Liverpool Lectures on String Theory
Problem Set 2
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Problem 1 The Nambu-Goto action.

The action for a relativistic string is given by

$$S_{\rm NG}[X] = \int d^2\sigma \mathcal{L} = -T \int_\Sigma d^2\sigma \sqrt{|\det(\partial_\alpha X^\mu \partial_\beta X_\mu)|} \; . \qquad (1)$$

1. Show that the action is invariant under (orientation preserving) reparametrisations of the world sheet Σ:

$$\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^0, \sigma^1) \; , \quad \text{where} \quad \det\left(\frac{\partial \tilde{\sigma}^\alpha}{\partial \sigma^\beta}\right) > 0 \; . \qquad (2)$$

2. Compute the momentum densities

$$P^0_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \; , \quad P^1_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu} \; . \qquad (3)$$

3. Show that the canonical momenta $\Pi^\mu = P^\mu_0$ are subject to the two constraints

$$\begin{aligned} \Pi^\mu X'_\mu &= 0 \\ \Pi^2 + T^2 (X')^2 &= 0 \end{aligned} \qquad (4)$$

and that the canonical Hamiltonian vanishes:

$$\mathcal{H}_{\rm can} = \dot{X}\Pi - \mathcal{L} = 0 \; . \qquad (5)$$

Problem 2 The Polyakov action.

The Polyakov action is given by:

$$S_{\rm P} = -\frac{T}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \; , \qquad (6)$$

where $h = -\det(h_{\alpha\beta}) = |\det(h_{\alpha\beta})|$.

1. Show that the Polyakov action is invariant under reparamerisations $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^0, \sigma^1)$. Use that reparametrisations act by

$$\tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma) \quad \text{and} \quad \tilde{h}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial \sigma^\gamma}{\partial \tilde{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \tilde{\sigma}^\beta} h_{\gamma\delta}(\sigma) \qquad (7)$$

on the fields.

2. Show that the Polyakov action is invariant under Weyl transformations

$$h_{\alpha\beta}(\sigma) \to e^{2\Lambda(\sigma)} h_{\alpha\beta}(\sigma) \ . \tag{8}$$

Why does this not work if you replace the string by a particle or membrane?

3. In the conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$, the energy-momentum tensor takes the form

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu \ . \tag{9}$$

Show that

$$\eta^{\alpha\beta} T_{\alpha\beta} = \ = \ 0 \quad \text{off shell} \tag{10}$$

$$\partial^\alpha T_{\alpha\beta} \ = \ 0 \quad \text{on shell, only} \ . \tag{11}$$

4. Redo the previous problem without imposing the conformal gauge, i.e., for a general world sheet metric $h_{\alpha\beta}$.

Problem 3 The Fourier modes of the energy-momentum tensor.

For a closed string, the Fourier expansion of the solution to the equations of motion is

$$X^\mu(\sigma) = x^\mu + \frac{1}{\pi T} p^\mu \sigma^0 + \frac{i}{2}\sqrt{\frac{1}{\pi T}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-} + \frac{i}{2}\sqrt{\frac{1}{\pi T}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+} \ . \tag{12}$$

The lightcone components of the energy-momentum tensor are $T_{\pm\pm} = \frac{1}{2}(\partial_\pm X^\mu)^2$. Use the Fourier expansion of X^μ to show that the Fourier modes of T_{--} at worldsheet time $\sigma^0 = 0$ are given by:

$$L_m = T\int_0^\pi d\sigma^1 \ e^{-2im\sigma^1} T_{--} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n \ . \tag{13}$$