
From Special Geometry to Black Hole Partition Functions

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Abstract: These notes are based on lectures given at the Erwin-Schrödinger Institut in Vienna in 2006/2007 and at the 2007 School on Attractor Mechanism in Frascati. Lecture I reviews special geometry from the superconformal point of view. Lecture II discusses the black hole attractor mechanism, the underlying variational principle and black hole partition function. Two versions of the OSV-conjecture are formulated, which are then tested in the following lectures. Lecture III discusses BPS black holes in $N = 4$ supergravity, both large and small, including the effects of higher curvature terms. Lecture IV is devoted to their microscopic description by $N = 4$ string compactifications, with emphasis on precision state counting. The lectures are accompanied by exercises, which encourage student readers to develop some of the key ideas for themselves. Solutions are available upon request. Appendix A reviews special geometry from the view point of differential geometry. Appendix B provides the necessary background in modular forms needed to understand S-duality and string state counting (and to solve the corresponding exercise).

1 Introduction

Recent years have witnessed a renewed interest in the detailed study of supersymmetric black holes in string theory. This has been triggered by the work of H. Ooguri, A. Strominger and C. Vafa [1], who introduced the so-called mixed partition function for supersymmetric black holes, and who formulated an intriguing conjecture about its relation to the partition function of the topological string. The ability to test these ideas in a highly non-trivial way relies on two previous developments, which have been unfolding over the last decade. The first is that string theory provides models of black holes at the fundamental or ‘microscopic’ level, where microstates can be identified and counted with high precision, at least for supersymmetric black holes [2, 3, 4]. The second development is that one can handle subleading contributions to the thermodynamical or ‘macroscopic’ black hole entropy. The macroscopic

description of black holes is provided by solutions to the equations of motion of effective, four-dimensional supergravity theories, which approximate the underlying string theory at length scales which are large compared to the string, Planck and compactification scale. In this framework subleading contributions manifest themselves as higher derivative terms in the effective action. For a particular class of higher derivative terms in $N = 2$ supergravity, which are usually referred to as ‘ R^2 -terms’, it is possible to construct exact near-horizon asymptotic solutions and to compute the black hole entropy to high precision [5, 6]. The subleading corrections to the macroscopic entropy agree with the subleading contributions to the microscopic entropy, provided that the area law for the entropy is replaced by Wald’s generalized formula, which applies to any diffeomorphism invariant Lagrangian [7].

The main tools which make it possible to handle the R^2 -terms are the superconformal calculus, which allows the off-shell construction of $N = 2$ supergravity coupled to vector multiplets, and the so-called special geometry, which highly constrains the vector multiplet couplings. The reason for this simplification is that scalars and gauge fields sit in the same supermultiplet, so that the electric-magnetic duality of the gauge fields imprints itself on the whole multiplet. As a result the complicated structure of the theory, including an infinite class of higher derivative terms, becomes manageable and transparent, once all quantities are organised such that they transform as functions or vectors under the symplectic transformations which implement electric-magnetic duality. This is particularly important if the $N = 2$ supergravity theory is the effective field theory of a string compactification, because string dualities form a subset of these symplectic transformations.

In these lectures we give a detailed account of the whole story, starting from the construction of $N = 2$ supergravity, proceeding to the definition of black hole partition functions, and ending with microscopic state counting. In more detail, the first lecture is devoted to special geometry, the superconformal calculus and the construction of $N = 2$ supergravity with vector multiplets, including the R^2 -terms. The essential concept of gauge equivalence is explained using non-supersymmetric toy examples. When reviewing the construction of $N = 2$ supergravity we focus on the emergence of special geometry and stress the central role of symplectic covariance. Appendix A, which gives an account of special geometry from the mathematical point of view, provides an additional perspective on the subject. Lecture II starts by reviewing the concept of BPS or supersymmetric states and solitons. Its main point is the black hole variational principle, which underlies the black hole attractor equations. Based on this, conjectures about the relation between the macroscopically defined black hole free energy and the microscopically defined black hole partition functions are formulated. We do not only discuss how R^2 -terms enter into this, but also give a detailed discussion of the crucial role played by the so-called non-holomorphic corrections, which are essential for making physical quantities, such as the black hole entropy, duality invariant.

The second half of the lectures is devoted to tests of the conjectures formulated in Lecture II. For concreteness and simplicity, I only discuss the simplest string compactification with $N = 4$ supersymmetry, namely the compactification of the heterotic string on T^6 . After explaining how the $N = 2$ formalism can be used to analyse $N = 4$ theories, we will see that $N = 4$ black holes are governed by a simplified, reduced variational principle for the dilaton. There are two different types of supersymmetric black holes in $N = 4$ compactifications, called ‘large’ and ‘small’ black holes, and we summarize the results on the entropy for both of them.

With Lecture IV we turn to the microscopic side of the story. While the counting of $\frac{1}{2}$ -BPS states, corresponding to small black holes, is explained in full detail, we also give an outline of how this generalises to $\frac{1}{4}$ -BPS states, corresponding to large black holes. With the state degeneracy at hand, the corresponding black hole partition functions can be computed and confronted with the predictions made on the basis of the macroscopically defined free energy. We give a critical discussion of the results and point out which open problems need to be addressed in the future. While Appendix A reviews Kähler and special Kähler geometry from the mathematical point of view, Appendix B collects some background material on modular forms.

The selection of the material and the presentation are based on two principles. The first is to give a pedagogical account, which should be accessible to students, postdocs, and researchers working in other fields. The second is to present this field from the perspective which I found useful in my own work. For this reason various topics which are relevant or related to the subject are not covered in detail, in particular the topological string, precision state counting for other $N = 4$ compactifications and for $N = 2$ compactifications, and the whole field of non-supersymmetric extremal black holes. But this should not be a problem, given that these topics are already covered by other excellent recent reviews and lectures notes. See in particular [9] for an extensive review of the entropy function formalism and non-supersymmetric black holes, and [10] for a review emphasizing the role of the topological string. The selection of references follows the same principles. I have not tried to give a complete account, but to select those references which I believe are most useful for the reader. The references are usually given in paragraphs entitled ‘Further reading and references’ at the end of sections or subsections.

At the ends of Lectures I and IV I formulate exercises which should be instructive for beginners. The solutions of these exercises are available upon request. In addition, some further exercises are suggested within the lectures.

2 Lecture I: Special Geometry

Our first topic is the so-called special geometry which governs the couplings of $N = 2$ supergravity with vector multiplets. We start with a review of the Stückelberg mechanism for gravity, explain how this can be generalized to the

gauge equivalence between gravity and a gauge theory of the conformal group, and then sketch how this can be used to construct $N = 2$ supergravity in the framework of the superconformal tensor calculus.

2.1 Gauge equivalence and the Stückelberg mechanism for gravity

The Einstein-Hilbert action

$$S[g] = -\frac{1}{2\kappa^2} \int d^n x \sqrt{-g} R \quad (1)$$

is not invariant under local dilatations

$$\delta g_{\mu\nu} = -2\Lambda(x)g_{\mu\nu} . \quad (2)$$

However, we can enforce local dilatation invariance at the expense of introducing a ‘compensator’. Let $\phi(x)$ be a scalar field, which transforms as

$$\delta\phi = \frac{1}{2}(n-2)\Lambda\phi . \quad (3)$$

Then the action

$$\tilde{S}[g, \phi] = - \int d^n x \sqrt{-g} \left(\phi^2 R - 4 \frac{n-1}{n-2} \partial_\mu \phi \partial^\mu \phi \right) \quad (4)$$

is invariant under local dilatations. If we impose the ‘dilatational gauge’

$$\phi(x) = a = \text{const.} , \quad (5)$$

we obtain the gauge fixed action

$$\tilde{S}_{\text{g.f.}} = -a^2 \int d^n x \sqrt{-g} R . \quad (6)$$

This is proportional to the Einstein-Hilbert action (1), and becomes equal to it if we choose the constant a to satisfy $a^2 = \frac{1}{2\kappa^2}$.

The actions $S[g]$ and $\tilde{S}[g, \phi]$ are said to be ‘gauge equivalent’. We can go from $S[g]$ to $\tilde{S}[g, \phi]$ by adding the compensator ϕ , while we get from $\tilde{S}[g, \phi]$ to $S[g]$ by gauge fixing the additional local scale symmetry. Both theories are equivalent, because the extra degree of freedom ϕ is balanced by the additional symmetry.

There is an alternative view of the relation between $S[g]$ and $\tilde{S}[g, \phi]$. If we perform the field redefinition

$$g_{\mu\nu} = \phi^{(n-2)/4} \tilde{g}_{\mu\nu} , \quad (7)$$

then

$$S[g] = \tilde{S}[\tilde{g}, \phi] . \quad (8)$$

Conversely, starting from $\tilde{S}[\tilde{g}, \phi]$, we can remove ϕ by a field-dependent gauge transformation with parameter $\exp(\Lambda) = \frac{b}{\phi}$, where $b = \text{const}$. The field redefinition (7) decomposes the metric into its trace (a scalar) and its traceless part (associated with the graviton). This is analogous to the Stückelberg mechanism for a massive vector field, which decomposes the vector field into a massless vector (the transverse part) and a scalar (the longitudinal part), and which makes the action invariant under $U(1)$ gauge transformations.

We conclude with some further remarks:

1. The same procedure can be applied in the presence of matter. The compensator field has to be added in such a way that it compensates for the transformation of matter fields under dilatations. Derivatives need to be covariantized with respect to dilatations (we will see how this works in section 2.2.)
2. It is possible to write down a dilatation invariant action for gravity, which only involves the metric and its derivatives, but this action is quadratic rather than linear in the curvature:¹

$$S[g] = \int d^4x \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right). \quad (9)$$

This action contains terms with up to four derivatives. These and other higher derivative terms typically occur when quantum or stringy corrections to the Einstein-Hilbert action are taken into account.

3. When looking at $\tilde{S}[g, \phi]$, one sees that the kinetic term for the scalar ϕ has the ‘wrong’ sign, meaning that the kinetic energy is not positive definite. This signals that ϕ is not a matter field, but a compensator.

2.2 Gravity as a constrained gauge theory of the conformal group

Let us recall some standard concepts of gauge theory. Given a reductive² Lie algebra with generators X_A and relations $[X_A, X_B] = f_{AB}^C X_C$, we define a Lie algebra valued gauge field (connection)

$$h_\mu = h_\mu^A X_A. \quad (10)$$

The corresponding covariant derivative (frequently also called the connection) is

$$D_\mu = \partial_\mu - i h_\mu, \quad (11)$$

where it is understood that h_μ operates on the representation of the field on which D_μ operates. The field strength (curvature) is

¹ In contrast to other formulae in this subsection, the following formula refers specifically to $n = 4$ dimensions.

² A direct sum of simple and abelian Lie algebras.

$$R_{\mu\nu}^A = 2\partial_{[\mu}h_{\nu]}^A + 2h_{[\mu}^B h_{\nu]}^C f_{BC}^A . \quad (12)$$

We now specialize to the conformal group, which is generated by translations P^a , Lorentz transformations M^{ab} , dilatations D and special conformal transformations K^a . Here $a, b = 0, 1, 2, 3$ are internal indices. We denote the corresponding gauge fields (with hindsight) by $e_\mu^a, \omega_\mu^{ab}, b_\mu, f_\mu^a$, where μ is a space-time index. The corresponding field strength are denoted $R(P)_{\mu\nu}^a, R(M)_{\mu\nu}^{ab}, R(D)_{\mu\nu}, R(K)_{\mu\nu}^a$.

So far the conformal transformations have been treated as internal symmetries, acting as gauge transformations at each point of space-time, but not acting on space-time. The set-up is precisely as in any standard gauge theory, except that our gauge group is not compact and wouldn't lead to a unitary Yang-Mills-type theory.

But now the so-called conventional constraints are imposed, which enforce that the local translations are identified with diffeomorphisms of space-time, while the local Lorentz transformations become Lorentz transformations of local frames.

1. The first constraint is

$$R(P)_{\mu\nu}^a = 0 . \quad (13)$$

It can be shown that this implies that local translations act as space-time diffeomorphisms, modulo gauge transformations. As a result, the M-connection ω_μ^{ab} becomes a dependent field, and can be expressed in terms of the P-connection e_μ^a and the D-connection b_μ :

$$\omega_\mu^{ab} = \omega(e)_\mu^{ab} - 2e_\mu^{[a} e^{b]\nu} b_\nu , \quad (14)$$

$$\omega(e)_{\mu b}{}^c = \frac{1}{2} e_\mu{}^a (-\Omega_{ab}{}^c + \Omega_b{}^c{}_a + \Omega^c{}_{ab}) \quad (15)$$

$$\Omega_{ab}^c = e_a^\mu e_b^\nu (\partial_\mu e_\nu^c - \partial_\nu e_\mu^c) , \quad (16)$$

$$\text{where } e_\mu^a e_a^\nu = \delta_\mu^\nu .$$

2. The second constraint imposes ‘Ricci-flatness’ on the M-curvature:

$$e_b^\nu R(M)_{\mu\nu}^{ab} = 0 . \quad (17)$$

This constraint allows to solve for the K-connection:

$$f_\mu^a = \frac{1}{2} e^{\nu a} \left(R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu} \right) , \quad (18)$$

where

$$R_{\mu\nu}^{ab} := R(\omega)_{\mu\nu}^{ab} := 2\partial_{[\mu}\omega_{\nu]}^{ab} - 2\omega_{[\mu}^{ac}\omega_{\nu]}^{db}\eta_{cd} \quad (19)$$

is the part of the M-curvature which does not involve the K-connection:

$$R(M)_{\mu\nu}^{ab} = R(\omega)_{\mu\nu}^{ab} - 4f_{[\mu}^{[a} e_{\nu]}^{b]} . \quad (20)$$

By inspection of (15) and (19) we can identify $\omega(e)_\mu^{ab}$ with the spin connections, $R_{\mu\nu}^{ab}$ with the space-time curvature, e_μ^a with the vielbein and $\Omega_{ab}{}^c$ with the anholonomy coefficients.³ While ω_μ^{ab} and f_μ^a are now dependent quantities, the D-connection b_μ is still an independent field. However, it can be shown that b_μ can be gauged away using K-transformations, and the vielbein e_μ^a remains as the only independent physical field. Thus we have matched the field content of gravity. To obtain the Einstein-Hilbert action, we start from the conformally invariant action for a scalar field ϕ :

$$S = - \int d^4x e \phi (D_c)^2 \phi, \quad (21)$$

where $(D_c)^2 = D_\mu D^\mu$ is the conformal D'Alambert operator. In the K-gauge $b_\mu = 0$ this becomes

$$S = \int d^4x e \left(\partial_\mu \phi \partial^\mu \phi - \frac{1}{6} R \phi^2 \right). \quad (22)$$

As in our discussion of the Stückelberg mechanism, we can now impose the D-gauge $\phi = \phi_0 = \text{const.}$ to obtain the Einstein-Hilbert action. Observe that the kinetic term for ϕ has again the ‘wrong’ sign, indicating that this field is a compensator. Note that the Einstein-Hilbert action is obtained from a conformal matter action, and not from a Yang-Mills-type action with Lagrangian $\sim (R(M)_{\mu\nu}^{ab})^2$. As we have seen already in the discussion of the Stückelberg mechanism, such actions are higher order in derivatives, and become interesting once we want to include higher order corrections to the Einstein-Hilbert action.

2.3 Rigid $N = 2$ vector multiplets

Before we can adapt the method of the previous section to the case of $N = 2$ supergravity, we need to review rigidly supersymmetric $N = 2$ vector multiplets. An $N = 2$ off-shell vector multiplet has the following components:

$$(X, \lambda_i, A_\mu | Y_{ij}) . \quad (23)$$

X is a complex scalar and λ_i is a doublet of Weyl spinors. The $N = 2$ supersymmetry algebra has the R-symmetry group $SU(2) \times U(1)$, and the index $i = 1, 2$ belongs to the fundamental representation of $SU(2)$. A_μ is a gauge field, and Y_{ij} is an $SU(2)$ -triplet ($Y_{ij} = Y_{ji}$) of scalars, which is subject to the reality constraint $\bar{Y}^{ij} = Y_{ij}$ ⁴. All together there are 8 bosonic and 8 fermionic degrees of freedom.

³ The anholonomy coefficients measure the deviation of a given frame (choice of basis of tangent space at each point) from a coordinate frame (choice of basis corresponding to the tangent vector fields of a coordinate system).

⁴ $SU(2)$ indices are raised and lowered with the invariant tensor $\varepsilon_{ij} = -\varepsilon_{ji}$.

If we build an action with abelian gauge symmetry, then the gauge field A_μ will only enter through its field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, which is part of a so-called restricted⁵ chiral $\mathcal{N} = 2$ multiplet

$$\underline{X} = (X, \lambda_i, F_{\mu\nu}^-, \dots | Y_{ij}, \dots) , \quad (24)$$

where the omitted fields are dependent. $F_{\mu\nu}^-$ is the anti-selfdual part of the field strength $F_{\mu\nu}$. The selfdual part $F_{\mu\nu}^+$ resides in the complex conjugate of the above multiplet, together with the complex conjugate scalar \overline{X} and fermions of the opposite chirality.

We take an arbitrary number $n + 1$ of such multiplets and label them by $I = 0, 1, \dots, n$. The general Lagrangian is given by a chiral integral over $N = 2$ superspace,

$$\mathcal{L}^{\text{rigid}} = \int d^4\theta F(\underline{X}^I) + \text{c.c.} , \quad (25)$$

where $F(\underline{X}^I)$ is a function which depends arbitrarily on the restricted chiral superfields \underline{X}^I but not on their complex conjugates. Restricting the superfield $F(\underline{X}^I)$ to its lowest component, we obtain a holomorphic function $F(X^I)$ of the scalar fields, called the prepotential. The bosonic part of the resulting component Lagrangian is given by the highest component of the same superfield and reads

$$\mathcal{L}^{\text{rigid}} = i(\partial_\mu F_I \partial^\mu \overline{X}^I - \partial_\mu \overline{F}_I \partial^\mu X^I) + \frac{i}{4} F_{IJ} F_{\mu\nu}^{-I} F^{-J|\mu\nu} - \frac{i}{4} \overline{F}_{IJ} F_{\mu\nu}^{+I} F^{+J|\mu\nu} . \quad (26)$$

Here \overline{X}^I is the complex conjugate of X^I , etc., and

$$F_I = \frac{\partial F}{\partial X^I} , \quad F_{IJ} = \frac{\partial^2 F}{\partial X^I \partial X^J} , \quad \text{etc.} \quad (27)$$

The equations of motion for the gauge fields are:⁶

$$\partial_\mu \left(G_I^{-|\mu\nu} - G_I^{+|\mu\nu} \right) = 0 , \quad (28)$$

$$\partial_\mu \left(F_I^{-|\mu\nu} - F_I^{+|\mu\nu} \right) = 0 . \quad (29)$$

Equations (28) are the Euler-Lagrange equations resulting from variations of the gauge fields A_μ^I . We formulated them using the dual gauge fields

⁵ While a general chiral $N = 2$ chiral multiplet has $16 + 16$ components, a restricted chiral multiplet is obtained by imposing additional conditions and has only $8 + 8$ (independent) components. Moreover, the anti-selfdual tensor field $F_{\mu\nu}^-$ of a restricted chiral multiplet is subject to a Bianchi identity, which allows to interpret it as a field strength.

⁶ As an additional exercise, convince yourself that you get the Maxwell equations if the gauge couplings are constant.

$$G_I^{\pm|\mu\nu} := 2i \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}^{I\pm}} . \quad (30)$$

Equations (29) are the corresponding Bianchi identities. The combined set of field equations is invariant under linear transformations of the $2n + 2$ field strength $(F^{I\pm}, G_I^{\pm})^T$. Since the dual field strength are dependent quantities, we would like to interpret the rotated set of field equations as the Euler-Lagrange equations and Bianchi identities of a ‘dual’ Lagrangian. Up to rescalings of the field strength, this restricts the linear transformations to the symplectic group $Sp(2n + 2, \mathbb{R})$. These symplectic rotations generalize the electric-magnetic duality transformations of Maxwell theory.⁷

Since $G_{\mu\nu}^{I-} \propto F_{IJ} F_{\mu\nu}^{J-}$, the gauge couplings F_{IJ} must transform fractionally linearly:

$$\mathbb{F} \rightarrow (W + V\mathbb{F})(U + Z\mathbb{F})^{-1} , \quad (31)$$

where $\mathbb{F} = (F_{IJ})$ and

$$\begin{pmatrix} U & Z \\ W & V \end{pmatrix} \in Sp(2n + 2, \mathbb{R}) . \quad (32)$$

This transformation must be induced by a symplectic rotation of the scalars. This is the case if $(X^I, F_I)^T$ transforms linearly, with the same matrix as the field strength.

Quantities which transform linearly, such as the field strength $(F_{\mu\nu}^{I\pm}, G_{I|\mu\nu}^{\pm})^T$ and the scalars $(X^I, F_I)^T$ are called symplectic vectors. A function $f(X)$ is called a symplectic function if

$$f(X) = \tilde{f}(\tilde{X}) . \quad (33)$$

Note that the prepotential $F(X)$ is *not* a symplectic function, but transforms in a rather complicated way. However, we can easily construct examples of symplectic functions, by contracting symplectic vectors. The following symplectic functions will occur in the following:

$$K = i \left(X^I \overline{F}_I - F_I \overline{X}^I \right) , \quad (34)$$

$$\mathcal{F}_{\mu\nu}^- = X^I G_{I|\mu\nu}^- - F_I F_{\mu\nu}^{I-} . \quad (35)$$

The scalar part of the action (26) can be rewritten as follows:

$$\mathcal{L}_{\text{scalar}}^{\text{rigid}} = -N_{IJ} \partial_\mu X^I \partial^\mu X^J , \quad (36)$$

where

⁷ To see this more clearly, take F_{IJ} to be constant and restrict yourself to one single gauge field. The resulting $Sp(2, \mathbb{R}) \simeq SL(2, \mathbb{R})$ mixes the field strength with its Hodge dual.

$$N_{IJ} = -i (F_{IJ} - \overline{F}_{IJ}) = \frac{\partial^2 K}{\partial X^I \partial \overline{X}^J} . \quad (37)$$

N_{IJ} can be interpreted as a Riemannian metric on the target manifold of the scalars X^I , which we denote M . In fact, N_{IJ} is a Kähler metric with Kähler potential (34). Thus the scalar manifold M is a Kähler manifold. Moreover, M is a non-generic Kähler manifold, because its Kähler potential can be expressed in terms of the holomorphic prepotential $F(X^I)$. Such manifolds are called ‘affine special Kähler manifolds.’

An intrinsic definition of affine special Kähler manifolds can be given in terms of the so-called special connection ∇ (which is different from the Levi-Civita connection of the metric N_{IJ}). This is explained in appendix A. Equivalently, an affine special Kähler manifold can be characterised (locally) by the existence of a so-called Kählerian Lagrangian immersion

$$\Phi : M \rightarrow T^*\mathbb{C}^{n+1} \simeq \mathbb{C}^{2n+2} . \quad (38)$$

In this construction the special Kähler metric of M is obtained by pulling back a flat Kähler metric from $T^*\mathbb{C}^{n+1}$. In other words, all specific properties of M are encoded in the immersion Φ . Since the immersion is Lagrangian, it has a generating function, which is nothing but the prepotential: $\Phi = dF$. The immersed manifold M is (generically) the graph of a map $X^I \rightarrow W_I = F_I(X)$, where (X^I, W_I) are symplectic coordinates on $T^*\mathbb{C}^{n+1}$. Along the immersed manifold, half of the coordinates of $T^*\mathbb{C}^{n+1}$ become functions of the other half: the X^I are coordinates on M while the W_I can be expressed in terms of the X^I using the prepotential as $W_I = \frac{\partial F}{\partial X^I}$. We refer the interested reader to appendix A for more details on the mathematical aspects of this construction.

2.4 Rigid superconformal vector multiplets

The superconformal calculus provides a systematic way to obtain the Lagrangian of $N = 2$ Poincaré supergravity by exploiting its gauge equivalence with $N = 2$ conformal supergravity. This proceeds in the following steps:

1. Construct the general Lagrangian for rigid superconformal vector multiplets.
2. Gauge the superconformal group to obtain conformal supergravity.
3. Gauge fix the additional transformations to obtain Poincaré supergravity.

One can use the gauge equivalence to study Poincaré supergravity in terms of conformal supergravity, which is useful because one can maintain manifest symplectic covariance. In practice one might gauge fix some transformations, while keeping others intact, or use gauge invariant quantities.

As a first step, we need to discuss the additional constraints resulting from rigid $N = 2$ superconformal invariance. Besides the conformal generators P^a , M^{ab} , D , K^a , the $N = 2$ superconformal algebra contains the generators A and V^A of the $U(1) \times SU(2)$ R-symmetry, the supersymmetry generators Q

and the special supersymmetry generators S . Note that the superconformal algebra has a second set of supersymmetry transformations which balances the additional bosonic symmetry transformations.

The dilatations and chiral $U(1)$ transformations naturally combine into complex scale transformations. The scalars have scaling weight $w = 1$ and $U(1)$ charge $c = -1$:

$$X^I \rightarrow \lambda X^I, \quad \lambda = |\lambda| e^{-i\phi} \in \mathbb{C}^*. \quad (39)$$

Scale invariance of the action requires that the prepotential is homogenous of degree 2:

$$F(\lambda X^I) = \lambda^2 F(X^I). \quad (40)$$

Geometrically, this implies that the scalar manifold M of rigid superconformal vector multiplets is a complex cone. Such manifolds are called ‘conical affine special Kähler manifolds’.

2.5 $N = 2$ conformal supergravity

The construction of $N = 2$ supergravity now proceeds along the lines of the $N = 0$ example given in section 2.2. Starting from (25), one needs to covariantize all derivatives with respect to superconformal transformations. The corresponding gauge fields are: e_μ^a (Translations), ω_μ^{ab} (Lorentz transformations), b_μ (Dilatations), f_μ^a (special conformal transformations), A_μ (chiral $U(1)$ transformations), $\mathcal{V}_{\mu i}^j$ ($SU(2)$ transformations), ψ_μ^i (supersymmetry transformations) and ϕ_μ^i (special supersymmetry transformations).

As in section 2.2 one needs to impose constraints, which then allow to solve for some of the gauge fields. The remaining, independent gauge fields belong to the Weyl multiplet,

$$\left(e_\mu^a, \psi_\mu^i, b_\mu, A_\mu, \mathcal{V}_{\mu i}^j | T_{ab}^-, \chi^i, D \right), \quad (41)$$

together with the auxiliary fields T_{ab}^- (anti-selfdual tensor), χ^i (spinor doublet) and D (scalar). The only physical degrees of freedom contributed to Poincaré supergravity from this multiplet are the graviton e_μ^a and the two gravitini ψ_μ^i . The other connections can be gauged away or become dependent fields upon gauge fixing.

While covariantization of (25) with respect to superconformal transformations leads to a conformal supergravity Lagrangian with up to two derivatives in each term, it is also possible to include a certain class of higher derivative terms. This elaborates on the previous observation that one can also construct a Yang-Mills like action quadratic in the field strength. The field strength associated with the Weyl multiplet form a reduced chiral tensor multiplet \underline{W}_{ab} , whose lowest component is the auxiliary tensor field T_{ab}^- . The highest component contains, among other terms, the Lorentz curvature,

which after superconformal gauge fixing becomes the anti-selfdual Weyl tensor ${}^{-}C_{\mu\nu\rho\sigma}^-$. By contraction of indices one can form the (unreduced) chiral multiplet $\underline{W}^2 = \underline{W}_{ab}\underline{W}^{ab}$, which is also referred to as ‘the’ Weyl multiplet. While its lowest component is $\hat{A} = (T_{ab}^-)^2$, the highest component contains, among other terms, the square of the anti-selfdual Weyl tensor. Higher curvature terms can now be incorporated by allowing the prepotential to depend *explicitly* on the Weyl multiplet: $F(X^I) \rightarrow F(X^I, \hat{A})$. Dilatation invariance requires that this (holomorphic) function must be (graded) homogenous of degree 2:

$$F(\lambda X^I, \lambda^2 \hat{A}) = \lambda^2 F(X^I, \hat{A}) . \quad (42)$$

We refrain from writing down the full bosonic Lagrangian. However it is instructive to note that the scalar part, which is the analogue of (21) reads

$$8\pi e^{-1} \mathcal{L}_{\text{scalar}} = i \left(\overline{F}_I D^a D_a X^I - F_I D^a D_a \overline{X}^I \right) . \quad (43)$$

Here D_a is the covariant derivative with respect to all superconformal transformations.

2.6 $N = 2$ Poincaré supergravity

Our goal is to construct the coupling of n vector multiplets to $N = 2$ Poincaré supergravity. The gauge equivalent superconformal theory involves the Weyl multiplet and $n + 1$ vector multiplets, one of which acts a compensator. Moreover, one needs to add a second compensating multiplet, which one can take to be a hypermultiplet. The second compensator does not contribute any physical degrees of freedom to the vector multiplet sector. This is different for the compensating vector multiplet. The physical fields in the $N = 2$ supergravity multiplet are the graviton e_μ^a , the gravitini ψ_μ^i and the graviphoton $\mathcal{F}_{\mu\nu}$. While the first two fields come from the Weyl multiplet, the graviphoton is a linear combination of the field strength of all the $n + 1$ superconformal vector multiplets:

$$\mathcal{F}_{\mu\nu}^- = X^I G_{I|\mu\nu}^- - F_I F_{\mu\nu}^{I-} . \quad (44)$$

At the two-derivative level, one obtains $T_{\mu\nu}^- = \mathcal{F}_{\mu\nu}^-$ when eliminating the auxiliary tensor by its equation of motion. Note, however, that once higher derivative terms have been added, this relation becomes more complicated, and can only be solved iteratively in derivatives.

While all $n + 1$ gauge fields of the superconformal theory correspond to physical fields of the Poincaré supergravity theory, one of the superconformal scalars acts as a compensator for the complex dilatations. Gauge fixing imposes one complex condition on $n + 1$ complex scalars, which leaves n physical complex scalars. Geometrically, the scalar manifold of the Poincaré supergravity theory arises by taking the quotient of the ‘superconformal’ scalar manifold by the action of the complex dilatations.

To see what happens with the scalars, we split the superconformal covariant derivative D_μ into the covariant derivative \mathcal{D}_μ , which contains the connections for $M, D, U(1), SU(2)$, and the remaining connections. Then the scalar term (43) becomes

$$\begin{aligned} 8\pi e^{-1} \mathcal{L}_{\text{scalar}} = & i \left(\overline{F}_I \mathcal{D}^a \mathcal{D}_a X^I - F_I \mathcal{D}^a \mathcal{D}_a \overline{X}^I \right) \\ & - i \left(F_I \overline{X}^I - \overline{F}_I X^I \right) \left(\frac{1}{6} R - D \right). \end{aligned} \quad (45)$$

In absence of higher derivative terms, the only other term containing the auxiliary field D is

$$8\pi e^{-1} \mathcal{L}_{\text{comp}} = \chi \left(\frac{1}{6} R + \frac{1}{2} D \right), \quad (46)$$

where χ depends on the compensating hypermultiplet. The equation of motion for D is solved by⁸

$$\frac{1}{2} \chi = i \left(F_I \overline{X}^I - \overline{F}_I X^I \right). \quad (47)$$

When substituting this back, D cancels out, and we obtain

$$\begin{aligned} 8\pi e^{-1} (\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{comp}}) = & i \left(\overline{F}_I \mathcal{D}^a \mathcal{D}_a X^I - F_I \mathcal{D}^a \mathcal{D}_a \overline{X}^I \right) \\ & + \left(i (F_I \overline{X}^I - \overline{F}_I X^I) \right) \left(-\frac{1}{2} R \right). \end{aligned} \quad (48)$$

The second line gives the standard Einstein-Hilbert term, in Planckian units $G_N = 1$,

$$8\pi e^{-1} \mathcal{L} = -\frac{1}{2} R + \dots, \quad (49)$$

once we impose the D-gauge

$$i \left(F_I \overline{X}^I - X^I \overline{F}_I \right) = 1. \quad (50)$$

Geometrically, imposing the D-gauge amounts to taking the quotient of the scalar manifold M with respect to the (real) dilatations $X^I \rightarrow |\lambda| X^I$. The chiral $U(1)$ transformations act isometrically on the quotient, and therefore we can take a further quotient by imposing a $U(1)$ gauge. The resulting manifold $\overline{M} = M/\mathbb{C}^*$ is the scalar manifold of the Poincaré supergravity theory. It is a Kähler manifold, whose Kähler potential can be expressed in terms of the prepotential $F(X^I)$. The target manifolds of vector multiplets of in $N = 2$ Poincaré supergravity are called ‘(projective) special Kähler manifolds.’

⁸ Thus, at the two-derivative level, D just acts as a Lagrange multiplier. This changes once higher-derivative terms are added, but we won’t discuss the implications here.

To see how the geometry of \overline{M} arises, consider the scalar sigma model given by the first line of (48)

$$8\pi e^{-1} \mathcal{L}_{\text{sigma}} = i(\mathcal{D}_\mu F_I \mathcal{D}^\mu \overline{X}^I - \mathcal{D}_\mu X^I \mathcal{D}^\mu \overline{F}_I) \quad (51)$$

$$= -N_{IJ} \mathcal{D}_\mu X^I \mathcal{D}^\mu \overline{X}^J, \quad (52)$$

where

$$N_{IJ} = 2\text{Im}F_{IJ} = -i(F_{IJ} - \overline{F}_{IJ}), \quad (53)$$

and

$$\mathcal{D}_\mu X^I = (\partial_\mu + iA_\mu)X^I, \quad \mathcal{D}_\mu \overline{X}^I = (\partial_\mu - iA_\mu)\overline{X}^I, \quad (54)$$

$$\mathcal{D}_\mu F_I = (\partial_\mu + iA_\mu)F_I, \quad \mathcal{D}_\mu \overline{F}_I = (\partial_\mu - iA_\mu)\overline{F}_I. \quad (55)$$

We imposed the K-gauge $b_\mu = 0$, so that only the $U(1)$ gauge field A_μ appears in the covariant derivative. This gauged non-linear sigma model is the only place where A_μ occurs in the Lagrangian. A_μ can be eliminated by solving its equation of motion

$$A_\mu = \frac{1}{2}(\overline{F}_I \overleftrightarrow{\partial}_\mu X^I - \overline{X}^I \overleftrightarrow{\partial}_\mu F_I). \quad (56)$$

Substituting this back, we obtain the non-linear sigma model

$$\begin{aligned} 8\pi e^{-1} \mathcal{L}_{\text{sigma}} &= -(N_{IJ} + e^{\mathcal{K}}(N\overline{X})_I(NX)_J) \partial_\mu X^I \partial^\mu \overline{X}^J \\ &=: -M_{IJ} \partial_\mu X^I \partial^\mu \overline{X}^J \end{aligned} \quad (57)$$

Here we suppress indices which are summed over:

$$(NX)_I := N_{IJ} X^J, \quad \text{etc.}$$

The scalar metric M_{IJ} has two null directions

$$X^I M_{IJ} = 0 = M_{IJ} \overline{X}^J. \quad (58)$$

This does not imply that the kinetic term for the physical scalars is degenerate, because M_{IJ} operates on the ‘conformal scalars’ X^I , which are subject to dilatations and $U(1)$ -transformations. We have already gauge-fixed the dilatations by imposing the D-gauge. We could similarly impose a gauge condition for the $U(1)$ transformations, but it is more convenient to introduce the gauge invariant scalars

$$Z^I = \frac{X^I}{X^0}. \quad (59)$$

One of these scalars is trivial, $Z^0 = 1$, while the others $z^i = Z^i$, $i = 1, \dots, n$ are the physical scalars of the Poincaré supergravity theory. Using the transversality relations (58) and the homogeneity of the prepotential, we can rewrite the Lagrangian in terms of the gauge-invariant scalars Z^I :

$$8\pi e^{-1} \mathcal{L}_{\text{sigma}} = -g_{IJ} \partial_\mu Z^I \partial^\mu \bar{Z}^J ,$$

where

$$g_{IJ} = -\frac{N_{IJ}}{(ZN\bar{Z})} + \frac{(N\bar{Z})_I (NZ)_J}{(ZN\bar{Z})^2} . \quad (60)$$

Note that we have used the homogeneity of the prepotential to rewrite it and its derivatives in terms of the Z^I :

$$F(X) = (X^0)^2 F(Z) , \quad F_I(X) = X^0 F_I(Z) , \quad F_{IJ}(X) = F_{IJ}(Z) , \quad \text{etc.}$$

One can show that g_{IJ} has the following properties:

1. g_{IJ} is degenerate along the complex direction Z^I , or, in other words, along the orbits of the \mathbb{C}^* -action. We will call this direction the vertical direction. As we will see below the vertical directions correspond to unphysical excitations.
2. g_{IJ} is non-degenerate along the horizontal directions, which form the orthogonal complement of the horizontal direction with respect to the non-degenerate metric N_{IJ} . As we will see below, this implies a non-degenerate kinetic term for the physical scalars.
3. g_{IJ} is positive definite along the horizontal directions if and only if N_{IJ} has signature $(2, 2n)$ or $(2n, 2)$. This corresponds to the case where N_{IJ} has opposite signature along the vertical and horizontal directions. We need to impose this to have standard kinetic terms for the physical scalars.
4. g_{IJ} can be obtained from a Kähler potential which in turn can be expressed by the prepotential of the underlying superconformal theory:

$$g_{IJ} = \frac{\partial^2 K}{\partial Z^I \partial \bar{Z}^J} , \quad K = -\log \left(i(F_I \bar{Z}^I - Z^I \bar{F}_I) \right) .$$

Here it is understood that we only set $Z^0 = 1$ at the end.

Since $Z^0 = 1$, and, hence, $\partial_\mu Z^0 = 0$, the Lagrangian only depends on in the physical scalars $z^i = Z^i$, $i = 1, \dots, n$. Following conventions in the literature, we distinguish holomorphic indices i and anti-holomorphic indices \bar{i} when using the physical scalars z^i , despite that we do not make such a distinction for X^I , Z^I , etc. Thus the complex conjugate of $z^i = Z^i$ is denoted $\bar{z}^{\bar{i}} = \bar{Z}^{\bar{i}}$.

To express the Lagrangian in terms of the physical scalars, we define

$$\mathcal{F}(z^1, \dots, z^n) := F(Z^0, Z^1, \dots, Z^n) .$$

The Lagrangian only depends on the horizontal part of g_{IJ} , which is denoted $g_{i\bar{j}}$, and which is given by

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{j}}} . \quad (61)$$

with Kähler potential

$$K = -\log \left(2i(\mathcal{F} - \overline{\mathcal{F}}) - i(z^i - \overline{z}^{\overline{i}})(\mathcal{F}_i + \overline{\mathcal{F}}_{\overline{i}}) \right), \quad (62)$$

where $\mathcal{F}_i = \frac{\partial \mathcal{F}}{\partial z^i}$. The Lagrangian takes the form

$$8\pi e^{-1} \mathcal{L}_{\text{sigma}} = -g_{i\overline{j}} \partial_\mu z^i \partial^\mu \overline{z}^{\overline{j}}.$$

Geometrically, we have performed a quotient of the rigid superconformal scalar manifold M by the \mathbb{C}^* -action and obtained the metric $g_{i\overline{j}}$ of the scalar manifold \overline{M} of the Poincaré supergravity theory in terms of special coordinates z^i . Metrics and manifolds obtained in this way are called ‘projective special Kähler metrics’ and ‘projective special Kähler manifolds,’ respectively. One can reformulate the theory in terms of general holomorphic coordinates, but we will not pursue this here. The special coordinates are physically distinguished, because they are the lowest components of Poincaré vector multiplets. They are also natural from the geometrical point of view, because they can be defined in terms of intrinsic properties of M , as explained in more details in appendix A.

Since the z^i are not part of a symplectic vector, the action of the symplectic transformations in the scalar sector is complicated. Therefore it is often more convenient to work on the rigid scalar manifold M using the ‘conformal scalars’ X^I and the symplectic vector $(X^I, F_I)^T$. As we have seen, the superconformal and the super Poincaré theory are gauge-equivalent, and we know how to go back and forth between the two. The advantage of the superconformal picture is that there is an equal number of gauge fields and scalars, which all sit in vector multiplets. Therefore symplectic transformations act in a simple way on the scalars.

Let us finally have a brief look at the higher derivative terms. We expand the function $F(X^I, \hat{A})$ in \hat{A} :

$$F(X^I, \hat{A}) = \sum_{g=0}^{\infty} F^{(g)}(X^I) \hat{A}^g. \quad (63)$$

While $F^{(0)}(X^I) = F(X^I)$ is the prepotential, the functions $F^{(g)}(X^I)$ with $g > 0$ are coupling functions multiplying various higher derivative terms. The most prominent class of such terms are

$$F^{(g)}(X^I) (-C_{\mu\nu\rho\sigma}^-)^2 (T_{\mu\nu}^-)^{2g-2} + \text{c.c.}, \quad (64)$$

where ${}^-C_{\mu\nu\rho\sigma}$ is the antiselfdual Weyl tensor and $T_{\mu\nu}^-$ is the antiselfdual auxiliary field in the Weyl multiplet. To lowest order in derivatives, this field equals the anti-selfdual graviphoton field strength $\mathcal{F}_{\mu\nu}^-$. Therefore such terms are related to effective couplings between two gravitons and $2g - 2$ graviphotons.

$N = 2$ supergravity coupled to vector multiplets (and hypermultiplets) arises by dimensional reduction of type-II string theory on Calabi Yau threefolds. Terms of the above form arise from loop diagrams where the external

states are two gravitons and $2g - 2$ graviphotons, while an infinite number of massive strings states runs in the loop. It turns out that in the corresponding string amplitudes only genus- g diagrams contribute, and that only BPS states make a net contribution. Moreover these amplitudes are ‘topological’: upon topological twisting of the world sheet theory the couplings $F^{(g)}(X^I)$ turn into the genus- g free energies (logarithms of the partition functions) of the topological type-II string. This means that the couplings $F^{(g)}(X^I)$ can be computed, at least in principle.

2.7 Further reading and references

Besides original papers, my main sources for this lecture are the 1984 Trieste lecture notes of de Wit [11], and an (unpublished) Utrecht PhD thesis [12]. Roughly the same material was covered in Chapter 3 of my review [13]. Readers who would like to study special geometry and $N = 2$ supergravity in the superconformal approach in detail should definitely look into the original papers, starting with [14, 15]. Electric-magnetic duality in the presence of R^2 -corrections was investigated in [16, 17], and is reviewed in [13]. Special geometry has been reformulated in terms of general (rather than special) holomorphic coordinates [18, 19, 20]. We will not discuss this approach in these lectures and refer the reader to [21] for a review of $N = 2$ supergravity within this framework. The intrinsic definition of special Kähler geometry in terms of the special connection ∇ was proposed in [22]. The equivalent characterisation by a Kählerian Lagrangian immersion into a complex symplectic vector space is described in [23]. Key references about the topological string and its role in computing couplings in the effective action are [24] and [25]. See also [10] for a review of the role of the topological string for black holes.

2.8 Problems

Problem 1 The Stückelberg mechanism for gravity.

Compute the variation of the Einstein-Hilbert action

$$S[g] = -\frac{1}{2\kappa^2} \int d^n x \sqrt{-g} R \quad (65)$$

and the variation of the action

$$\tilde{S}[g, \phi] = - \int d^n x \sqrt{-g} \left(\phi^2 R - 4 \frac{n-1}{n-2} \partial_\mu \phi \partial^\mu \phi \right) \quad (66)$$

under local dilatations

$$\delta g_{\mu\nu} = -2\Lambda(x)g_{\mu\nu}, \quad \delta\phi = \frac{1}{2}(n-2)\Lambda\phi. \quad (67)$$

You can use that

$$\begin{aligned}\delta\sqrt{-g} &= -n\Lambda\sqrt{-g}, \\ g^{\mu\nu}R_{\mu\nu} &= -2(n-1)\nabla^2\Lambda.\end{aligned}\tag{68}$$

You should find that (66) is invariant while (65) is not, as explained in Lecture I. Convince yourself that you can obtain (65) from (66) by gauge fixing.

If you are not familiar with the Stückelberg mechanism, use what you have learned to make the action of a free massive vector field invariant with respect to local $U(1)$ transformations.

Problem 2 Einstein-Hilbert action from conformal matter.

Show that the Einstein-Hilbert action (65) can be obtained from the conformally invariant matter action

$$S = - \int d^4x e \phi D_c^2 \phi, \tag{69}$$

where $D_c^2 = D_\mu D^\mu$ is the conformal D'Alambert operator, by gauge fixing the K- and D-transformations.

Instruction: the scalar field ϕ is neutral under K-transformations and transforms with weight $w = 1$ under D_μ . Its first and second conformally covariant derivatives are:

$$D_\mu \phi = \partial_\mu \phi - b_\mu \phi, \tag{70}$$

$$D_\mu D^a \phi = (\partial_\mu - 2b_\mu) D^a \phi - \omega_\mu^{ab} D_b \phi + f_\mu^a \phi. \tag{71}$$

The K-connection f_μ^a appears in the second line because the D-connection b_μ transforms non-trivially under K. Note that b_μ is the only field in the problem which transforms non-trivially under K, and that $D^2\phi$ is invariant under K. The K-transformations can be gauged fixed by setting $b_\mu = 0$. (In fact, it is clear that b_μ will cancel out of (69). Why?) Use this together with the result of Problem 1 to obtain the Einstein-Hilbert action (65) by gauge fixing (69).

3 Lecture II: Attractor Mechanism, Variational Principle, and Black Hole Partition Functions

We are now ready to look at BPS black holes in $N = 2$ supergravity with vector multiplets. First we review the concept of a BPS state.

3.1 BPS states

The N -extended four-dimensional supersymmetry algebra has the following form:

$$\begin{aligned}\{Q_\alpha^A, Q_{\dot{\beta}}^{+B}\} &= 2\sigma_{\alpha\dot{\beta}}^\mu \delta^{AB} P_\mu, \\ \{Q_\alpha^A, Q_\beta^B\} &= \epsilon_{\alpha\beta} Z^{AB}.\end{aligned}$$

$A, B, \dots = 1, \dots, N$ label the supercharges, which we have taken to be Weyl spinors. The generators $Z^{AB} = -Z^{BA}$ are central, i.e. they commute with all generators of the Poincaré Lie superalgebra. On irreducible representations they are complex multiples of the unit operators. One can then skew-diagonalise the antisymmetric constant matrix Z^{AB} , and the skew eigenvalues Z_1, Z_2, \dots are known as the central charges carried by the representation. The eigenvalue of the Casimir operator $P^\mu P_\mu$ is $-M^2$, where M is the mass. Using the algebra one can derive the BPS inequality

$$M^2 \geq |Z_1|^2 \geq |Z_2|^2 \geq \dots \geq 0,$$

where we have labeled the central charges according to the size of their absolute values. Thus the mass is bounded from below by the central charges. Whenever a bound on the mass is saturated, some of the supercharges operate trivially on the representation, and therefore the representation is smaller than a generic massive representation. Such multiplets are called shortened multiplets or BPS multiplets. The extreme case is reached when all bounds are saturated, $M = |Z_1| = |Z_2| = \dots$. In these representations half of the supercharges operate trivially, and the representation has as many states as a massless one. These multiplets are called short multiplets or $\frac{1}{2}$ -BPS multiplets.

Here are some examples of $N = 2$ multiplets.

1. $M > |Z|$: these are generic massive multiplets. One example is the ‘long’ vector multiplet, which has $8 + 8$ on-shell degrees of freedom.
2. $M = |Z|$: these are short or $\frac{1}{2}$ -BPS multiplet. Examples are hypermultiplets and ‘short’ vector multiplets, which both have $4 + 4$ on-shell degrees of freedom. The short vector multiplet is the ‘Higgsed’ version of the massless vector multiplet discussed earlier in these lectures.⁹ The long vector multiplet combines the degrees of freedom of a hypermultiplet and a short vector multiplet. This shows that one cannot expect that the number of BPS multiplets is conserved when deforming the theory (by moving through its moduli space of vacua), because BPS multiplets can combine into non-BPS multiplets. However the difference between the number of hypermultiplets and short vector multiplets is preserved under multiplet recombination and has the chance of being an ‘index’.

Let us give some examples of $N = 4$ multiplets.

1. $M > |Z_1| > |Z_2|$: these are generic massive multiplets. The number of states is 2^8 .¹⁰

⁹ This has $8 + 8$ off-shell degrees of freedom and $4 + 4$ on-shell degrees of freedom.

¹⁰ We are referring here to representations of the algebra generated by the supercharges. Irreducible representations of the full Poincaré Lie superalgebra are ob-

2. $M = |Z_1| > |Z_2|$: these are called intermediate or $\frac{1}{4}$ -BPS multiplets. One quarter of the supercharges operate trivially, and they have (a multiple of) 2^6 states.
3. $M = |Z_1| = |Z_2|$: these are short or $\frac{1}{2}$ -BPS multiplets, with (a multiple of) 2^4 states. One example are short $\bar{N} = 4$ vector multiplets which have $8 + 8$ states, as many as a massless $N = 4$ vector multiplet. Short or massless multiplets have the same field content as an large $N = 2$ vector multiplet, or, equivalently, as a short or massless $N = 2$ vector multiplet plus a hypermultiplet.

Finally, there can of course also be singlets under the supersymmetry algebra, states which are completely invariant. Such states are maximally supersymmetric and can therefore be interpreted as supersymmetric ground states.

Further reading and references

This section summarises basic facts about the representation theory of Poincaré Lie superalgebras, which can be found in textbooks on supersymmetry, i.p. in Chapter II of [26] and Chapter 8 of [27].

3.2 BPS solitons and BPS black holes

One class of BPS states are states in the Hilbert space which sit in BPS representations. They correspond to fundamental fields in the Lagrangian, which transform in BPS representations of the supersymmetry algebra. Another class of BPS states is provided by non-trivial static solutions of the field equations, which have finite mass and are non-singular. Such objects are called solitons and interpreted as extended particle-like collective excitations of the theory.

Because of the finite mass condition they have to approach Minkowski space at infinity¹¹ and can be classified according to their transformation under the asymptotic Poincaré Lie superalgebra generated by the Noether charges. If this representation is BPS, the soliton is called a BPS soliton. The corresponding field configuration admits Killing spinors, i.e. there are choices of the supersymmetry transformation parameters $\epsilon(x)$ such that the field configuration is invariant:

$$\delta_{\epsilon(x)}\Phi(x)|_{\Phi_0(x)} = 0 .$$

Here Φ is a collective notation for all fundamental fields, and Φ_0 is the invariant field configuration. The maximal number of linearly independent Killing spinors equals the number N of supercharges. Solutions with N Killing spinors

tained by replacing the lowest weight state by any irreducible representation of the little group. Their dimension is therefore a multiple of 2^8 .

¹¹ We only consider theories where Minkowski space is a supersymmetric ground state.

are completely invariant under supersymmetry and qualify as supersymmetric ground states.¹² Generic solitonic solutions of the field equations do not have Killing spinors and correspond to generic massive representations. Solitonic solutions with $\frac{N}{n}$ Killing spinors are invariant under $\frac{1}{n}$ of the asymptotic symmetry algebra and correspond to $\frac{1}{n}$ -BPS representations.¹³

The particular type of solitons we are interested in are black hole solutions of $N = 2$ supergravity. Black holes are asymptotically flat, have a finite mass, and are ‘regular’ in the sense that they do not have naked singularities. For static four-dimensional black holes in Einstein-Maxwell type theories with matter, the BPS bound coincides with the extremality bound. Therefore BPS black holes are extremal black holes, with vanishing Hawking temperature. Since this makes them stable against decay through Hawking radiation, the interpretation as a particle-like solitonic excitation appears to be reasonable.

We will restrict ourselves in the following to static, spherically symmetric $\frac{1}{2}$ -BPS solutions of $N = 2$ supergravity with n vector multiplets. Such solutions describe single black holes.¹⁴ As a first step, let us ignore higher derivative terms and work with a prepotential of the form $F(X)$.

In an asymptotically flat space-time, we can define electric and magnetic charges by integrating the flux of the gauge fields over an asymptotic two-sphere at infinity:

$$\begin{pmatrix} p^I \\ q_I \end{pmatrix} = \begin{pmatrix} \oint F_{\mu\nu}^I d^2 \Sigma^{\mu\nu} \\ \oint G_{I|\mu\nu} d^2 \Sigma^{\mu\nu} \end{pmatrix}. \quad (72)$$

By construction, the charges form a symplectic vector $(p^I, q_I)^T$. The central charge under the asymptotic Poincaré Lie superalgebra is given by the charge associated with the graviphoton:

$$Z = \oint \mathcal{F}_{\mu\nu}^- d^2 \Sigma^{\mu\nu} = \oint (F_{\mu\nu}^{I-} F_I - G_{I|\mu\nu}^- X^I) d^2 \Sigma^{\mu\nu} = p^I F_I(\infty) - q_I X^I(\infty). \quad (73)$$

This is manifestly invariant under symplectic transformations. By common abuse of terminology, the symplectic function

$$Z = p^I F_I - q_I X^I$$

is also called the central charge, despite that it is actually a function of the scalars which are in turn functions on space-time.

A static, spherically symmetric metric can be brought to the following form:¹⁵

¹² Minkowski space is a trivial example. Here all Killing spinors are constant (in linear coordinates).

¹³ More precisely, the collective modes generated by the broken supersymmetries fall into such representations.

¹⁴ There are also static multi-black hole solutions, which we will not discuss here.

¹⁵ The solution can be constructed without fixing the coordinate system, but we present it in this way for pedagogical reasons.

$$ds^2 = -e^{2g(r)} dt^2 + e^{2f(r)} (dr^2 + r^2 d\Omega^2), \quad (74)$$

with two arbitrary functions $f(r)$, $g(r)$ of the radial variable r . We also impose that the solution has four Killing spinors. In this case one can show that $g(r) = -f(r)$. For the gauge fields and scalars we impose the same symmetry requirements as for the metric. Therefore each gauge field has only two independent components, one electric and one magnetic, which are functions of r :

$$F_{\underline{t}\underline{r}}^I = F_E^I(r), \quad F_{\underline{\theta}\underline{\phi}}^I = F_M^I(r).$$

Here $\underline{t}, \underline{r}, \underline{\theta}, \underline{\phi}$ are tangent space indices.¹⁶

The physical scalar fields z^i can be functions of the radial variable r , $z^i = z^i(r)$. In order to maintain symplectic covariance, we work in the gauge-equivalent superconformal theory and use the conformal scalars X^I . It turns out to be convenient to rescale the scalars and to define

$$Y^I(r) = \overline{Z}(r) X^I(r),$$

where $Z(r)$ is the ‘central charge’. Note that

$$|Z|^2 = \overline{Z}Z = \overline{Z}(p^I F_I(X) - q_I X^I) = p^I F_I(Y) - q_I Y^I,$$

where we used that F_I is homogenous of degree one.

In the following we will focus on the near-horizon limit. In the isotropic coordinates used in (74), the horizon is located at $r = 0$. The scalar fields show a very particular behaviour in this limit: irrespective of their ‘initial values’ $z^i(\infty)$ at spatial infinity, they approach fixed point values $z_*^i = z^i(p^I, q_I)$ at the horizon. This behaviour was discovered by Ferrara, Kallosh and Strominger and is called the black hole attractor mechanism. The fixed point values are determined by the attractor equations, which can be brought to the following, manifestly symplectic form:

$$\left(\begin{array}{c} Y^I - \overline{Y}^I \\ F_I - \overline{F}_I \end{array} \right)_* = i \left(\begin{array}{c} p^I \\ q_I \end{array} \right).$$

Here and in the following ‘*’ indicates the evaluation of a quantity on the horizon. Depending on the explicit form of the prepotential it may or may not be possible to solve this set of algebraic equations to obtain explicit formulae for the scalars as functions of the charges. The remaining data of the near-horizon solution are the metric and the gauge fields. The near-horizon metric takes the form

$$ds^2 = -\frac{r^2}{|Z_*|^2} dt^2 + \frac{|Z_*|^2}{r^2} dr^2 + |Z_*|^2 d\Omega_{(2)}^2,$$

¹⁶ If we use world indices, F_M^I depends on the angular variables. This dependence is trivial in the sense that it disappears when the tensor components are evaluated in an orthonormal frame.

where Z_* is the horizon value of the central charge,

$$|Z_*|^2 = (p^I F_I(Y) - q_I Y^I)_* .$$

The near horizon geometry is therefore $AdS^2 \times S^2$, with curvature radius $R = |Z_*|^2$. This is a maximally symmetric space, or more precisely the product of two maximally symmetric spaces. The gauge fields become covariantly constant in the near horizon limits, i.e., they become fluxes whose strength is characterized by the charges (p^I, q_I) . In suitable coordinates¹⁷ one simply has

$$F_E^I = q_I , \quad F_M^I = p^I .$$

$AdS^2 \times S^2$, supported by fluxes and constant scalars is a generalisation of the Bertotti-Robinson solution of Einstein-Maxwell theory.

This generalised Bertotti-Robinson solution is not only the near horizon solution of BPS black holes, but also an interesting solution in its own right. It can be shown that it is the most general static fully supersymmetric solution (8 Killing spinors) of $N = 2$ supergravity with vector multiplets. Note that the attractor equations follow from imposing full supersymmetry, or, equivalently, the field equations. Thus in a Bertotti-Robinson background the scalars cannot take arbitrary values. This is easily understood by interpreting the solution as a flux compactification of four-dimensional supergravity on S^2 . Since S^2 is not Ricci flat, flux must be switched on to solve the field equations. The dimensionally reduced theory is a gauged supergravity theory with a non-trivial scalar potential with a non-degenerate AdS^2 ground state and fixed moduli.

The BPS black hole solution, which has only four Killing spinor, interpolates between two supersymmetric ground states with eight Killing spinors. At infinity it approaches Minkowski space, and in this limit the values of the scalars are arbitrary, because the four-dimensional supergravity theory has no scalar potential and a moduli space of vacua, parametrised by the scalars. At the horizon we approach another supersymmetric ground state, but here the scalars have to flow to the fixed point values dictated by the attractor equations. The black hole solution can be viewed as a dynamical system for the radial evolution of the scalars¹⁸ from arbitrary initial values at $r = \infty$ to fixed point values at $r = 0$.

For completeness we mention that not all flows correspond to regular black holes. For non-generic choices of the charges (typically when switching off sufficiently many charges) the scalar fields can run off to the boundary of moduli space. In these cases $|Z_*|^2$ becomes zero or infinity, so that there is no black hole horizon. The original derivation of the attractor equations was

¹⁷ Essentially, $r \rightarrow \frac{1}{r}$ combined with a rescaling of t . In these coordinates it becomes manifest that the metric is conformally flat.

¹⁸ The other non-trivial data, namely metric and gauge fields can be expressed in terms of the scalars.

in fact motivated by this observation: if one imposes that the scalars do not run off to infinity at the horizon, this implies that the solution must approach a supersymmetric ground state, which in turn implies that the geometry is Bertotti-Robinson and that the scalars take fixed point values. In this context the attractor equations were called stabilisation equations, because they forbid that the moduli run off.

There can also be more complicated phenomena if the flow crosses, at finite r , a line of marginal stability, where the BPS spectrum changes, or if it runs into a boundary point or other special point in the moduli space. We will concentrate on regular black hole solutions here, and make some comments on so-called small black holes later.

The attractor behaviour of the scalars is important for the consistency of black hole thermodynamics. The laws of black hole mechanics, combined with the Hawking effect, suggest that a black hole has a macroscopic (thermodynamical) entropy proportional to its area A :

$$S_{\text{macro}} = \frac{A}{4} .$$

The corresponding microscopic (statistical) entropy is given by the state degeneracy¹⁹

$$S_{\text{micro}} = \log \#\{\text{Microstates corresponding to given macrostate}\} .$$

Both entropies should be equal, at least asymptotically in the semi-classical limit (which, for non-rotating black holes, is the limit of large mass and charges). Therefore it should not be possible to change the area continuously. This is precisely what the attractor mechanism guarantees.

From the near horizon geometry we can read off that the area of the black hole is $A = 4\pi|Z_*|^2$. The entropy is given by the following symplectic function of the charges:

$$S_{\text{macro}} = \frac{A}{4} = \pi|Z_*|^2 = \pi|p^I F_I(X) - q_I X^I|_*^2 = \pi(p^I F_I(Y) - q_I Y^I)_* .$$

Further reading and references

For a general introduction to solitons (and instantons), see for example the book by Rajaraman [28]. The idea to interpret extremal black holes as supersymmetric solitons is due to Gibbons [29] (see also [30]). There are many good reviews on BPS solitons in string theory, in particular [31] and [32]. The black hole attractor mechanism was discovered by Ferrara, Kallosh and Strominger [33]. This section is heavily based on a paper written jointly with Cardoso, de Wit and Käppeli [6], where we proved that the attractor mechanism is not

¹⁹ The macrostate of a black hole is given by its mass, angular momentum and conserved charges.

only sufficient, but also necessary for $\frac{1}{2}$ -BPS solution, and that the Bertotti-Robinson solution is the only static solution preserving full supersymmetry. We refer to Sen's recent review [9] for the discussion of non-supersymmetric attractors.

We mentioned that not all attractor flows correspond to regular black holes solutions. One phenomenon which can occur is that the solution becomes singular before the horizon is reached (i.e. the solution becomes singular at finite values of r .) In string theory such singularities can usually be explained by a breakdown of the effective field theory. In particular, for domain walls and black holes in five-dimensional string compactifications it has been shown that one always reaches an internal boundary of moduli space before the singularity forms [43, 44]. When the properties of the internal boundary are taken into account, the solutions becomes regular.²⁰ In four dimensions the variety of phenomena appears to be more complex. There are so-called split attractor flows, which correspond to situations where the flow crosses a line of marginal stability [45]. This has the effect that solutions which look like single-centered black hole solutions when viewed from infinity, turn out to be complicated composite objects when viewed from nearby. The role of lines of marginal stability has been studied recently in great detail in [71].

3.3 The black hole variational principle

Almost immediately after the black hole attractor mechanism was discovered, it was observed that the attractor equations follow from a variational principle. More recently it has been realized that this variational principle plays an important role in black hole thermodynamics and can be used to relate macrophysics (black hole solutions of effective supergravity) to microphysics (string theory, and in particular BPS partition functions and the topological string) in an unexpectedly direct way.

To explain the variational principle we start by defining the 'entropy function'

$$\Sigma(Y, \bar{Y}, p, q) := \mathcal{F}(Y, \bar{Y}) - q_I(Y^I + \bar{Y}^I) + p^I(F_I + \bar{F}_I),$$

where $\mathcal{F}(Y, \bar{Y})$ is the 'free energy'

$$\mathcal{F}(Y, \bar{Y}) = -i(F_I \bar{Y}^I - Y^I \bar{F}_I).$$

The terminology will become clear later. If we extremize the entropy function with respect to the scalars, the equations characterising critical points of Σ are precisely the attractor equations:

$$\frac{\partial \Sigma}{\partial Y^I} = 0 = \frac{\partial \Sigma}{\partial \bar{Y}^I} \iff \begin{pmatrix} Y^I - \bar{Y}^I \\ F_I - \bar{F}_I \end{pmatrix}_* = i \begin{pmatrix} p^I \\ q_I \end{pmatrix}.$$

²⁰ At internal boundaries one typically encounters additional massless states, and this changes the flow corresponding to the solution.

And if we evaluate the entropy function at its critical point, we obtain the entropy, up to a conventional factor:

$$\pi\Sigma_* = S_{\text{macro}}(p, q) .$$

The geometrical meaning of the entropy function becomes clear if we use the special affine coordinates

$$\begin{aligned} x^I &= \text{Re}Y^I , \\ y_I &= \text{Re}F_I(Y) , \end{aligned} \tag{75}$$

instead of the special coordinates $Y^I = x^I + iu^I$. The special affine coordinates $(q^a) = (x^I, y_I)^T$ have the advantage that they form a symplectic vector. In special affine coordinates, the special Kähler metric can be expressed in terms of a real Kähler potential $H(x^I, y_I)$, called the Hesse potential. The Hesse potential is related to the prepotential by a Legendre transform, which replaces $u^I = \text{Im}Y^I$ by $y_I = \text{Re}(Y^I)$ as an independent field:

$$H(x^I, y_I) = 2 \left(\text{Im}F(x^I + iu^I(x, y)) - y^I u_I(x, y) \right) ,$$

where

$$y_I = \frac{\partial \text{Im}F}{\partial u^I} .$$

If we express the entropy function in terms of special affine coordinates, we find:

$$\Sigma(x, y, q, p) = 2H(x, y) - 2q_I x^I + 2p^I y_I ,$$

where

$$2H(x, y) = \mathcal{F}(Y, \bar{Y}) = -i(F_I \bar{Y}^I - Y^I \bar{F}_I) .$$

Thus, up to a factor, the Hesse potential is the free energy. The critical points of the entropy function satisfy the black hole attractor equations, which in special affine coordinates take the following form:

$$\frac{\partial H}{\partial x^I} = q_I , \quad \frac{\partial H}{\partial y_I} = -p^I .$$

The black hole entropy is obtained by substituting the critical values into the entropy function:

$$S_{\text{macro}}(p, q) = 2\pi \left(H - x^I \frac{\partial H}{\partial x^I} - y_I \frac{\partial H}{\partial y_I} \right)_* .$$

This shows that, up to a factor, the macroscopic black hole entropy is Legendre transform of the Hesse potential. Note that at the horizon the scalar fields are determined by the charges, so that the charges provide coordinates on the scalar manifold. More precisely, the charges are not quite coordinates, because they can only take discrete values, but by the attractor equations

they are proportional to continuous quantities which provide coordinates. The attractor equations can be rewritten in the form

$$\begin{pmatrix} 2u^I \\ 2v_I \end{pmatrix} = \begin{pmatrix} p^I \\ q_I \end{pmatrix}, \quad (76)$$

where $u^I = \text{Im}Y^I$ and $v_I = \text{Im}F_I$. It can be shown that (u^I, v_I) is another system of special affine coordinates. Thus the attractor equations specify a point on the scalar manifold in terms of the coordinates (u^I, v_I) . The extremisation of the entropy function can be viewed as a Legendre transform from one set of special affine coordinates to another.

The special affine coordinates (x^I, y_I) also have a direct relation to the gauge fields, which even holds away from the horizon. By the gauge field equations of motion in a static (or stationary) background the scalars (x^I, y_I) are proportional to the electrostatic and magnetostatic potentials (ϕ^I, χ_I) :

$$\begin{pmatrix} 2x^I \\ 2y_I \end{pmatrix} = \begin{pmatrix} \phi^I \\ \chi_I \end{pmatrix}.$$

Thermodynamically, the electrostatic and magnetostatic potentials are the chemical potentials associated with the electric and magnetic charges in a grand canonical ensemble.

Further reading and references

The black hole variational principle described in this section was formulated by Behrndt et al in [34]. The reformulation in terms of real coordinates is relatively recent [35]. The relation of the black hole variational principle to the work of Ooguri, Strominger and Vafa [1] will be explained in the following sections. Sen's entropy function (see [9] for a review and references), which can be used to establish the attractor mechanism for general extremal black holes, irrespective of supersymmetry and details of the Lagrangian, can be viewed as a generalisation of the entropy function discussed here, in the sense that the two entropy functions differ by terms which vanish in BPS backgrounds [36].²¹

For completeness, we need to mention that there is yet another 'variational' approach to the black hole entropy. The concept of a black hole effective potential was already introduced in [46]. The idea is to use the symmetries of static, spherically symmetric black holes to reduce the dynamics to the one of particle moving in an effective potential. This does not rely on supersymmetry and has become, besides Sen's entropy function, the second approach for studying the attractor mechanism for non-BPS black holes [47]. The two

²¹ To be precise, Sen's formalism is based on an entropy function which is based on the 'mixed' rather than the 'canonical' ensemble. This is explained in the next section.

approaches are not completely unrelated. Dimensional reduction along Killing vectors plays a role in both of them, since Sen's entropy function is obtained by dimensionally reducing the action, evaluated on the horizon, along the S^2 factor of the horizon geometry $AdS^2 \times S^2$, and then taking a Legendre transform.²²

3.4 Canonical, microcanonical and mixed ensemble

For a grand canonical ensemble, the first law of thermodynamics takes the following form:

$$\delta E = T\delta S - p\delta V + \mu_i\delta N_i .$$

Here E is the energy, T the temperature, S the entropy, p the pressure, V the volume, μ_i the chemical potential and N_i the particle number of the i -th species of particles. In relativistic systems the particle number is replaced by the conserved charge under a gauge symmetry. For a general stationary black hole, the first law of black hole mechanics has the same structure:

$$\delta M = \frac{\kappa_S}{2\pi}\delta A + \omega\delta J + \phi^I\delta q_I + \chi_I\delta p^I .$$

Here M is the mass, κ_S the surface gravity, A the area, ω the rotation velocity, J the angular momentum, and ϕ^I, χ_I, p^I, q_I are the electric and magnetic potentials and charges. The Hawking effect and the generalized second law of thermodynamics suggest to take the formal analogy between thermodynamics and black hole physics seriously. In particular, the Hawking temperature of a black hole is $T = \frac{\kappa_S}{2\pi}$, which fixes the relation between area and entropy to be $S = \frac{A}{4}$.

In thermodynamics we consider other ensembles as well. The canonical ensemble is obtained by freezing the particle number while the microcanonical ensemble is obtained by freezing the energy as well. In general, the result for a thermodynamical quantity will depend on the ensemble one uses. However, all ensembles give the same result in the thermodynamical limit.

We will only discuss non-rotating black holes, $\omega = 0$. The analogous ensemble in thermodynamics does not seem to have a particular name, but, by common abuse of terminology, we will call this the canonical ensemble. Moreover, we only consider extremal black holes, with zero temperature. For $\kappa_S = 0$ the first law does not give directly a relation between mass and entropy, but we can interpret extremal black holes as limits of non-extremal ones. The independent variables in the canonical ensemble are the potentials $(\phi^I, \chi_I) \propto (x^I, y_I)$. This ensemble corresponds to a situation where the electric and magnetic charge is allowed to fluctuate, while the corresponding chemical potentials are prescribed. The ensemble obtained by fixing the electric and

²² This is a partial Legendre transform, because the mixed ensemble is used. See the next section for explanation.

magnetic charges is called the microcanonical ensemble. Here the independent variables are $(p^I, q_I) \propto (u^I, v_I)$.

At the microscopic ('statistical mechanics') level, all three ensembles are characterised by a corresponding partition function. The microcanonical partition function is simply given by the microscopic state degeneracy:

$$Z_{\text{micro}}(p, q) = d(p, q) ,$$

where $d(p, q)$ is the number of microstates of a BPS black hole with charges p^I, q_I . The microscopic (statistical) entropy of the black hole is

$$S_{\text{micro}}(p, q) = \log d(p, q) .$$

The partition function of the canonical ensemble is obtained by a formal discrete Laplace transform:

$$Z_{\text{can}}(\phi, \chi) = \sum_{p, q} d(p, q) e^{\pi(q\phi - p\chi)} . \quad (77)$$

This relation can be inverted (formally):

$$d(p, q) = \oint d\phi d\chi Z_{\text{can}}(\phi, \chi) e^{-\pi(q\phi - p\phi)} .$$

These partition functions are supposed to provide the microscopic description of BPS black holes. The macroscopic description is provided by black hole solutions of the effective supergravity theory, through the attractor equations, the macroscopic entropy and the entropy function. The variational principle suggests that the Hesse potential should be interpreted as the BPS black hole free energy with respect to the microscopic ensemble. This leads to the conjecture

$$e^{2\pi H(\phi, \chi)} \approx Z_{\text{can}} = \sum_{p, q} d(p, q) e^{\pi[q_I \phi^I - p^I \chi_I]} , \quad (78)$$

or, using special coordinates instead of special affine coordinates:

$$e^{\pi \mathcal{F}(Y, \bar{Y})} \approx Z_{\text{can}} = \sum_{p, q} d(p, q) e^{\pi[q_I (Y^I + \bar{Y}^I) - p^I (F_I + \bar{F}_I)]} . \quad (79)$$

Here ' \approx ' means asymptotic equality in the limit of large charges, which is the semiclassical and thermodynamic limit. Ideally, one would hope to find an exact relation between macroscopic and microscopic quantities, but so far there is only good evidence for a weaker, asymptotic relation. We can formally invert (78), (79) to obtain a prediction for the state degeneracy in terms of the macroscopically defined free energy:

$$d(p, q) \approx \int dx dy e^{\pi \Sigma(x, y)} \approx \int dY d\bar{Y} |\det[\text{Im} F_{KL}]| e^{\pi \Sigma(Y, \bar{Y})}$$

Observe that this formula is manifestly invariant under symplectic transformations, because

$$dx dy := \prod_{I,J} dx^I dy_J = (dx^I \wedge dy_I)^{\text{top}}$$

is the natural volume form on the scalar manifold (the top exterior power of the symplectic form $dx^I \wedge dy_I$), and $\Sigma(x, y)$ is a symplectic function.²³ Note that there is a non-trivial Jacobian if we go to special coordinates.

By the variational principle, the saddle point value of $\pi\Sigma$ is the macroscopic entropy. Therefore it is obvious that microscopic and macroscopic entropy agree to leading order in a saddle point evaluation of the integral:

$$e^{S_{\text{micro}}(p,q)} = d(p, q) \approx e^{S_{\text{macro}}(p,q)(1+\dots)} .$$

However, in general the microscopic entropy (defined through state counting) and the macroscopic entropy (defined geometrically through the area law) will be different. The reason is that the macroscopic entropy is the Legendre transform of the canonical free energy, while the microcanonical and canonical partition functions are related by the Laplace transform (77). The Legendre transform between canonical free energy and macroscopic entropy provides the leading order approximation of this Laplace transform. In other words, the macroscopic entropy is not computed in the microcanonical ensemble, and we can only expect it to agree with the microscopic entropy in the thermodynamical limit.

The mixed ensemble

We will now consider the so-called mixed ensemble, where the independent variables are p^I and ϕ^I . This corresponds to a situation where the magnetic charge is fixed while the electric charge fluctuates and the electrical potential is prescribed. This ensemble has the disadvantage that the independent variables do not form a symplectic vector, which obscures symplectic covariance. However, the mixed ensemble is natural in the functional integral framework, and one obtains a direct relation between black hole thermodynamics and the topological string.

The partition function of the mixed ensemble is obtained from the microcanonical partition function through a Laplace transform with respect to half of the variables:

$$Z_{\text{mix}}(p, \phi) = \sum_q d(p, q) e^{\pi q \phi} ,$$

$$d(p, q) = \oint Z_{\text{mix}}(p, \phi) e^{-\pi q \phi} .$$

²³ Observe that the relevant scalar manifold is M rather than \overline{M} .

Let us discuss this ensemble from the macroscopic point of view. In our previous treatment of the variational principle, we extremized the entropy function with respect to all scalar fields/potentials at once. This extremisation process can be broken up into several steps. The ‘magnetic’ attractor equations

$$Y^I - \bar{Y}^I = ip^I$$

fix the imaginary parts of the Y^I :

$$Y^I = \frac{1}{2}(\phi^I + ip^I) .$$

If we substitute this into Σ we obtain a reduced entropy function:

$$\Sigma(\phi, p, q)_{\text{mix}} = \mathcal{F}_{\text{mix}}(p, \phi) - q_I \phi^I ,$$

where

$$\mathcal{F}_{\text{mix}}(p, \phi) = 4\text{Im}F(Y, \bar{Y})$$

is interpreted as the free energy in the mixed ensemble. Σ_{mix} can be interpreted as the entropy function in the mixed ensemble, because there is a new, reduced variational principle in the following sense: if we extremize Σ_{mix} with respect to the remaining scalars $\phi^I = \frac{1}{2}\text{Re}Y^I$, then we obtain the remaining ‘electric’ attractor equations:

$$F_I - \bar{F}_I = q_I .$$

If this is substituted back into the mixed entropy function, we obtain the macroscopic entropy:

$$S_{\text{macro}}(p, q) = \pi \Sigma_{\text{mix},*} .$$

The extremisation of the mixed entropy function defines a Legendre transform between the mixed free energy and the entropy. Note that the mixed free energy is the imaginary part of the prepotential.

The mixed free energy should be related to the mixed partition function. One conceivable relation is the original ‘OSV-conjecture’

$$e^{\pi \mathcal{F}_{\text{mix}}(p, \phi)} \approx Z_{\text{mix}}(p, \phi) . \quad (80)$$

To leading order in a saddle point approximation the variational principle guarantees that macroscopic and microscopic entropy agree. But one disadvantage of the mixed ensemble is that the independent variables p^I, ϕ^I do not form a symplectic vector. Therefore symplectic covariance is obscure.

Let us then compare (80) to the symplectically covariant conjecture (79) involving the canonical ensemble. Since the variational principle can be broken up into two steps, we can perform a partial saddle point approximation of (79) with respect to the imaginary parts of the scalars and obtain

$$d(p, q) \approx \int d\phi \sqrt{|\det \text{Im} F_{IJ}|} e^{\pi[\mathcal{F}_{\text{mix}}(p, \phi) - q\phi]} .$$

This can be formally inverted with the result:

$$\sqrt{\Delta^-} e^{\pi \mathcal{F}_{\text{mix}}(p, \phi)} \approx Z_{\text{mix}} = \sum_q d(p, q) e^{\pi q_I \phi^I} . \quad (81)$$

Thus by imposing symplectic covariance we predict the presence of a non-trivial ‘measure factor’ in the mixed ensemble.

Further reading and references

The idea to interpret the (partial) Legendre transform of the black hole entropy as a free energy (in the mixed ensemble) is due to Ooguri, Strominger and Vafa [1] and has triggered an immense number of publications which elaborate on their observation. Our presentation, which is based on [36], uses the variational principle of [34] to reformulate the ‘OSV-conjecture’ in a manifestly symplectically covariant way.

3.5 R^2 -corrections

Non-trivial tests of conjectures about state counting and partition functions depend on the ability to compute subleading corrections to the macroscopic entropy. Such corrections are due to quantum and stringy corrections to the effective action, which manifest themselves as higher derivative terms. Within the superconformal calculus one class of such terms can be handled by giving the prepotential an explicit dependence on the lowest component of the Weyl multiplet. Incidentally, in type-II Calabi-Yau compactifications the same class of terms is controlled by the topologically twisted world sheet theory. Therefore these higher derivative couplings can be computed, at least in principle.

It is possible to find the most general stationary $\frac{1}{2}$ -BPS solution for a general prepotential of the form $F(X^I, \hat{A})$, at least iteratively. Here we restrict ourselves to the near-horizon limit of static, spherically symmetric single black hole solutions. It is convenient to introduce rescaled variables $Y^I = \bar{Z} X^I$ and $\mathcal{Y} = \bar{Z}^2 \hat{A}$, and by homogeneity we get a rescaled prepotential $F(Y^I, \mathcal{Y}) = \bar{Z}^2 F(X^I, \hat{A})$. The near horizon solution is completely determined by the generalized attractor equations

$$\left(\begin{array}{c} Y^I - \bar{Y}^I \\ F_I(Y, \mathcal{Y}) - \bar{F}_I(\bar{Y}, \bar{\mathcal{Y}}) \end{array} \right)_* = i \begin{pmatrix} p^I \\ q_I \end{pmatrix}, \quad \mathcal{Y}_* = -64 . \quad (82)$$

This is symplectically covariant, because $(Y^I, F_I(Y, \mathcal{Y}))^T$ is a symplectic vector. The variable \mathcal{Y} is invariant and takes a particular numerical value at the horizon. The geometry is still $AdS^2 \times S^2$, but the radius and therefore the area is modified by the higher derivative corrections:

$$A = 4\pi |p^I F_I(X, \hat{A}) - q_I X^I|_*^2 = 4\pi (p^I F_I(Y, \mathcal{Y}) - q_I Y^I)_* .$$

But this is not the only modification of the entropy, because in theories with higher curvature terms the entropy is not determined by the area law. Wald has shown by a careful derivation of the first law of black hole mechanics for generally covariant Lagrangians (admitting higher curvature terms) that the definition of the entropy must be modified, if the first law is still to be valid. Entropy, mass, angular momentum and charges can be defined as surface charges, which are the Noether charges related to the Killing vectors of the space-time. The entropy is given by the integral of a Noether two-form over the event horizon:

$$S = \oint Q .$$

The symmetry associated with this Noether charge is the one generated by the so-called horizontal Killing vector field. For static black holes this is the timelike Killing vector field associated with the time-independence of the background, while for rotating black holes it is a linear combination of the timelike and the axial Killing vector field. In practice the Noether charge can be expressed in terms of variational derivatives of the Lagrangian with respect to the Riemann tensor:

$$S = \oint \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \varepsilon_{\mu\nu} \varepsilon_{\rho\sigma} \sqrt{h} d^2 \Omega_2 .$$

Here $\varepsilon_{\mu\nu}$ is the normal bivector, normalized to $\varepsilon_{\mu\nu} \varepsilon^{\mu\nu} = -2$ and $\sqrt{h} d^2 \Omega$ is the induced volume element of the horizon. If one evaluates this formula for $N = 2$ supergravity with prepotential $F(Y^I, \mathcal{Y})$, the result is

$$S_{\text{macro}} = \pi \left((p^I F_I(Y, \mathcal{Y}) - q_I Y^I) - 256 \operatorname{Im} \left(\frac{\partial F}{\partial \mathcal{Y}} \right) \right) . \quad (83)$$

This is the sum of two symplectic functions. The first term corresponds to the area law while the second is an explicit modification. This modification is crucial for the matching of subleading contributions to the macroscopic and microscopic entropy in string theory.

R^2 -corrections can be incorporated into the variational principle in a straightforward way. One defines a generalized Hesse potential as the Legendre transform of (two times the imaginary part of) the prepotential $F(Y^I, \mathcal{Y})$:

$$H(x, y, \mathcal{Y}, \bar{\mathcal{Y}}) = 2 (\operatorname{Im} F(Y^I, \mathcal{Y}) - y_I u^I) ,$$

where

$$y_I = \operatorname{Re} F_I(Y^I, \mathcal{Y}) = \frac{\partial \operatorname{Im} F(Y^I, \mathcal{Y})}{\partial u^I} .$$

The canonical free energy is

$$\mathcal{F}(Y, \bar{Y}) = 2H(x, y) = -i(\bar{Y}^I F_I - Y^I \bar{F}_I) - 2i(\mathcal{Y} F_{\mathcal{Y}} - \bar{\mathcal{Y}} \bar{F}_{\bar{\mathcal{Y}}}) .$$

Here and in the following we adopt a notation where we usually suppress the dependence on \mathcal{Y} , unless where we want to emphasize that R^2 -corrections have been taken into account. The entropy function takes the form

$$\Sigma(x, y, p, q) = 2(H - qx + py) ,$$

where H is now the generalized Hesse potential. It is straightforward to show that the extremization of this entropy function gives the attractor equations (82), and that its critical values gives the entropy (83): $S_{\text{macro}} = \pi \Sigma_*$.

Further reading and references

R^2 -corrections to BPS solutions of $N = 2$ supergravity with vector (and hyper) multiplets were first obtained in [5] in the near horizon limit. The comparison with subleading corrections to state counting in $N = 2$ string compactifications [3, 4] showed that is crucial to use Wald's modified definition of the black hole entropy [7]. This approach assumes a Lagrangian which is covariant under diffeomorphisms, and identifies the correct definition of the entropy by imposing the validity of the first law of black hole mechanics. The entropy is found to be a Noether surface charge, which can be expressed in terms of variational derivatives of the Lagrangian [8]. The full derivation is quite intricate, and while no concise complete review is available, some elements of it have been reformulated in [37] from a more conventional gauge theory perspective. Otherwise, see [13] for a more detailed account on Wald's entropy formula and its merits in string theory. Sen's entropy function formalism [9] is based on Wald's definition of black hole entropy.

The general class of stationary $\frac{1}{2}$ -BPS solutions in $N = 2$ supergravity with R^2 -terms was described in [6]. The generalisation of the black hole variational principle to include R^2 terms was found in [35].

3.6 Non-holomorphic corrections

There is a further type of corrections which need to be taken into account, the so-called non-holomorphic corrections. One way of deducing that such corrections must be present is to investigate the transformation properties of the entropy under string dualities, specifically under S-duality and T-duality. We will discuss an instructive example in section 4.3. The consequence is that the entropy and the attractor equations can only be duality invariant, if there are additional contributions to the entropy and to the symplectic vector $(Y^I, F_I(Y, \mathcal{Y}))$, which cannot be derived from a holomorphic prepotential $F(Y, \mathcal{Y})$. This is related to a generic feature of string-effective actions and their couplings. One has to distinguish between two types of effective actions. The Wilsonian action is always local and the corresponding Wilsonian couplings are holomorphic functions of the moduli (in supersymmetric theories). The other type of effective action is the generating functional of the scattering

amplitudes. If massless modes are present this is in general non-local, and the associated physical couplings have a more complicated, non-holomorphic dependence on the moduli. Both types of actions differ by threshold corrections associated with the massless modes, which can be computed by field theoretic methods. The supergravity actions which we have constructed and discussed so far are based on a holomorphic prepotential and have to be interpreted as Wilsonian actions. Their couplings are holomorphic, and they are different from the physical couplings, which can be extracted from string scattering amplitudes. The Wilsonian couplings are not necessarily invariant under symmetries, such as string dualities, whereas the physical couplings are. The same distinction between holomorphic, but non-covariant quantities and non-holomorphic, but covariant quantities occurs for the topological string, which is the tool used to compute the couplings. Here the non-holomorphicity arises from the integration over the world-sheet moduli space, and it is encoded in the holomorphic anomaly equations.

In the following we will describe a general formalism for incorporating non-holomorphic corrections to the attractor equations and the entropy. This formalism is model-independent (as such), but we should stress that it is inspired by the example which we are going to discuss in section 4.3. While it has been shown to work in $N = 4$ compactifications, it is not clear a priori whether the non-holomorphic modifications that are introduced are general enough to cover generic $N = 2$ compactifications. Moreover, it should be interesting to investigate the relation between this formalism and the holomorphic anomaly equation of the topological string in more detail.

The basic assumption underlying the formalism is that all non-holomorphic modifications are captured by a single real-valued function $\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$, which is required to be (graded) homogenous of degree two:

$$\Omega(\lambda Y^I, \bar{\lambda} \bar{Y}^I, \lambda^2 \Upsilon, \bar{\lambda}^2 \bar{\Upsilon}) = |\lambda|^2 \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) .$$

We then define a generalized Hesse potential by taking the Legendre transform of $\text{Im}F + \Omega$:

$$\hat{H}(x, y) = 2 (\text{Im}F(x + iu, \Upsilon) + \Omega(x, y, \Upsilon, \bar{\Upsilon}) - qx + p\hat{y}) , \quad (84)$$

where

$$\hat{y}_I = y_I + i(\Omega_I - \Omega_{\bar{I}}) . \quad (85)$$

Clearly, this modification is only non-trivial if Ω is not a harmonic function, because otherwise it could be absorbed by redefining the holomorphic function F .

We now take the generalized Hesse potential as our canonical free energy and define the entropy function

$$\Sigma = 2(\hat{H} - qx + p\hat{y}) . \quad (86)$$

By variation of the entropy function with respect to x, \hat{y} we obtain the attractor equations

$$\frac{\partial \hat{H}}{\partial x} = q, \quad \frac{\partial \hat{H}}{\partial \hat{y}} = -p, \quad (87)$$

and by substituting the critical values back into the entropy function we obtain the macroscopic black hole entropy

$$S_{\text{macro}} = \pi \Sigma_* = 2\pi \left(\hat{H} - x \frac{\partial \hat{H}}{\partial x} - \hat{y} \frac{\partial \hat{H}}{\partial \hat{y}} \right)_*. \quad (88)$$

In practice, one works with special coordinates rather than special affine coordinates, because explicit expressions for subleading contributions to the couplings are only known in terms of complex coordinates. In special coordinates the entropy function has the following form:

$$\Sigma(Y, \bar{Y}, p, q) = \mathcal{F}(Y, \bar{Y}, \mathcal{Y}, \bar{\mathcal{Y}}) - q_I(Y^I + \bar{Y}^I) + p^I(F_I + \bar{F}_I + 2i(\Omega_I - \Omega_{\bar{I}})),$$

with canonical free energy

$$\mathcal{F}(Y, \bar{Y}, \mathcal{Y}, \bar{\mathcal{Y}}) = -i(\bar{Y}^I F_I - Y^I \bar{F}_I) - 2i(\mathcal{Y} F_{\mathcal{Y}} - \bar{\mathcal{Y}} \bar{F}_{\bar{\mathcal{Y}}}) + 4\Omega - 2(Y^I - \bar{Y}^I)(\Omega_I - \Omega_{\bar{I}}).$$

The attractor equations are

$$\begin{pmatrix} Y^I - \bar{Y}^I \\ F_I - \bar{F}_I + 2i(\Omega_I + \Omega_{\bar{I}}) \end{pmatrix} = \begin{pmatrix} p^I \\ q_I \end{pmatrix},$$

and the entropy is

$$S_{\text{macro}} = \pi (|Z|^2 - 256 \text{Im}(F_{\mathcal{Y}} + i\Omega_{\mathcal{Y}}))_*. \quad (89)$$

By inspection, the net effect of the non-holomorphic corrections is to replace $F \rightarrow F + 2i\Omega$ in the entropy function and in the attractor equations, but $F \rightarrow F + i\Omega$ in the definition of the Hesse potential and in the entropy.²⁴

As before we can impose half of the attractor equations and go from the canonical to the mixed ensemble. The modified mixed free energy is found to be

$$\mathcal{F}_{\text{mix}} = 4(\text{Im}F + \Omega).$$

Since the non-holomorphic modifications are enforced by duality invariance, they are relevant for the conjectures about the relation between macroscopic quantities (free energy and macroscopic entropy) and microscopic quantities (partition functions and microscopic entropy).

Our basic conjecture is that the canonical free energy, including non-holomorphic modifications, is related to the canonical partition function by:

²⁴ As an exercise, the curious reader is encouraged to verify this statement by himself, starting from the definition of the generalized Hesse potential and re-deriving all the formulae step by step.

$$e^{2\pi H(x,y)} \approx Z_{\text{can}} = \sum_{p,q} d(p,q) e^{2\pi[q_I x^I - p^I \hat{y}_I]} . \quad (90)$$

In special coordinates, this reads

$$e^{\pi \mathcal{F}(Y, \bar{Y})} \approx Z_{\text{can}} = \sum_{p,q} d(p,q) e^{\pi[q_I(Y^I + \bar{Y}^I) - p^I(\hat{F}_I + \bar{\hat{F}}_I)]} . \quad (91)$$

We can formally invert these formulae to get a prediction of the state degeneracy in terms of macroscopic quantities:

$$d(p,q) \approx \int dx d\hat{y} e^{\pi \Sigma(x, \hat{y})} \approx \int dY d\bar{Y} \Delta^-(Y, \bar{Y}) e^{\pi \Sigma(Y, \bar{Y})} , \quad (92)$$

where we defined

$$\Delta^\pm(Y, \bar{Y}) = |\det [\text{Im} F_{KL} + 2\text{Re}(\Omega_{KL} \pm \Omega_{K\bar{L}})]| . \quad (93)$$

In saddle point approximation, we predict the following relation between the microscopic and the macroscopic entropy:

$$e^{S_{\text{micro}}(p,q)} = d(p,q) \approx e^{S_{\text{macro}}(p,q)} \sqrt{\frac{\Delta^-}{\Delta^+}} \approx e^{S_{\text{macro}}(p,q)(1+\dots)} .$$

Here we used that both the measure factor Δ^- and the fluctuation determinant Δ^+ are subleading in the limit of large charges.

We can also perform a partial saddle point approximation

$$d(p,q) \approx \int d\phi \sqrt{\Delta^-(p, \phi)} e^{\pi[\mathcal{F}_{\text{mix}}(\phi, p) - q_I \phi^I]}$$

and get a conjecture for the relation between the mixed free energy and the mixed partition function:

$$\sqrt{\Delta^-} e^{\pi \mathcal{F}_{\text{mix}}(p, \phi)} \approx Z_{\text{mix}} = \sum_q d(p,q) e^{\pi q_I \phi^I} . \quad (94)$$

The conjecture put forward by Ooguri, Strominger and Vafa is:

$$e^{\pi \mathcal{F}_{\text{mix}}^{\text{hol}}(p, \phi)} \approx Z_{\text{BH}}^{(\text{mix})} = \sum_q d(p,q) e^{\pi q_I \phi^I} . \quad (95)$$

This differs from (94) in two ways: (i) the measure factor Δ^- is absent, and (ii) the mixed free energy does not include contributions from non-holomorphic terms. Since these modifications are subleading, the black hole variational principle guarantees that both formulae agree to leading order for large charges. As indicated by our presentation, we expect that the measure factor and the non-holomorphic contributions to the free energy are present, because they are needed for symplectic covariance and duality invariance. In fact, the presence of subleading modifications in (94) has been verified, and we will review this later.

The relation to the topological string

One nice feature of (95) is that provides a direct link between the mixed black hole partition function and the partition function of the topological string. The coupling functions $F^{(g)}(X)$ in the effective action of type-II strings compactified on a Calabi-Yau threefold are related to particular set of ‘topological’ amplitudes. If one performs a topological twist of the worldsheet conformal field theory, the function $F^{(g)}(X)$ becomes the free-energies of the twisted theory on a world-sheet of genus g . The generalized prepotential $F(X, \hat{A})$ is therefore proportional to the all-genus free energy, i.e., to the logarithm of the all-genus partition function Z_{top} of the topological string. As we have seen, the mixed free energy $\mathcal{F}_{\text{mix}}^{\text{hol}}$ is proportional to the imaginary part of $F(X, \hat{A})$. Taking into account conventional normalization factors, (95) can be rewritten in the following, suggestive form:

$$Z_{\text{BH}}^{\text{mix}} \approx |Z_{\text{top}}|^2 . \quad (96)$$

However, general experience with holomorphic quantities in supersymmetric theories suggests that such a relation should not be expected to be exact, but should be modified by a non-holomorphic factor.²⁵ And indeed, work done over the last years on state counting and partition functions in $N = 2$ compactifications, has established that the holomorphic factorisation of the black hole partition function holds to leading order, but is spoiled by subleading corrections. The underlying microscopic picture is that the black hole corresponds, modulo string dualities, to a system of branes and antibranes. To leading order, when interactions can be neglected, this leads to the holomorphic factorisation.

Currently, the detailed microscopic interpretation of the modified conjecture (91), (94) and its relation to the topological string is still an open question. In the following two lectures, we will discuss how the general ideas explained in this lecture can be tested in concrete examples.

Further reading and references

This section is mostly based on [35], where we used the results of [38] to formulate a modified version of the ‘OSV conjecture’ [1]. The relation between Wilsonian and physical couplings in string effective actions was worked out in [39] and is reviewed in [40]. Concrete examples for the failure of physical

²⁵ One example is the mass formula $M^2 = e^{-K} |\mathcal{M}|^2$ for orbifold models, where K is the Kähler potential and \mathcal{M} is the chiral mass which depends holomorphically on the moduli. In this case the presence of the non-holomorphic factor e^{-K} can be inferred from T-duality. Another example, which has been pointed out to me by S. Shatashvili, is the path integral measure for strings. While it shows holomorphic factorisation for critical strings, this is spoiled by a correction factor, namely the exponential of the Liouville action, for the generic, non-critical case.

quantities of supersymmetric theories to show holomorphic factorisation are provided by mass formulae (see e.g. [41]) and by the path integral measure of the non-critical string (see e.g. [42] for a discussion). The topological string can be used to derive the physical couplings of $N = 2$ compactifications [24, 25]. In this case the non-holomorphic corrections are captured by the holomorphic anomaly equations. The relation between these and symplectic covariance in supergravity have been discussed in [48], while the relevance of non-holomorphic corrections for black hole entropy was explained in [38]. The role of non-holomorphic corrections for the microscopic aspects of the OSV conjecture has been addressed in [49]. The ramifications of the OSV conjecture for ‘topological M-theory’, and the role of non-holomorphic corrections in this context have been discussed in [42, 50].

References for tests of the OSV conjecture will be given in Lecture IV.

4 Lecture III Black holes in $N = 4$ supergravity

4.1 $N = 4$ compactifications

The dynamics of string compactifications with $N = 4$ supersymmetry is considerably more restricted than the dynamics of $N = 2$ compactifications. In particular, the classical S- and T-duality symmetries are exact, and there are fewer higher derivative terms. Therefore $N = 4$ compactifications can be used to test conjectures by precision calculations. We consider the simplest example, the compactification of the heterotic string on a six-torus. This is equivalent to the compactification of the type-II string on $K3 \times T^2$, but we will mostly use the heterotic language.

The massless spectrum consists of the $N = 4$ supergravity multiplet (graviton, four gravitini, six graviphotons, four fermions, one complex scalar, which is, in heterotic $N = 4$ compactifications, the dilaton) together with 22 $N = 4$ vector multiplets (one gauge boson, four gaugini, six scalars). Since the gravity multiplet contains six graviphotons, the resulting gauge group is $U(1)^{28}$ (at generic points of the moduli space). The corresponding electric and magnetic charges each live on a copy of the Narain lattice $\Gamma = \Gamma_{22,6}$, which is an even self-dual lattice of signature $(22, 6)$:

$$(p, q) \in \Gamma \oplus \Gamma .$$

Locally, the moduli space is

$$\mathcal{M} \simeq \frac{SL(2, \mathbb{R})}{SO(2)} \otimes \frac{SO(22, 6)}{SO(22) \otimes SO(6)} ,$$

where the first factor is parametrised by the (four-dimensional, heterotic) dilaton S ,

$$S = e^{-2\phi} + ia .$$

The vacuum expectation value of ϕ is related to the four-dimensional heterotic string coupling g_S by $e^{\langle\phi\rangle} = g_S$, and a is the universal axion (the dual of the universal antisymmetric tensor field). The global moduli space is obtained by modding out by the action of the duality group

$$SL(2, \mathbb{Z})_S \otimes SO(22, 6, \mathbb{Z})_T .$$

The T-duality group $SO(22, 6, \mathbb{Z})_T$ is a perturbative symmetry under which the dilaton S is inert, and which acts linearly on the Narain lattice Γ . The S-duality group $SL(2, \mathbb{Z})_S$ is a non-perturbative symmetry, which acts on the dilaton by fractional linear transformations,

$$S \rightarrow \frac{aS + ib}{-icS + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad (97)$$

while it acts linearly on the charge lattice $\Gamma \oplus \Gamma$ by

$$\begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \begin{pmatrix} a \mathbb{I}_{28} & b \mathbb{I}_{28} \\ c \mathbb{I}_{28} & d \mathbb{I}_{28} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \quad (98)$$

Using the Narain scalar product, we can form three quadratic T-duality invariants out of the charges: $p^2, q^2, p \cdot q$. Under S-duality these quantities form a ‘vector’, i.e., they transform in the $\mathbf{3}$ -representation, which is the fundamental representation of $SO(2, 1) \simeq SL(2)$. The scalar product of two such S-duality vectors is an S-duality singlet. One particularly important example is the S- and T-duality invariant combination of charges

$$p^2 q^2 - (p \cdot q)^2,$$

which discriminates between different types of BPS multiplets. Recall that the $N = 4$ algebra has two complex central charges. Short ($\frac{1}{2}$ -BPS) multiplets satisfy

$$M = |Z_1| = |Z_2| \Leftrightarrow p^2 q^2 - (p \cdot q)^2 = 0,$$

whereas intermediate ($\frac{1}{4}$ -BPS) multiplets satisfy

$$M = |Z_1| > |Z_2| \Leftrightarrow p^2 q^2 - (p \cdot q)^2 \neq 0.$$

4.2 $N = 4$ supergravity in the $N = 2$ formalism

In constructing BPS black hole solutions, we can make use of the $N = 2$ formalism. The $N = 4$ gravity multiplet decomposes into the $N = 2$ gravity multiplet, one vector multiplet (which contains the dilaton), and 2 gravitino multiplets (each consisting of a gravitino, two graviphotons, and one fermion). Each $N = 4$ vector multiplet decomposes into an $N = 2$ vector multiplet plus a hypermultiplet. We will truncate out the gravitino and hypermultiplets and work with the resulting $N = 2$ vector multiplets. This means that we ‘loose’

four electric and four magnetic charges, corresponding to the four gauge fields in the gravitino multiplets. But as we will see we can use T-duality to obtain the entropy formula for the full $N = 4$ theory.

At the two-derivative level, the effective action is an $N = 2$ vector multiplet action with prepotential

$$F(Y) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0}, \quad (99)$$

where

$$Y^a \eta_{ab} Y^b = Y^2 Y^3 - (Y^4)^2 - (Y^5)^2 - \dots .$$

The dilaton is given by

$$S = -i \frac{Y^1}{Y^0} .$$

The corresponding scalar manifold is (locally)

$$\overline{M} \simeq \frac{SL(2, \mathbb{R})}{SO(2)} \otimes \frac{SO(22, 2)}{SO(22) \otimes SO(2)},$$

with duality group $SL(2, \mathbb{Z})_S \otimes SO(22, 2, \mathbb{Z})_T$.

The prepotential (99) corresponds to a choice of the symplectic frame where the symplectic vector of the scalars is $(Y^I, F_I(Y))^T$. The magnetic and electric charges corresponding to this frame are denoted (p^I, q_I) . This symplectic frame is called the supergravity frame in the following. Heterotic string perturbation theory distinguishes a different symplectic frame, called the heterotic frame, which is defined by imposing that all gauge coupling go to zero in the limit of weak string coupling $g_S \rightarrow 0$ (equivalent to $S \rightarrow \infty$). In this frame p^1 is an electric charges while q_1 is a magnetic charge. An alternative way of defining the heterotic frame is to impose that the electric charges are those which are carried by heterotic strings, while magnetic and dyonic charges are carried by solitons (wrapped five-branes). The heterotic frame has the particular property that ‘there is no prepotential’ (see also appendix A). The symplectic transformation relating the heterotic frame and the supergravity frame is $p^1 \rightarrow q_1, q_1 \rightarrow -p^0$. If one applies this transformation to $(Y^I, F_I)^T$, then the transformed Y^I are dependent and do not form a coordinate system on M (the complex cone over \overline{M} , while the transformed F_I do not form a gradient.

Since one frame is not adapted to string perturbation theory while the other is inconvenient, one uses a hybrid formalism, where calculations are performed in the supergravity frame but interpreted in the heterotic frame. The vectors of physical electric and magnetic charges are:

$$\begin{aligned} q &= (q_0, p^1, q_a) \in \Gamma, \\ p &= (p_0, -q_1, p^a) \in \Gamma, \end{aligned} \quad (100)$$

where $a, b = 2, \dots$. In this parametrisation, the explicit expressions for the T-duality invariant scalar products are:

$$\begin{aligned}
q^2 &= 2(q_0 p^1 - \frac{1}{4} q_a \eta^{ab} q_b) \\
p^2 &= 2(-p^0 q_1 - p^a \eta_{ab} p^b) \\
p \cdot q &= q_0 p^0 - q_1 p^1 + q_2 p^2 + q_3 p^3 + \dots,
\end{aligned} \tag{101}$$

where

$$\begin{aligned}
p^a \eta_{ab} p^b &= p^2 p^3 - (p^4)^2 - (p^5)^2 - \dots, \\
q_a \eta^{ab} q_b &= 4q_2 q_3 - (q_4)^2 - (q_5)^2 - \dots.
\end{aligned} \tag{102}$$

In the heterotic frame, S-duality acts according to (98), and the three quadratic T-duality invariants transform in the vector representation of $SO(2,1) \simeq SL(2)$, where $SO(2,1)$ is realised as the invariance group of the indefinite bilinear form $a_1 a_2 - a_3^2$. The scalar product of two S-duality vectors is a scalar, and the quartic S- and T-duality invariant of the charges is

$$q^2 p^2 - (p \cdot q)^2 = (q^2, p^2, p \cdot q) \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} q^2 \\ p^2 \\ p \cdot q \end{pmatrix}.$$

For a prepotential of the form (99) the attractor equations can be solved in closed form, and the resulting formula for the entropy is

$$S_{\text{macro}} = \pi \sqrt{p^2 q^2 - (p \cdot q)^2}. \tag{103}$$

This formula is manifestly invariant under $SL(2, \mathbb{Z})_S \otimes SO(22, 2, \mathbb{Z})_T$, and we can reconstruct the eight missing charges by passing to the corresponding invariant of the full duality group $SL(2, \mathbb{Z})_S \otimes SO(22, 6, \mathbb{Z})_T$. This result agrees with the direct derivation of the solution within $N = 4$ supergravity.

When using the prepotential (99) we neglect higher derivative corrections to the effective action. Therefore the solution is only valid if both the string coupling and the curvature are small at the event horizon. This is the case if the charges are uniformly large in the following sense:

$$q^2 p^2 \gg (p \cdot q)^2 \gg 1.$$

Note that if the scalars take values inside the moduli space²⁶ then $q^2 < 0$ and $p^2 < 0$ in our parametrisation.

From the entropy formula (103) it is obvious that there are two different types of BPS black holes in $N = 4$ theories.

- If $p^2 q^2 - (p \cdot q)^2 \neq 0$ the black hole is $\frac{1}{4}$ -BPS and has a finite horizon. These are called large black holes.

²⁶ The moduli space is realised as an open domain in \mathbb{R}^n , which is given by a set of inequalities. In our parametrisation one of these inequalities is $\text{Re}S = e^{-2\phi} > 0$, which implies that the dilaton lives in a half plane (the right half plane). Solutions where $\text{Re}S < 0$ at the horizon are therefore unphysical. Similar remarks apply to the other moduli.

- If $p^2 q^2 - (p \cdot q)^2 = 0$ the black hole is $\frac{1}{2}$ -BPS and has a vanishing horizon. These are called small black holes. They are null singular, which means that the event horizon coincides with the singularity.

Further reading and references

The conventions used in this section are those of [38]. See there for more information and references about the relation between $N = 4$ and $N = 2$ compactifications. The entropy for large black holes in $N = 4$ compactifications was computed in [51, 52] and rederived using the $N = 2$ formalism in [38].

4.3 R^2 -corrections for $N = 4$ black holes

Let us now incorporate higher derivative corrections. Since no treatment within $N = 4$ supergravity is available, it is essential that we can fall back onto the $N = 2$ formalism. One simplifying feature of $N = 4$ compactifications is that all higher coupling functions $F^{(g)}(Y)$ with $g > 1$ vanish. The only higher derivative coupling is $F^{(1)}(Y)$, which, moreover, only depends on the dilaton S . The generalized prepotential takes the following form:

$$F(Y, \mathcal{Y}) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + F^{(1)}(S) \mathcal{Y} .$$

In order to find duality covariant attractor equations and a duality invariant entropy, we must incorporate the non-holomorphic corrections to the Wilsonian coupling $F^{(1)}(Y)$, which are encoded in a homogenous, real valued, non-harmonic function $\Omega(Y, \bar{Y}, \mathcal{Y}, \bar{\mathcal{Y}})$.

One way to find this function is to compute the physical coupling of the curvature-squared term in string theory. Since this coupling depends on the dilaton (but not on the other moduli), it can receive non-perturbative corrections (though no perturbative ones). At this point one has to invoke the duality between the heterotic string on T^6 and the type-IIA string on $K3 \times T^2$. Since the heterotic dilaton corresponds to a geometric type-IIA modulus, the exact result can be found by a perturbative calculation in the IIA theory. This calculation is one-loop, and can be done exactly in α' , because there is no dependence on the K3-moduli.

Alternatively, one can start with the perturbative heterotic coupling and infer the necessary modifications of the attractor equations and of the entropy by imposing S-duality invariance. It turns out that there is a minimal S-duality invariant completion, which in principle could differ from the full result by further subleading S-duality invariant terms. But for the case at hand the minimal S-duality completion turns out to give complete result.

At tree level, the coupling function $F^{(1)}$ is given by

$$F_{\text{tree}}^{(1)}(S) = c_1 i S , \quad \text{where } c_1 = -\frac{1}{64} .$$

We know a priori that there can be instanton corrections $\mathcal{O}(e^{-S})$. The function $F^{(1)}(S)$ determines the ‘ R^2 -couplings’

$$\mathcal{L}_{R^2} \simeq \frac{1}{g^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \Theta C_{\mu\nu\rho\sigma} \tilde{C}^{\mu\nu\rho\sigma} , \quad (104)$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, through $g^{-2} \simeq \text{Im}F^{(1)}$ and $\theta \simeq \text{Re}F^{(1)}$. Therefore $\text{Im}F^{(1)}$ must be an S-duality invariant function, whereas $\text{Re}F^{(1)}$ must only be invariant up to discrete shifts. According to (97), the linear tree-level piece is not invariant. Restrictions on the functional dependence of $F^{(1)}$ on S result from the requirement that the S-duality transformation (97) of the dilaton induces the symplectic transformation (98) of the symplectic vector $(Y^I, F_I)^T$. This implies that

$$f(S) := -i \frac{\partial F^{(1)}}{\partial S}$$

must transform with weight 2:

$$f\left(\frac{aS - ib}{icS + d}\right) = (icS + d)^2 f(S) .$$

A classical result in the theory of modular forms²⁷ implies that $f(S)$, (and, hence, $F^{(1)}$) cannot be holomorphic. The holomorphic object which comes closest to transforming with weight 2 is the holomorphic second Eisenstein series

$$G_2(S) = -4\pi \partial_S \eta(S) ,$$

where $\eta(S)$ is the Dedekind η -function.²⁸ To obtain a function which transforms with weight 2 one needs to add a non-holomorphic term and obtains the non-holomorphic second Eisenstein series:

$$G_2(S, \bar{S}) = G_2(S) - \frac{2\pi}{S + \bar{S}} .$$

This is the only candidate for $f(S)$. We will write $f(S, \bar{S})$ in the following, to emphasize that this function is non-holomorphic. We need to check that we get the correct asymptotics in the weak coupling limit $S \rightarrow \infty$. Since $F^{(1)} \rightarrow c_1 iS$, we know that $f(S, \bar{S})$ must go to a constant. This is indeed true for the second Eisenstein series (the non-holomorphic term is subleading):

$$G_2(S, \bar{S}) \rightarrow \frac{\pi^2}{3} ,$$

and therefore the minimal choice for $f(S, \bar{S})$ is

²⁷ We refer the reader to appendix B for a brief review of modular forms and references.

²⁸ Here $G_2(S)$ is short for $G_2(iS)$, etc.

$$f(S, \bar{S}) = c_1 \frac{3}{\pi^2} G_2(S, \bar{S}) .$$

This can be integrated, and we obtain the non-holomorphic function

$$F^{(1)}(S, \bar{S}) = -ic_1 \frac{6}{\pi} (\log \eta^2(S) + \log(S + \bar{S})) . \quad (105)$$

This function generates a symplectic vector $(Y^I, F_I(Y, \bar{Y}))^T$ with the correct behaviour under S-duality. Moreover, the function $p^I F_I(Y, \bar{Y}) - q_I Y^I$, which is proportional to the area, is S-duality invariant. However $F^{(1)}(S, \bar{S})$ is not S-duality invariant, but transforms as follows:

$$F^{(1)}(S, \bar{S}) \rightarrow F^{(1)}(S, \bar{S}) + ic_1 \frac{6}{\pi} \log(-ic\bar{S} + d) .$$

This was to be expected, because derivatives (and, hence, integrals) of modular forms are not modular forms but transform with additional terms. The function $F^{(1)}(S, \bar{S})$ was constructed by requiring that its derivative is a modular form of weight 2. Therefore it does not quite transform as a modular form of weight zero (modular function). In order to get an S-duality invariant function, we need to add a further non-holomorphic piece:

$$F_{\text{phys}}^{(1)}(S, \bar{S}) = F^{(1)}(S, \bar{S}) + ic_1 \frac{3}{\pi} \log(S + \bar{S}) = F^{(1)}(S)_{\text{hol}} + ic_1 \frac{6}{\pi} \log(S + \bar{S}) ,$$

where

$$F_{\text{hol}}^{(1)}(S) = -ic_1 \frac{6}{\pi} \log \eta^2(S) .$$

The invariant function $F_{\text{phys}}^{(1)}$ is the minimal S-duality completion of the R^2 -coupling (104). An explicit calculation of this coupling in string theory shows that this is in fact the full R^2 -coupling.

Since the entropy must be S-duality invariant, it is also clear that the correct way of generalizing the holomorphic function $F^{(1)}(S)$ in the entropy formula is:²⁹

$$S_{\text{macro}} = \pi \left[(p^I F_I(Y, \bar{Y}) - q_I Y^I) + 4\text{Im} \left(\gamma F_{\text{phys}}^{(1)}(S, \bar{S}) \right) \right]_* .$$

Note that the non-holomorphic modifications are purely imaginary. Therefore they only modify the R^2 -coupling $g^{-2} \simeq \text{Im} F^{(1)}$ and reside in a real-valued, non-harmonic function Ω . In the following we find it convenient to absorb the holomorphic function $\gamma F^{(1)}(S)$ into Ω :

$$\begin{aligned} \Omega(S, \bar{S}, \gamma, \bar{\gamma}) &= \text{Im} \left(\gamma F^{(1)}(S, \bar{S}) + \gamma ic_1 \frac{3}{\pi} \log(S + \bar{S}) \right) \\ &= \text{Im} \left(\gamma F^{(1)}(S) - \gamma ic_1 \frac{3}{\pi} \log(S + \bar{S}) \right) . \end{aligned} \quad (106)$$

²⁹ Remember $\gamma_* = -64$.

This function encodes all higher derivative corrections to the tree-level prepotential.

We already mentioned that the holomorphic R^2 -corrections correspond to instantons. To make this explicit we expand $F_{\text{hol}}^{(1)}(S)$ for large S :

$$F_{\text{hol}}^{(1)}(S) \simeq \log \eta^{24}(S) = -2\pi S - 24e^{-2\pi S} + \mathcal{O}(e^{-4\pi S}).$$

This shows that the R^2 -coupling has a classical piece proportional to S , followed by an infinite series of instanton corrections, which correspond to wrapped five-branes.

Further reading and references

This section is based on [38]. The treatment of the non-holomorphic corrections illustrates the general formalism introduced in [35]. In fact, the formalism is modelled on this example, and it is not excluded that generic $N = 2$ compactifications need more general modifications. The R^2 -term in the effective action for $N = 4$ compactifications was computed in [53].

4.4 The reduced variational principle for $N = 4$ theories

It is possible and in fact instructive to analyse the attractor equations and entropy without using the explicit form of Ω . Using that Ω depends on the dilaton S , but not on the other moduli $T^a \simeq \frac{Y^a}{Y^0}$, one can solve all but two of the attractor equations explicitly. The remaining two ‘dilaton attractor equations’ are the only ones which involve Ω , and they determine the dilaton as a function of the charges. Substituting the solved attractor equations into the entropy function, we obtain the following, reduced entropy function:

$$\Sigma(S, \bar{S}, p, q) = -\frac{q^2 - ip \cdot q(S - \bar{S}) + p^2 |S|^2}{S + \bar{S}} + 4\Omega(S, \bar{S}, \gamma, \bar{\gamma}). \quad (107)$$

Extremisation of this function yields the remaining dilatonic attractor equations

$$\partial_S \Sigma = 0 = \partial_{\bar{S}} \Sigma \quad \Leftrightarrow \quad \text{Dilaton attractor equations,}$$

and its critical value gives the entropy:

$$\begin{aligned} S_{\text{macro}}(p, q) &= \pi \Sigma_*(p, q) \\ &= \left(-\frac{q^2 - ip \cdot q(S - \bar{S}) + p^2 |S|^2}{S + \bar{S}} + 4\Omega(S, \bar{S}, \gamma, \bar{\gamma}) \right)_{|\partial_S \Sigma=0} \end{aligned} \quad (108)$$

The entropy function is manifestly S-duality and T-duality invariant, provided that Ω is an S-duality invariant function.³⁰

³⁰ $\frac{1}{S+\bar{S}}(1, |S|^2, -i(S-\bar{S}))$ transforms as an $SO(2, 1)$ (co-)vector under S-duality, and therefore the contraction with the vector $(q^2, p^2, p \cdot q)$ gives an invariant.

Further reading and references

The observation that all but two of the $N = 4$ attractor equations can be solved, even in presence of R^2 -terms, was already made in [38] and exploited in [54] and [35].

4.5 Small $N = 4$ black holes

Let us now have a second look at small black holes. For convenience we take them to be electric black holes, $p = 0$. By this explicit choice, S-duality is no longer manifest, but T-duality remains manifest. As we saw above, as long as $\Omega = 0$ the area of a $\frac{1}{2}$ -BPS black hole vanishes, $A = 0$, and therefore the Bekenstein-Hawking entropy is zero, too. In fact, the moduli also show singular behaviour, and, in particular, the dilaton runs off to infinity at the horizon $S_* = \infty$. Thus small black holes live on the boundary of moduli space.

The lowest order approximation to the R^2 -coupling is to take its classical part,

$$F^{(1)} \simeq \log \eta^{24}(S) = -2\pi S + \mathcal{O}(e^{-2\pi S}),$$

and to neglect all instanton and non-holomorphic corrections. In this approximation one can solve the dilatonic attractor equations explicitly. This results in the following, non-vanishing and T-duality invariant area:

$$A = 8\pi \sqrt{\frac{1}{2}|q^2|} \neq 0.$$

Thus the R^2 -corrections smooth out the null-singularity and create a finite horizon. We need to impose that $|q^2| \gg 1$ ³¹ in order that the dilaton S is large, which we need to impose because we neglect subleading corrections to the R^2 -coupling. Note that in contrast to the two-derivative approximation the dilaton is now finite at the horizon. Thus not only the metric but also the moduli are smoothed by the higher derivative corrections. The horizon area is small in string units, even though it is large in Planck units. This motivates the terminology ‘small black holes.’

The resulting Bekenstein-Hawking entropy is:

$$S_{\text{Bekenstein-Hawking}} = \frac{A}{4} = 2\pi \sqrt{\frac{1}{2}|q^2|}.$$

However, since the area law does not apply to theories with higher curvature terms, the correct way to compute the macroscopic black hole entropy is (83). Evaluating this for the case at hand gives

$$S_{\text{macro}} = \frac{A}{4} + \text{correction} = \frac{A}{4} + \frac{A}{4} = \frac{A}{2} = 4\pi \sqrt{\frac{1}{2}|q^2|}.$$

³¹ In our parametrisation $q^2 < 0$, if the horizon values of the scalars are inside the moduli space.

In this particular case the correction is as large as the area term itself. Later we will have the opportunity to confront both formulae with string microstate counting.

In the limit of large S the next subleading correction comes from the non-holomorphic corrections $\propto \log(S + \bar{S})$. We can still find an explicit formula for the entropy:

$$S_{\text{macro}} = 4\pi \sqrt{\frac{1}{2}|q^2|} - 6 \log |q^2| ,$$

which we will compare to microstate counting later.

If we include further corrections, ultimately the full series of instanton corrections encoded in $\log \eta^{24}(S)$, we cannot find an explicit formula for the entropy anymore. However, we know that the exact macroscopic entropy is given as the solution of the extremisation problem for the dilatonic entropy function (107). This can be used for a comparison with state counting.

Further reading and references

The observation that R^2 -corrections smooth or ‘cloak’ the null singularity of small black holes was made in [55]. This result follows immediately from [38].

5 Lecture IV: $N = 4$ state counting and black hole partition functions

The BPS spectrum of the heterotic string on T^6 consists of the excited modes of the heterotic string itself, and solitons. Heterotic string states are labeled by 28 quantum numbers: six winding numbers, six discrete momenta around T^6 and 16 charges of the unbroken $U(1)^{16} \subset E_8 \otimes E_8$ gauge group. They combine into 22 left- and 6 right-moving momenta, which take values in the Narain lattice:

$$(p_L; p_R) \in \Gamma .$$

Modular invariance of the world sheet conformal field theory implies that the lattice Γ must be even and selfdual with respect to the bilinear form $p_L^2 - p_R^2$, which has signature $(+)^{22}(-)^6$. From the four-dimensional point of view, the 28 left- and rightmoving momenta are the 28 electric charges with respect to the generic gauge group $U(1)^{16+6+6}$: $q = (p_L; p_R) \in \Gamma$.

Similarly, the winding states of heterotic five-branes carry magnetic charges $p \in \Gamma^* = \Gamma$. If purely electric or purely magnetic states satisfy a BPS bound, they must be $\frac{1}{2}$ -BPS states, because $p^2 q^2 - p \cdot q = 0$ if either $p = 0$ or $q = 0$. However, there are also dyonic solitonic states with $q^2 p^2 - p \cdot q \neq 0$, which are $\frac{1}{4}$ -BPS. By string–black hole complementarity, the BPS states with charges $(p, q) \in \Gamma \oplus \Gamma$ should be the microstates of $N = 4$ black holes with the same charges. We will now discuss how these states are counted and compare our results to the macroscopic black hole entropy and free energy.

5.1 Counting $\frac{1}{2}$ -BPS states

Without loss of generality, we take the $\frac{1}{2}$ -BPS states to be electric, $p = 0$. Such states correspond to excitations of the heterotic string, and are called Dabholkar-Harvey states. Recall that the world-sheet theory of the heterotic string has two different sectors. The left-moving sector consists of 24 world sheet bosons (using the light cone gauge), namely the left-moving projections of the eight coordinates transverse to the world sheet, and 16 bosons with values in the maximal torus of $E_8 \otimes E_8$. The right-moving sector consists of the right-moving projections of the eight transverse coordinates, together with eight right-moving fermions. This sector is supersymmetric in the two-dimensional, world-sheet sense. World-sheet supersymmetry combined with a condition on the spectrum of charges implies the existence of an extended chiral algebra on the world-sheet, which is equivalent to $N = 4$ supersymmetry in the ten-dimensional, space-time sense. The generators of the space-time supersymmetry algebra are built exclusively out of right-moving degrees of freedom. To obtain BPS states one needs to put the right-moving sector into its ground state, but still has the freedom to excite the left-moving sector. A basis of such states is

$$\alpha_{-m_1}^{i_1} \alpha_{-m_2}^{i_2} \cdots |q\rangle \otimes \mathbf{16}, \quad (109)$$

where $\alpha_{-m_i}^{i_k}$ are creation operators for the oscillation modes of the string. The indices $i_k = 1, \dots, 24$ label the directions transverse to the world-sheet of the string, while $m_k = 1, 2, 3, \dots$ label the oscillation modes. $q = (p_L; p_R) = \Gamma$ are the electric charges, which correspond to the winding modes, momentum modes and $U(1)^{16}$ charges. $\mathbf{16}$ denotes the ground state of the right-moving sector, which carries the degrees of freedom of an $N = 4$ vector multiplet (with 16 on-shell degrees of freedom). States of this form are invariant under as many supercharges as the right-moving groundstate, and therefore they belong to $\frac{1}{2}$ -BPS multiplets. To be physical, the state must satisfy the level matching condition,

$$N - 1 + \frac{1}{2}p_L^2 = \tilde{N} + \frac{1}{2}p_R^2, \quad (110)$$

where N, \tilde{N} are the total left-moving and right-moving excitation numbers. BPS states have $\tilde{N} = 0$, and therefore the excitation level is fixed by the charges:

$$N - 1 + \frac{1}{2}p_L^2 = \tilde{N} + \frac{1}{2}p_R^2 \Rightarrow N = \frac{1}{2}p_R^2 - \frac{1}{2}p_L^2 - 1 = -q^2 - 1 = |q^2| - 1. \quad (111)$$

This is equivalent to the statement that the mass saturates the BPS bound. Note that $q^2 < 0$ for physical BPS states. For large charges we can use $N \approx |q^2|$.

The problem of counting $\frac{1}{2}$ -BPS states amounts to counting in how many ways a given total excitation number $N \approx |q^2|$ can be distributed among the creation operators α_{-m}^i . If we ignore the additional space-time index

$i = 1, \dots, 24$, this becomes the classical problem of counting the partitions of an integer N , which was studied by Hardy and Ramanujan. The space-time index i adds an additional 24-fold degeneracy, and we might say that we have to count partitions of N into integers with 24 different ‘colours’. Incidentally exactly the same problems arises (up to the overall factor 16 from the degeneracy of the right-moving ground state) when counting the physical states of the open bosonic string. From the world-sheet perspective, both problems amount to finding the partition function of 24 free bosons, which is a standard problem in quantum statistics and conformal field theory.

The reader is encouraged to solve Problem 3, which is to derive the following formula for the state degeneracy:

$$d(q) = d(q^2) = 16 \oint d\tau \frac{e^{i\pi\tau q^2}}{\eta^{24}(\tau)}, \quad (112)$$

where $\tau = \tau_1 + i\tau_2 \in \mathcal{H}$, where $\mathcal{H} = \{\tau \in \mathbb{C} | \tau_2 > 0\}$ is the upper half plane and where $\eta(\tau)$ is the Dedekind η -function. The integration contour runs through a strip of width one in the upper half plane, i.e., it connects two points $\tau^{(1)}$ and $\tau^{(2)} = \tau^{(1)} + 1$. Since the integrand is periodic under $\tau \rightarrow \tau + 1$ (which is a general property of modular forms), this integration contour is effectively closed. (It becomes a closed contour when going to the new variable $e^{2\pi i\tau}$, which takes values in the interior of the unit disc.)

In its present form this expression is not very useful, because we want to know $d(q)$ explicitly, at least asymptotically for large values of $|q^2|$. This type of problem was studied already by Hardy and Ramanujan, and a method for solving it exactly was found by Rademacher. For our specific problem with 24 ‘colours’ the Rademacher expansion takes the following form:

$$d(q^2) = 16 \sum_{c=1}^{\infty} c^{-14} \text{Kl} \left(\frac{1}{2} |q^2|, -1; c \right) \hat{I}_{13} \left(\frac{4\pi}{c} \sqrt{\frac{1}{2} |q^2|} \right), \quad (113)$$

where \hat{I}_{13} is the modified Bessel function of index 13, and $\text{Kl}(l, m; c)$ are the so-called Kloosterman sums.

Modified Bessel functions have the following integral representation:

$$\hat{I}_{\nu}(z) = -i(2\pi)^{\nu} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{dt}{t^{\nu+1}} e^{t + \frac{z^2}{4t}},$$

and their asymptotics for $\text{Re}(z) \rightarrow \infty$ is:

$$\hat{I}_{\nu}(z) \approx \frac{e^z}{\sqrt{2}} \left(\frac{z}{4\pi} \right)^{-\nu - \frac{1}{2}} \left(1 - \frac{2\nu^2 - 1}{8z} + \mathcal{O}(z^{-2}) \right).$$

We will not need the values of the Kloostermann sums, except that $\text{Kl}(l, m; 1) = 1$.

In the limit of large $|q^2|$ the term with $c = 1$ is leading, while the terms with $c > 1$ are exponentially suppressed:

$$d(q^2) = 16 \hat{I}_{13}(4\pi\sqrt{\frac{1}{2}|q^2|}) + \mathcal{O}(e^{-|q^2|}).$$

Using the asymptotics of Bessel functions, this can be expanded in inverse powers of $|q^2|$:

$$S_{\text{micro}}(q^2) = \log d(q^2) \approx 4\pi\sqrt{\frac{1}{2}|q^2|} - \frac{27}{4} \log |q^2| + \frac{15}{2} \log(2) - \frac{675}{32\pi|q^2|} + \dots$$

The first two terms correspond to a saddle point evaluation of the integral representation (112): The first term is the value of integrand at its saddle point, while the second term is the ‘fluctuation determinant’. The derivation of the first two terms using a saddle point approximation of (112) is left to the reader as Problem 4. A derivation of the full Rademacher expansion (113) can be found in the literature.

Further reading and references

An excellent and accessible account on the Rademacher expansion can be found in [56]. See in particular the appendix of this paper for two versions of the proof of the Rademacher expansion. We have also borrowed some formulae from [57, 58], who have studied the state counting for $\frac{1}{2}$ -BPS states in great detail, including various $N = 4$ and $N = 2$ orbifolds of the toroidal $N = 4$ compactification considered in this lecture.

5.2 State counting for $\frac{1}{4}$ -BPS states

For the problem of counting $\frac{1}{4}$ -BPS states the dual type-II picture of the $N = 4$ compactification is useful. Here all the heterotic $\frac{1}{2}$ - and $\frac{1}{4}$ -BPS states arise as winding states of the NS-five-brane. It is believed that the dynamics of an NS-five-brane is described by a string field theory whose target space is the world volume of the five-brane. If one assumes that the counting of BPS states is not modified by interactions, the problem of state counting reduces to counting states in a multi-string Fock space. For $\frac{1}{2}$ -BPS states the resulting counting problem is found to be equivalent to the one described in the last section, as required by consistency. For $\frac{1}{4}$ -BPS states the counting problem is more complicated, but one can derive the following integral representation:

$$d(p, q) = \oint d\rho d\sigma dv \frac{e^{i\pi[\rho p^2 + \sigma q^2 + (2v-1)pq]}}{\Phi_{10}(\rho, \sigma, v)}. \quad (114)$$

This formula requires some explanation. Essentially it is a generalisation of (112), where the single complex variable τ has been replaced by three complex

variables ρ, σ, τ , which live in the so called rank-2 Siegel upper half space \mathcal{S}_2 . In general the rank- n Siegel upper half space is the space of all symmetric $(n \times n)$ -matrices with positive definite imaginary part. This is a symmetric space,

$$\mathcal{S}_n \simeq \frac{Sp(2n)}{U(n)},$$

which can be viewed as a generalisation of the upper half plane

$$\mathcal{H} = \frac{Sp(2)}{U(1)} = \mathcal{S}_1.$$

The group $Sp(2n, \mathbb{Z})$ acts by fractional linear transformations on the $(n \times n)$ matrices $\Omega \in \mathcal{S}_n$,

$$\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1}, \quad \text{where} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n).$$

The discrete subgroup $Sp(2n, \mathbb{Z})$ is a generalisation of the modular group $Sp(2, \mathbb{Z}) \simeq SL(2, \mathbb{Z})$, and there is a corresponding theory of Siegel modular forms. A Siegel modular form is said to have weight $2k$, if it transforms as

$$\Phi(\Omega) \rightarrow \Phi((A\Omega + B)(C\Omega + D)^{-1}) = (\det(C\Omega + D))^k \Phi(\Omega).$$

In the rank-2 case, we can parametrise the matrix Ω as

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix},$$

and positive definiteness of the imaginary part implies that

$$\rho_2 > 0, \quad \sigma_2 > 0, \quad \rho_2 \sigma_2 - v^2 > 0,$$

where $\rho = \rho_1 + i\rho_2$, etc.

In the theory of rank-2 Siegel modular forms, the analogon of the weight-12 cusp form $\eta^{24}(\tau)$ is the weight-10 Siegel cusp form $\Phi_{10}(\rho, \sigma, v)$, which enters into the state counting formula (114). Like modular forms, Siegel modular forms are periodic in the real parts of the variables ρ, σ, v . The integration contour in the Siegel half space is along a path of the form $\rho \rightarrow \rho + 1, \sigma \rightarrow \sigma + 1, v \rightarrow v + 1$, which is effectively a closed contour since the integrand is periodic.³² The state counting formula (114) is manifestly T-duality invariant. It is also formally S-duality invariant, in the sense that S-duality transformations can be compensated by $Sp(4, \mathbb{Z})$ transformations of the integration variables.

As in the $\frac{1}{2}$ -BPS case one would like to evaluate (114) asymptotically, in the limit of large charges $q^2 p^2 - (p \cdot q)^2 \gg 1$. One important difference between Φ_{10} and η^{24} is that the Siegel cusp form has zeros in the interior of the Siegel

³² In the numerator one has to use that the Narain lattice is even selfdual.

half space \mathcal{S}_2 , namely at $v = 0$ and its images under $Sp(4, \mathbb{Z})$. The v -integral therefore evaluates the residues of the integrand. At $v = 0$, the asymptotics of Φ_{10} is

$$\Phi \simeq_{v=0} v^2 \eta^{24}(\rho) \eta^{24}(\sigma) .$$

The asymptotics at the other zeros can be found by applying $Sp(4, \mathbb{Z})$ transformations.

If one sets the magnetic charges to zero, the residue at $v = 0$ is the only one which contributes to (114). This can be used to derive the $\frac{1}{2}$ -BPS formula (112) as a special case of (114).³³

For $\frac{1}{4}$ -BPS states it can be shown that for large charges the dominant contribution comes from the residue at

$$D = v + \rho\sigma - v^2 = 0 ,$$

while all other residues are exponentially suppressed. Neglecting the subleading residues, one can perform the v -integral. The remaining integral has the following structure:

$$d(p, q) = \oint dp d\sigma e^{i\pi(X_0+X_1)(\rho, \sigma)} \Delta(\rho, \sigma) . \quad (115)$$

The parametrisation has been chosen such that X_1 and Δ are subleading for large charges.

This integral can be evaluated in a saddle point approximation, analogous to (112). The leading term for large charges is given by the approximate saddle point value of the integrand,

$$d(p, q) \approx e^{i\pi X_0|_*} = e^{\pi \sqrt{p^2 q^2 - (pq)^2}} . \quad (116)$$

This result is manifestly T- and S-duality invariant.

A refined approximation can be obtained as follows: one identifies the exact critical point of $e^{i\pi X} = e^{i\pi(X_0+X_1)}$, expands the integrand $e^{i\pi X} \Delta$ to second order and performs a Gaussian integral. This is different from a standard saddle point approximation, where one would expand around the critical point of the full integrand $e^{i\pi X} \Delta$. This modification is motivated by the observation that the critical point of $i\pi X$ agrees exactly with the the critical point of the reduced dilatonic entropy function (107), which gives the exact macroscopic entropy:

$$i\pi X_* = \pi \Sigma_* = S_{\text{macro}}(p, q) .$$

At the critical point one has the following relation between the critical values of ρ, σ and the fixed point value of the dilaton:

$$\rho_* = \frac{i|S_*|^2}{S_* + \bar{S}_*} , \quad \sigma_* = \frac{i}{S_* + \bar{S}_*} .$$

³³ Incidentally, the problem is equivalent to the factorisation of a genus-2 string partition function into two genus-1 string partition functions.

One might think that the subleading contributions from Δ spoil the resulting equality between microscopic and macroscopic entropy. However, these cancel against the contributions from the Gaussian integration (the ‘fluctuation determinant’), at least to leading order in an expansion in inverse powers of the charges:

$$e^{S_{\text{micro}}(p,q)} = d(p,q) \approx e^{\pi\Sigma_*+\dots} = e^{S_{\text{macro}}(p,q)+\dots} . \quad (117)$$

This shows that the modified saddle point approximation is compatible with a systematic expansion in large charges. Moreover, there is an intriguing direct relation between the saddle point approximation of the exact microscopic state degeneracy (114) and the black hole variational principle.

Further reading and references

The state counting formula for $\frac{1}{4}$ -BPS states in $N = 4$ compactifications was proposed in [59]. There several ways of deriving it were discussed, which provide very strong evidence for the formula. Further evidence was obtained more recently in [60], by using the relation between four-dimensional and five-dimensional black holes [61]. While the leading order matching between state counting and black hole entropy was already observed in [59], the subleading corrections were obtained in [54] by using the modified saddle point evaluation explained above.

Another line of development is the generalisation from toroidal compactifications to a class of $N = 4$ orbifolds, the so-called CHL-models [62, 63]. The issue of choosing integration contours is actually more subtle than apparent from our review, see [64, 65] for a detailed account. For a comprehensive account of Siegel modular forms, see [86].

5.3 Partition functions for large black holes

The strength of this result becomes even more obvious when we use it to compare the (microscopically defined) black hole partition function to the (macroscopically defined) free energy.

One way of doing this is to evaluate mixed partition function $Z_{\text{mix}}(p, \phi) = \sum_q d(p, q) e^{\pi q_I \phi^I}$ using integral representation (114) of $d(p, q)$. The result can be brought to the following form

$$Z_{\text{mix}}(p, \phi) = \sum_{\text{shifts}} \sqrt{\tilde{\Delta}(p, \phi)} e^{\pi \mathcal{F}_{\text{mix}}(p, \phi)} . \quad (118)$$

\mathcal{F}_{mix} is the black hole free energy, including all, both the holomorphic and the non-holomorphic corrections:

$$\mathcal{F}_{\text{mix}}(p, \phi) = \frac{1}{2}(S + \bar{S}) (p^a \eta_{ab} p^b - \phi^a \eta_{ab} \phi^b) - i(S - \bar{S}) p^a \eta_{ab} \phi^b + 4\Omega(S, \bar{S}, \Upsilon, \bar{\Upsilon}) .$$

By imposing the magnetic attractor equations in the transition to the mixed ensemble, the dilaton has become a function of the electric potentials and the magnetic charges:

$$S = \frac{-i\phi^1 + p^1}{\phi^0 + ip^0}.$$

The mixed partition functions is by construction invariant under shifts $\phi \rightarrow \phi + 2i$. The mixed free energy is found to have a different periodicity, and this manifests itself by the appearance of a finite sum over additional shifts of ϕ in (118). As predicted on the basis of symplectic covariance, the relation between the partition function and the free energy is modified by a ‘measure factor’ $\tilde{\Delta}^-$, which we do not need to display explicitly. This factor agrees with the measure factor Δ^- in (94), which we found by imposing symplectic covariance in the limit of large charges:

$$\tilde{\Delta}^- \approx \Delta^-.$$

Since we already made a partial saddle point approximation when going from the canonical to the mixed ensemble, we could not expect an exact agreement. It is highly non-trivial that we can match the full mixed free energy, including the infinite series of instanton corrections. Moreover, we have established that there is a non-trivial measure factor, which agrees to leading order with the one constructed by symmetry considerations.

Further reading, references, and some comments

The idea to evaluate the mixed partition function using microscopic state counting in order to check the OSV conjecture for $N = 4$ compactifications was first used in [66]. This confirmed the expectation that the OSV conjecture needs to be modified by a measure factor once subleading corrections are taken into account. This result was generalized in [35], where we showed that the measure factor agrees asymptotically with our conjecture which is based on imposing symplectic covariance. Above, we pointed out that in (118) we obtain the full mixed \mathcal{F}_{mix} , including the non-holomorphic corrections. Of course, this way of organising the result is motivated by our approach to non-holomorphic corrections, and it is consistent to regard these contributions as part of the measure factor, as other authors appear to do. Further work is needed, in particular on the role played by the non-holomorphic corrections in the microscopic description, before we can decide which way interpreting the partition function is more adequate. Let us also mention that while we specifically considered toroidal $N = 4$ compactifications in this section, all results generalise to CHL models.

There has also been much activity on $N = 2$ compactifications over the last years. Much of this work has focussed on establishing and explaining the asymptotic factorisation

$$Z_{\text{mix}} \simeq |Z_{\text{top}}|^2$$

predicted by the OSV conjecture [67, 68, 69, 70]. The strategy pursued in these papers is to use string-dualities, in particular the AdS_3/CFT_2 -correspondence, to reformulate the problem in terms of two-dimensional conformal field theory. In comparison to the simpler $N = 4$ models, the relevant microscopic partition functions are related to the so-called elliptic genus of the underlying CFT. Roughly, the elliptic genus is a ‘BPS partition function’, i.e. a partition function which has been modified by operator insertions such that it only counts BPS states. The main problem is to find a suitable generalisation of the Rademacher expansion which allows to evaluate these BPS partition functions asymptotically for large charges. The picture emerging from this treatment is that the black hole can be described microscopically (modulo string dualities) as a non-interacting state of branes and anti-branes. This explains the asymptotic factorisation.

But as we have stressed throughout, non-holomorphic corrections are expected to manifest themselves at the subleading level, which microscopically correspond to interactions between branes and antibranes. And indeed, a more recent refined analysis [71] has revealed the presence of a measure factor, which agrees with the one found in [66] and [35] in the limit of large charges.

There is one further point which we need to comment on. During this lecture we have tentatively assumed that ‘state counting’ means literally to count all the BPS states. But, as we have mentioned previously, the BPS spectrum changes when crossing a line of marginal stability. This is a possible cause for discrepancies between state counting and thermodynamical entropy, because they are computed in different regions of the parameter space. In their original work [1] therefore conjectured that the microscopical entropy entering into the OSV conjecture is an ‘index’, i.e. a weighted sum over states which remains invariant when crossing lines of marginal stability. The detailed study of [57, 58] showed that it is very hard in practice to discriminate between absolute versus weighted state counting. While one example appeared to support absolute state counting, it was pointed out later that there are several candidates for the weighted counting [71]. One intriguing proposal is that the correct absolute state counting is in fact captured by an index, once it is taken into account that states which are stable in the free limit become unstable once interactions are taken into account [71].

In conclusion, the OSV conjecture appears to work well in the semi-classical approximation, if supplemented by a measure factor. The concrete proposal discussed in these lectures works correctly in this limit. It is less clear what is the status of the original, more ambitious goal of finding an exact relation [1], which would have various ramifications, such as helping to find a non-perturbative definition of the topological string [1], formulating a mini-superspace approximation of stringy quantum cosmology [72], studying $N = 1$ compactifications via ‘topological M-theory’ [42, 50], and approaching the vacuum selection problem of string theory by invoking an ‘entropic principle’ [73, 74, 75].

5.4 Partition functions for small black holes

The counting of $\frac{1}{2}$ -BPS states gave rise to the following microscopic entropy:

$$S_{\text{micro}} \approx \log \hat{I}_{13} \left(4\pi \sqrt{\frac{1}{2}|q^2|} \right) \approx 4\pi \sqrt{\frac{1}{2}|q^2|} - \frac{27}{4} \log |q^2| + \dots \quad (119)$$

This is to be compared with the macroscopic entropy. Including the classical part of the R^2 -coupling and the non-holomorphic corrections, but neglecting instantons, this is:

$$S_{\text{macro}} = 4\pi \sqrt{\frac{1}{2}|q^2|} - 6 \log |q^2| + \dots \quad (120)$$

While the leading terms agree, the first subleading term comes with a slightly different coefficient. However, we have seen that both entropies belong to different ensembles, so that we can only expect that they agree in the thermodynamical limit. Since we have a conjecture about the exact (or at least asymptotically exact) relation between both entropies, we can check whether the shift in the coefficient of the subleading term is predicted correctly. Our conjecture about the relation between the canonical free energy and the canonical partition function predicts the following relation (see section 3.6):

$$S_{\text{micro}} = S_{\text{macro}} + \log \sqrt{\frac{\Delta^-}{\Delta^+}} .$$

This shows that both entropy are indeed different if the measure factor Δ^- and the fluctuation determinant Δ^+ are different. For dyonic black holes we found that both were equal, up to subleading contribution. Unfortunately our relation is not useful for small black holes, because

$$\begin{aligned} \Delta^- &= 0, \quad \text{up to non-holomorphic terms and instantons,} \\ \Delta^+ &= 0, \quad \text{up to instantons.} \end{aligned}$$

Since the measure factor and the fluctuation determinant are degenerate (up to subleading contributions) the saddle point approximation is not well defined. This reflect that small black hole live on the boundary of moduli space.

We can still test our conjecture about the relation between the mixed partation and the mixed free energy, in particular the presence of a measure factor and the role of non-holomorphic contributions. This requires to evaluate

$$\exp(S_{\text{micro}}) = d(p^1, q) \approx \int d\phi \sqrt{\Delta^-(p^1, \phi)} e^{\pi[\mathcal{F}_{\text{mix}}(p^1, \phi) - q_I \phi^I]} ,$$

where a non-vanishing Δ^- is obtained by including the non-holomorphic corrections.³⁴ We still neglect the contributions of the instantons.

³⁴ Remember that p^1 is an electric charge for the heterotic string. We take $q_1 = 0$, because this is a magnetic charge.

The integral can be evaluated, with the result:

$$d(p^1, q) \approx \int \frac{dS d\bar{S}}{(S + \bar{S})^{14}} \sqrt{S + \bar{S} - \frac{12}{2\pi}} \exp \left[-\frac{\pi q^2}{S + \bar{S}} + 2\pi(S + \bar{S}) \right]. \quad (121)$$

Here the integrals over $\phi^a = \phi^2, \phi^3, \dots, \phi^{27}$ have been performed and the remaining integrals over ϕ^0 and ϕ^1 have been expressed in terms of the dilaton S . If we approximate

$$\sqrt{S + \bar{S} - \frac{12}{2\pi}} \approx \sqrt{S + \bar{S}},$$

this becomes the integral representation of a modified Bessel function.

Then our conjecture predicts

$$S_{\text{micro}}^{(\text{predicted})} \approx \log \hat{I}_{13-\frac{1}{2}}(4\pi \sqrt{\frac{1}{2}|q^2|}) \approx 4\pi \sqrt{\frac{1}{2}|q^2|} - \frac{13}{2} \log |q^2| + \dots, \quad (122)$$

while the entropy obtained from state counting is:

$$S_{\text{micro}} \approx \log \hat{I}_{13}(4\pi \sqrt{\frac{1}{2}|q^2|}) \approx 4\pi \sqrt{\frac{1}{2}|q^2|} - \frac{27}{4} \log |q^2| + \dots \quad (123)$$

Thus there is a systematic mismatch in the index of the Bessel function, and while the leading terms agree, the coefficients of the log-terms and all the following inverse-power terms mismatch.

This result can be compared with the original OSV-conjecture, where one does not have a measure factor, and where only holomorphic contributions to the free energy are taken into account:

$$\begin{aligned} d(p^1, q) &\approx \int d\phi e^{\pi[\mathcal{F}_{OSV}(p^1, \phi) - q_I \phi^I]} \approx (p^1)^2 \hat{I}_{15}(4\pi \sqrt{\frac{1}{2}|q^2|}) \\ S_{\text{micro}}^{(\text{predicted})} &= 4\pi \sqrt{\frac{1}{2}|q^2|} - \frac{31}{4} \log |q^2| + \log(p^1)^2 + \dots \end{aligned} \quad (124)$$

In this case the index of the Bessel function deviates even more, and in addition there is an explicit factor $(p^1)^2$ which spoils T-duality. This clearly shows that the OSV conjecture needs to be modified by a measure factor.

When deriving (124), we have integrated over 28 potentials ϕ^I , as we have done in our discussion of large black holes, and in (121). There is one subtlety to be discussed here. The full $N = 4$ theory has 28 gauge fields, but we have used the $N = 2$ formalism. Since we disregard the gravitini multiplets (and the hypermultiplets), we work with a truncation to a subsector consisting of the $N = 2$ gravity multiplet and 23 vector multiplets. This theory only has 24 gauge fields, and therefore it only has 24 electrostatic potentials ϕ^I . However, at the end we should reconstruct the missing 4 gauge potentials, and as we have seen when recovering the $N = 4$ entropy formula using the

$N = 2$ formalism, this extension is uniquely determined by T-duality. As we have seen this prescription works for large black holes, but for small black holes we do not quite obtain the right index for the Bessel function.

However, the correct index for the Bessel function is obtained when using the unmodified OSV conjecture, but integrating only over 24 instead of 28 electrostatic potentials:

$$d(p^1, q) \approx \int d\phi e^{\pi[\mathcal{F}_{OSV}(p^1, \phi) - q_I \phi^I]} \approx (p^1)^2 \hat{I}_{13}(4\pi \sqrt{\frac{1}{2}|q^2|}), \quad (125)$$

$$S_{\text{micro}}^{(\text{predicted})} = 4\pi \sqrt{\frac{1}{2}|q^2|} - \frac{27}{4} \log |q^2| + \log(p^1)^2 + \dots \quad (126)$$

Note that this does not cure the problem with the prefactor $(p^1)^2$, which is incompatible with T-duality. It is intriguing, but at the same time puzzling that the correct value for the index is obtained by reducing the number of integrations. However, it is not clear how to interpret this restriction. Moreover, it is unavoidable to include a measure factor to implement T-duality, and this is very likely to have an effect of the index.

Further reading and references

In this section, we followed [35], and compared the result with the calculation based on the original OSV conjecture [57, 58]. Both approaches find agreement for the leading term, but disagreement for the subleading terms. Moreover, when sticking to the original OSV conjecture, the result is not compatible with T-duality. Further problems and subtleties with the OSV conjecture for $\frac{1}{2}$ -BPS black holes have been discussed in detail in [57, 58]. One obvious explanation for these difficulties is that in the ‘would-be leading’ order approximation small black holes are singular: they have a vanishing horizon area and the moduli take values at the boundary of the moduli space. While the higher curvature smooth the null singularity, leading to agreement between macroscopic and microscopic entropy to leading order in the charges, the semi-classical expansion is still ill defined, since one attempts to expand around a singular configuration. Apparently one needs to find a different way of organising the expansion, if some version of the OSV conjecture is to hold at the semi-classical level. A more drastic alternative is that the OSV conjecture simply does not apply to small black holes. But since the mismatch of the subleading corrections appears to follow some systematics, there is room for hope. The situation is less encouraging for the non-perturbative corrections coming from instantons. As observed both in [57, 58] and in [35] the analytical structure of the terms observed in microscopic state counting is different from the one expected on the basis of the OSV conjecture.

5.5 Problems

Problem 3 Counting states of the open bosonic string.

In the light cone gauge, a basis for the Hilbert space of the open bosonic string (neglecting the center of mass momentum) is given by

$$\alpha_{-m_1}^{i_1} \alpha_{-m_2}^{i_2} \cdots |0\rangle, \quad (127)$$

where $i_k = 1, \dots, 24$ and $m_k = 1, 2, 3, \dots$. States with the same (total) excitation number $n = m_1 + m_2 + \dots$ have the same mass. Incidentally, the problem of counting states of the open bosonic string with given mass, is the same as counting the number of $\frac{1}{2}$ -BPS states for the heterotic string, compactified on T^6 , with given charges $q \in \Gamma_{\text{Narain}}$.

The number of states with given excitation number n is encoded in the partition function

$$Z(q) = \text{Tr } q^N, \quad (128)$$

where the trace is over the Hilbert space of physical string states (light cone gauge), $q \in \mathbb{C}$, and N is the number operator with eigenvalues $n = 0, 1, 2, 3, \dots$. Evaluation of the trace gives

$$Z(q) = \left(\prod_{l=1}^{\infty} (1 - q^l) \right)^{-c}, \quad |q| < 1, \quad (129)$$

where $c = D - 2 = 24$ is the number of space-time dimensions transverse to the string world sheet (the physical excitations). The number d_n of string states at level n is encoded in the Taylor expansion

$$Z(q) = \sum_{n=0}^{\infty} d_n q^n. \quad (130)$$

Verify that d_n counts string states, for small $n = 0, 1, 2, \dots$. Do this either for the critical open bosonic string, $c = 24$, or for just one string coordinate, $c = 1$. The latter is the classical problem of counting partitions of an integer. It is instructive to evaluate d_n both directly, by reorganising the product representation (129) into a Taylor series, and by the integral representation of d_n obtained by inverting (130).

Hints: Note that

$$Z(q) = q \Delta^{-1}(q), \quad (131)$$

where $\Delta(q) = \eta^{24}(q)$ is the cusp form (η is the Dedekind eta-function). $\Delta(q)$ is a modular form of weight 12 and has the following expansion around the cusp $q = 0$:

$$\Delta(q) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \mathcal{O}(q^6). \quad (132)$$

$\Delta(q)$ has no zeros for $0 < |q| < 1$.

d_n can be computed by a contour integral in the unit disc $|q| < 1$.

Problem 4 The asymptotic state density of the open bosonic string.

Given the information provided in Problem 3, compute the asymptotic number of open bosonic string states d_n for $n \rightarrow \infty$. (You may restrict yourself to the case $c = 24$, which corresponds to the critical dimension.)

Instructions:

1. The unit disc $|q| < 1$ can be mapped to the semi-infinite strip $-\frac{1}{2} < \tau_1 < \frac{1}{2}$, $\tau_2 > 0$ in the complex τ -plane, $\tau = \tau_1 + i\tau_2$ by

$$q = e^{2\pi i\tau} . \quad (133)$$

(Like other modular forms, Δ extends to a holomorphic function on the whole upper half plane by periodicity in τ_1 .)

Rewrite the contour integral for d_n as a contour integral over τ .

2. Use the modular properties of $\Delta(\tau)$ to find the behaviour of the integrand close to $\tau = 0$ from the known behaviour of $\Delta(\tau)$ at $\tau = i\infty \Leftrightarrow q = 0$. Show that for $n \rightarrow \infty$ the integrand has a sharp saddle point. Use this to evaluate the contour integral in saddle point approximation. (Expand the integrand to second order around the saddle point and perform the resulting Gaussian integral.)
3. The correct result is

$$d_n \approx \text{Const.} \cdot e^{4\pi\sqrt{n}} n^{-\frac{27}{4}} . \quad (134)$$

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A Kähler manifolds and special Kähler manifolds

In this appendix we review Kähler manifolds and special Kähler manifolds from the mathematical perspective. The first part is devoted to the basic

definitions and properties of complex, hermitean and Kähler manifolds. For a more extensive review we recommend the book by Nakahara [76], and, for readers with a stronger mathematical inclination, the concise lecture notes by Ballmann [77]. The characterisation of complex and Kähler manifolds in terms of holonomy groups can be found in [78]. The second part reviews special Kähler manifolds and is mostly based on [22, 23] with supplements from [79, 80, 81]. A review of special geometry from a modern perspective can also be found in [82].

A para-complex variant of special geometry, which applies to the target manifolds of Euclidean $N = 2$ theories has been developed in [79, 80]. The framework of ϵ -Kähler manifolds, which will be employed in [81], is particularly suitable for treating Euclidean supersymmetry and standard (Lorentzian) supersymmetry in parallel.

A.1 Complex and almost complex manifolds

Let M be a differentiable manifold of dimension $2n$.

Definition 1 An *almost complex structure* I on M is tensor field of type $(1, 1)$ with the property that (pointwise)

$$I^2 = -\text{Id} .$$

In components, using real coordinates $\{x^m | m = 1, \dots, 2n\}$, this condition reads:

$$I_p^m I_n^p = -\delta_n^m . \quad (135)$$

Definition 2 An almost complex structure is called *integrable* if the associated Nijenhuis tensor N_I vanishes for all vector fields X, Y on M :

$$N_I(X, Y) := [IX, IY] + [X, Y] - I[X, IY] - I[IX, Y] = 0 .$$

The expression for N_I in terms of local coordinates $\{x^m\}$ can be found by substituting the coordinate expressions $X = X^m \partial_m$, $Y = Y^m \partial_m$ for the vector fields.³⁵ We will not need this explicitly.

Remark: The integrability of an almost complex structure is equivalent to the existence of local complex coordinates $\{z^i | i = 1, \dots, n\}$. An integrable almost complex structure is therefore also simply called a *complex structure*.

Definition 3 A manifold which is equipped with an (almost) complex structure is called an (almost) complex manifold.

Remark: The existence of an (almost) complex structure can be rephrased in terms of holonomy. An almost complex structure is a $GL(n, \mathbb{C})$ structure, and a complex structure is a torsion-free $GL(n, \mathbb{C})$ structure.

³⁵ In fact, it is sufficient to substitute a basis of coordinate vector fields $\{\partial_m\}$ to obtain the components $N_{mn} = N(\partial_m, \partial_n)$.

A.2 Hermitean manifolds

Let (M, I) be a complex manifold and let g be a (pseudo-)Riemannian metric on M .

Definition 4 (M, g, I) is called a *hermitean* manifold, if I generates isometries of g :

$$I^*g = g . \quad (136)$$

Remark: Condition (136) is equivalent to saying that

$$g(IX, IY) = g(X, Y) ,$$

for all vector fields X, Y on M . In local coordinates the condition reads:

$$g_{pq}I_m^p I_n^q = g_{mn} . \quad (137)$$

Remark: If the metric is indefinite, (M, g, I) is called pseudo-hermitean, but we will usually drop the prefix ‘pseudo-’.

On a hermitean manifold one can define the so-called *fundamental two-form*:

$$\omega(X, Y) := g(IX, Y) ,$$

or, in coordinates,

$$\omega_{mn} = -g_{mp}I_n^p . \quad (138)$$

Note that $\omega_{mn} = -\omega_{nm}$, because g_{mn} is symmetric, while I satisfies (135) and (137). Moreover the two-form ω is non-degenerate, because g is.

Equation (138) can be solved for the metric g or for the complex structure I :

$$\begin{aligned} g_{mn} &= \omega_{mk}I_n^k , \\ I_n^m &= -g^{mk}\omega_{kn} . \end{aligned} \quad (139)$$

Thus, if any two of the three data g (metric), I (complex structure) or ω (fundamental two-form) are given on a hermitean manifold, the third is already determined.

When we use complex coordinates $\{z^i\}$, the complex structure only has ‘pure’ components:

$$I_j^i = i\delta_j^i , \quad I_{\bar{j}}^{\bar{i}} = -i\delta_{\bar{j}}^{\bar{i}} .$$

For a hermitean metric the pure components vanish, $g_{ij} = 0$ and $g_{\bar{i}\bar{j}} = 0$. Only the ‘mixed’ components $g_{i\bar{j}}$ and $g_{\bar{i}j} = \overline{g_{i\bar{j}}}$ remain. Note that the matrix $g_{i\bar{j}}$ is hermitean. The fundamental two-form also only has mixed components, and $\omega_{i\bar{j}} = ig_{i\bar{j}}$. Thus in complex coordinates the matrices representing the metric and the fundamental two-form are hermitean and anti-hermitean, respectively, while in real coordinates they are symmetric and antisymmetric, respectively.

On a hermitean manifold the metric

$$g = g_{i\bar{j}}(dz^i \otimes \bar{z}^{\bar{j}} + d\bar{z}^{\bar{j}} \otimes dz^i)$$

and the fundamental two-form

$$\omega = ig_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}} = ig_{i\bar{j}}(dz^i \otimes d\bar{z}^{\bar{j}} - d\bar{z}^{\bar{j}} \otimes dz^i)$$

can be combined into the hermitean form

$$\tau = g_{i\bar{j}}dz^i \otimes d\bar{z}^{\bar{j}} = \frac{1}{2}(g - i\omega).$$

The hermitean form defines a hermitean metric on the complexified tangent bundle $TM_{\mathbb{C}}$ of M . All statements and formulae in this section apply irrespective of g being positive definit or indefinit (but non-degenerate).

A.3 Kähler manifolds

Definition 5 A *Kähler manifold* (M, g, I) is a hermitean manifold where the fundamental form is closed:

$$d\omega = 0.$$

Remark: Equivalently, one can impose that the complex structure is parallel with respect to the Levi-Civita connection,

$$\nabla^{(g)}I = 0.$$

Comment: Hermitean manifolds are characterised by ‘pointwise’ compatibility conditions between metric, complex structure and fundamental form. For Kähler manifolds one imposes a stronger compatibility condition: the complex structure I must be parallel (covariantly constant) with respect to the Levi-Civita connection $\nabla^{(g)}$. Since the metric g itself is parallel by definition of $\nabla^{(g)}$, parallelity of I is equivalent to the parallelity of the fundamental form ω . Moreover, it can be shown that if ω is closed, it is automatically parallel with respect to $\nabla^{(g)}$.

The fundamental form of a Kähler manifold is called its *Kähler form*. It can be shown that a Kähler metric can be expressed in terms of a real-analytic function, the Kähler potential, by³⁶

$$g_{i\bar{j}} = \frac{\partial^2 K(z, \bar{z})}{\partial z^i \partial \bar{z}^{\bar{j}}}.$$

The Kähler form can also be expressed as the second derivative of the Kähler potential:

³⁶ In fact, this might serve as yet another equivalent definition of a Kähler manifold.

$$\omega = i\partial\bar{\partial}K = ig_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}, \quad \text{where } \partial = dz^i\partial_i, \quad \bar{\partial} = d\bar{z}^{\bar{j}}\partial_{\bar{j}}$$

are the Dolbeault operators (holomorphic exterior derivatives).

Remark: If the metric g is positive definit, a Kähler manifold can be defined equivalently as a $2n$ -dimensional manifold with a torsion-free $U(n)$ structure. Note that $U(n) \simeq GL(n, \mathbb{C}) \cap SO(2n) \subset GL(2n, \mathbb{R})$, which shows that $U(n)$ holonomy implies that there is a connection such that both the metric and the complex structure are parallel.

Remark: If the metric is not positive definit, $U(n)$ is replaced by a suitable non-compact form. Pseudo-hermitean manifolds with closed fundamental form are called pseudo-Kähler manifolds. We have seen in the main text that the (conical affine special) Kähler manifolds occuring in the construction of supergravity theories within the superconformal calculus always have indefinite signature, because the compensator of complex dilatations has a kinetic term with an inverted sign. We usually omit the prefix ‘pseudo-’ in the following and in the main text.

A.4 Affine special Kähler manifolds

Special Kähler manifolds are distinguished by the fact that the Kähler potential $K(z, \bar{z})$ can itself be expressed in terms of a holomorphic prepotential $F(z)$. The intrinsic definition of such manifolds is as follows [22].

Definition 6 An *affine special Kähler manifold* (M, g, I, ∇) is a Kähler manifold (M, g, I) equipped with a flat, torsion-free connection ∇ , which has the following properties:

1. The connection is symplectic, i.e., the Kähler form is parallel

$$\nabla\omega = 0 .$$

2. The complex structure satisfies:

$$d^\nabla I = 0 ,$$

which means, in local coordinates, that

$$\nabla_{[m} I_{n]}^p = 0 .$$

Remark: The complex structure is not parallel with respect to the special connection ∇ , but only ‘closed’ (regarding I as a vector-valued one-form). This, together with the fact that ∇ is flat shows that the connections ∇ and $\nabla^{(g)}$ are different, except for the trivial case of a flat Levi-Civita connection.

It can be shown that the existence of a special connection ∇ is equivalent to the existence of a Kählerian Lagrangian immersion of M into a model vector

space, namely the standard complex vector space of doubled dimension [23]. Let us review this construction in some detail.

The standard complex symplectic vector space of complex dimension $2n$ is $V = T^*\mathbb{C}^n$. As a vector space, this is isomorphic to \mathbb{C}^{2n} . Let z^i be linear coordinates on \mathbb{C}^n and w_i be coordinates on $T_z\mathbb{C}^n$. Then we can take (z^i, w_i) as coordinates on $T^*\mathbb{C}^n$, and the symplectic form is

$$\Omega_V = dz^i \wedge dw_i .$$

If we interpret V as a phase space, then the z^i are the coordinates and the w_i are the associated momenta. Symplectic rotations of (z^i, w_i) give rise to different ‘polarisations’ (choices of coordinates vs momenta) of V .

The vector space V can be made a Kähler manifold in the following way: starting from the antisymmetric complex bilinear form Ω_V one can define an hermitean sesquilinear form γ_V by applying complex conjugation in the second argument of Ω , plus multiplication by i :

$$\gamma_V = i (dz^i \otimes d\bar{w}_i - dw_i \otimes d\bar{z}^i) .$$

The real part of γ_V is a flat Kähler metric of signature $(2n, 2n)$:

$$g_V = \text{Re}(\gamma_V) = i (dz^i \otimes_{\text{sym}} d\bar{w}_i - dw_i \otimes_{\text{sym}} d\bar{z}^i) ,$$

while the imaginary part is the associated Kähler form:

$$\omega_V = \text{Im}(\gamma_V) = dz^i \wedge d\bar{w}_i - dw_i \wedge d\bar{z}^i .$$

Now consider the immersion of a manifold M into V . An immersion is a map with invertible differential. An immersion need not be an invertible map, but it can be made invertible by restriction. Invertible immersions are called embeddings. (Intuitively, the difference between immersions and embeddings is that embeddings are not allowed to have self-intersections, or points where two image points come arbitrarily close.)

Definition 7 An immersion Φ of a complex manifold M into a Kähler manifold is called *Kählerian*, if it is holomorphic and if the pullback $g = \Phi^*g_V$ of the Kähler metric is nondegenerate.

Remark: Equivalently, one can require that the pullback of the hermitean form or of the Kähler form is non-degenerate.

Definition 8 An immersion Φ of a complex manifold M into a complex symplectic manifold is called *Lagrangian* if the pullback of the complex symplectic form vanishes, $\Phi^*\Omega_V = 0$.

Remark: For generic choices of coordinates, a Lagrangian immersion Φ is generated by a holomorphic function F on M , i.e. $\Phi = dF$.

It has been shown that for any affine special Kähler manifold of complex dimensions n there exists³⁷ a Kählerian Lagrangian immersion into $V = T^*\mathbb{C}^n$. Moreover every Kählerian Lagrangian immersion of an n -dimensional complex manifold M into V induces on it the structure of an affine special Kähler manifold.

By the immersion Φ , the special Kähler manifold M is mapped into V as the graph³⁸ of a map $z^i \rightarrow w_i = \frac{\partial F}{\partial z^i}$, where F is the prepotential of the special Kähler metric, which is the generating function of the immersion: $\Phi = dF$. Using the immersion, one obtains ‘special’ coordinates on M by picking half of the coordinates (z^i, w_i) of V (say, the z^i). Along the graph, the other half of the coordinates of V are dependent quantities, and can be expressed through the prepotential: $w_i = w_i(z) = \frac{\partial F}{\partial z^i}$. The special Kähler metric g , the Kähler form ω and the hermitean form γ on M are the pullbacks of the corresponding data g_V, ω_V, τ_V of V under the immersion.

Remark: For non-generic choices of Φ the immersed M may be not a graph. Then the z^i do not provide local coordinates, the w_i are not the components of a gradient, and Φ does not have a generating function, i.e., ‘there is no prepotential’.³⁹ This is not a problem, since one can work perfectly well by using only the symplectic vector (z^i, w_i) . Moreover, by a symplectic transformation one can always make the situation generic and go to a symplectic basis (polarisation of V) which admits a prepotential.

Remark: In the main text we denoted the component expression for the affine special Kähler metric on M by N_{IJ} instead of $g_{i\bar{j}}$. The scalar fields X^I correspond to the special coordinates z^i . More precisely, the scalar fields can be interpreted as compositions of maps from space-time into M with coordinate maps $M \supset U \rightarrow \mathbb{C}^n$. The key formulae which express the Kähler potential and the metric in terms of the prepotential are (34) and (37).

Special affine coordinates and the Hesse potential

Kähler manifolds are in particular symplectic manifolds, because the fundamental form is both non-degenerate and closed. The additional structure on affine special Kähler manifolds is the special connection ∇ , which is both flat and symplectic (i.e. the symplectic form ω is parallel with respect to ∇).⁴⁰ As a consequence, there exist ∇ -affine (real) coordinates $x^i, y_i, i = 1, \dots, n$ on M ,

³⁷ locally, and if the manifold is simply connected even globally,

³⁸ More precisely, the image is *generically* the graph of map. We comment on non-generic immersions below.

³⁹ In the physics literature, this phenomenon and its consequences have been discussed in detail in [83, 84].

⁴⁰ It is of course also parallel with respect to the Levi-Civita connection $\nabla^{(g)}$, but the Levi-Civita connection is not flat (except in trivial cases).

$$\nabla dx^i = 0, \quad \nabla dy_i = 0,$$

which are adapted to the symplectic structure,

$$\omega = 2dx^i \wedge dy_i.$$

The relation between these special affine coordinates and the special coordinates z^i can be elucidated by using the immersion of M into V . We can decompose z^i, w_i into their real and imaginary parts:

$$z^i = x^i + iu^i, \quad w_i = y_i + iv_i.$$

Then the Kähler form ω_V takes the form

$$\omega_V = dx^i \wedge dy_i + du^i \wedge dv_i.$$

Using that the pullback of the complex symplectic form Ω_V vanishes, one finds that the pullback of ω_V is⁴¹

$$\omega = \Phi^* \omega_V = 2dx^i \wedge dy_i.$$

Thus the special real coordinates form the real part of the symplectic vector (z^i, w_i) . The real and imaginary parts of $z^i = x^i + iu^i$ also form a system of real coordinates on M , which is induced by the complex coordinate system z^i , but not adapted to the symplectic structure (since x^i, u^i do not form a symplectic vector). The change of coordinates

$$(x^i, u^i) \rightarrow (x^i, y_i)$$

can be viewed as a Legendre transform, because

$$y_i = \operatorname{Re} \left(\frac{\partial F}{\partial z^i} \right) = \frac{\partial \operatorname{Im} F}{\partial \operatorname{Im} z^i} = \frac{\partial \operatorname{Im} F}{\partial u_i}. \quad (140)$$

The Legendre transform maps the imaginary part of the prepotential to the Hesse potential

$$H(x, y) = 2 (\operatorname{Im} F(x + iu(x, y)) - u_i y^i).$$

A Hesse potential is a real Kähler potential, i.e., a potential for the metric, but based on real rather than complex coordinates. Denoting the affine special coordinates by $\{q^a | a = 1, \dots, 2n\} = \{x^i, y_i | i = 1, \dots, n\}$, the special Kähler metric on M is given by

$$g = \frac{\partial^2 H}{\partial q^a \partial q^b} dq^a \otimes_{\text{sym}} dq^b.$$

⁴¹ For notational simplicity, we denote the pulled back coordinates $\Phi^* x^i, \Phi^* y^i$ by x^i, y_i .

The special connection present on an affine special Kähler manifold is not unique. The $U(1)$ action generated by the complex structure generates a one-parameter family of such connections. Each of these comes with its corresponding special affine coordinates. The imaginary part (u^i, v_i) of the symplectic vector (z^i, w_i) provides one of these special affine coordinate systems. The coordinate systems (x^i, y_i) and (u^i, v_i) both occur naturally in the construction of BPS black hole solutions.

A.5 Conical affine special Kähler manifolds and projective special Kähler manifolds

Definition 9 A *conical* affine special Kähler manifold (M, g, I, ∇, ξ) is an affine special Kähler manifold endowed with a vector field ξ such that

$$\nabla^{(g)}\xi = \nabla\xi = \text{Id} . \quad (141)$$

The condition $\nabla^{(g)}\xi = \text{Id}$ implies that ξ is a homothetic Killing vector field, and that it is hypersurface orthogonal. Then one can introduce adapted coordinates $\{r, v^a\}$ such that

$$\xi = r \frac{\partial}{\partial r}$$

and

$$g = dr^2 + r^2 g_{ab}(v) dv^a dv^b .$$

Thus M is a real cone. However, in our case M carries additional structures, and ξ satisfies the additional condition $\nabla\xi = \text{Id}$. It can be shown that this implies that M has a freely acting $U(1)$ isometry, with Killing vector field $I\xi$. The level surfaces $r = \text{const.}$ of the cone coincide with the level surfaces of the moment map of the isometry. Therefore the isometry preserves the level surfaces, and $M \subset T^*\mathbb{C}^{n+1}$ has the structure of a complex cone, with \mathbb{C}^* -action generated by $\{\xi, I\xi\}$.

One can choose special affine coordinates such that ξ has the form⁴²

$$\xi = q^a \frac{\partial}{\partial q^a} = x^i \frac{\partial}{\partial x^i} + y_i \frac{\partial}{\partial y_i} . \quad (142)$$

Moreover, it can be shown that the existence of a vector field ξ which satisfies (141) is equivalent to the condition that the prepotential is homogenous of degree 2:

$$F(\lambda z^i) = \lambda^2 F(z^i) ,$$

where $z^i \rightarrow \lambda z^i$ is the action of \mathbb{C}^* on the (conical) special coordinates $\{z^i\}$ associated with the (conical) special affine coordinates $\{x^i, y_i\}$. In special coordinates, ξ takes the form⁴³

⁴² These are called conical special affine coordinates, but we will usually drop ‘conical’.

⁴³ Note that this is equivalent to (142) if and only if the prepotential is homogenous of degree 2.

$$\xi = z^i \frac{\partial}{\partial z^i} .$$

The quotient $\overline{M} = M/\mathbb{C}^*$ is a Kähler manifold which inherits its metric from M . Manifolds which are obtained from conical affine special Kähler manifolds in this way are called *projective special Kähler manifolds*. These are the scalar manifolds of vector multiplets in $N = 2$ Poincaré supergravity. The corresponding conical affine special Kähler manifold is the target space of a gauge equivalent theory of superconformal vector multiplets. As we have seen from the physical perspective one can go back and forth between M and \overline{M} . Geometrically, M can be regarded as a \mathbb{C}^* -bundle over \overline{M} . In turn M itself is embedded into $V = T^*\mathbb{C}^{n+1}$, where $n + 1$ is the complex dimensions of M . In the main text the D-gauge is fixed by imposing

$$-i(X^I \overline{F}_I - F_I \overline{X}^I) = 1$$

on the symplectic vector (X^I, F_I) . Geometrically, this means that (X^I, F_I) is required to be a unitary section of the so-called universal line bundle over \overline{M} . Instead of using unitary sections, one can also reformulate the theory in terms of holomorphic sections of the universal bundle. This is frequently done when working with general (in contrast to special) coordinates, see [21]. For a more detailed account on the universal bundle, see [81].

In the main text we gave explicit formulae for various quantities defined on projective special Kähler manifolds in the notation used in the supergravity literature. In particular, (61) and (62) are the expressions for the metric and Kähler potential in terms of special coordinates on \overline{M} . There we also discussed the relation between the signatures of the special Kähler metrics on M and \overline{M} . The ‘horizontal’ metric g_{IJ} (60) vanishes along the vertical directions (those orthogonal to \overline{M} under the natural projection, with respect to the special Kähler metric of M), but it is non-degenerate (and positive definite, if the signature of the metric of M is complex Lorentzian) along the horizontal directions, which are those which project orthogonally onto \overline{M} . This defines a projective special Kähler metric on \overline{M} , for which an explicit formula is given in (61), (62) in special coordinates.

B Modular forms

Here we summarize some standard results on modular forms. See [85] for a more detailed account. As we mentioned in the main text, the theory of Siegel modular forms is a generalisation of the theory of ‘standard’ modular forms reviewed here. Some facts are stated in the main text. For a detailed account on Siegel modular forms see for example [86].

The action of the modular group $PSL(2, \mathbb{Z}) \simeq SL(2, \mathbb{Z})/\mathbb{Z}_2$ on the upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} | \text{Im}\tau > 0\}$ is:

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

The modular group is generated by the two transformations⁴⁴

$$T : \tau \rightarrow \tau + 1, \quad S : \tau \rightarrow -\frac{1}{\tau}.$$

The interior of the standard fundamental domain for this group action is

$$\mathcal{F} = \left\{ \tau \in \mathcal{H} \mid -\frac{1}{2} < \text{Re}\tau < \frac{1}{2}, |\tau| > 1 \right\}.$$

The full domain is obtained by adding a point at infinity, denoted $i\infty$, and identifying points on the boundary which are related by the group action. The point $i\infty$ is called the cusp point.

A function on \mathcal{H} is said to transform with (modular) weight k :

$$\phi(\tau') = (c\tau + d)^k \phi(\tau)$$

A function on \mathcal{H} is called a modular function, a modular form, a cusp form, if it is meromorphic, holomorphic, vanishing at the cusp, respectively.

The ring of modular forms is generated by the Eisenstein series G_4, G_6 , which have weights 4 and 6 respectively. The (normalized⁴⁵) Eisenstein series of weight k is defined by

$$G_k(\tau) = \frac{(k-1)!}{2(2\pi i)^k} \sum'_{m,n} \frac{1}{(m\tau + n)^k},$$

where the sum is over all pairs of integers (m, n) except $(0, 0)$. The sum converges absolutely for $k > 2$ and vanishes identically for odd k . For $k = 2$ the sum is only conditionally convergent, and one can define two functions with interesting properties. The holomorphic second Eisenstein series is defined by

$$G_k(\tau) = \frac{(k-1)!}{(2\pi i)^k} \sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{m=1}^{\infty} \left(\frac{(k-1)!}{(2\pi i)^k} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k} \right),$$

with $k = 2$ (the same organisation of the sum can be used for $k > 2$). The non-holomorphic second Eisenstein series is defined by

$$\bar{G}_2(\tau, \bar{\tau}) = -\frac{1}{8\pi^2} \lim_{\epsilon \rightarrow 0^+} \left(\sum'_{m,n} \frac{1}{(m\tau + n)|m\tau + n|^\epsilon} \right).$$

⁴⁴ The notation T and S is standard in the mathematical literature, and does not refer to T- or S-duality. However, there are several examples where either T-duality or S-duality acts by $PSL(2, \mathbb{Z})$ transformations on complex fields.

⁴⁵ With these prefactors, the coefficients of an expansion in $q = e^{2\pi i\tau}$ are rational numbers. In fact, they are related to the Bernoulli numbers.

Both are related by

$$\overline{G}_2(\tau, \bar{\tau}) = G_2(\tau) + \frac{1}{8\pi\tau_2} .$$

While the non-holomorphic $\overline{G}_2(\tau, \bar{\tau})$ transforms with weight two, the holomorphic function $G_2(\tau)$ transforms with an extra term:

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - \frac{c(c\tau + d)}{4\pi i} .$$

There is no modular form of weight two: $G_2(\tau)$ is holomorphic but does not strictly transform with weight two, while $G_2(\tau, \bar{\tau})$ transforms with weight two but is not holomorphic.

There is a unique cusp form Δ_{12} of weight 12, which can be expressed in terms of the Dedekind η -function by

$$\Delta(\tau) = \eta^{24}(\tau) ,$$

where

$$\begin{aligned} \Delta(\tau) = \eta^{24}(\tau) &= q \prod_{l=1}^{\infty} (1 - q^l)^{-24} , \\ \eta(\tau) &= q^{\frac{1}{24}} \prod_{l=1}^{\infty} (1 - q^l)^{-1} . \end{aligned} \tag{143}$$

The Dedekind η -function is a modular form of weight $\frac{1}{2}$ with multiplier system, i.e. a ‘modular form up to phase’:

$$\eta(\tau + 1) = e^{\frac{2\pi i}{24}} \eta(\tau) , \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau) .$$

Modular forms are periodic under $\tau \rightarrow \tau + 1$ and therefore they have a Fourier expansion in $\tau_1 = \text{Re}\tau$. It is convenient to introduce the variable

$$q = e^{2i\pi\tau} .$$

In the main text we avoid using the variable q , because it might be confused with the electric charge vector $q \in \Gamma$. The transformation $\tau \rightarrow q$ maps the semi-infinite strip $\{\tau \in \mathbb{C} \mid |\tau_1| \leq 1, \tau_2 > 0\} \subset \mathcal{H}$ onto the unit disc $\{q \in \mathbb{C} \mid |q| < 1\} \subset \mathbb{C}$. In particular, the cusp $\tau = i\infty$ is mapped to the origin $q = 0$. The Fourier expansion in τ_1 maps to a Laurent expansion in q , known as the q -expansion.

The q -expansion of the cusp form $\Delta_{12} = \eta^{24}$ is

$$\eta^{24}(q) = q - 24q^2 + 252q^3 + \dots$$

In the main text we express modular forms in terms of variables which live in right half plane rather than in the upper half plane, e.g., the heterotic dilaton S , where $\tau = iS$. For notational simplicity we then write $\eta(S)$ instead of $\eta(iS)$.

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