2} & \psi_{D-2,D-1} \\ \psi_{D-1,0} & \psi_{D-1,1} & \psi_{D-1,2} & \cdots & \psi_{D-1,D-2} & \psi_{D-1,D-1} \end{pmatrix} \\ &= \begin{pmatrix} \psi_{00} & \psi_{01} & \psi_{02} & \cdots & \psi_{0,D-2} & -\psi_{00} \\ \psi_{01} & \psi_{11} & \psi_{12} & \cdots & \psi_{1,D-2} & -\psi_{01} \\ \vdots & & & & & \vdots \\ \psi_{0,D-2} & \psi_{1,D-2} & \psi_{2,D-2} & \cdots & \psi_{D-2,D-2} & -\psi_{0,D-2} \\ -\psi_{00} & -\psi_{01} & -\psi_{02} & \cdots & -\psi_{0,D-2} & \psi_{00} \end{pmatrix} \end{aligned} \tag{173}$$

Next, we have to work out the spurious part $k_\mu \zeta_\nu + \zeta_\mu k_\nu$ in the basis. Since $k^\mu \zeta_\mu = 0$, we have

$$(k^\mu) = (k^0, 0, \ldots, 0, k^0) \tag{174}$$
$$(k_\mu) = (k_0, 0, \ldots, 0, -k_0) \tag{175}$$
$$(\zeta_\mu) = (\zeta_0, \zeta_1, \ldots, \zeta_{D-2}, -\zeta_0) \tag{176}$$

[5]In contrast the 'trace' $\frac{1}{D} s^\rho_\rho \eta_{\mu\nu}$ is also a scalar, but does not satisfy the physical state condition.

Note that $k^0 = -k_0$. Thus

$$(k_\mu \zeta_\nu + \zeta_\mu k_\nu) = \begin{pmatrix} 2k_0\zeta_0 & k_0\zeta_1 & k_0\zeta_2 & \cdots & k_0\zeta_{D-2} & -2k_0\zeta_0 \\ k_0\zeta_1 & 0 & 0 & \cdots & 0 & -k_0\zeta_1 \\ k_0\zeta_2 & 0 & 0 & \cdots & 0 & -k_0\zeta_2 \\ \vdots & & & & & \vdots \\ k_0\zeta_{D-2} & 0 & 0 & \cdots & 0 & -k_0\zeta_{D-2} \\ -2k_0\zeta_0 & -k_0\zeta_1 & -k_0\zeta_2 & \cdots & -k_0\zeta_{D-2} & 2k_0\zeta_0 \end{pmatrix}$$

We can add an arbitrary multiple of a spurious state to a physical state. This can be used to decompose a physical state in the following way into a transverse state ψ_{transv} and a spurious state ψ_{spur}:

$$\psi_{\mu\nu} = \psi_{\mu\nu}^{\text{transv.}} + \psi_{\mu\nu}^{\text{spur.}}$$

where

$$(\psi_{\mu\nu}^{\text{transv.}}) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \psi_{11} & \psi_{12} & \dots & \psi_{1,D-2}, 0 & \\ \vdots & & & & & \vdots \\ 0 & \psi_{1,D-2} & \psi_{2,D-2} & \dots & \psi_{D-2,D-2} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and

$$(\psi_{\mu\nu}^{\text{transv.}}) = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{D-2} & -\lambda_0 \\ \lambda_1 & 0 & 0 & \cdots & 0 & -\lambda_1 \\ \vdots & & & & & \vdots \\ \lambda_{D-2} & 0 & 0 & \cdots & 0 & -\lambda_{D-2} \\ -\lambda_0 & -\lambda_1 & -\lambda_2 & \cdots & -\lambda_{D-2} & \lambda_0 \end{pmatrix}$$

with arbitrary λ_μ (subject to $\lambda_{D-1} = -\lambda_0$).

The transversality conditions eliminate D of the $\frac{D(D+1)}{2}$ components of $\psi_{\mu\nu}$. Among the remaining $\frac{D(D+1)}{2} - D$ physical components $D-1$ are spurious. The number of independent physical degrees of freedoms is

$$\frac{D(D+1)}{2} - D - (D-1) = \frac{(D-2)(D-1)}{2}$$

In our parametrisation we see that they form a symmetric second rank tensor of the little group $SO(D-2)$. This representation is reducible: it decomposes into the trace, which is a scalar, and the traceless part. The traceless, symmetric, second rank tensor representation of the little group has dimension $\frac{(D-2)(D-1)}{2} - 1$ and is the representation of the D-dimensional graviton.

In $D = 4$ the little group $SO(2)$ is abelian and its irreducible representations are one-dimensional. A traceless, symmetric, second rank tensor transforms as follows:

$$\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{12} & -\phi_{11} \end{pmatrix} \rightarrow \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{12} & -\phi_{11} \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

To see the decomposition into two irreducible representations, introduce complex components (corresponding to a circular polarisation basis)

$$\phi_{\pm\pm} = \phi_{11} \pm i\phi_{12}$$

Then

$$\phi_{++} \to e^{2i\ \varphi}\phi_{++}\ , \quad \phi_{--} \to e^{-2i\varphi}\phi_{--}\ .$$

These sates carry helicity $h = \pm 2$.

3.3.7 Review of the Pauli-Fierz Langrangian and of linearized gravity