Liverpool Lectures on String Theory
Problem Set 3
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Problem 1 A solution for open strings. Or: why do the ends of an open string move with the speed of light?

The equations of motion for a relativistic string in the conformal gauge are

$$(\partial_0^2 - \partial_1^2) X^\mu = 0 \; .$$

Solutions must satisfy the constraints, which take the following form in the conformal gauge:

$$\partial_0 X^\mu \partial_1 X_\mu = 0 \; , \quad \partial_0 X^\mu \partial_0 X_\mu + \partial_1 X^\mu \partial_1 X_\mu = 0 \; .$$

We consider open strings with boundary conditions

$$\partial_1 X_\mu = 0 \text{ for } \sigma^1 = 0, 1 \; .$$

1. Show that the equations of motions, the constraints and the boundary conditions are satisfied by the following solution:
$$\begin{aligned} X^0 &= L\sigma^0 \; , \\ X^1 &= L \cos\sigma^1 \cos\sigma^0 \; , \\ X^2 &= L \cos\sigma^1 \sin\sigma^0 \; , \\ X^i &= 0 \text{ for } i > 2 \; . \end{aligned}$$

2. Explain in words how the open string moves in this solution.

3. Compute the mass, momentum and angular momentum of the string. (The relevant formulae are in the lecture notes.)

4. Compute the speed of the endpoints of the string. Explain why the result you find holds for *any* solution of the open string equations of motion.

Problem 2 A solution for closed strings. Or: why is T called the string tension?

The equations of motion for a relativistic string in the conformal gauge are

$$(\partial_0^2 - \partial_1^2) X^\mu = 0 \; .$$

Solutions must satisfy the constraints, which in the conformal gauge take the following form:

$$\partial_0 X^\mu \partial_1 X_\mu = 0 \; , \quad \partial_0 X^\mu \partial_0 X_\mu + \partial_1 X^\mu \partial_1 X_\mu = 0 \; .$$

We consider closed strings with boundary conditions

$$X^\mu(\sigma^0, \sigma^1) = X^\mu(\sigma^0, \sigma^1 + \pi) \; .$$

1. Show that equations of motion and constraints are solved by the following solution:

$$\begin{aligned} X^0 &= 2R\sigma^0 \ , \\ X^1 &= R\cos(2\sigma^1)\cos(2\sigma^0) \ , \\ X^2 &= R\sin(2\sigma^1)\cos(2\sigma^0) \ , \\ X^i &= 0 \text{ for } i > 2 \ . \end{aligned}$$

2. Describe in words how the closed string moves.

3. Compute the length, the mass and the momentum of the string. (The relevant formulae are given in the lecture notes.)

4. Show that the string is at rest at time $X^0 = 0$. Express the total energy in terms of its length at time $X^0 = 0$ and interpret the result.

Problem 3 Commutation relations.
Compute

$$[L_m, \alpha_n^\nu] \ .$$

The basic commutation relations and the definition of the L_m are given in the lecture notes.

Problem 4 The Witt algebra.

Show that the operators

$$l_m = \frac{i}{2} e^{2im\sigma^+} \partial_+$$

satisfy the Witt algebra

$$[l_m, l_n] = (m-n) l_{m+n} \ .$$

Problem 5 Show that the classically ordered Fourier modes of the world-sheet energy momentum tensor,

$$L_m^{\mathrm{CO}} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n$$

satisfy the Witt algebra.

Then, show that the normally ordered Fourier modes

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_{m-n} \cdot \alpha_n :$$

satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12} m(m+1)(m-1) \delta_{m+n,0} \ .$$

where $D =$ number of space-time dimensions.