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Black Hole Laws and the Attractor Mechanism

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Abstract

We discuss black hole physics, beginning with an explanation of general relativity from a variational principle. The Schwarzschild and Reissner-Nordström metrics are derived with discussion of singularities and coordinate systems. The computer program 'Maple' is employed in this work.

An analysis of the Kerr metric follows before discussion of black holes and thermodynamics. The zeroth and first laws are derived with a sketch of the proof of the second law. Finally we review more recent work by Ferrera and Kallosh (1996) on the 'Attractor Mechanism' by deriving the field equations for this model of a black hole and verifying that they are satisfied by the given solution.

Contents

1 Introduction

The term 'black hole' was first used by Wheeler in 1967. However, such objects, characterised by the complete collapse of matter, have been studied theoretically long before the advent of relativity theory. In the 18th century Michell and Laplace studied Newtonian theory and showed that it was possible for matter to completely collapse on itself.

In Newtonian theory the collapse of matter to a point is completely analogous to the singularity in the Coulomb potential. However, as is popularly known, relativity theory has far deeper implications for the physics of collapsed matter. It implies that there exists a singularity in spacetime itself.

General relativity is based on extending special relativity (to include non-inertial observers) and on the priciple of equivalence: A frame linearly accelerated relative to an inertial frame in special relativity is locally identical to a frame at rest in a gravitational field. Einstein made use of the fact that the motion of a particle in a gravitational field is independent of it's mass. Therefore, there exists special paths in spacetime, determined by the gravitational field of massive objects, on which all inertial particles move. In general the spacetime will be curved and these special paths, called *geodesics*, are the curved space analogues of straight lines.

Of course the equations describing the motion of particles will be highly non-linear since in principle each particle moving on a geodesic also has it's own gravitational field. In this project we consider the gravitational field due to a single massive body only and denote it's mass M the *geometrical mass*, i.e the mass that affects the geometry. We neglect the mass of particles/observers moving in the field of M.

In the spirit of relativity all observers are equivalent in the sense that each can 'know' the laws of physics. However, in curved spacetime there is no preffered coordinate system, as different to Minkowski space. We must consider the most general physical manifolds (see [3] for an introduction to manifolds), which in general are covered by any number of *coordinate patches*.

The principle of general covariance states that the laws of physics should be invariant under a coordinate transformation. However, there often exist coordinate systems in which the symmetries of the system are most apparent and can be used to simplify the mathematics.

After formulating the physical principles, Einstein realised that a great deal of the mathematics of Riemannian geometry and tensor calculus must be used to describe a theory of curved spacetime. In the appendix of this project we have included some of the basic mathematical relations used in black hole physics. For an introduction to this material see [2] and [1].

General relativity has had much success in it's verification by experiment. For example, it successfully explains the advance of the perihelion of Mercury, see [2], and gravitational lensing effects. However, most of the phenomena present in the relatively weak gravitational fields of the solar system can be explained by Newtonian theory.

The curvature of spacetime in the region of a completely collapsed body of infinite density is so large that Newtonian theory is completely inadaquate to describe the physics of this region. So, for the description of black holes, relativity is employed to the full. General relativity relies upon the language of tensors. The appendix of the project has a collection of tensorial relations for the reader to refer to.

A black hole is viewed as a point of infinite curvature surrounded by an event horizon, from which nothing can escape. So called 'naked singularities', with no event horizon, are thought to be unphysical.

During the 1970s remarkable properties of black holes were studied theoretically resulting in the *black hole laws* that are analogous to the thermodynamical laws. In 1975 Hawking studied the quantum field theory in the region of a black hole event horizon. He found that black holes emit radiation and that this radiation had a thermal spectrum. This is a thermodynamical phenomenon resulting from a combination of classical general relativity and quantum theory. Thus, black hole physics remains an important topic in theories that unite general relativity and quantum theory up to the present day. Issues such as how one explains the non-zero entropy of a black hole in terms of Boltzmanns law microstate counting are currently being addressed in theories of quantum gravity such as superstring theory.

The theoretical work, up to the 1980s, was undertaken without physical confirmation that black holes existed. However, today there is numerous astrophysical data, which indicates that there is indeed a so called supermassive black hole at the centre of our galaxy and many other possible black hole candidates.

2 General Relativity from a variational principle

For many purposes it is useful to express General Relativity in a Lagrangian formulation. As well as providing an easy method for computing the field equations, this approach is fundamental in any theory of quantum mechanics in curved spacetime. The idea is the same as nonrelativistic mechanics; that we have a collection of tensor fields ψ_i, and we define a funtional $I(\psi_i)$, which maps these field configurations, and their derivatives, to a number. $I(\psi_i)$ is called the *action* and, as usual, we extremise the action to obtain the field equations. As in flat space we express the action in terms of a *Lagrangian density* $\mathcal{L}(\psi_i, \nabla\psi_i)$:

$$I(\psi_i) = \int_M \mathcal{L}\,\Omega\,, \tag{2.0.1}$$

where Ω is a volume element of our spacetime manifold M. However, in general relativity the field variable is the metric $g_{\mu\nu}$ and we realise that the volume element in the above is not constant over the spacetime. We must be more precise when defining integration over a more general manifold and we now introduce the mathematical tools to do this.

2.1 Differential forms and integration

Here we give an outline of the mathematics of differential forms relevant to our discussion. For a more in depth treatment refer to [1] and [3]. A *differential p-form* is a totally antisymmetric tensor of type (0,p) (see Appendix). If $\omega_{a_1\cdots a_p}$ are the components of a p-form then

$$\omega_{a_1\cdots a_p} = \omega_{[a_1\cdots a_p]}. \tag{2.1.1}$$

If the dimension of our manifold is n then any p-form, where $p > n$, vanishes. Note that a one-form is simply a *dual vector*. The space of dual vectors is isomorphic to the space of vectors at any point on a pseudo-Riemannian manifold. The isomorphism is given by the metric

$$g : v^\mu \to g_{\nu\mu} v^\mu = v_\nu. \tag{2.1.2}$$

On a manifold we define the basis vectors e_μ of the tangent space as

$$e_\mu = \frac{\partial}{\partial x^\mu}, \tag{2.1.3}$$

where x^μ are the coordinates of a *coordinate neighbourhood* of the vector space. Similarly the basis $e^{*\mu}$ of dual vectors is given by

$$e^{*\mu} = dx^\mu. \tag{2.1.4}$$

One can verify that these basis obey the correct coordinate transformation laws for covariant/contravariant vectors. We can then define the inner product between a vector $V = V^\mu \frac{\partial}{\partial x^\mu}$ and a dual vector $\omega = \omega_\nu dx^\nu$ by

$$\langle \omega, V \rangle = \omega_\nu V^\mu \left\langle dx^\nu, \frac{\partial}{\partial x^\mu} \right\rangle = \omega_\nu V^\mu \delta^\nu_\mu = \omega_\nu V^\nu. \tag{2.1.5}$$

We can construct a basis for arbitrary rank p-forms by taking the *wedge product* between one-forms. We define the wedge product as

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r} = \sum_{P \in S_r} \text{sgn(P)} dx^{\mu_{P(1)}} \otimes dx^{\mu_{P(2)}} \otimes \ldots \otimes dx^{\mu_{P(r)}}, \tag{2.1.6}$$

where $P(r)$ is a permutation belonging to the permutation group S_r of r elements. Then sgn(P) is +1 (-1) depending on whether $P(r)$ is an even (odd) permutation. Then a general p-form can be written

$$\omega_p = \frac{1}{p!} \omega_{\mu_1 \mu_2 \ldots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}, \tag{2.1.7}$$

where $\omega_{\mu_1 \mu_2 \ldots \mu_p}$ is totally antisymmetric. We define the *exterior derivative* d, which maps a p-form to a (p+1)-form. It can be easily shown (see [1]) that d is a derivative operator independent of the metric. We can therefore express it in terms of the ordinary derivative ∂_a. Then

$$d\omega_p = \frac{1}{p!} \left(\frac{\partial}{\partial x^\nu} \omega_{\mu_1 \mu_2 \ldots \mu_p} \right) dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}. \tag{2.1.8}$$

ω_0 is just an ordinary function, f say, ω_1 is a dual-vector and in three-dimensional space we have

$$d\omega_1 = \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) dx \wedge dy + \left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z} \right) dy \wedge dz + \left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x} \right) dz \wedge dx. \tag{2.1.9}$$

Hence, d acting on ω_1 is the curl of a (dual) vector. It is easy to verify that $d\omega_0$ is the gradient of a function, $d\omega_2$ the divergence and $d\omega_3 = 0$ because the resulting (3+1)-form has dimension greater than that of the manifold, $d = 3$.

We have seen how forms are used to reproduce the usual differential calculus in Euclidean space. Their generality means that they can be applied to any manifolds and hence are important in studies of general relativity. We now look at how they are used to provide a well-defined volume element on a manifold and hence a means of integration.

Firstly we must define the *orientation* of a manifold as the integration of a differential form is only defined when the manifold is *orientable*. If p is a point of a manifold such that two coordinate patches x^μ, y^ν overlap at this point then we transform the basis vectors as

$$\tilde{e}_\nu = \left(\frac{\partial x^\mu}{\partial y^\nu}\right) e_\mu \,. \tag{2.1.10}$$

If $J = \det(\partial x^\mu / \partial y^\nu) > 0$ then two coordinate patches are said to define the same orientation at p. If $J < 0$ they define opposite orientations. We then define an orientable manifold to be one covered by coordinate patches that everywhere define the same orientation.

If we have an n-dimensional orientable manifold M then there exists an n-form ω which vanishes nowhere. This n-form can be used to find the *invariant volume element*, which we use to integrate a function on M.

Two orientations, ω and $\tilde{\omega}$, are said to be equivalent if $\tilde{\omega} = h\omega$ where h is a strictly positive function. We now have the volume element, up to a positive function. We are able to define the *invariant volume element*, on an n-dimensional manifold M, by

$$\Omega_M \equiv \sqrt{|g|} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n, \tag{2.1.11}$$

where $g = \det g_{\mu\nu}$. We now show that Ω_M is indeed invariant as we move from one coordinate patch x^μ to another y^λ on the manifold. The invariant volume element is then

$$\begin{aligned} & \sqrt{\left|\det\left(\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\lambda} g_{\mu\nu}\right)\right|} dy^1 \wedge dy^2 \wedge \ldots \wedge dy^n \\ = \quad & \left|\det\left(\frac{\partial x^\mu}{\partial y^\alpha}\right)\right| \sqrt{|g|} \det\left(\frac{\partial y^\lambda}{\partial x^\nu}\right) dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \\ = \quad & \pm\sqrt{|g|} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n. \end{aligned} \tag{2.1.12}$$

If the manifold is orientable then $\det(\partial y^\lambda / \partial x^\nu) > 0$ and we see that Ω_M is invariant under the coordinate change. The signature of our metric is -2 so we write $\sqrt{|g|} = \sqrt{-g}$. We now define the integration of a function f on a manifold to be

$$\int_M f\, \Omega_M \equiv \int_M f \sqrt{-g}\, dx^1 dx^2 \ldots dx^n. \tag{2.1.13}$$

If an object can be written in the form

$$T^{abc\ldots}_{\mu\nu\rho\ldots} = \sqrt{-g}\, \tilde{T}^{abc\ldots}_{\mu\nu\rho\ldots}\,, \tag{2.1.14}$$

where $\tilde{T}^{abc\ldots}_{\mu\nu\rho\ldots}$ is a tensor, then we call it a *tensor density*. We see that a Lagrangian for gravity must be a scalar density.

Finally, for later use, we include here *Stokes' theorem* for an arbitrary manifold, for a proof see [3]. For oriented manifold M, dimension d, with d-dimensional submanifold N with boundary ∂N, and (d-1)-form ω, Stokes' theorom states

$$\int_N d\omega = \int_{\partial N} \omega \,. \tag{2.1.15}$$

Notice that the integration over a volume, on the left hand side, is over a d-form as discussed above.

2.2 The Einstein-Hilbert Lagrangian

In the last section we have seen that our Lagrangian must contain the factor $\sqrt{-g}$. Then the simplest scalar density we can build out of the metric and it's derivatives is

$$\mathcal{L}_G = \sqrt{-g}\, R, \tag{2.2.1}$$

where R is the *Ricci scalar* (see Appendix). We call this the *Einstein-Hilbert Lagrangian* and we now show that varying this Lagrangian with respect to g_{ab} leads to the Einstein field equations. Firstly, we show some preliminary results. Let g_{ab} be the metric with inverse g^{ab}. Then we have

$$g^{ab} = \frac{1}{g} A^{ba} \,. \tag{2.2.2}$$

where A^{ba} is the transpose of the matrix of cofactors of g_{ab} and g is the determinant. Summing only over the column index, the determinant is given by

$$g = \sum_{b=1}^{n} g_{ab} A^{ab} \,. \tag{2.2.3}$$

This gives

$$\frac{\partial g}{\partial g_{ab}} = A^{ab} = g g^{ba} \,. \tag{2.2.4}$$

We can then deduce that

$$\begin{aligned} \frac{\partial (-g)^{\frac{1}{2}}}{\partial g_{ab}} &= \frac{\partial g}{\partial g_{ab}} \frac{\partial (-g)^{\frac{1}{2}}}{\partial g} \\ &= -\frac{1}{2} g g^{ab} (-g)^{-\frac{1}{2}} \\ &= \frac{1}{2} g^{ab} (-g)^{\frac{1}{2}} \,, \end{aligned} \tag{2.2.5}$$

recalling that the metric is symmetric. Notice that $g^{ab} g_{ab}$ is a constant tensor so that

$$\begin{aligned} \delta(g^{ab} g_{ab}) &= g_{ab} \delta g^{ab} + g^{ab} \delta g_{ab} = 0 \\ &\Rightarrow g_{ab} \delta g^{ab} = -g^{ab} \delta g_{ab} \,, \end{aligned} \tag{2.2.6}$$

and (2.2.5) gives

$$\begin{aligned} \delta(-g)^{1/2} &= \frac{1}{2} (-g)^{1/2} g^{ab} \delta g_{ab} \\ &= -\frac{1}{2} (-g)^{1/2} g_{ab} \delta g^{ab} \,. \end{aligned} \tag{2.2.7}$$

It follows that

$$\frac{\partial(-g)^{\frac{1}{2}}}{\partial g^{ab}} = -\frac{1}{2} g_{ab}(-g)^{\frac{1}{2}} \,. \tag{2.2.8}$$

Note the different sign to (2.2.5).

We now prove the *Palatini equation*, which gives an expression for the variation of the *Riemann tensor* (see Appendix). We use geodesic coordinates, where the connection vanishes at some point p, so that $\Gamma^a_{bc} = 0$ and, from (A.0.46), the Riemann tensor reduces to

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} \,. \tag{2.2.9}$$

The connection is not a tensor so this is only valid at p. However, we can form a tensor by varying the connection,

$$\Gamma^a_{bc} \to \overline{\Gamma}^a_{bc} = \Gamma^a_{bc} + \delta\Gamma^a_{bc} \,. \tag{2.2.10}$$

$\delta\Gamma^a_{bc}$ is the difference of two connections, and therefore a tensor (see [2] chapter 6). The above variation induces a variation in the Riemann tensor:

$$R^a{}_{bcd} \to \overline{R}^a{}_{bcd} = R^a{}_{bcd} + \delta R^a{}_{bcd} \,. \tag{2.2.11}$$

In geodesic coordinates we have

$$\begin{aligned} \delta R^a{}_{bcd} &= \partial_c(\delta\Gamma^a_{bc}) - \partial_d(\delta\Gamma^a_{bc}) \\ &= \nabla_c(\delta\Gamma^a_{bc}) - \nabla_d(\delta\Gamma^a_{bc}) \,, \end{aligned} \tag{2.2.12}$$

where ∇_a is the *covariant derivative* (see Appendix). Notice that $\delta R^a{}_{bcd}$ is the difference of two tensors so is itself, a tensor and (2.2.12), the Palatini equation, holds in any coordinates. Contracting a and c gives the useful result

$$\delta R_{bd} = \nabla_a(\delta\Gamma^a_{bd}) - \nabla_d(\delta\Gamma^a_{ba}) \,, \tag{2.2.13}$$

where R_{bd} is the *Ricci tensor* (see Appendix).

We now compute the partial and covariant derivatives of the metric determinant g. Using (2.2.4) we have

$$\begin{aligned} \frac{\partial g}{\partial x^c} &= \frac{\partial g}{\partial g_{ab}} \frac{\partial g_{ab}}{\partial x^c} \\ &= g g^{ba} \frac{\partial g_{ab}}{\partial x^c} \,. \end{aligned} \tag{2.2.14}$$

The metric is symmetric so we can write

$$\begin{aligned} \partial_c g &= g g^{ab} \partial_c g_{ab} \\ &= g g^{ab} (\Gamma^d_{ac} g_{db} + \Gamma^d_{bc} g_{ad}) \\ &= g \delta^a_d \Gamma^d_{ac} + g \delta^b_d \Gamma^d_{bc} \\ &= 2g\Gamma^a_{ac} \,, \end{aligned} \tag{2.2.15}$$

where we have used $\nabla_c g_{ab} = 0$ in the second line (see section 2.5).

It is easy to verify that the covariant derivative of g vanishes. It can be written

$$g = \det(g_{ab}) = \epsilon^{\mu\nu\rho\sigma} g_{\mu 1} g_{\nu 2} g_{\rho 3} g_{\sigma 4} \,, \tag{2.2.16}$$

where ϵ is the alternating symbol. Each $g_{\mu 1}$ is covariantly constant so $\nabla_c g = 0$. This implies

$$\nabla_c \sqrt{-g} = 0 \,. \tag{2.2.17}$$

It is interesting to note that $\nabla_c \sqrt{-g} \neq \partial_c \sqrt{-g}$ so we see that $\sqrt{-g}$ is not a scalar but a scalar density.

Now, writing $\mathcal{L}_G$ from (2.2.1) in terms of the Ricci tensor the action is

$$I = \int_\Omega \sqrt{-g} g^{ab} R_{ab} \, d\Omega \,. \tag{2.2.18}$$

We now perform the variation, giving

$$\delta I = \int_\Omega \left(\delta(\sqrt{-g}\, g^{ab}) R_{ab} + \sqrt{-g}\, g^{ab} \delta R_{ab} \right) \, d\Omega \,. \tag{2.2.19}$$

Using (2.2.13) and (2.2.17) the second term on the right hand side becomes

$$\begin{aligned} \int_\Omega \sqrt{-g}\, g^{ab} \delta R_{ab} d\Omega &= \int_\Omega \sqrt{-g}\, g^{ab} \left(\nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ac} \right) \, d\Omega \\ &= \int_\Omega \left(\nabla_c (\sqrt{-g}\, g^{ab} \delta \Gamma^c_{ab}) - \nabla_b (\sqrt{-g}\, g^{ab} \delta \Gamma^c_{ac}) \right) \,. \end{aligned} \tag{2.2.20}$$

The last line is the integral of a divergence so is equal to a surface integral by Stokes' theorem. Therefore, if all variations are zero on the boundary of the spacetime, (2.2.20) vanishes and we are left with

$$\begin{aligned} \delta I &= \int_\Omega \delta(\sqrt{-g}\, g^{ab}) R_{ab} \, d\Omega \\ &= \int_\Omega \left(R_{ab} g^{ab} \delta \sqrt{-g} + R_{ab} \sqrt{-g} \delta g^{ab} \right) \, d\Omega \\ &= \int_\Omega \left(R_{ab} g^{ab} (-\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab}) + R_{ab} \sqrt{-g} \delta g^{ab} \right) \, d\Omega \\ &= \int_\Omega \sqrt{-g} \left(R_{ab} - \frac{1}{2} R g_{ab} \right) \delta g^{ab} \, d\Omega \\ &= \int_\Omega \sqrt{-g} G_{ab} \delta g^{ab} \, d\Omega \,, \end{aligned} \tag{2.2.21}$$

where we have used (2.2.7) and the definition of the Ricci scalar (A.0.53). G_{ab} is the Einstein tensor. δI must vanish and δg^{ab} is arbitrary, so we see that $G_{ab} = 0$ and we have the vacuum field equations.

2.3 The full field equations

To obtain the full field equations we include the *matter Lagrangian* $\mathcal{L}_M$ in the expression for the action:

$$I = \int_\Omega \left(\mathcal{L}_G + \kappa \mathcal{L}_M \right) \, d\Omega \,, \tag{2.3.1}$$

where κ is the coupling constant. We then set

$$\frac{\delta \mathcal{L}_G}{\delta g^{ab}} = \sqrt{-g} G_{ab} \tag{2.3.2}$$

and

$$\frac{\delta \mathcal{L}_M}{\delta g^{ab}} = -\sqrt{-g} T_{ab} \,, \tag{2.3.3}$$

where the last equation defines the *energy-momentum tensor* T_{ab} of the matter fields. The variation of the action must vanish so we have reproduced the full field equations

$$G_{ab} = \kappa T_{ab} \,. \tag{2.3.4}$$

For a discussion of the numerical constant κ see [2], section 12.10, where it is argued that $\kappa = 8\pi$. Here, and henceforth, we use *geometric units* with $G_N = c = 1$, where G_N is Newtons constant and c is the velocity of light in vacuum.

2.4 Geodesics

Curves in spacetime can be timelike, traversed by massive particles; lightlike, for photons; or spacelike which connect points not in physical contact (communication). Further discussion can be found in books on special (general) relativity.

The infinitesimal length ds along a timelike curve can be defined

$$ds^2 = g_{ab} dx^a dx^b \,, \tag{2.4.1}$$

where dx^a is the change in coordinate along the curve and g_{ab} is the metric. Let u be a parameter along the curve and write

$$\left(\frac{ds}{du}\right)^2 = g_{ab} \frac{dx^a}{du} \frac{dx^b}{du} \,. \tag{2.4.2}$$

If u is linearly related to s then it is an *affine parameter*. For example, if

$$s = \alpha u + \beta \tag{2.4.3}$$

then (2.4.2) becomes

$$\alpha^2 = g_{ab} \alpha^2 \dot{x}^a \dot{x}^b \tag{2.4.4}$$

and

$$g_{ab} \dot{x}^a \dot{x}^b = 1 \,, \tag{2.4.5}$$

where a dot is differentiation with respect to affine parameter s. Similarly for a spacelike curve

$$g_{ab} \dot{x}^a \dot{x}^b = -1 \,. \tag{2.4.6}$$

Null curves are possible on a manifold with metric of indeterminate form. These curves connect points such that $ds = 0$. Summarising

$$g_{ab} \dot{x}^a \dot{x}^b = \begin{cases} 1 & \text{if timelike} \\ 0 & \text{if null} \\ -1 & \text{if spacelike} \end{cases} \tag{2.4.7}$$

Returning to (2.4.2) we have

$$s = \int ds = \int \frac{ds}{du} du = \int \left(g_{ab} \frac{dx^a}{du} \frac{dx^b}{du} \right)^{1/2} du \,. \tag{2.4.8}$$

A geodesic is an extremal curve, so that $\delta s = 0$, giving the Euler-Lagrange equation

$$\frac{\partial L}{\partial x^a} - \frac{d}{du} \left(\frac{\partial L}{\partial \dot{x}^a} \right) = 0 \,, \tag{2.4.9}$$

where $L = (g_{ab} \dot{x}^a \dot{x}^b)^{1/2}$. We can evaluate the Euler-Lagrange equation explicitly to give the geodesic equation (see [2] chapter 6). However, we will not need this for our purposes. We will be mostly interested in null geodesics, which are useful to elucidate the structure of a spacetime and it will be enough for us to state that they must obey equations (2.4.7) and (2.4.9).

2.5 Killing's equation

We include here a discussion of *Killing vectors*. A spacetime, in general, will possess continuous and discrete symmetries, collectively called *isometries*. The tangents to continuous isometries are Killing vectors. Killing vectors are useful for understanding the structure of a spacetime. For example a Killing vector could be timelike in which case the spacetime would be stationary, i.e 'looks' the same for all times.

We can derive a useful equation satisfied by Killing vectors. The reader is referred to the Appendix for definitions in the following. Firstly note that

$$X^b \nabla_b Y^a - Y^b \nabla_b X^a = X^b \partial_b Y^a + X^b \Gamma^a_{cb} Y^c - Y^b \partial_b X^a - Y^b \Gamma^a_{cb} X^c \,, \tag{2.5.1}$$

where we have used (A.0.43). If the connection is torsion free (see Appendix) then the right hand side is

$$X^b \partial_b Y^a - Y^b \partial_b X^a = L_X Y^a \,, \tag{2.5.2}$$

just the *Lie derivative* of Y^a. So in any expression for the Lie derivative of a vector one can replace partial derivatives by covariant derivatives. Next we show that the covariant derivative of the metric vanishes. Using (A.0.41), we have

$$\begin{aligned}
\nabla_c g_{ab} &= \partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} \\
&= \partial_c g_{ab} - \frac{1}{2} g^{de} (\partial_a g_{ec} + \partial_c g_{ea} - \partial_e g_{ac}) g_{db} \\
&\quad - \frac{1}{2} g^{df} (\partial_b g_{fc} + \partial_c g_{fb} - \partial_f g_{bc}) g_{ad} \\
&= \partial_c g_{ab} - \frac{1}{2} (\partial_a g_{bc} + \partial_c g_{ba} - \partial_b g_{ac}) - \frac{1}{2} (\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}) \\
&= 0 \,.
\end{aligned} \tag{2.5.3}$$

Now, under any isometric coordinate change the metric will be invariant. That is, as $x^a \to x'^a$:

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a} \frac{\partial x'^d}{\partial x^b} g_{cd}(x') \,. \tag{2.5.4}$$

Note here that the metric components remain the same functions of their coordinates under the isometry. The continuous isometry is connected to the identity so we can write

$$x^a \to x'^a = x^a + \epsilon X^a(x)\,, \tag{2.5.5}$$

where ϵ is a small parameter. This gives

$$\frac{\partial x'^c}{\partial x^a} = \delta^c_a + \epsilon \partial_a X^c\,. \tag{2.5.6}$$

Substituting this into (2.5.4) we have

$$\begin{aligned} g_{ab}(x) &= (\delta^a_c + \epsilon\partial_a X^c)(\delta^d_b + \epsilon\partial_b X^d) g_{cd}(x^e + \epsilon X^e) \\ &= (\delta^a_c + \epsilon\partial_a X^c)(\delta^d_b + \epsilon\partial_b X^d)(g_{cd}(x) + \epsilon X^e \partial_e g_{cd}(x)) \\ &= g_{ab} + \epsilon(g_{ad}\partial_b X^d + g_{bd}\partial_a X^d + X^e \partial_e g_{ab}) \\ &= g_{ab} + \epsilon L_X g_{ab} \\ &\Rightarrow \; L_X g_{ab} = 0\,. \end{aligned} \tag{2.5.7}$$

We can replace partial derivatives by covariant derivatives in this Lie derivative and the first term vanishes due to (2.5.3). We are left with *Killing's equation*:

$$\nabla_b X_a + \nabla_a X_b = 0\,, \tag{2.5.8}$$

where X^a is a Killing vector field which generates isometries.

3 Static, spherically symmetric black holes

In this section we discuss the well known Schwarzschild solution (1916). This is the simplest static, spherically symmetric solution of Einstein's equation and is the basic model used to describe the orbits of the planets in the solar sytem. This solution has provided experimental verification of general relativity by explaining such phenomena as the advance of the perihelion of Mercury (see [2], section 15.3).

We will see how the Schwarzschild solution exemplifies the concept of a coordinate, and intrinsic, singularity. The latter is seen to be a point of infinite curvature and thus a singularity in spacetime itself. However, the singularity is not visible to an observer at infinity because it is surrounded by the event horizon. We shall introduce solutions that remove the coordinate singularity and so are well defined everywhere except at the intrinsic, curvature singularity. We then investigate properties of the Schwarzschild black hole using spacetime diagrams.

3.1 Static, spherically symmetric spacetimes

Here we define the terms static and spherically symmetric. For a spacetime to be static it necessarily must be stationary. A stationary spacetime is invariant under time translation $t \to t + c$, but a static spacetime must also have the stronger condition of time reversal invariance $t \to -t$.

A stationary spacetime is defined to be one which admits a timelike Killing vector field. That is, there exists a continuous, timelike symmetry of the metric, agreeing with the concept of time translation invariance. It is, therefore, possible to make a coordinate transformation so that the metric becomes independent of the timelike coordinate, x^0.

To understand static spacetimes we introduce the concept of hypersurface orthogonality. A hypersurface is a submanifold with dimension one less than the spacetime manifold, for example, the surface of the Earth in three-dimensional space. In general, the hypersurface can be thought of as the submanifold given by placing one constraint on the coordinates x^a:

$$f(x^a) = 0\,, \tag{3.1.1}$$

where f is the constraining function. Consider two points on the hypersurface x^a and nearby $x^a + dx^a$. Then, to first order, (3.1.1) gives

$$f(x^a + dx^a) = f(x^a) + \frac{\partial f}{\partial x^a} dx^a = 0\,, \tag{3.1.2}$$

and so,

$$\frac{\partial f}{\partial x^a} dx^a = 0\,. \tag{3.1.3}$$

Here we see that $f_{,a}$ is orthogonal to dx^a and, therefore, orthogonal to the hypersurface. Any vector field, X_a, proportional to $f_{,a}$ is called a *hypersurface orthogonal vector field* and characterised by

$$X_a = \lambda(x) f_{,a}\,, \tag{3.1.4}$$

where $\lambda(x)$ is an arbitrary function of the coordinates.

The antisymmetric part of

$$X_a \partial_b X_c = \lambda f_{,a} \lambda_{,b} f_{,c} + \lambda^2 f_{,a} f_{,cb} \tag{3.1.5}$$

vanishes because the first term on the right hand side is symmetric in a and c whereas the second term is symmetric in b and c, giving

$$X_{[a} \partial_b X_{c]} = 0\,. \tag{3.1.6}$$

This is, in fact, an equation of Frobenius' theorem which gives the above as necessary and sufficent for X_a to be hypersurface orthogonal; we have shown necessity. See [1], Appendix B for discussion of Frobenius' theorem.

The time reflection invariance of a static spacetime implies that there are no cross terms, involving the timelike coordinate, in the metric; no terms like $dx^0 dx^2$. Following [2], section 14.3, we consider a two-dimensional region of the spacetime described by the metric

$$ds^2 = g_{00}(dx^0)^2 + 2g_{01} dx^0 dx^1 + g_{11}(dx^1)^2\,, \tag{3.1.7}$$

and send $t \to -t$. The metric goes to

$$ds'^2 = g_{00}(dx^0)^2 - 2g_{01} dx^0 dx^1 + g_{11}(dx^1)^2\,. \tag{3.1.8}$$

Invariance implies $ds^2 = ds'^2$, so we see that the term linear in dx^0 must vanish. The same approach can be used to show that the other cross terms with dx^0 must vanish.

The mathematical definition of a static spacetime is that there exists a hypersurface orthogonal, timelike Killing vector. We now argue that this implies that there are no cross terms with the timelike coordinate in the metric.

Consider a hypersurface, Σ, of the spacelike coordinates, x^j, described by the metric

$$ds^2 = h_{ij}(x^1, x^2, x^3)dx^i dx^j \,, \tag{3.1.9}$$

where h is a matrix of functions. Picture point p on Σ mapped to p' on Σ'. The Killing vector field X^a is orthogonal to Σ so for every such p on Σ there is a bijective map to p'. Now consider the family of hypersurfaces Σ_t, where t is the timelike coordinate labelling them. X^a will be orthogonal to *all* of them and the metric on each will be independent of t. So we can write down the following ansatz

$$ds^2 = F(x^1, x^2, x^3)dt^2 + h_{ij}(x^1, x^2, x^3)dx^i dx^j \,, \tag{3.1.10}$$

where there are no cross terms with the timelike coordinate.

Finally, we discuss spherical symmetry. We introduce the familiar angles from three-dimensional space; θ, the polar angle, ranging $[0, \pi]$ and ϕ, the azimuthal angle, ranging $[0, 2\pi]$. In four-dimensional spacetime we imagine a surface at constant radius, a, and time, t, and ascribe the line element for the unit two-sphere

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \,, \tag{3.1.11}$$

which we recognise as the solid angle. Analogously to the time reflection invariance for staticity; for spherical symmetry we require the metric to be invariant under

$$\begin{aligned} \theta \to \theta' &= \pi - \theta \\ \phi \to \phi' &= -\phi \,. \end{aligned} \tag{3.1.12}$$

Similarly to (3.1.7) it is easily shown that the above demands that there be no cross terms in $d\theta$ and $d\phi$, in the metric. In a spherically symmetric spacetime all variation of θ and ϕ must leave the solid angle, (3.1.11), invariant and we impose that they only enter into the metric in this combination. We are now ready to formulate an ansatz for a general, spherically symmetric spacetime. Here it is

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{3.1.13}$$

where $\nu = \nu(t, r)$, $\lambda = \lambda(t, r)$. Note that we have not imposed that the metric is independent of the time coordinate. We have included the factor r^2 in analogy to the usual volume element for a sphere. The first two metric components are expressed as exponentials because, as they are positive definite, they preserve the Lorentzian signature, -2, of the metric. The task, therefore, is reduced to finding the form of the functions ν and λ.

3.2 The Schwarzschild solution

Computing the Einstein vacuum field equations from the spherically symmetric ansatz we have the following non-zero components (Worksheet 1):

$$
\begin{aligned}
{G_0}^0 &= \frac{1}{e^\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0 && (3.2.1)\\
{G_0}^1 &= \frac{\dot\lambda}{re^\lambda}=0 && (3.2.2)\\
{G_1}^1 &= \frac{1}{e^\lambda}\left(\frac{1}{r^2}+\frac{\lambda'}{r}\right)-\frac{1}{r^2}=0 && (3.2.3)\\
{G_2}^2 &= {G_3}^3\\
&= \frac{1}{2e^\lambda}\left(\frac{\nu'\lambda'}{2}+\frac{\lambda'}{r}-\frac{\nu'}{r}-\frac{\nu'^2}{2}-\nu''\right)+\frac{1}{2e^\nu}\left(\ddot\lambda+\frac{\dot\lambda^2}{2}-\frac{\dot\lambda\dot\nu}{2}\right)\\
&= 0\,. && (3.2.4)
\end{aligned}
$$

By (3.2.2) we see that $\lambda=\lambda(r)$ and (3.2.1) is an ode, which we can write

$$e^{-\lambda}-\lambda' re^{-\lambda}=1\,, \tag{3.2.5}$$

which gives

$$
\begin{aligned}
&(re^{-\lambda})'=1\\
\Rightarrow\quad & e^\lambda=\left(1+\frac{C}{r}\right)^{-1}\,, && (3.2.6)
\end{aligned}
$$

where C is a constant of integration.

Adding (3.2.1) and (3.2.3) gives

$$\lambda'+\nu'=0\,. \tag{3.2.7}$$

Integrating

$$\lambda+\nu=f(t)\,. \tag{3.2.8}$$

From [6] section 8.1, we can add an arbitrary function of t to ν. Consider the first component of (3.1.13). If we make a transformation $t\to t'$ such that $t=t(t')$ and

$$\nu\to\nu'=\nu+2\ln\left(\frac{dt}{dt'}\right) \tag{3.2.9}$$

then

$$
\begin{aligned}
e^\nu dt^2\to e^{\nu'}dt'^2 &= \left(\frac{dt}{dt'}\right)^2 e^\nu dt'^2\\
&= e^\nu dt^2\,. && (3.2.10)
\end{aligned}
$$

We have seen that the line element (3.1.13) is invariant if we add an arbitrary function of t to ν so we can write

$$\lambda+\nu=0\,. \tag{3.2.11}$$

From (3.2.11) and (3.2.6) we have

$$e^{\nu} = 1 + \frac{C}{r} \,. \tag{3.2.12}$$

To interpret the constant C, we must compare Newton's theory with the Newtonian limit of general relativity. Therefore, considering weak, slowly varying gravitational fields produced by a slowly moving source we obtain (see [2], section 12.9)

$$g_{00} \approx 1 + \frac{2\phi}{c^2} \,, \tag{3.2.13}$$

where ϕ is the Newtonian gravitational potential. For a spherically symmetric system, solution of Poisson's equation for ϕ gives

$$\phi = -\frac{GM}{r} \,. \tag{3.2.14}$$

Then (3.2.13) and (3.2.12) give

$$1 + \frac{C}{r} \approx 1 + \frac{2\phi}{c^2 r} \,. \tag{3.2.15}$$

Therefore, in geometric units we set $C = -2M$ and call M the *geometric mass*, the distribution of which determines the spacetime geometry. We have now arrived at the Schwarzschild solution

$$ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{3.2.16}$$

This is a static solution describing the exterior of a star or planet, where there is no matter. In our ansatz we only postulated spherical symmetry. The functions ν λ were defined as functions of r *and* t. Instead we have *shown* that it is static. Crucially, we were able to absorb the arbitrary function of t in (3.2.8). Thus we have arrived at *Birkhoff's theorem* (1923): A spherically symmetric vacuum solution in the exterior region is necessarily static.

3.3 Singularities

In general, coordinate patches are only well defined over portions of the manifold and one coordinate system need not be well defined everywhere. For example, in the spherically symmetric spacetime we have degeneracy at $\theta = 0, \pi$ but this problem is easily removed using Cartesian coordinates. Such points are called *coordinate singularites* and are said to be *removable*. We can cover a manifold with many coordinate patches, requiring that they overlap and we can make transformations between them with smooth functions. It is easy to see from (3.2.16) that the Schwarzschild solution is not well defined at $r = 0$ and $r = 2M$. The question is: are these removable singularities?

The concept of a singularity in spacetime is difficult to precisely define. This is because the metric is not well defined at the singularity and we can not really think of it in terms of a place in spacetime. The metric, being the distance function, can not be used to measure a distance in space and time between the singularity and another point.

A useful conceptual approach has been to think of a singularity as a hole in the manifold and characterised by causal geodesics which never reach future null/timelike infinity (see section 6.4, page 34, for a discussion of these terms). It is interesting to note that if the universe began in a singular state then we can not consider the instant of the 'big bang' as belonging to our spacetime manifold.

More practically we can analyse the behaviour of scalar invariants built from the metric, such as the Ricci scalar (A.0.53). These have the useful property that they are invariant under coordinate transformations. Of course, we can not evaluate these exactly at the singularity (as we consider it as a point with no metric structure) but we can analyse the asymptotic behaviour as the singularity is approached. We expect that if the singularity is removable then the Ricci scalar will remain finite, but if it is a true singularity of the spacetime then it will diverge.

The Ricci scalar of the vacuum Schwarzschild solution is identically zero as Einstein's equation is traceless in the absence of sources. However, we can also compute the Riemann curvature scalar, which gives

$$R^{abcd}R_{abcd} = \frac{48m^2}{r^6}\,. \tag{3.3.1}$$

This is obviously finite at $r = 2M$, indicating a coordinate/removable singularity, but diverges at $r = 0$. Therefore, the singularity at the origin is irremovable and an *intrinsic* singularity of the spacetime.

At the end of section 3.2 we concluded that the Schwarzschild solution was applicable to the exterior of a spherical body. The interior solution would demand a non-zero energy-momentum tensor. In section 3.6 we will see that at the radius $r = 2M$ there is a surface of infinite gravitational red shift. This is called the *Schwarzschild radius* and photons can not escape from this distance. However, for most masses this radius is deep inside the body, for example, the Schwarzschild radius of the Sun is approximately 3km.

However, for sufficiently massive stars, over ten times the mass of the Sun, it is predicted that, after the nuclear fuel is spent, the gravitational forces will pull all the matter within the Schwarzschild radius and inevitably result in a singularity of infinite density. A Schwarzschild black hole is formed. Note that this solution is an idealisation because the contracting body must be perfectly spherical. The result is not really too surprising since Newtonian theory also predicts that a perfect sphere of matter can collapse on itself. If the collapse continues through the Schwarzschild radius then light will not be able to escape the enclosed region. However, there is no resulting spacetime singularity in Newtonian theory.

A lot of work has been done to determine the outcome of the general, and more physically realistic, case of a non spherically symmetric collapsing body. We will discuss in section 5 that this also leads to a black hole solution called the *Kerr solution* (1963), which is independent of the details of the collapse.

3.4 The event horizon

At $r = 2M$ the radial component of (3.2.16) vanishes. This implies that the radial coordinate is null on this surface and the Schwarzschild radius is a *null hypersurface*. This means that a particle must travel in a null (light-like) direction in order to remain in this surface. In fact, in this section and the next we will argue that light can remain on this surface but massive particles (on timelike geodesics) inevitably proceed to decreasing values of r and to the singularity at $r = 0$. We proceed by analysing spacetime diagrams for *radial null geodesics* and the trajectory of a radially infalling particle.

The spacetime diagrams of null geodesics allow us to examine the light cone structure of a region. In Minkowski space we draw the light cones with angles of 45°. In a curved space the light cone angle will vary from point to point.

The radial null geodesics are defined by requiring

$$ds^2 = \dot{\theta}^2 = \dot{\phi}^2 = 0\,. \tag{3.4.1}$$

We use the variational method for computing geodesics. From (2.4.7) we have

$$(1 - 2M/r)\dot{t}^2 - (1 - 2M/r)^{-1}\dot{r}^2 = 0\,, \tag{3.4.2}$$

The Euler-Lagrange equation (2.4.9), for $a = 0$, gives

$$(1 - 2M/r)\dot{t} = k\,, \tag{3.4.3}$$

where k is constant. Substituting into (3.4.2) gives

$$\dot{r} = \pm k\,. \tag{3.4.4}$$

From (3.4.3) and (3.4.4) we have

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = \frac{r}{r - 2M}\,, \tag{3.4.5}$$

which leads to

$$t = r + 2M \ln |r - 2M| + \text{constant}\,. \tag{3.4.6}$$

Or taking the negative sign in (3.4.4)

$$t = -(r + 2M \ln |r - 2M| + \text{constant})\,. \tag{3.4.7}$$

In the region $r > 2M$ (3.4.6) indicates that r increases as t increases so we define the curves governed by (3.4.6) to be the *radial null outgoing geodesics*. Similarly (3.4.7) describes *radial null ingoing geodesics*. Figure 1 displays the computed trajectories of these curves. It is easy to see the light cones 'closing' as the horizon is approached and beyond the horizon the curves 'tip over' so that r becomes a timelike coordinate and the geodesic inevitably leads to the singularity.

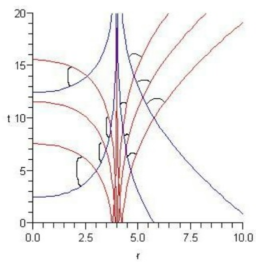

Figure 1: Radial null geodesics of Schwarzschild spacetime

The diagram also seems to suggest that an ingoing radial null geodesic (curve of increasing t, decreasing r) only approaches the horizon asymptotically and never reaches it. We will now see that this is due to the difference between the Schwarzschild world time t, and the proper time τ.

Consider a radially infalling particle. From the variational principle we have

$$\begin{aligned} (1 - 2M/r)\dot{t} &= k \\ (1 - 2M/r)\dot{t}^2 - (1 - 2M/r)^{-1}\dot{r}^2 &= 1\,, \end{aligned} \tag{3.4.8}$$

where a dot denotes differentiation with respect to proper time. Setting $k = 1$ we find, at large r, $\dot{t} \approx 1$, so $t \approx \tau$ towards infinity. Then (3.4.8) gives

$$\left(\frac{d\tau}{dr}\right)^2 = \frac{r}{2M}\,, \tag{3.4.9}$$

which integrates to give

$$\tau - \tau_0 = \frac{2}{3(2M)^{1/2}}(r_0^{3/2} - r^{3/2})\,, \tag{3.4.10}$$

where r_0 is the radius at initial proper time τ_0. Now, instead we describe the motion in terms of the Schwarzschild world time t givng

$$\frac{dt}{d\tau} = -\left(\frac{r}{2M}\right)^{1/2}(1 - 2M/r)^{-1}\,, \tag{3.4.11}$$

which integrates to give

$$\begin{aligned} t - t_0 &= -\frac{2}{3(2M)^{1/2}}(r^{3/2} - r_0^{3/2} + 6Mr^{1/2} - 6Mr_0^{1/2}) \\ &+ 2M\ln\frac{(r^{1/2} + (2M)^{1/2})(r_0^{1/2} - (2M)^{1/2})}{(r_0^{1/2} + (2M)^{1/2})(r^{1/2} - (2M)^{1/2})}\,. \end{aligned} \tag{3.4.12}$$

For convenient values of the constants, r_0, τ_0 and t_0, the equations are plotted in Figure 2 .

The graph clearly shows that in world time t (upper curve) the particle approaches the horizon asymptotically but proceeds through the horizon and to the singularity at $r = 0$ when measured in proper time τ (lower curve). We therefore conclude that an observer at infinity would never see a particle disappearing through the horizon whereas a comoving observer would reach the horizon in a finite proper time.

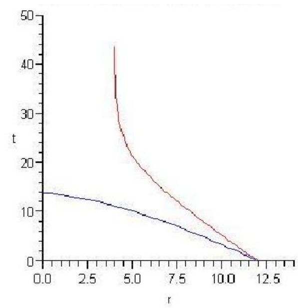

Figure 2: Comparison of trajectories in world time and proper time

3.5 Eddington-Finkelstein coordinates.

In section 3.4, page 16, we mentioned that at $r = 2M$ there is a removable singularity and in the previous section we have seen how this surface splits the manifold into two distinct parts. In this section we change to a coordinate system which is adapted to radial infall and which removes the coordinate singularity at $r = 2M$.

We make the transformation

$$t \to \bar{t} = t + 2M\ln(r - 2M)\,, \tag{3.5.1}$$

so that (3.4.7) becomes

$$\bar{t} = -r + \text{constant}\,, \tag{3.5.2}$$

which is a straight line of $-45°$ with the r-axis. From (3.5.1) we have

$$d\bar{t} = dt + \frac{2M}{r-2M}dr\,, \tag{3.5.3}$$

which leads to

$$\begin{aligned}
dt^2 &= \left(d\bar{t} - \frac{2M}{r-2M}dr\right)^2 \\
&= d\bar{t}^2 + \frac{4M^2}{(r-2M)^2}dr^2 - \frac{4M}{r-2M}d\bar{t}dr \\
\Rightarrow ds^2 &= \left(1-\frac{2M}{r}\right)\left[d\bar{t}^2 + \frac{4M^2}{(r-2M)^2}dr^2 - \frac{4M}{r-2M}d\bar{t}dr\right] \\
&\quad - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\
&= \left(1-\frac{2M}{r}\right)d\bar{t}^2 + \frac{(r-2M)}{r}\frac{4M^2}{(r-2M)^2}dr^2 - \frac{(r-2M)}{r}\frac{4M}{(r-2M)}d\bar{t}dr \\
&\quad - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\
&= \left(1-\frac{2M}{r}\right)d\bar{t}^2 + \left(\frac{4M^2}{r(r-2M)} - \frac{r}{r-2M}\right)dr^2 \\
&\quad - \frac{4M}{r}d\bar{t}dr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\
&= \left(1-\frac{2M}{r}\right)d\bar{t}^2 - \frac{4M}{r}d\bar{t}dr - \left(1+\frac{2M}{r}\right)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,.
\end{aligned} \tag{3.5.4}$$

This is the *advanced Eddington-Finkelstein* form of the line element. The spacetime diagram for this coordinate system is shown in figure 3. The coordinate singularity at $r = 2M$ no longer exists in this form. The solution is regular for the whole range $0 < r < \infty$. However, the solution is no longer time symmetric. The transformation (3.5.1) has 'straightened out' the ingoing radial null geodesics (3.5.2) but the outgoing radial null geodesics (3.5.1) are curves. Also shown is the trajectory of an ingoing particle. Notice that it must 'follow' the light cones as they 'tip over'.

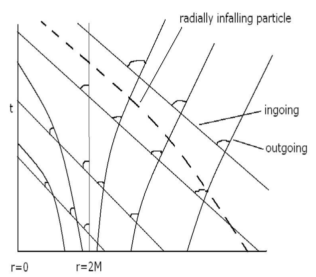

Figure 3: Advanced Eddington-Finkelstein coordinates

We can make further simplification of the metric (3.5.4) by transforming to the null coordinate

$$v + \bar{t} + r\,. \tag{3.5.5}$$

The ingoing geodesics are then given by $v = \text{const}$. The metric transforms as

$$\begin{aligned}
d\bar{t} &= dv - dr \\
\Rightarrow ds^2 &= \left(1 - \frac{2M}{r}\right)(dv^2 + dr^2 - 2dvdr) - \frac{4M}{r}(dv - dr)dr \\
&\quad - \left(1 + \frac{2M}{r}\right) dr^2 - r^2 d\Omega^2 \\
&= \left(1 - \frac{2M}{r}\right) dv^2 + \left(-\frac{4M}{r} + \frac{4M}{r}\right) dr^2 \\
&\quad - \left(2\left(1 - \frac{2M}{r}\right) + \frac{4M}{r}\right) dvdr - r^2 d\Omega^2 \\
&= \left(1 - \frac{2M}{r}\right) dv^2 - 2dvdr - r^2 d\Omega^2 , \qquad (3.5.6)
\end{aligned}$$

which is a simple form known as *advanced Eddington-Finkelstein* coordinates.

Alternatively, if we make the transformation

$$t \to t^* = t - 2M \ln |r - 2M| \qquad (3.5.7)$$

then, from (3.4.7), we see that the outgoing radial null geodesics are 'straightened' out and the ingoing are just given by (3.4.7). The spacetime is similar to Figure 3 except with time coordinate reversed.

We can further make the transformation

$$w = t^* - r , \qquad (3.5.8)$$

which leads to the *retarded Eddington-Finkelstein* form:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dw^2 + 2dwdr - r^2 d\Omega^2 . \qquad (3.5.9)$$

The above shows how coordinate singularities can be removed by a coordinate change. However, the radius $2M$ still has physical significance due to the radial coordinate becoming timelike when $r < 2M$.

3.6 Gravitational red shift

The surface $r = 2M$ can also be distinguised by the behaviour of the physical phenomenon, *gravitational red shift* ([1], section 6.3). As in ordinary red shift this signifies light losing energy as it's frequency is shifted to the 'red' end of the spectrum, except in this case the effect is due to the presence of a massive body causing the spacetime to be non-flat.

We consider two static observers in a stationary spacetime with 4-velocities u_1^a and u_2^a. Static observers necessarily move along time translation Killing vector field.

If we have Killing vector field ξ^a and tangent to a geodesic u^a then $\xi^a u_a$ is constant on the geodesic. We can see this from

$$\begin{aligned}
u^b \nabla_b (\xi_a u^a) &= u^b u^a \nabla_b \xi_a + \xi_a u^b \nabla_b u^a \\
&= 0 , \qquad (3.6.1)
\end{aligned}$$

where the first term on the right is zero from Killing's equation and the second term also zero because u is tangent to a geodesic.

If u_1^a is the tangent of observer 1's timelike geodesic and k^a is the tangent to the light emitted by the observer, then

$$\omega_1 = k_a u_1^a\big|_{p_1}\,, \tag{3.6.2}$$

where ω_1 is the frequancy of the light and p_1 is the position of the observer. Similarly for observer 2:

$$\omega_2 = k_a u_2^a\big|_{p_2}\,. \tag{3.6.3}$$

However, u_1^a and u_2^a are unit vectors and point in the direction of ξ^a, so can be written

$$u_1^a = \xi^a/(\xi^b\xi_b)^{1/2}\big|_{p_1} \tag{3.6.4}$$
$$u_2^a = \xi^a/(\xi^b\xi_b)^{1/2}\big|_{p_2}\,. \tag{3.6.5}$$

From (3.6.1) we see that $k_a\xi^a\big|_{p_1} = k_a\xi^a\big|_{p_2}$ so, for the ratio of frequencies, we obtain

$$\frac{\omega_1}{\omega_2} = \frac{(\xi^b\xi_b)^{1/2}\big|_{p2}}{(\xi^b\xi_b)^{1/2}\big|_{p1}}\,. \tag{3.6.6}$$

For the Schwarzschild metric we have $\xi^a\xi_a = g_{tt} = (1-2M/r)$ so that for $p_2 = r_2$ and $p_1 = r_1$ we have

$$\frac{\omega_1}{\omega_2} = \frac{(1-2M/r_2)^{1/2}}{(1-2M/r_1)^{1/2}}\,. \tag{3.6.7}$$

So if $r_2 > r_1$ then $\omega_2 < \omega_1$. Furthermore, as $r_1 \to 2M$ then $\omega_1/\omega_2 \to \infty$ and we see that $r = 2M$, in the Schwarzschild solution, is a surface of infinite gravitational red shift. Notice that the event horizon and surface of infinite gravitational red shift coincide for the Schwarzschild solution. As we shall see this isn't always the case.

4 Charged black holes

In this section we consider black holes with net charge resulting in the Reissner-Nordström solution. Unlike the previous section for the Schwarzschild solution we impose staticity from the outset. We also search for a spherically symmetric solution so we can work from (3.1.13) with ν and λ functions of r only.

We assume that there is a radial electric field $E(r)$ so that the Maxwell tensor, defined as

$$F_{ab} = \partial_a A_b - \partial_b A_a\,, \tag{4.0.8}$$

has the form

$$F_{ab} = E(r)\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.0.9}$$

We will consider source free regions only, so that, in general relativity, the usual Lagrangian for the electromagnetic field becomes

$$\mathcal{L}_E = \frac{(-g)^{\frac{1}{2}}}{8\pi} F^{ab} F_{ab} \,, \tag{4.0.10}$$

where we have included the term for the invariant volume element as described in section 2. We have defined the energy-momentum tensor in (2.3.3). So varying the Lagrangian with respect to the metric gives

$$\begin{aligned} \frac{\delta \mathcal{L}_E}{\delta g^{\mu\nu}} &= \frac{\partial \mathcal{L}_E}{\partial g^{\mu\nu}} \\ &= \frac{1}{8\pi}\left((-g)^{\frac{1}{2}} \left(g^{bd} F_{\mu b} F_{\nu d} + g^{ac} F_a \mu F_{c\nu}\right) - \frac{1}{2} g_{\mu\nu} (-g)^{\frac{1}{2}} F_{ab} F^{ab}\right) \\ &= \frac{(-g)^{\frac{1}{2}}}{8\pi}\left(2 g^{bd} F_{\mu b} F_{\nu d} - \frac{1}{2} g_{\mu\nu} F_{ab} F^{ab}\right) , \end{aligned} \tag{4.0.11}$$

where we have used (2.2.8). From (2.3.3), the Einstein equation is

$$G_{ab} = -2 g^{cd} F_{ac} F_{bd} + \frac{1}{2} g_{ab} F_{cd} F^{cd} \,. \tag{4.0.12}$$

In source free regions the Maxwell equations are

$$\nabla_a F_{bc} = 0 \,, \tag{4.0.13}$$

$$\partial_{[a} F_{bc]} = 0 \,. \tag{4.0.14}$$

The single, non-zero component of (4.0.13) is

$$\frac{1}{2} r^2 \lambda' E(r) + r^2 \nu' E(r) - 2 r^2 E(r)' - 4 r E(r) = 0 \,, \tag{4.0.15}$$

which gives

$$E(r) = \frac{Q e^{\frac{1}{2}(\nu+\lambda)}}{r^2} \,, \tag{4.0.16}$$

where Q is the integration constant. We assume that far from the black hole the metric (3.1.13) becomes flat so that for large r: $\nu, \lambda \to 0$. This implies that $E \approx Q/r^2$ asymptotically and we regain the usual electric field with charge Q.

Computing the Einstein equation gives four relations. The 00 and 11 equations are (see Worksheet 2)

$$\frac{Q^2}{r^2 e^{\lambda}} = -e^{\nu} \frac{(\lambda' r + e^{\lambda} - 1)}{r^2 e^{\lambda}} \tag{4.0.17}$$

$$-\frac{Q^2}{r^2 e^{\nu}} = \frac{-\nu' r + e^{\lambda} - 1}{r^2} \tag{4.0.18}$$

Adding, we find that

$$\lambda' + \nu' = 0 \,, \tag{4.0.19}$$

which implies $\lambda = -\nu$, using the asymptotic condition above. The 22 component gives

$$E^2 r e^{\nu} = -\frac{1}{2}(2\nu' + r\nu'' + r\nu'^2) \,, \tag{4.0.20}$$

which solves to give

$$e^{\nu} = \frac{Q^2 + Cr^2 - Dr}{r^2} . \qquad (4.0.21)$$

After absorbing the constant C and identifying $D = 2M$ by analogy to the Schwarzschild solution, we find

$$e^{\nu} = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) . \qquad (4.0.22)$$

Inserting this into (3.1.13) we obtain the *Reissner-Nordström solution*

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\phi^2) . \qquad (4.0.23)$$

It is possible to derive this solution without initially demanding it be static. Then Birkhoff's theorem extends to the case of the charged black hole: *A spherically symmetric, charged black hole is necessarily static.*

4.1 The Reissner-Nordström metric

For r satisfying

$$r^2 - 2Mr + Q^2 = 0 , \qquad (4.1.1)$$

the solution (4.0.23) is not well defined. The solutions to this quadratic are

$$r_{\pm} = M \pm (M^2 - Q^2)^{1/2} . \qquad (4.1.2)$$

Here we only consider the case where $M^2 \geq Q^2$ because, according to the singularity theorem of Penrose, the alternative would give rise to naked singularities which are forbidden by the *cosmic censor conjecture* (see [1], section 12.1).

As well as the singularity at $r = 0$ the two cases of (4.1.2) imply that the spacetime is split into three regular regions.:

1. $r_+ < r < \infty$,
2. $r_- < r < r_+$,
3. $0 < r < r_-$.

We now make a coordinate transformation analogous to the Eddington-Finkelstein transformation. We can then analyse the behaviour of radial null geodesics in the three regions. To compute the radial null geodesics we start with (2.4.7):

$$\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \dot{t}^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} \dot{r}^2 = 0 , \qquad (4.1.3)$$

and the Euler-Lagrange equation (2.4.9) for $a = 0$

$$\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \dot{t} = C , \qquad (4.1.4)$$

where C is a constant. Combining (4.1.3) and (4.1.4) gives

$$\begin{aligned} \dot{r}^2 &= C^2 \\ \dot{r} &= \pm C\,. \end{aligned} \tag{4.1.5}$$

Then, taking the positive sign:

$$\begin{aligned} \frac{\dot{t}}{\dot{r}} = \frac{dt}{dr} &= \pm\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} \\ &= \frac{r^2}{r^2 - 2Mr + Q^2} \\ &= \frac{r^2}{(r - r_+)(r - r_-)}\,, \end{aligned} \tag{4.1.6}$$

where $r_\pm$ are the solutions of (4.1.2).

We are searching for straight lines so dt/dr is constant and we can write

$$r^2 = (r - r_+)(r - r_-)a + b\,, \tag{4.1.7}$$

where a, b are constant. If we set $a = 1$ and expand

$$\begin{aligned} r^2 &= r^2 - r_+ r - r_- r + r_+ r_- + b \\ \Rightarrow b &= (r_+ + r_-)r - r_+ r_-\,. \end{aligned} \tag{4.1.8}$$

Then

$$\begin{aligned} \frac{r^2}{(r - r_+)(r - r_-)} &= \frac{dt}{dr} \\ &= 1 + \frac{(r_+ + r_-)r - r_+ r_-}{(r - r_+)(r - r_-)} \\ &= 1 + \frac{{r_+}^2}{(r_+ - r_-)(r - r_+)} + \frac{{r_-}^2}{(r_- - r_+)(r - r_-)} \\ \Rightarrow t &= r + \frac{{r_+}^2}{(r_+ - r_-)}\ln(r - r_+) + \frac{{r_-}^2}{(r_- - r_+)}\ln(r - r_-) + \text{const}\,. \end{aligned}$$

To 'straighten' the ingoing, radial null geodesics we define null coordinate $\bar{t} = -r + \text{const}$, which gives

$$\bar{t} = t + \frac{r_+^2}{(r_+ - r_-)}\ln(r - r_+) - \frac{{r_-}^2}{(r_+ - r_-)}\ln(r - r_-)\,, \tag{4.1.9}$$

and

$$\begin{aligned} dt &= d\bar{t} - \frac{r_+^2\,dr}{(r_+ - r_-)(r - r_+)} + \frac{r_-^2\,dr}{(r_+ - r_-)(r - r_-)} \\ dt^2 &= \left(d\bar{t} + \left(\frac{r_-^2}{(r_+ - r_-)(r - r_-)} - \frac{r_+^2}{(r_+ - r_-)(r - r_+)}\right)dr\right)^2 \\ &= d\bar{t}^2 + \left(\frac{r_-^4}{(r_+ - r_-)^2(r - r_-)^2} + \frac{r_+^4}{(r_+ - r_-)^2(r - r_+)^2}\right. \\ &\quad \left. - \frac{2r_-^2 r_+^2}{(r_+ - r_-)(r - r_-)(r - r_+)}\right)dr^2 + 2\left(\frac{r_-^2}{(r_+ - r_-)(r - r_-)} - \frac{r_+^2}{(r_+ - r_-)(r - r_+)}\right)dr d\bar{t}\,. \end{aligned} \tag{4.1.10}$$

Substituting this into metric (4.0.23):

$$\begin{aligned} ds^2 &= \frac{(r-r_-)(r-r_+)}{r^2}\left(d\bar{t}^2 + \left(\frac{r_-^4}{(r_+-r_-)^2(r-r_-)^2} + \frac{r_+^4}{(r_+-r_-)^2(r-r_+)}\right.\right. \\ &- \frac{2r_-^2 r_+^2}{(r_+-r_-)(r-r_-)(r-r_+)}dr^2 + 2\left(\frac{r_-^2}{(r_+-r_-)(r-r_-)} - \frac{r_+^2}{(r_+-r_-)(r-r_+)}\right)drd\bar{t} \\ &- \frac{r^2}{(r-r_-)(r-r_+)}dr^2 - r^2 d\Omega^2 \end{aligned}$$

and, as in [2], defining $f = 1 - g_{00} = (rr_- + rr_+ - r_-r_+)/r^2$ we obtain (see Worksheet 2)

$$\begin{aligned} ds^2 &= \frac{(r-r_-)(r-r_+)}{r^2}d\bar{t}^2 - 2\frac{rr_+ + rr_- - r_+r_-}{(r-r_+)(r-r_-)}d\bar{t}dr \\ &\quad - \frac{r^2 + rr_- + rr_+ - r_+r_-}{r^2}dr^2 - r^2 d\Omega^2 \\ &= (1-f)d\bar{t}^2 - 2fd\bar{t}dr - (1+f)dr^2 - r^2 d\Omega^2 \,. \end{aligned} \tag{4.1.11}$$

This form is regular for all r, retaining the intrinsic singularity at $r = 0$. We now show that this metric 'straightens' ingoing radial null geodesics and so is the advanced Eddington-Finkelstein form of the Reissner-Nordström metric.

Calculating radial null geodesics for (4.1.11):

$$\dot{\theta} = \dot{\phi} = ds^2 = 0 \,. \tag{4.1.12}$$

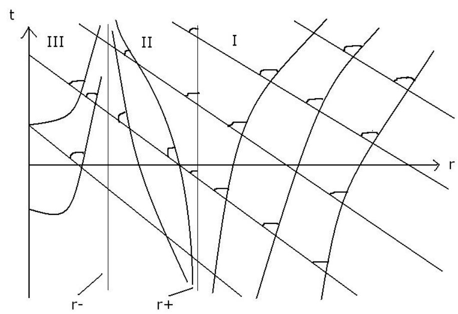

Figure 4: Advanced Eddington-Finkelstein form of the Reissner-Nordström metric

Then (2.4.7) gives

$$\begin{aligned} & (1-f)\dot{\bar{t}}^2 - 2f\dot{\bar{t}}\dot{r} - (1+f)\dot{r}^2 = 0 \\ \Rightarrow \quad & (1-f)\left(\frac{d\bar{t}}{dr}\right)^2 - 2f\frac{d\bar{t}}{dr} - (1+f) = 0 \,. \end{aligned} \tag{4.1.13}$$

Solving as a quadratic in $d\bar{t}/dr$ gives

$$\begin{aligned} \frac{d\bar{t}}{dr} &= \frac{2f \pm \sqrt{4f^2 + 4(1-f)(1+f)}}{2(1-f)} \\ &= \frac{f \pm 1}{1-f} = -1 \text{ or } \frac{1+f}{1-f} \,. \end{aligned} \tag{4.1.14}$$

The first case describes the ingoing radial null geodesics, which are the straight lines

$$\bar{t} + r = \text{const} \,. \tag{4.1.15}$$

The second gives a differential equation for the outgoing family

$$\frac{d\bar{t}}{dr} = \frac{1+f}{1-f} \,. \tag{4.1.16}$$

As in [2], section 18.3, we can obtain all the information we need to draw a spacetime diagram from this equation, without solving it, by analysing the slope for different values of r i.e plotting functions $1+f$ and $1-f$. The resulting radial null geodesics are shown in Figure 4.1.

Returning to (4.1.2) we see that the minimum possible mass of a charged black hole is $M = Q$. A smaller mass would imply a naked singularity, which is thought to be unphysical. Black holes that have the minimum possible mass are called *extremal*. The extremal black hole is important in contemporary black hole physics from the point of unified theories such as string theory. For the extremal black hole, string theories correctly account for the microscopic degrees of freedom that should be present if the black hole has entropy proportional to it's surface area. We will discuss the entropy of a black hole in section 8. The black holes of the attractor mechanism, section 9, are also extremal black holes except they also have magnetic charge.

5 Stationary, axisymmetric black holes

In this section we discuss a particular black hole solution known as the *Kerr metric* (1963). This is a stationary, axisymmetric solution and can be considered as the description of the gravitational field due to a rotating, massive collapsed body. It is necessary to include this case here because, as we shall see, we must consider black holes with angular momentum in order to make analogies with thermodynamics.

Also, the Kerr metric is important because there are theorems due to Israel, Carter, Hawking and Robinson, between 1967 and 1975 [1], which virtually prove that the Kerr metric is the most general stationary, vacuum black hole solution. All collapsing matter is thought to settle to a final stationary state. So the Kerr black hole solution is physically realistic and, in fact, there is evidence suggesting that the 'supermassive' black hole at the center of the galaxy can be described by this model.

Here we give the standard form of the solution, a detailed derivation of which can be found in [10]:

$$\begin{aligned} ds^2 &= \left(1 - \frac{2Mr}{\Sigma_1}\right) dt^2 - \frac{4aMr\sin^2\theta}{\Sigma_1} d\phi dt \\ &\quad - \left(\frac{2Ma^2 r \sin^2\theta}{\Sigma_1} + r^2 + a^2\right)\sin^2\theta d\phi^2 - \Sigma_1\left(\frac{dr^2}{\Sigma_2} + d\phi^2\right), \end{aligned} \tag{5.0.17}$$

where $\Sigma_1 = r^2 + a^2\cos^2\theta$ and $\Sigma_2 = r^2 - 2Mr + a^2$. Notice that the coordinate time t does not enter into the metric components so the solution is stationary with Killing vector field $\partial/\partial t$. The metric is also independent of the coordinate ϕ so is invariant under $\phi \to \phi + a$. Therefore, we assume that there exists a Killing vector field $\partial/\partial\phi$. Indeed, Hawking and Ellis (1973) have shown that, for the Kerr black hole, there exists a one parameter group of isometries, which commute with the stationary isometry $\partial/\partial t$ and whose tangents coincide with the null geodesic generators of the horizon.

The Killing vector field χ^a, which is null, $\chi_a\chi^a = 0$, on the event horizon is given by the linear combination

$$\chi^a = \xi^a + \Omega_{BH}\psi^a\,, \tag{5.0.18}$$

where ψ^a is a Killing vector field whose orbits are closed i.e an axial Killing vector associated with invariance under translation of ϕ. We then associate Ω_{BH} with the angular velocity of the black hole.

The solution (5.0.17) contains two parameters, a and M. It is easy to see that setting $a = 0$ we obtain the Schwarzschild solution and so we identify M as the geometric mass of the spacetime. Also notice that the second term in (5.0.17) causes the solution to lose invariance under $t \to -t$. However, the combined transformation $t \to -t$ and $\phi \to -\phi$ leaves the solution invariant. A rotating body of matter in Newtonian theory also has this property and so the Kerr metric is believed to describe a rotating black hole.

However, we have not yet discussed the singularities of the Kerr metric but will see next that it does describe a black hole solution except one with a very different structure to the Schwarzschild case. Calculation of the Riemann invariant $R^{abcd}R_{abcd}$ reveals that there is an intrinsic singularity when

$$r^2 + a^2\cos^2\theta = 0\,. \tag{5.0.19}$$

This defines a circle of radius a and indicates that the Kerr metric has a *ring singularity.*

In analogy with the Schwarzschild solution we search for the event horizon by finding when $r =$ const. becomes null. This is when $g^{11} = 0$ and given by

$$\frac{\Sigma_2}{\Sigma_1} = \frac{r^2 - 2Mr + a^2}{r^2 + a^2\cos^2\theta} = 0 \tag{5.0.20}$$

or

$$r_\pm = M \pm (M^2 - a^2)^{1/2}\,. \tag{5.0.21}$$

Therefore, assuming $a^2 < m^2$, we have two event horizons, Σ_+ (outer) and Σ_- (inner) and can divide the spacetime into three regions where the metric is regular:

1. $r_+ < r < \infty$,
2. $r_- < r < r_+$,
3. $0 < r < r_-$.

No outgoing timelike or null geodesic can cross Σ_+ so it is the event horizon for the Kerr metric.

Calculation of the norm of the timelike Killing vector gives

$$\xi_a\xi^a = g_{tt} = 1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\,. \tag{5.0.22}$$

This defines two surfaces of infinite gravitational red shift (see section 3.6) when

$$r^2 + \cos^2\theta - 2Mr = 0\,, \tag{5.0.23}$$

which gives

$$R_\pm = M \pm (M^2 - a^2\cos^2\theta)^{1/2}\,. \tag{5.0.24}$$

Therefore, comparing (5.0.22) and (5.0.24) we have always $R_+ \geq r_+$ and $R_- \leq r_-$.

At $r > R_+$ there exists timelike Killing vector ξ^a. However, (5.0.22) reveals that for $R_- < r < R_+$, $\xi_a \xi^a < 0$ and so ξ^a is spacelike in this region and timelike again when $r < R_-$.

At R_+, R_- there are surfaces S_+ and S_- respectively, as shown in Figure 5, sourced from [10]. The region of $\Sigma_+ < r < S_+$ is called the *ergosphere* of the black hole. ξ^a is spacelike in this region and so a particle would have to travel faster than light to remain on an orbit of ξ^a. Thus it is impossible for a particle to remain stationary in this region and so S_+ is called the *stationary limit surface*. However, it is still possible for the particle to escape to infinity from the ergosphere. The various surfaces of the Kerr black hole can be seen in Figure 5.

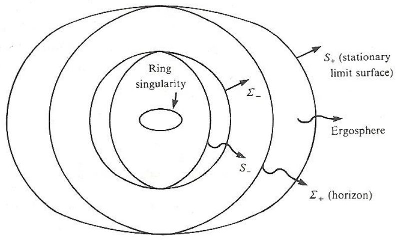

Figure 5: structure of the Kerr black hole

In the Schwarzschild limit $a \to 0$; $S_+ \to 2M$, $S_- \to 0$, $\Sigma_+ \to 2M$ and $\Sigma_- \to 0$. Thus the event horizons and surfaces of infinite gravitational red shift are recombined as we should expect.

6 Geodesics and Causality

In this section we discuss *geodesic congruences* and derive results used in the formulation of the black hole laws based on geodesic relations. We also briefly discuss arguments based on the causal structure of spacetime, which are used, in particular, to derive the second law.

As discussed in section 2.4 geodesics are curves of extremal length. Given the tangent vector ξ^a of a curve we say that the curve is a geodesic if the tangent vector is parallely propagated along the curve

$$\xi^a \nabla_a \xi^b \propto \xi^b \,. \tag{6.0.25}$$

However, it is always possible to reparametrise the curve so that

$$\xi^a \nabla_a \xi^b = 0 \,, \tag{6.0.26}$$

and in this case the parameter is said to be *affine* on the geodesic. For example, the proper time is an affine parameter.

6.1 The Equation of Geodesic Deviation.

A congruence is a family of curves such that precisely one curve passes through each point of the manifold. We parametrise a congruence of geodesics by proper time, τ, and ν, where ν labels distinct, adjacent geodesics. We define two vector fields on a geodesic with coordinates x^a,

$$\xi^b = \frac{dx^b}{d\tau} \tag{6.1.1}$$

and

$$\eta^c = \frac{dx^c}{d\nu} \,. \tag{6.1.2}$$

Then ξ^b is the tangent vector and η^c is a vector field, orthogonal to the geodesic, that we shall call the deviation vector.

We work in a manifold with a symmetric connection (torsion free) $\Gamma^a_{bc} = \Gamma^a_{cb}$ so we can replace partial derivatives by covariant derivatives in the Lie derivative and we have

$$\begin{aligned} L_\xi \eta^a &= \xi^b \partial_b \eta^a - \eta^b \partial_b \xi^a \\ &= \frac{dx^b}{d\tau}\frac{\partial}{\partial x^b}\frac{dx^a}{d\nu} - \frac{dx^b}{d\nu}\frac{\partial}{\partial x^b}\frac{dx^a}{d\tau} \\ &= \frac{d^2x^a}{d\tau d\nu} - \frac{d^2x^a}{d\nu d\tau} \\ &= 0 \\ &= \xi^b \nabla_b \eta^a - \eta^b \nabla_b \xi^a \end{aligned} \tag{6.1.3}$$

We define the absolute derivative along a geodesic as

$$\frac{D}{D\tau} = \xi^b \nabla_b \,, \tag{6.1.4}$$

and interpret

$$\frac{D^2\eta^a}{D\tau^2} = \xi^c \nabla_c (\xi^b \nabla_b \eta^a) \,, \tag{6.1.5}$$

as being the relative acceleration of neighbouring geodesics. Using (6.1.3) and the definition of the Riemann tensor (A.0.46) in the third line, we have

$$\begin{aligned} \xi^b \nabla_b (\xi^c \nabla_c \eta^a) &= \xi^c \nabla_c (\eta^b \nabla_b \xi^a) \\ &= (\xi^c \nabla_c \eta^b) \nabla_b \xi^a + \eta^b \xi^c \nabla_c \nabla_b \xi^a \\ &= (\eta^c \nabla_c \xi^b) \nabla_b \xi^a + \eta^b \xi^c \nabla_b \nabla_c \xi^a - R_{cbd}{}^a \eta^b \xi^c \xi^d \\ &= \eta^c \nabla_c (\xi^b \nabla_b \xi^a) - \eta^c \xi^b \nabla_c \nabla_b \xi^a + \eta^b \xi^c \nabla_b \nabla_c \xi^a - R_{cbd}{}^a \eta^b \xi^c \xi^d \\ &= -R_{cbd}{}^a \eta^b \xi^c \xi^d \end{aligned} \tag{6.1.6}$$

This gives

$$\frac{D^2\eta^a}{D\tau^2} = -R_{cbd}{}^a \eta^b \xi^c \xi^d \,, \tag{6.1.7}$$

which we call the *equation of geodesic deviation* and will use later on.

6.2 Geodesic congruences

We can define a vector field ξ^a of tangents to the geodesic congruence. We assume that the congruence is a family of timelike geodesics parametrised by proper time, τ. For a timelike geodesic congruence we normalise the tangents as $\xi^a \xi_a = 1$. We define the tensor field

$$B_{ab} = \nabla_b \xi_a \,. \tag{6.2.1}$$

This gives

$$B_{ab}\xi^a = \nabla_b \xi_a \xi^a = 0 \tag{6.2.2}$$

and

$$B_{ab}\xi^b = -\nabla_a \xi_b \xi^b = 0\,. \tag{6.2.3}$$

Therefore, this tensor field is everywhere orthogonal to ξ^a, i.e it is purely spatial.

Let η^a be the orthogonal deviation vector from one of the geodesics in our congruence to another. We then have

$$L_\xi \eta^a = 0\,, \tag{6.2.4}$$

where L_ξ is the Lie derivative with respect to ξ. This gives

$$\xi^b \nabla_b \eta^a = \eta^b \nabla_b \xi^a = B^a{}_b \eta^b\,. \tag{6.2.5}$$

Thus, $B^a{}_b$ is a linear map measuring the failure of η^a to be parallelly transported along ξ^b.

We use the spatial projection operator,

$$p^a{}_b \equiv \delta^a{}_b - \xi^a \xi_b\,, \tag{6.2.6}$$

to define the spatial metric

$$\begin{aligned} h_{ab} &= \delta^c{}_a g_{cb} - \xi^c \xi_a g_{cb} \\ &= g_{ab} - \xi_a \xi_b\,. \end{aligned} \tag{6.2.7}$$

We now introduce the expansion θ, shear σ_{ab} and twist ω_{ab} of the congruence by

$$\theta = B^{ab} h_{ab}\,, \tag{6.2.8}$$

$$\sigma_{ab} = B_{(ab)} - \frac{1}{3}\theta h_{ab}\,, \tag{6.2.9}$$

$$\omega_{ab} = B_{[ab]}\,, \tag{6.2.10}$$

We can then write

$$B_{ab} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}\,. \tag{6.2.11}$$

We now turn our attention towards an equation for the rate of change of θ, σ_{ab}, and ω_{ab}. We have

$$\begin{aligned} \xi^c \nabla_c B_{ab} &= \xi^c \nabla_c \nabla_b \xi_a = \xi^c \nabla_b \nabla_c \xi_a + R_{cba}{}^d \xi^c \xi_d \\ &= \nabla(\xi^c \nabla_c \xi_a) - (\nabla_b \xi^c)(\nabla_c \xi_a) + R_{cba}{}^d \xi^c \xi_d \\ &= -B^c{}_b B_{ac} + R_{cba}{}^d \xi^c \xi_d\,. \end{aligned} \tag{6.2.12}$$

where in the first line we have used the definition of the Riemann tensor.

We are predominantely interested in the rate of change of the trace of B_{ab}, i.e θ. So taking the trace of (6.2.12), and using (6.2.8)-(6.2.10), we have

$$\begin{aligned}
B^c{}_b B_{ac} &\rightarrow B^{ca}B_{ac} \\
&= (\frac{1}{3}\theta h^{ca} + \sigma^{ca} + \omega^{ca})(\frac{1}{3}\theta h_{ac} + \sigma_{ac} + \omega_{ac}) \\
&= \frac{1}{9}\theta^2 h^{ca}h_{ac} + \frac{1}{3}\theta h^{ac}\sigma_{ac} + \frac{1}{3}\theta h^{ac}\omega_{ac} + \frac{1}{3}\theta h_{ac}\sigma^{ca} + \sigma^{ca}\sigma_{ac} + \sigma^{ca}\omega_{ac} \\
&\quad + \frac{1}{3}\omega^{ca}\theta h_{ac} + \omega^{ca}\sigma_{ac} + \omega^{ca}\omega_{ac} \\
&= \frac{1}{9}\theta^2 h^{ca}h_{ac} + \frac{1}{3}\theta h^{ac}(B_{(ac)} - \frac{1}{3}\theta h_{ac}) + \frac{1}{3}\theta(B^{(ca)} - \frac{1}{3}\theta h^{ca})h_{ac} \\
&\quad + \sigma^{ca}\sigma_{ac} + \omega^{ca}\omega_{ac} \\
&= \frac{1}{3}\theta^2 + \sigma^{ac}\sigma_{ac} - \omega^{ac}\omega_{ac}\,, \qquad (6.2.13)
\end{aligned}$$

where, in the second line, all the antisymmetric tensors, fully contracted with a symmetric tensor, vanish. In the last line we have used the trace of the identity matrix in 3 dimensions and equation (6.2.8). Therefore, we have, from (6.2.12),

$$\xi^c \nabla_c \theta = \frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{ac}\sigma_{ac} + \omega^{ac}\omega_{ac} - R_{cd}\xi^c\xi^d\,, \qquad (6.2.14)$$

where, for the last term, we have used antisymmetry property (A.0.50) of the Riemann tensor and the definition of the Ricci tensor (A.0.53). The above equation is known as *Raychaudhuri's equation.* The rates of change of σ_{ab} and ω_{ab} can be found by taking the trace-free symmetric and antisymmetric parts of (6.2.12) respectively. However, it was Raychaudhuri's equation that was key to the proof of the singularity theorems, given by Hawking (1967), and it is the equation we shall use in proving the Zeroth Law and Second Law of black hole mechanics.

We now investigate the last term of (6.2.12). Using Einstein's equation we have

$$R_{ab}\xi^a\xi^b = 8\pi\left(T_{ab} - \frac{1}{2}Tg_{ab}\right)\xi^a\xi^b = 8\pi\left(T_{ab}\xi^a\xi^b - \frac{1}{2}T\right). \qquad (6.2.15)$$

Here we shall make some physical assumptions. Firstly, $T_{ab}\xi^a\xi^b$ physically represents the energy density of matter as seen by an observer with 4-velocity ξ^a. It is generally believed that, for all physically reasonable matter,

$$T_{ab}\xi^a\xi^b \geq 0\,. \qquad (6.2.16)$$

This is known as the *weak energy condition.* We further assume that physically reasonable matter will not cause the right hand side of (6.2.15) to become negative, so that

$$T_{ab}\xi^a\xi^b \geq \frac{1}{2}T\,. \qquad (6.2.17)$$

This is known as the *strong energy condition*, so called because it gives a stronger mathematical requirement. From this last requirement we can see that the last term of Raychaudhuri's equation is nonpositive. This is interpreted as a manifestation of the attractiveness of gravity. Finally, we shall state another energy condition for physical matter. If ξ^a is a future directed, timelike vector then $T^a{}_b\xi^b$ will be a future directed timelike or null vector. This quantity physically represents the energy-momentum 4-vector of matter as seen by an observer with 4-velocity ξ^a. Known as the *dominant energy condition*, this is interpreted as requiring that the speed of energy flow be always less than the speed of light.

We return to Raychaudhuri's equation (6.2.14) in the case where our congruence is hypersurface orthogonal, then by Frobenius' theorem (see [1], Appendix B) the term $\omega^{ab} = 0$. We have seen that the last term is nonpositive if the strong energy condition holds and the second term is always nonpositive. This gives

$$\begin{aligned} & \frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0 \\ \Rightarrow \qquad & -\frac{1}{\theta^2}\frac{d\theta}{d\tau} \leq -\frac{1}{3} \\ \Rightarrow \qquad & \frac{d\theta^{-1}}{d\tau} \geq \frac{1}{3}\,, \end{aligned} \tag{6.2.18}$$

and finally

$$\theta^{-1}(\tau) \geq \theta_0^{-1} + \frac{1}{3}\tau\,. \tag{6.2.19}$$

Consider the situatiuon where the congruence is initially converging so that $\theta_0 \leq 0$. Then (6.2.19) implies that θ^{-1} must pass through zero and, consequently, $\theta \to -\infty$. This result proves interesting when the congruence is emanating from a black hole.

Although we do not derive it here, refer to [1], we state the null vector analogue to Raychaudhuri's equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}k^ck^d\,, \tag{6.2.20}$$

where here k^c is a null vector. The difficulty with the case of the null vector is that, because they are self-orthogonal, there is no natural way for them to be normalised. Therefore, it is not as straight forward to compare points on neighbouring geodesics in a congruence of null geodesics.

6.3 Conjugate points

The solution of the geodesic deviation equation, η^a, is called a Jacobi field. Consider a geodesic, γ, with points $p, q \in \gamma$. Then p and q are said to be *conjugate* if η vanishes at both of these points (η^a must not be identically zero along γ). We can picture p and q as being points where a neighbouring geodesic crosses γ and state, without proof (see [1], section 9.3), that they characterise a geodesic failing to be a local maximum of proper time. Our interest is in deriving a relation between the expansion $\theta = \mathrm{tr}(B)$, discussed in the previous section, and the appearance of conjugate points.

Consider a point p, where $\eta^a = 0$. We use the geodesic deviation equation, parametrised so that $\eta^a(\tau)|_p = \eta^a(0)$, to find a conjugate point. Notice that this is a second order, linear ordinary differential equation so the solution must depend linearly upon the initial data $\eta^a(0)$ and $D\eta^a/D\tau(0)$. However, $\eta^a(0) = 0$

by construction so we find

$$\eta^a(\tau) = A^a_b(\tau)\frac{D\eta^b}{D\tau}(0)\,, \tag{6.3.1}$$

where $A^a_b(\tau)$ is a matrix. Substituting the above into (6.1.7) we have

$$\begin{aligned} \frac{D^2}{D\tau^2}A^a_b(\tau)\frac{D\eta^b}{D\tau}(0) &= -R_{cbd}{}^a\xi^c\xi^d A^b_e(\tau)\frac{D\eta^e}{D\tau}(0) \\ \Rightarrow \frac{D^2}{D\tau^2}A^a_b(\tau) &= -R_{cbd}{}^a\xi^c\xi^d A^b_e(\tau) \end{aligned} \tag{6.3.2}$$

From (6.3.1), $A^a_b(0) = 0$ and $DA^a_b/D\tau(0) = \delta^a_b$. Then by (6.3.1), q will be conjugate to p if $\det A^a_b = 0$ at q, provided that $D\eta^b/D\tau(0) \neq 0$ because otherwise η^a would be identically zero along γ.

We look for a relation between $\det A$ and θ. First notice, using (6.1.3), that

$$\begin{aligned} \frac{D\eta^a}{D\tau} &= \xi^b\nabla_b\eta^a \\ &= \eta^b\nabla_b\xi^a \\ &= \eta^a B^b{}_a\,, \end{aligned} \tag{6.3.3}$$

where we have used (6.2.1). However, from (6.3.1) we find

$$\frac{D\eta^a}{D\tau} = \frac{DA^a_b}{D\tau}\frac{D\eta^b}{D\tau}(0)\,, \tag{6.3.4}$$

which gives

$$\frac{DA^a_b}{D\tau}\frac{D\eta^b}{D\tau}(0) = A^a_c B^c_b\frac{D\eta^b}{D\tau}(0)\,. \tag{6.3.5}$$

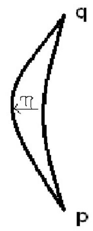

Figure 6: Conjugate points p and q

As matrices we can write

$$\frac{D\,A}{D\tau} = AB\,, \tag{6.3.6}$$

and

$$B = A^{-1}\frac{D\,A}{D\tau}\,. \tag{6.3.7}$$

This gives our equation for θ

$$\theta = \mathrm{tr}(B) = \mathrm{tr}\left[A^{-1}\frac{D\,A}{D\tau}\right]\,. \tag{6.3.8}$$

Along γ, between p and q, we have $\det A \neq 0$ so A^{-1} exists and, for any nonsingular matrix A, we have the following formula

$$\mathrm{tr}\left[A^{-1}\frac{D\,A}{D\tau}\right] = \frac{1}{\det A}\frac{D}{D\tau}(\det A)\,, \tag{6.3.9}$$

which gives

$$\theta = \frac{D}{D\tau}(\ln|\det A|)\,. \tag{6.3.10}$$

We have seen from (6.3.1) that $\eta^a(\tau) \to 0$ when $\det A \to 0$ thus producing a conjugate point. From (6.3.10) we see that this implies $\theta \to -\infty$ at a conjugate point. The converse is also true, namely, $\det A \to 0$ as $\theta \to -\infty$. So we see that a necessary and sufficient condition for there to be a conjugate point at q is that the expansion, θ of the geodesic congruence goes to negative infinity. This is our desired result.

6.4 Causality

In this section we give a brief discussion of the causal structure of spacetime, focusing upon those results that we shall use deriving the second law of black holes. The theorems are mostly derived from intricate toplogical arguments and set theory. For a broader and more detailed discussion we refer the reader to [1] and [5].

Firstly, we make some definitions. We define a manifold with metric, (M, g_{ab}), to be time orientable if there exists a smooth, timelike vector field, t^a, which vanishes nowhere. The smoothness of this vector field implies that we can always distinguish between the notions of forwards and backwards in time, i.e that the light cones do not suddenly 'flip' and reverse orientation. Note that this definition is distinct from general manifold orientability, defined in section 2, where we searched for an invariant volume element on a manifold.

We also require that, in order to be physical, the manifold contain no closed causal curves, which would violate causality. However, the above requirement does not forbid light cones to be on the verge of violating causality. This suggests that an infinitesimal perturbation of the metric would lead to light cones 'flipping over', and violating causality. If no such infinitesimal perturbation exists then we call the spacetime *stably causal.*

We define $I^+(p)$ to be the *chronological future* of point p. This is the set of points that can be reached from p by timelike geodesics. We extend the definition to include a set of points that are the chronological future of set, S, i.e $I^+(S)$. Analogous definitions are listed below (see Figure 7):

$I^-(S)$;	points in the chronological past of S,
$J^+(S)$;	points in the causal future of S ($I^+(S)$+points reached by null geodesics),
$J^-(S)$;	points in the causal past of S,
$D^+(S)$;	points completely dependent on S ($D^+(S)$ can not be reached by points outside S),
$D^-(S)$;	the time reversed of the above.

The definition for $D^+(s)$ implies that given initial data on S, we can completely predict events in $D^+(s)$ and similarly for the time reversed case $D^-(S)$. We call $D(S) = D^+(S) \cup D^-(S)$ the *complete domain of dependence* for S.

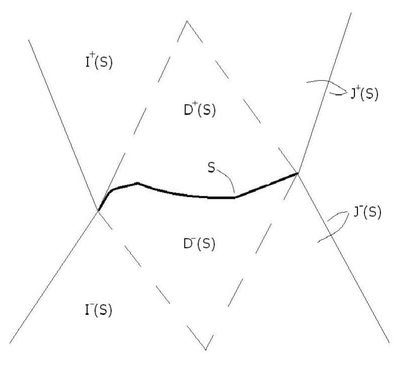

Figure 7: Causal sets of S

An *achronal set* S is such that no points $p, q \in S$ satisfy $q \in I^+(p)$, i.e no two points can be joined by a timelike curve. A closed, achronal set Σ for which $D(\Sigma) = M$, the whole manifold, is called a *Cauchy surface.* It may be thought of as an instant of time throughout the whole universe. If the spacetime possesses a Cauchy surface then it is called *Globally Hyperbolic.* All physically reasonable spacetimes are thought to be globally hyperbolic implying that one can know or predict all events from data on Σ (ignoring chaotic phenomena where Cauchy data would need to be infinitely accurate).

Here, a few more definitions. Let I be a set, then $\dot{I}$ is the boundary of I and $\overline{I}$ is the closure of I, i.e $\overline{I} = I \cup \dot{I}$. We call $\mathcal{J}^+$ future null infinity, points reached by future directed null curves, and $\mathcal{J}^-$ past null infinity for the time reversed case.

We now discuss the second black hole law due to Hawking (1971), [5]. The reasoning we adopt closely follows that of [1] and we state here two theorems from this reference without proof. First, Theorem (9.3.11) of [1]. *Let K be a compact, orientable, two-dimensional spacelike submanifold of M. Then every $p \in \dot{I}^+(K)$ lies on a future directed null geodesic starting from K, which is orthogonal to K and has no point conjugate to points in K.* Secondly, resulting from Theorem (8.3.11) in [1], we have that for K a compact set in a globally hyperbolic spacetime, $J^+(K) = \overline{I^+(K)}$. The above theorems merely state that, for a certain class of spacetimes, the closure of the chronological future is equal to the causal future and the boundary of the chronological future of a set describes null geodesics emenating from that set. So we have: for every $p \in \dot{I}^+(K)$ there exists a past directed null geodesic connecting p to K.

Proceeding, we define a general black hole region B to be

$$B = M - J^-(\mathcal{J}^+)\,. \tag{6.4.1}$$

It consists of all points that can not be connected with future null infinity. Then, let $B \subset (M, g_{ab})$; an asymtotically flat, globally hyperbolic spacetime, where Einstein's equation holds and the weak and strong energy conditions are satisfied by the matter. Consider two Cauchy surfaces Σ_1 and Σ_2, with $\Sigma_2 \subset I^+(\Sigma_1)$ and let $\mathcal{H}_1 = H \cap \Sigma_1$, $\mathcal{H}_2 = H \cap \Sigma_2$, where H is the three dimensional boundary of B, i.e the event horizon. Then $\mathcal{H}$ would be the two dimensional event horizon we picture at an instant of time. The theorem states that $\mathcal{H}_2 \geq \mathcal{H}_1$, the area of the event horizon can only increase or remain constant.

To corroborate the theorem we consider the expansion, θ, of the null geodesic generators of H. Consider, $\theta < 0$ at $p \in H$. Suppose, also, that $p \in \Sigma_1 \Rightarrow p \in \mathcal{H}_1$. We can deform $\mathcal{H}_1$ outward around p to become $\mathcal{H}_2$, so that it enters into $J^-(\mathcal{J}^+)$, maintaining that $\theta < 0$. Let $K \subset \Sigma$ be the compact, spacelike, two-dimensional region that would be found between $\mathcal{H}_1$ and $\mathcal{H}_2$. Let $q \in \mathcal{J}^+$ be a point at future null infinity implying that $q \in \dot{J}^+(K)$. From above we have that the null geodesic generators enter K orthogonally and that there are no conjugate points between K and q. However, (6.2.19) implies that if $\theta_0 < 0$, as we have assumed, then $\theta(\tau) \to -\infty$ in a finite time. But (6.3.10) implies that, as $\theta(\tau) \to -\infty$, a conjugate point arises along the geodesic. Hence, we have a contradiction and conclude that $\theta_0 \geq 0$; the null geodesic congruence expands as it leaves K. Each point of $\mathcal{H}_1$ is naturally mapped to a point on $\mathcal{H}_2$ by the null geodesic congruence. However, this map need not be surjective because $\theta \geq 0$ and we conclude that it is only possible for the area of the event horizon to increase or remain constant.

It is also possible to show, by similar arguments, that a black hole can never bifurcate (see [1], [5]) and that 'naked' singularities can not exist. These type of arguments were employed by Hawking and Penrose to prove the so called Singularity Theorems. These state the general requirements for singularities to appear in a spacetime and they postulate that, under certain physical assumptions, the universe began in a singular state.

7 The Surface Gravity of a Black Hole

For a stationary, axisymmetric black hole we have the two Killing vector fields, ξ^a and ψ^a. The former defines a time translation isometry and the latter a rotational isometry. For the Kerr solution there is a Killing field, χ^a, normal to the horizon, that can be expressed as (see [1])

$$\chi^a = \xi^a + \Omega_{BH}\psi^a \,, \tag{7.0.2}$$

where Ω_{BH}, a constant, is the angular velocity of the black hole. The horizon is a null surface such that $\chi^a\chi_a = 0$ everywhere on the horizon. Therefore, $\nabla^a(\chi^b\chi_b)$ is normal to the horizon and, consequently, proportional to the Killing field such that

$$\nabla^a\left(\chi^b\chi_b\right) = -2\kappa\chi^a \,, \tag{7.0.3}$$

where κ is some function of the coordinates. We can write (7.0.3) as

$$\chi^b\nabla_a\chi_b = -\chi^b\nabla_b\chi_a = -\kappa\chi_a \,. \tag{7.0.4}$$

Comparison with (6.0.26) we see that this is the geodesic equation in a non-affine parametrisation.

Our Killing field is hypersurface orthogonal on the horizon. Frobenius' theorem states that the necessary and sufficient condition for a vector field to be hypersurface orthogonal is

$$\chi_{[a}\nabla_b\chi_{c]} = 0 \,. \tag{7.0.5}$$

If we expand this expression and use Killing's equation $\nabla_a\chi_b = -\nabla_b\chi_a$ then we find

$$\begin{aligned} \chi_c\nabla_a\chi_b &= -\chi_a\nabla_b\chi_c - \chi_b\nabla_c\chi_a \\ &= -2\chi_{[a}\nabla_{b]}\chi_c \,. \end{aligned} \tag{7.0.6}$$

If we contract with $\nabla^a\chi^b$, we have

$$\chi_c(\nabla^a\chi^b)(\nabla_a\chi_b) = -\chi_a(\nabla^a\chi^b)\nabla_b\chi_c - \chi_b(\nabla^a\chi^b)\nabla_c\chi_a \,. \tag{7.0.7}$$

The second term on the right hand side can be rearranged, using Killing's equation, as

$$\begin{aligned} -\chi_b(\nabla^a\chi^b)\nabla_c\chi_a &= \chi_b(\nabla^a\chi^b)\nabla_a\chi_c \\ &= -\chi_b(\nabla^b\chi^a)\nabla_a\chi_c \\ &= -\chi_a(\nabla^a\chi^b)\nabla_b\chi_c \,. \end{aligned} \tag{7.0.8}$$

This gives

$$\begin{aligned} \chi_c(\nabla^a\chi^b)(\nabla_a\chi_b) &= -2\chi_a(\nabla^a\chi^b)\nabla_b\chi_c \\ &= -2\kappa\chi^b\nabla_b\chi_c \\ &= -2\kappa^2\chi_c \,, \end{aligned} \tag{7.0.9}$$

where we have used (7.0.4). We then have for κ, on the horizon,

$$\kappa^2 = -\frac{1}{2}(\nabla^a\chi^b)(\nabla_a\chi_b) \,. \tag{7.0.10}$$

We now look for a physical interpretation of κ. Firstly, we have everywhere in the spacetime

$$\begin{aligned}
3(\chi^{[a}\nabla^{b}\chi^{c]})(\chi_{[a}\nabla_{b}\chi_{c]}) &= 3\cdot 2\cdot 2\cdot\left(\frac{1}{3!}\right)^{2}(\chi^{a}\nabla^{b}\chi^{c}+\chi^{b}\nabla^{c}\chi^{a}+\chi^{c}\nabla^{a}\chi^{b}) \\
&\quad\times(\chi_{a}\nabla_{b}\chi_{c}+\chi_{b}\nabla_{c}\chi_{a}+\chi_{c}\nabla_{a}\chi_{b}) \\
&= \frac{1}{3}(3\chi^{a}\chi_{a}(\nabla^{b}\chi^{c})(\nabla_{b}\chi_{c})+6\chi^{a}(\nabla^{b}\chi^{c})\chi_{b}(\nabla_{c}\chi_{a})) \\
&= \chi^{a}\chi_{a}(\nabla^{b}\chi^{c})(\nabla_{b}\chi_{c}) \\
&\quad -2(\chi^{a}\nabla_{a}\chi_{c})(\chi^{b}\nabla_{b}\chi^{c})\,. \qquad (7.0.11)
\end{aligned}$$

Here we have used the fact that there are 3! permutations of three indices and Killings equation has been used to convert negative permutations to positive giving the factor 2 for each bracket. Rearranging we have

$$-\frac{1}{2}(\nabla^{b}\chi^{c})(\nabla_{b}\chi_{c}) = -\frac{3(\chi^{[a}\nabla^{b}\chi^{c]})(\chi_{[a}\nabla_{b}\chi_{c]})}{2\chi^{d}\chi_{d}} - \frac{(\chi_{b}\nabla^{b}\chi^{c})(\chi_{a}\nabla^{a}\chi_{c})}{\chi^{d}\chi_{d}}\,. \qquad (7.0.12)$$

Upon approaching the horizon we recognise the limiting value of the left hand side to equal κ^2. However, we know that $\chi^d\chi_d \to 0$ in this limit, and from (7.0.5) we see that the first term on the right hand side attains the indeterminate form 0/0. Therefore, we use l'Hopital's rule to proceed to the limit. The numerator is constant on the horizon and so it's gradient vanishes, whereas the gradient of the denominator is non-zero by (7.0.3). So, in the limit, we have remaining

$$\kappa^{2} = \lim\left(-(\chi_{b}\nabla^{b}\chi^{c})(\chi_{a}\nabla^{a}\chi_{c})/\chi^{d}\chi_{d}\right)\,. \qquad (7.0.13)$$

Recall that in a static spacetime, with $\chi^a = \xi^a$, the gravitational red shift factor is $V = (-\chi^a\chi_a)^{1/2}$ and that the acceleration of an orbit of χ^a is

$$a^{c} = (\chi^{b}\nabla_{b}\chi^{c})/V^{2} = (\chi^{b}\nabla_{b}\chi^{c})/(-\chi^{a}\chi_{a})\,. \qquad (7.0.14)$$

We then have

$$\begin{aligned}
-(\chi_{b}\nabla^{b}\chi^{c})(\chi_{a}\nabla^{a}\chi_{c})/\chi^{d}\chi_{d} &= -a^{c}a_{c}\chi^{d}\chi_{d} \\
&= a^{2}V^{2} \qquad (7.0.15)
\end{aligned}$$

and

$$\kappa = \lim(Va)\,. \qquad (7.0.16)$$

We call κ the surface gravity of the black hole. It can be shown that this is the force that must be exerted to keep a particle stationary on the horizon by an observer at infinity. Now compute

$$\begin{aligned}
\nabla_{\mu}\log V &= \frac{1}{V}\nabla_{\mu}V \\
&= \frac{1}{2V^{2}}\nabla_{\mu}(\chi\cdot\chi) \\
&= -\frac{1}{V^{2}}\chi\nabla_{\rho}\chi_{\mu} \\
\Rightarrow a_{\mu} &= -\nabla_{\mu}\log V\,. \qquad (7.0.17)
\end{aligned}$$

We can now compute the surface gravity of a Schwarzschild black hole. We have

$$V = \sqrt{1 - \frac{2m}{r}} \quad (7.0.18)$$

$$a = \sqrt{a^\mu a_\mu} = \frac{1}{V}\sqrt{\nabla_\mu V \nabla^\mu V} \quad (7.0.19)$$

$$V a = \sqrt{g^{\nu\gamma}\partial_\nu V\, \partial_\gamma V} \quad (7.0.20)$$

$$\partial_\nu V = \frac{1}{2}\frac{2M}{r^2} \quad (7.0.21)$$

$$\begin{aligned} g^{\nu\gamma}\partial_\nu V\, \partial_\gamma V &= g^{00}\partial_0 V\, \partial_0 V \\ &= V^2 \left(\frac{1}{2V}\right)^2 \left(\frac{2M}{r^2}\right)^2 \end{aligned} \quad (7.0.22)$$

$$\Rightarrow V a = \frac{1}{2}\frac{2M}{r^2}\,. \quad (7.0.23)$$

Now we take the limit to the horizon and we find

$$\lim_{r \to 2M} (V a) = \frac{1}{4M}\,. \quad (7.0.24)$$

7.1 Constancy of the surface gravity

We now show that the surface gravity of a black hole is constant on the black hole horizon. Again we follow the outline given in [1]. Firstly, we derive a useful relationship between the second derivative of the Killing field and the Riemann tensor. By definition of the Riemann tensor we have

$$\nabla_a \nabla_b \chi_c - \nabla_b \nabla_a \chi_c = R_{abc}{}^d \chi_d \quad (7.1.1)$$

and using Killing's equation

$$\nabla_a \nabla_b \chi_c + \nabla_b \nabla_c \chi_a = R_{abc}{}^d \chi_d\,.$$

To this add

$$\nabla_b \nabla_c \chi_a + \nabla_c \nabla_a \chi_b = R_{bca}{}^d \chi_d$$

and subtract

$$\nabla_c \nabla_a \chi_b + \nabla_a \nabla_b \chi_c = R_{cab}{}^d \chi_d\,.$$

This gives

$$\begin{aligned} 2\nabla_b \nabla_c \chi_a &= (R_{abc}{}^d + R_{bca}{}^d - R_{cab}{}^d)\chi_d \\ &= -2R_{cab}{}^d \chi_d\,, \end{aligned} \quad (7.1.2)$$

where we have used property (A.0.52). We then have

$$\nabla_a \nabla_b \chi_c = -R_{bca}{}^e \chi_d \quad (7.1.3)$$

as the required result.

Now, applying the operator $\chi_{[d}\nabla_{c]}$ to (7.0.4) we find

$$\begin{aligned} \chi_a \chi_{[d}\nabla_{c]}\kappa + \kappa\chi_{[d}\nabla_{c]}\chi_a &= \chi_{[d}\nabla_{c]}(\chi^b \nabla_b \chi_a) \\ &= (\chi_{[d}\nabla_{c]}\chi^b)(\nabla_b \chi_a) + \chi^b \chi_{[d}\nabla_{c]}\nabla_b \chi_a \\ &= (\chi_{[d}\nabla_{c]}\chi^b)(\nabla_b \chi_a) - \chi^b R_{ba[c}{}^e \chi_{d]}\chi_e\,, \end{aligned} \quad (7.1.4)$$

where we have used (7.1.3) in the last equality. Using (7.0.6) and (7.0.4) the first term on the right hand side is

$$\begin{aligned}(\chi_{[d}\nabla_{c]}\chi^b)(\nabla_b\chi_a) &= -\frac{1}{2}(\chi^b\nabla_d\chi_c)\nabla_d\chi_a \\ &= -\frac{1}{2}\kappa\chi_a\nabla_d\chi_c \\ &= \kappa\chi_{[d}\nabla_{c]}\chi_a\,. \end{aligned} \tag{7.1.5}$$

So we see that (7.1.4) becomes

$$\xi_a\chi_{[d}\nabla_{c]}\kappa = \chi_b R_{ab[c}{}^e\chi_{d]}\chi_e\,, \tag{7.1.6}$$

where we have also used the antisymmetry property of the first pair of indices of the Riemann tensor.

If we now apply $\chi_{[d}\nabla_{e]}$ to (7.0.6) we obtain

$$(\chi_{[d}\nabla_{e]}\chi_c)\nabla_a\chi_b + \chi_c\chi_{[d}\nabla_{e]}\nabla_a\chi_b = -2(\chi_{[d}\nabla_{e]}\chi_{[a})\nabla_{b]}\chi_c - 2(\chi_{[d}\nabla_{e]}\nabla_{[b}\chi_{|c|})\chi_{a]}\,. \tag{7.1.7}$$

Using (7.0.6) and Killing's equation, the first term on the left hand side

$$\begin{aligned}(\chi_{[d}\nabla_{e]}\chi_c)\nabla_a\chi_b &= -\frac{1}{2}\chi_c\nabla_d\chi_e\nabla_a\chi_b \\ &= \chi_{[a}\nabla_{b]}\chi_c\nabla_d\chi_e \\ &= \chi_a\nabla_b\chi_c\nabla_d - \chi_b\nabla_a\chi_c\nabla_d\chi_e \\ &= -2\chi_{[d}\nabla_{e]}\chi_a\nabla_b\chi_c + 2\chi_{[d}\nabla_{e]}\chi_b\nabla_a\chi_c \\ &= -2(\chi_{[d}\nabla_{e]}\chi_{[a})\nabla_b\chi_c\,, \end{aligned} \tag{7.1.8}$$

which cancels the first term on the right hand side. Using (7.1.3) the 2nd term on the left

$$\chi_c\chi_{[d}\nabla_{e]}\nabla_a\chi_b = -\chi_c R_{ab[e}{}^f\chi_{d]}\chi_f \tag{7.1.9}$$

and the 2nd term on the right

$$-2(\chi_{[d}\nabla_{e]}\nabla_{[b}\chi_{|c|})\chi_{a]} \quad = \quad -2\chi_{[a}R_{b]|c|[e}{}^f\chi_{d]}\chi_f\,. \tag{7.1.10}$$

So we have

$$\chi_c R_{ab[e}{}^f\chi_{d]}\chi_f = 2\chi_{[a}R_{b]c[e}{}^f\chi_{d]}\chi_f\,. \tag{7.1.11}$$

Multiplying by g^{ce}, and again using the antisymmetry property of the last two indices of the Riemann tensor, the left hand side becomes

$$\begin{aligned} g^{ce}\chi_c R_{ab[e}{}^f\chi_{d]}\chi_f &= \chi^e\chi_f R_{ab[e}{}^f\chi_{d]} \\ &= \frac{1}{2}\left(\chi^e\chi_f R_{ab[e}{}^f\chi_{d]} - \chi_e\chi^f R_{ab[f}{}^e\chi_{d]}\right) \\ &= 0 \end{aligned}$$

because the summed indices can be relabelled in the last line. So (7.1.11) gives

$$\begin{aligned} & g^{ce}\chi_{[a}R_{b]c[e}{}^f\xi_{d]}\chi_f = 0 \\ \Rightarrow\quad & g^{ce}\left(\chi_{[a}R_{b]ce}{}^f\chi_d\chi_f - \chi_{[a}R_{b]cd}{}^f\chi_e\chi_f\right) = 0 \\ \Rightarrow\quad & -\chi_{[a}R_{b]cf}{}^c\chi_d\chi^f - \chi_{[a}R_{b]cd}{}^f\chi^c\chi_f = 0 \\ \Rightarrow\quad & -\chi_{[a}R_{b]}{}^f\chi_d\chi_f = \chi_{[a}R_{b]cd}{}^f\chi^c\chi_f\,, \end{aligned} \tag{7.1.12}$$

where we have used the definition of the Ricci tensor in the last line. Using (7.1.6) the right hand side is

$$\begin{aligned} \chi_{[a} R_{b]cd}{}^{f} \chi^{c} \chi_{f} &= \chi_{c} R_{df[b}{}^{c} \chi_{a]} \chi^{f} \\ &= \chi_{d} \chi_{[a} \nabla_{b]} \kappa \,. \end{aligned} \tag{7.1.13}$$

Substituting into (7.1.12) gives

$$\chi_{[a} \nabla_{b]} \kappa = -\chi_{[a} R_{b]}{}^{f} \chi_{f} \,. \tag{7.1.14}$$

Now, consider the geodesic congruence generated by χ^a on the black hole horizon. In the last section we defined the expansion θ, twist ω_{ab} and shear σ_{ab} of the congruence. χ^a is always normal to the horizon to that θ, ω_{ab} and σ_{ab} all vanish for the congruence generated by χ^a. Also, the rate of change of θ on the horizon vanishes and from (6.2.20) we have

$$R_{ab} \chi^a \chi^b = 0 \tag{7.1.15}$$

on the horizon. The dominant energy condition states that $T^a{}_b \chi^b$ must be a future directed timelike or null vector. However, using Einstein's equation with (7.1.15) we have

$$\begin{aligned} & 8\pi \left(T_{ab} - \frac{1}{2} T g_{ab} \right) \chi^a \chi^b = 0 \\ & \Rightarrow T^a{}_b \chi_a \chi^b = 0 \,. \end{aligned} \tag{7.1.16}$$

This implies that $T^a{}_b \chi^b$ points in the direction of χ^a and also

$$\begin{aligned} & \chi_{[c} T_{a]b} \chi^b = 0 \\ & \Rightarrow \chi_c \left(\frac{1}{8\pi} R_{ab} + \frac{1}{2} T g_{ab} \right) \chi^b - \xi_a \left(\frac{1}{8\pi} R_{cb} + \frac{1}{2} T g_{cb} \right) \chi^b \\ & = \frac{1}{8\pi} \chi_c R_{ab} \chi^b - \frac{1}{8\pi} \chi_a R_{cb} \chi^b \\ & \Rightarrow \frac{1}{8\pi} \chi_{[c} R_{a]b} \chi^b = 0 \,, \end{aligned} \tag{7.1.17}$$

where we have again used Einstein's equation. Therefore, the right hand side of (7.1.14) vanishes and we conclude

$$\chi_{[a} \nabla_{b]} \kappa = 0 \,, \tag{7.1.18}$$

which states that the surface gravity is constant on the horizon.

8 Black holes and thermodynamics

In this section we discuss the remarkable mathematical similarity between the laws of black hole mechanics and the laws of thermodynamics. In this section we follow the arguments of [7] and [8]. We consider uncharged rotating black holes in vacuum but the arguments can be generalised to include charge and non-zero energy momentum tensor outside the horizon, see [7].

We begin by presenting the outline of the derivation of an expression for the mass of a black hole. We then derive the formula for the differential mass. Next we discuss particle creation by black holes. For the black hole this leads to the concept of Hawking radiation. Finally, we will collect the black hole laws, from this section and those previous, to examine the close analogy to the thermodynamical laws. Further discussion will indicate that perhaps the laws are not only an analogy but are essential to preserve our old thermodynamical concepts in the vicinity of a black hole.

8.1 The mass of a rotating black hole

Since in the vicinity of the black hole observers will, in general, disagree about the mass of a black hole we consider the mass as seen from infinity, where the gravitational field is weak. We have seen in section 5 that a stationary, axisymmetric black hole admits two Killing vector fields, ξ^a, associated with time translation invariance, and ψ^a, associated with azimuthal rotation invariance, which posseses closed orbits of parameter length 2π. Since ξ^a is a Killing vector, and according to (A.0.47), we have

$$\begin{aligned} \nabla_\beta \nabla_\alpha \xi_\mu &= R_{\gamma\beta\alpha\mu}\xi^\gamma \\ \Rightarrow \nabla_\beta \nabla^\beta \xi^\mu &= -R^\mu{}_\gamma \xi^\gamma \,, \end{aligned} \tag{8.1.1}$$

by contracting in α and β. If we integrate (8.1.1) over a spacelike, asymptotically flat hypersurface, Σ, and use Stokes' theorem (2.1.15) we have

$$\int_{\partial\Sigma} \nabla^\beta \xi^\mu \, d\sigma_{\mu\nu} = -\int_\Sigma R^\mu{}_\gamma \xi^\gamma \, d\sigma_\mu \,. \tag{8.1.2}$$

Here, $d\sigma_\mu$ and $d\sigma_{\mu\nu}$ are surface elements of Σ and its boundary, $\partial\Sigma$, respectively.

The hypersurface, Σ, is chosen so that it's boundary, $\partial\Sigma$, consists of the two-dimensional black hole boundary, ∂B, and a two-dimensional surface, $\partial\Sigma_\infty$, at spatial infinity.

Now consider the four-acceleration of the distant observer. As in (7.0.14), we have

$$a^\mu = -\frac{\xi^\nu \nabla_\nu \xi^\mu}{\xi^\alpha \xi_\alpha} \,. \tag{8.1.3}$$

According to [8] if the four-velocity $u^\mu = \xi^\mu / |\xi^\alpha \xi_\alpha|^{1/2}$ then in the asymptotic region we can approximate $a^\mu \approx u^\nu \nabla_\nu \xi^\mu$. In this region we expect u^ν to be orthogonal to Σ and to have only the three components of Newtonian theory. Thus we can relate the acceleration to a gravitational potential; $a_i = \partial_i U$. It is then possible to use Gauss' theorem to associate a source, M_∞, of the gravitational field with the flux, $4\pi M_\infty$, through a surface at infinity. If $\tilde{n}_\mu$ is the outward unit normal vector to $\partial\Sigma_\infty$ this gives

$$M_\infty = \frac{1}{4\pi} \int_{\partial\Sigma_\infty} u_\nu \nabla^\nu \xi^\mu \tilde{n}_\mu \, d^2\sigma \,, \tag{8.1.4}$$

where $d^2\sigma$ is an element of $\partial\Sigma_\infty$. By Killing's equation we see that μ and ν are antisymmetric indices so we can express $n_\mu u_\nu d^2\sigma$ as differential form, $d\sigma_{\mu\nu}$, obtaining

$$M_\infty = \frac{1}{4\pi} \int_{\partial\Sigma_\infty} \nabla^\nu \xi^\mu d\sigma_{\mu\nu} \,. \tag{8.1.5}$$

Returning to (8.1.2) we have

$$\begin{aligned} M_\infty &= \frac{1}{4\pi}\int_{\partial B} \nabla^\nu \xi^\mu \, d\sigma_{\mu\nu} - \int_\Sigma R^\mu{}_\gamma \xi^\gamma \, d\sigma_\mu \\ &= \frac{1}{4\pi}\int_{\partial\Sigma_\infty} \nabla^\nu \xi^\mu \, d\sigma_{\mu\nu} - \int_\Sigma (2T^\mu{}_\gamma - Tg^\mu{}_\gamma)\xi^\gamma \, d\sigma_\mu \,, \end{aligned} \qquad (8.1.6)$$

where the first term is the mass of the black hole, M_{BH}. The second term results from taking the trace of the Einstein equation (A.0.55).

If instead we replace ξ^μ by ψ^μ in (8.1.1), by an analogous argument, we obtain

$$J_\infty = -\int_\Sigma R^\mu{}_\gamma \psi^\gamma \, d\sigma_\mu - \frac{1}{8\pi}\int_{\partial B} \,, \qquad (8.1.7)$$

where J_∞ is the total angular momentum, measured from infinity. Then the first integral on the right hand side may be regarded as the angular momentum of the matter outside the black hole and the second to be the angular momentum, J_{BH}, of the black hole itself.

As in section 5 we can write

$$\chi^a = \xi^a + \Omega_{BH}\psi^a \,, \qquad (8.1.8)$$

where χ^a is normal to the black hole horizon. Using (8.1.8) and the definition of J_{BH}, we find

$$\int_{\partial B} \nabla^\nu \xi^\mu \, d\sigma_{\mu\nu} = 8\pi J_{BH}\Omega_{BH} + \int_{\partial B} \nabla^\nu \xi^\mu \, d\sigma_{\mu\nu} \,. \qquad (8.1.9)$$

As in (8.1.5) for the last term we can write $d\sigma_{\mu\nu} = \chi_{[\mu} n_{\nu]} dA$, where dA is an area element of ∂B and n_ν is a unit null vector, normal to ∂B. The null vectors are normalised as $l^\mu n_\mu = 1$.

Using (7.0.4) and multiplying both sides by n^a gives, for the surface gravity,

$$\kappa = \nabla_\nu \xi_\mu n^\mu \chi^\nu \,. \qquad (8.1.10)$$

So κ replaces the integrand of the last integral in (8.1.9). On the horizon κ is constant so, taking it out of the intgral, we find

$$\int_{\partial B} \nabla^\nu \xi^\mu \, d\sigma_{\mu\nu} = 8\pi J_{BH}\Omega_{BH} + \kappa A \qquad (8.1.11)$$

and

$$M_\infty = 2J_{BH}\Omega_{BH} + \frac{\kappa A}{4\pi} - \int_\Sigma (T^\mu{}_\gamma - Tg^\mu{}_\gamma)\xi^\gamma \, d\sigma_\mu \,. \qquad (8.1.12)$$

This is our expression for the mass of a stationary, axisymmetric spacetime. We will be interested in the vacuum solution so will give less attention to the term involving the energy-momentum tensor. To make a comparison with the thermodynamical laws we need an expression for the infinitesimal change in mass, dM.

8.2 Differential formula for the mass of a rotating black hole

In the previous section we derived an expression for the mass of a rotating black hole. To obtain a differential formula Bardeen, Carter and Hawking (1973) [7] peformed a variation of (8.1.12). This was a lengthy calculation including the details of a perfect fluid matter field outside of the horizon. For simplicity we follow an approach to obtain the vacuum solution (Hawking and Hartle, 1972), outlined in [9].

We consider a stationary black hole. We perturb it by some small influx of matter, across the horizon, represented by the variation, ΔT_{ab}, of the energy-momentum tensor. We assume that the black hole settles back to a stationary state in finite proper time, λ. We also assume that, to first order in Δ_{ab}, there is no change in the black hole geometry.

To begin, notice from (7.0.4) that, if we define the Killing parameter, v, as

$$\chi^a \nabla_a v = 1\,, \tag{8.2.1}$$

then κ measures the failure of v to be an affine parameter along the null geodesic generators of the horizon. Define a new vector field, k^a, by

$$k^a = e^{-\kappa v} \chi^a\,, \tag{8.2.2}$$

then, using (7.0.4) and the fact that κ is constant on χ^a, we have

$$\begin{aligned}
k^b \nabla_b k^a &= e^{-2\kappa v}(\chi^b \nabla_b \chi^a - \chi^a \chi^b \nabla_b(\kappa v)) \\
&= e^{-\kappa v}(\kappa \chi^a - \chi^a \chi^b (\nabla_b \kappa) v - \chi^a \chi^b (\nabla_b v)\kappa) \\
&= e^{-\kappa v}(\kappa \chi^a - \kappa \chi^a) \\
&= 0\,.
\end{aligned} \tag{8.2.3}$$

So k^a is the affinely parametrised tangent to the generators of the horizon. Between affine parameter λ and Killing parameter v, we have the relation

$$\frac{d\lambda}{dv} \propto e^{\kappa v} \tag{8.2.4}$$

and, if $\kappa \neq 0$,

$$\lambda \propto e^{\kappa v} \tag{8.2.5}$$

so that

$$v \propto \frac{1}{\kappa} \ln \lambda\,. \tag{8.2.6}$$

Returning to (8.1.12), our assumption that there is no change in the horizon geometry implies that, upon small variation of the matter, the first term will give no contribution. If λ is affine parameter and k^a tangent to the null geodesic generators of the horizon then we have

$$\Delta M = \int_0^\infty d\lambda \int d\sigma^2 \Delta T_{\mu\gamma} \xi^\gamma k^\mu\,, \tag{8.2.7}$$

where $d\sigma^2$ is an area element of the horizon at 'time' λ as in [9]. Also, we have the same form for the total angular momentum (8.1.7)

$$\Delta J = -\int_0^\infty d\lambda \int d\sigma^2 \Delta T_{\mu\gamma} \psi^\gamma k^\mu\,. \tag{8.2.8}$$

The change of area of the horizon can be analysed using Raychaudhuri's equation (6.2.14). To first order the quadratic terms can be neglected to leave

$$\frac{d\theta}{d\lambda} = -8\pi \Delta T_{ab} k^a k^b \,. \tag{8.2.9}$$

As different from (6.2.14), we use affine parameter λ opposed to τ because τ is associated with time translation invariance Killing vector ξ^a. We may substitute for k^a using (8.2.6):

$$\begin{aligned} k^a &= \left(\frac{\partial}{\partial \lambda}\right)^a = \frac{1}{\kappa\lambda}\left(\frac{\partial}{\partial v}\right)^a \\ &= \frac{1}{\kappa\lambda}\chi^a = \frac{1}{\kappa\lambda}(\xi^a + \Omega_{BH}\psi^a)\,. \end{aligned} \tag{8.2.10}$$

Now, multiplying (2) by $\kappa\lambda$ and integrating over the horizon we have

$$\begin{aligned} \kappa \int_0^\infty d\lambda \int d\sigma^2 \lambda \frac{d\theta}{d\lambda} &= -8\pi \int_0^\infty d\lambda \int d\sigma^2 \Delta T_{ab}(\xi^a + \Omega_{BH}\psi^a) k^b \\ &= -8\pi(\Delta M - \Omega_{BH}\Delta J)\,, \end{aligned} \tag{8.2.11}$$

using (8.2.7) and (8.2.8). The left hand side gives

$$\kappa \int d\sigma^2 \int_0^\infty d\lambda \lambda \frac{d\theta}{d\lambda} = \kappa \int d\sigma^2 \left[\lambda\theta\right]_0^\infty - \kappa \int d\sigma^2 \int_0^\infty d\lambda\theta\,, \tag{8.2.12}$$

using integration by parts. As discussed in section 6, θ is the expansion of the null geodesics with tangent k^a. It measures the local rate of change of cross-sectional area of the horizon. Therefore, if A is the cross-sectional area, we can write

$$\theta = \frac{1}{A}\frac{dA}{d\lambda}\,. \tag{8.2.13}$$

Then, in the limit, the last integral of (8.2.12) is just $-dA$.

At the beginning of this section we assumed that the black hole returns to a stationary state in finite λ. Therefore, $\theta \to 0$ before $\lambda \to \infty$ so the first term on the right hand side of (8.2.12) is 0. In the infinitesimal limit, this leaves

$$-\kappa dA = -8\pi(dM - \Omega_{BH} dJ) \tag{8.2.14}$$

or

$$dM = \frac{\kappa}{8\pi} dA + \Omega_{BH} dJ\,. \tag{8.2.15}$$

This is the required differential expression.

8.3 Particle creation by black holes

In this section we discuss, qualitatively, the physics of particle creation by black holes. The principle of the phenomenon can be easily understood by considering the nature of the Killing vector field in the neighbourhood of the horizon. However, for discussion of the technical details of a quantum theory in curved spacetime the reader is refferred to [9] and, for the argument relating the temperature of the black hole to the surface gravity, consult the original paper by Hawking (1975) [11]. Whereas here we follow arguments from [8].

We know that the physical vacuum is a complex entity. Virtual (off mass shell) particles are continually created and annihilated within time $\Delta t \leq \hbar/\Delta E$, where ΔE is their combined energy.

However, it is possible for these virtual particles to absorb energy from an external field and thus become real. In particular this can occur in a very strong gravitational field.

Consider a static spacetime, where there exists Killing field ξ^μ. A particle of four-momentum p^μ will have energy given by

$$E = \xi^\mu p_\mu \,. \tag{8.3.1}$$

This implies that the energy of the particle is always positive in regions where ξ^μ is timelike. A pair of virtual particles could never become real in this region because this would violate energy conservation.

However, now consider a static spacetime where there exists a Killing horizon, implying $\xi^\mu \xi_\mu$ on this surface (this is just the event horizon for the Schwarzschild black hole). This implies that the Killing field becomes spacelike in a certain region.

If a virtual pair is created near the Killing horizon then there is a finite probability that one of the virtual particles will move through the horizon while the other remains outside. There would then be a virtual particle in both timelike and spacelike regions of the Killing field. This means that they have oppositely signed $\xi^\mu p_\mu$ and energy can be conserved.

One of the particles can, therefore, escape to infinity as radiation. This phenomenon is not possible in an arbitrary static spacetime. The effect requires the existence of a Killing horizon and thus a black hole. Similar arguments can be applied to a general stationary spacetime. Therefore, this particle emission, called *Hawking radiation*, is a general black hole phenomenon.

Hawking [11] calculated the exact rate that the radiation is emitted and showed that it had the exact form as would be emitted by a black body of temperature

$$T_H = \frac{\hbar\kappa}{2\pi c k} \,, \tag{8.3.2}$$

where κ is the surface gravity and k is Boltzman's constant (included to give T_H the units of temperature) This is the *Hawking temperature.*

In geometric units

$$T_H = \frac{\kappa}{2\pi} \tag{8.3.3}$$

and, from (7.0.24), for the Schwarzschild black hole $T_H = 1/8\pi M$.

8.4 The laws of black hole mechanics

In the previous section we saw that a black hole could be considered as a black body with temperature proportional to the surface gravity. The zeroth law of thermodynamics states that a body in thermal equilibrium with it's environment has a uniform temperature. In section 7.1 we have seen that the surface gravity is constant over the horizon of a stationary black hole and so the association of surface gravity with temperature is indeed physical.

Now consider the first law of thermodynamics for a rotating body

$$dE = TdS + \Omega dJ \,, \tag{8.4.1}$$

where T is the temperature, S is the entropy and the second term is the 'work' associated with the rotation. Compare this equation to (8.2.15) at the end of section 8.2. The dM term is obviously analogous to dE and if we enter the Hawking temperature into (8.2.15) we have the analogy $dA \leftrightarrow \frac{1}{4}dS$.

In section 6.4 we demonstrated that the surface area A of the horizon could only ever increase or remain constant in complete analogy to the second law of thermodynamics. Now, consider some matter falling into a black hole so that all of its entropy is lost from observation. Unless we are willing to claim that the second law of thermodynamics is violated when a black hole is present, this entropy should be manifest in a different form. The obvious candidate is the surface area of the horizon so that all of the entropy of the black hole is given by

$$S = \frac{1}{4}A \,. \tag{8.4.2}$$

We can then consider the black hole laws and laws of thermodynamics as more than just an analogy but are transcended by laws which encompass them both. The generalised second law then states that the entropy, or horizon surface area, of an enclosed system can only increase or remain constant. Also the laws can be viewed as more than an analogy since E, in (8.4.1), and M, in (8.2.15) represent the same physical quantity.

It can be shown that the surface gravity κ vanishes for the extremal black hole. However, calculations due to Wald (1974) show that the closer one gets to an extremal black hole the harder it is to get closer. This is in the form of the third law of thermodynamics which states that it is impossible for a system to reach absolute zero temperature. We can now summarise the black hole laws [1]:

Zeroth:	κ constant over the horizon of a stationary black hole
First:	$dM = \frac{1}{8\pi}\kappa dA + \Omega_{BH} dJ$
Second:	$\delta A \geq 0$ for any process
Third:	It is impossible to reach $\kappa = 0$ by any physical process

9 The Attractor Mechanism

In this section we review work by Ferrara and Kallosh [4]. In section 4 of this paper the authors investigate black hole solutions in a supersymmetric spacetime. They are interested in the 'attractor mechanism', where the magnitude of a matter field ϕ obtains definite values on the black hole horizon, dependent only upon the electric $|q|$ and magnetic $|p|$ charges of the black hole. The matter field on the event horizon is completely independent of the field at infinity. This is another example of a no-hair theorem and is important for the black hole laws to hold for black holes in more general spacetimes. In this project we have so far only considered electrovac spacetimes, i.e no matter outside the black hole, just an electric field. The attractor mechanism implies that the black hole laws can be generalised to the case where there is external matter.

Our aim here is to verify that the solution in [4] satisfies the field equations derived from the action. The action is built out of the electromagnetic field tensor $F^{\mu\nu}$ and an electromagnetic dual field tensor given by

$$\widetilde{G}^{\mu\nu} = \frac{i}{2}\frac{1}{\sqrt{-g}} e^{-2\phi} \epsilon^{\mu\nu\lambda\delta} \widetilde{F}_{\lambda\rho} \,, \tag{9.0.3}$$

where $F^{\mu\nu}$ and $\widetilde{F}^{\mu\nu}$ are independent fields. The action is

$$I = \frac{1}{16\pi} \int d^4x \, (-g)^{\frac{1}{2}} \left[R - 2\partial^\mu \phi \, \partial_\mu \phi + \frac{1}{2} \left(e^{-2\phi} F^{\mu\nu} F_{\mu\nu} + e^{2\phi} \widetilde{G}^{\mu\nu} \widetilde{G}_{\mu\nu} \right) \right] . \tag{9.0.4}$$

Here, $e^{-2\phi}$ is a coupling constant, built from the scalar matter field, called the *dilaton*. We see the usual terms, in the Lagrangian, for the scalar field and electromagnetic field, plus a similar extra term for the dual field. We now compute the field equations from this action using the variational principle described in section 2.3. The matter Lagrangian, written more explicitly is

$$\begin{aligned} \mathcal{L}_M &= \frac{1}{16\pi}(-g)^{\frac{1}{2}} \Bigg(-2g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + \frac{1}{2} e^{-2\phi} \left(g^{\mu\nu} F_{\mu\gamma} F_\nu{}^\gamma + g^{\mu\nu} F_{\gamma\mu} F^\gamma{}_\nu \right) \\ &\quad + \frac{1}{2} e^{2\phi} \left(g^{\mu\nu} \widetilde{G}_{\mu\gamma} \widetilde{G}_\nu{}^\gamma + g^{\mu\nu} \widetilde{G}_{\gamma\mu} \widetilde{G}^\gamma{}_\nu \right) \Bigg) . \end{aligned} \tag{9.0.5}$$

Variation with respect to $g^{\mu\nu}$, using (2.2.8)

$$\begin{aligned} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} &= \frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} \\ &= \frac{1}{16\pi}(-g)^{\frac{1}{2}} \Bigg(g_{\mu\nu} \partial_\rho \phi \, \partial^\rho \phi - \frac{1}{4} g_{\mu\nu} \left(e^{-2\phi} F^{\rho\sigma} F_{\rho\sigma} + e^{2\phi} \widetilde{G}^{\rho\sigma} \widetilde{G}_{\rho\sigma} \right) \\ &\quad + e^{-2\phi} F_\mu{}^\gamma F_{\nu\gamma} + e^{2\phi} \widetilde{G}_\mu{}^\gamma \widetilde{G}_{\nu\gamma} - 2\partial_\mu \phi \, \partial_\nu \phi \Bigg) . \end{aligned} \tag{9.0.6}$$

So the energy-momentum tensor is

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{16\pi} \Bigg(g_{\mu\nu} \partial_\rho \phi \, \partial^\rho \phi - \frac{1}{4} g_{\mu\nu} \left(e^{-2\phi} F^{\rho\sigma} F_{\rho\sigma} + e^{2\phi} \widetilde{G}^{\rho\sigma} \widetilde{G}_{\rho\sigma} \right) \\ &\quad + e^{-2\phi} F_\mu{}^\gamma F_{\nu\gamma} + e^{2\phi} \widetilde{G}_\mu{}^\gamma \widetilde{G}_{\nu\gamma} - 2\partial_\mu \phi \, \partial_\nu \phi \Bigg) , \end{aligned} \tag{9.0.7}$$

which is proportional to the Einstein tensor with proportionality constant 8π (see [2]).

Now we compute the field equations for the scalar field ϕ. Varying the action with respect to ϕ

$$\begin{aligned} \partial_\mu \frac{\partial \mathcal{L}_M}{\partial(\partial_\mu \phi)} &= -\partial_\mu \left(\frac{4}{16\pi} (-g)^{\frac{1}{2}} g^{\mu\rho} \, \partial_\rho \phi \right) \\ &= -\frac{1}{4\pi} \left((-g)^{\frac{1}{2}} \, \partial^\rho \, \partial_\rho \phi + \partial^\mu \phi \, \partial_\mu (-g)^{\frac{1}{2}} \right) \\ &= -\frac{1}{4\pi} \left((-g)^{\frac{1}{2}} \partial^\rho \, \partial_\rho \phi + \partial^\mu \phi \cdot -\frac{1}{2} (-g)^{-\frac{1}{2}} \cdot 2g \Gamma^\nu_{\nu\mu} \right) \\ &= -\frac{1}{4\pi} \left((-g)^{\frac{1}{2}} \partial^\rho \, \partial_\rho \phi + (-g)^{\frac{1}{2}} \Gamma^\nu_{\nu\mu} \partial^\mu \phi \right) \\ &= -\frac{(-g)^{\frac{1}{2}}}{4\pi} \nabla_\rho \partial^\rho \phi , \end{aligned} \tag{9.0.8}$$

where we have used (2.2.4) in the third line and the definition of the covariant derivative of a contravariant vector in the last line. Next

$$\frac{\partial \mathcal{L}_M}{\partial \phi} = \frac{(-g)^{\frac{1}{2}}}{16\pi} \left(e^{2\phi} \widetilde{G}^{\mu\nu} \widetilde{G}_{\mu\nu} - e^{-2\phi} F^{\mu\nu} F_{\mu\nu} \right) . \tag{9.0.9}$$

So the field equation for ϕ is

$$\nabla_\rho \partial^\rho \phi = \frac{1}{4}\left(e^{-2\phi} F^{\mu\nu} F_{\mu\nu} - e^{2\phi} \widetilde{G}^{\mu\nu} \widetilde{G}_{\mu\nu}\right) . \tag{9.0.10}$$

We now derive the field equation for the gauge field A^μ. We have

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu . \tag{9.0.11}$$

So varying the action with A^γ gives

$$\begin{aligned}
\frac{\delta \mathcal{L}_M}{\delta A^\gamma} &= -\partial_\rho \frac{\partial \mathcal{L}_M}{\partial(\partial_\rho A^\gamma)} \\
&= -\partial_\rho \frac{\partial}{\partial(\partial_\rho A^\gamma)} \left(\frac{(-g)^{\frac{1}{2}}}{2 \cdot 16\pi} e^{-2\phi} (\partial^\mu A^\nu - \partial^\nu A^\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \right) \\
&= -\frac{e^{-2\phi}}{32\pi}\, \partial_\rho \left((-g)^{\frac{1}{2}} \frac{\partial}{\partial(\partial_\rho A^\gamma)} (g^{\mu\sigma} \partial_\sigma A^\nu g_{\nu\lambda} \partial_\mu A^\lambda \right. \\
&\quad \left. - g^{\mu\sigma} \partial_\sigma A^\mu g_{\mu\lambda} \partial_\nu A^\lambda - g^{\nu\sigma} \partial_\sigma A^\mu g_{\nu\lambda} \partial_\mu A^\lambda + g^{\nu\sigma} \partial_\sigma A^\mu g_{\mu\lambda} \partial_\nu A^\lambda) \right) \\
&= -\frac{1}{32\pi}\, \partial_\rho \left((-g)^{\frac{1}{2}} e^{-2\phi} (4\partial^\rho A_\gamma - 4\partial_\gamma A^\rho \right) .
\end{aligned} \tag{9.0.12}$$

The Euler-Lagrange equation then gives

$$\partial_\rho \left(e^{-2\phi} (-g)^{\frac{1}{2}} F^\rho_\gamma \right) = 0 . \tag{9.0.13}$$

The equivalent equation for the vector potential B^μ associated with $G^{\mu\nu}$ is

$$\partial_\rho \left(e^{2\phi} (-g)^{\frac{1}{2}} \widetilde{G}^\rho_\gamma \right) = 0 . \tag{9.0.14}$$

In [4] all functions depend only upon the radial coordinate $r = x_1^2 + x_2^2 + x_3^2$. The following isotropic form is postulated as an ansatz for the metric

$$ds^2 = e^{2U} dt^2 - e^{-2U} d\mathbf{x}^2 , \tag{9.0.15}$$

where $U = U(r)$. The gauge potentials are stipulated to have one non-vanishing component each, namely $A^0 = \psi(r)$ and $B^0 = \chi(r)$. It is stated that all the fields depend upon only two functions H_1 and H_2:

$$e^{-2U} = H_1 H_2 \quad e^{2\phi} = H_2/H_1 \tag{9.0.16}$$

$$\psi = \pm H_1^{-1} \qquad \chi = \pm H_2^{-1} \tag{9.0.17}$$

It is our aim to test this assertion by requiring that all the field equations are satisfied.

Firstly, we use the gauge field equations to derive the form of H_1 and H_2. The restrictions above imply that there is only one non-vanishing component for (9.0.13) and (9.0.14). Also, the ansatz for the metric gives

$$(-g)^{\frac{1}{2}} = \sqrt{e^{-4U}} = H_1 H_2 . \tag{9.0.18}$$

Using (9.0.17), (9.0.13) becomes

$$
\begin{aligned}
&\partial_\rho (H_1/H_2) H_1 H_2 (\partial^\rho A_\gamma - \partial_\gamma A^\rho) \\
&= \partial_i H_1^2 \partial^i A^t \\
&= \partial_i H_1^2 \left(\frac{\partial^i H_1}{H_1^2} \right) \\
&= \partial_i \partial^i H_1 = 0 \,,
\end{aligned} \tag{9.0.19}
$$

and (9.0.14) gives

$$
\begin{aligned}
&\partial_i H_2^2 \partial^i B^t \\
&= \partial_i \partial^i H_2 = 0 \,,
\end{aligned} \tag{9.0.20}
$$

We see that H_1 and H_2 are both harmonic functions agreeing with [4]. In spherical coordinates, where $\partial_i \partial^i = 1/r^2 \partial_r r^2 \partial^r$, the solutions can be written

$$
H_1 = e^{-\phi_0} + \tfrac{|q|}{r} \,, \quad H_2 = e^{+\phi_0} + \tfrac{|p|}{r} \,, \tag{9.0.21}
$$

where ϕ_0 is the matter field at infinity. As in [4], the constants are associated with the electric/magnetic charges to give the following form for the metric components

$$
g_{tt}^{-1} = g_{ii} = e^{-2U} = \left(e^{-\phi_0} + \frac{|q|}{r} \right) \left(e^{+\phi_0} + \frac{|p|}{r} \right) = 1 + \frac{e^{-\phi_0}|p| + e^{+\phi_0}|q|}{r} + \frac{|pq|}{r^2} \,. \tag{9.0.22}
$$

Notice here that the metric components have the form of the extremal Reisnerr-Nordstrøm solution with $Q \to |q| + |p|$ and $Q^2 \to |pq|$. The two harmonic functions then give the dilaton as

$$
e^{-2\phi} = \frac{e^{-\phi_0} + \frac{|q|}{r}}{e^{+\phi_0} + \frac{|p|}{r}} \,. \tag{9.0.23}
$$

To check all the above field equation we derive the Einstein tensor from the metric

$$
g_{tt}^{-1} = -g_{ii} = H_1(r) H_2(r) \,. \tag{9.0.24}
$$

In [4] the field strength tensors are written as differential forms:

$$
F = d\psi \wedge dt \,, \, \widetilde{G} = d\chi \wedge dt \,. \tag{9.0.25}
$$

From these relations and (9.0.17) the non-zero components are

$$
F_{ti} = \pm \frac{H_1'}{H_1^2} \frac{\partial r}{\partial x^i} \tag{9.0.26}
$$

$$
\widetilde{G}_{ti} = \pm \frac{H_2'}{H_2^2} \frac{\partial r}{\partial x^i} \,. \tag{9.0.27}
$$

So, for example

$$
F_{t2} = \frac{H_1'}{H_1^2} \frac{x_2}{r} \,. \tag{9.0.28}
$$

From this we then compute the energy-momentum tensor of (9.0.7). The outputs are extremely large and complex and it would be impractical to display the results here but can be seen in Worksheet 3 at the end of this project.

As an example, the scalar field equation produces

$$(r_x^2 + r_y^2 + r_z^2)\phi'' + (r_{xx} + r_{yy} + r_{zz})\phi' = \frac{1}{2}(r_x^2 + r_y^2 + r_z^2)\frac{(H_1')^2 H_2^2 - (H_2')^2 H_1^2}{H_1^2 H_2^2}\,, \tag{9.0.29}$$

where, for example r_x is r differentiated with respect to x. Using $r_x^2+r_y^2+r_z^2 = 1$ and $r_{xx} + r_{yy} + r_{zz} = 2/r$ we get

$$\phi'' + \frac{2\phi'}{r} = \frac{1}{2}\frac{(H_1')^2 H_2^2 - (H_2')^2 H_1^2}{H_1^2 H_2^2}\,. \tag{9.0.30}$$

This equation, along with the outputs from the Einstein equation, are entered into a seperate worksheet with (from [4]) the scalar field

$$\phi = \frac{1}{2}\ln H_2(r) - \frac{1}{2}\ln H_1(r) \tag{9.0.31}$$

and H_1, H_2 given by (9.0.21). The reader is referred to Worksheet 4 to see that the field equations are satisfied and (9.0.23) is a solution for the dilaton field.

Returning to (9.0.23) notice that in the limit $r \to 0$:

$$e^{-2\phi} \to \frac{|q|}{|p|}\,. \tag{9.0.32}$$

Figure 8 is a plot of (9.0.23). The different curves represent different values of the field at infinity. All the curves converge to the value $|q|/|p|$ on the horizon, which in this case is

$$e^{-2\phi}\big|_{r=0} = 2\,. \tag{9.0.33}$$

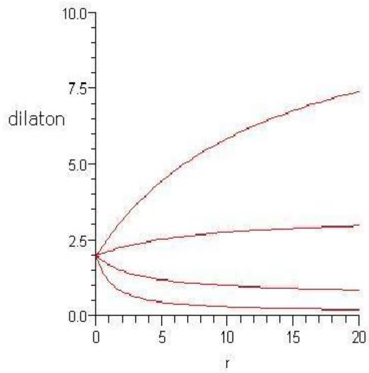

Figure 8: The Attractor Mechanism

This is another example of a black hole no-hair theorem. The attractor mechanism is also important as it implies that the black hole does not continuously depend upon external scalar fields. Then in string theories one can count microstates associated with the black hole and compare to the macroscopic entropy via Boltzmanns law.

10 Astrophysical black holes

In this section, as a conclusion to the project, we briefly discuss the method of detecting black holes as astrophysical objects. For further discussion of this topic see [8], section 9 and any book on astrophysics.

As mentioned in the introduction, black holes were studied theoretically long before there was any evidence of their physical existence. Since the 1980's groups have been reporting indirect evidence of black holes and by the 1990s only the most conservative astrophysicist would deny that there are black holes in our own galaxy.

By their very nature black holes are some of the most elusive objects in the universe. According to a calculation by Novikov and Thorne (1973) stars with mass 12-30 times the solar mass can form a black hole after a supernova explosion. The uncertainty is due to the unknown physics of the final stages of star evolution [8]. The late stages could produce a steady mass loss before the supernova and the mass remaining may not be large enough to collapse.

The mass spectrum of stars is roughly known so it has been possible to calculate the number of black hole progenitors in our galaxy. It is estimated that no more than 1% of the matter in the galaxy is in the form of a black hole (not including dark matter). However, this gives the total number of black holes as $N \approx 10^8$ [8].

Of course it is not possible to detect a black hole as one would detect a star, for example. The most forthcoming method has been to detect the emitted x-rays from gas accretion into the black hole. The accretion rate can be calculated in Newtonian theory by considering the likely black hole mass and the properties of the gas in the nebula surrounding the black hole.

One can calculate theoretically the rate at which matter falls through the horizon based on physical assumptions about the mass of the black hole and density of surrounding gas. The case of the Kerr black hole is most physically realistic and the accretion of matter detectable by the synchroton radiation emitted as charged particles are accelerated to the horizon moving through interstellar magnetic fields. For black holes resulting from stellar collapse this radiation is calculated to be too small for detection and to date there have not been any stellar black holes detected by this method.

Another method of detection results from the existence of binary systems; two stars orbiting each other. If one star is a collapsed object then one can observe the orbit of the single star around it's invisible partner. It is also predicted that matter from the star will gradually fall into the accompanying black hole. This mass loss should be detectable.

The most promising black hole candidates are within the so called 'Active Galactic Nuclei' at the center of galaxies. These objects have mass of the order of ten million solar masses and are known as *supermassive black holes.* Although firm evidence is elusive the x-ray spectrum of matter accretion into such objects is thought to be detectable.

There exists evidence (see [8] for original references) that these enormous masses are concentrated in a region of order 10^{-12}. Furthermore, the radiation is measured to originate from a region of a few Schwarzschild radii outside the predicted horizon. The X-rays have a broad spectrum indicating that the radiation has been red shifted gravitationally. The supermassive black hole at the center of our galaxy has been named 'Sagittarius A*' because it lies in the direction of the constellation Sagittarius.

Black holes are fascinating physical objects theoretically and observationally. They are perhaps the best arena to test physics of quantum gravity, space and time.

A Appendix

Here we collect some general properties of tensors and their use in general relativity. A contravariant index is up, X^a and a covariant index is down, X_a. A tensor of type (p, q) has covariant rank q and contravariant rank p. A covariant index transforms as

$$X'_a = \frac{\partial x^b}{\partial x^{a\prime}} X_b \tag{A.0.34}$$

and for the contravariant index

$$X^{a\prime} = \frac{\partial x^{a\prime}}{\partial x^b} X^b \,. \tag{A.0.35}$$

Repeated indices are to be summed over and for tensors, the order of indices is important, e.g $T^a{}_b \neq T_b{}^a$. The metric/metric inverse is used to lower/raise indices respectively

$$X_a = g_{ab} X^b \tag{A.0.36}$$

$$X^a = g^{ab} X_b \,. \tag{A.0.37}$$

The Lie derivative is defined as

$$L_X Y^a = X^b \partial_b Y^a - Y^b \partial_b X^a \tag{A.0.38}$$

$$L_X Y_a = X^b \partial_b Y_a + Y_b \partial_a X^b \,. \tag{A.0.39}$$

The covariant derivative of a covariant vector is

$$\nabla_c x_a = \partial_c X_a - \Gamma^b_{ac} X_b \,, \tag{A.0.40}$$

where Γ^b_{ac} are Christoffel symbols of the second kind (metric connection), given by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}) \,. \tag{A.0.41}$$

In general relativity the connection is torsion free, so that

$$\Gamma^a_{bc} = \Gamma^a_{cb} \,. \tag{A.0.42}$$

The Christoffel symbols are not tensors because they do not have the correct transformation properties. The covariant derivative of a general tensor is

$$\nabla_c T^{a\cdots}_{b\cdots} = \partial_c T^{a\cdots}_{b\cdots} + \Gamma^a_{dc} T^{d\cdots}_{b\cdots} + \cdots - \Gamma^d_{bc} T^{a\cdots}_{d\cdots} \,. \tag{A.0.43}$$

Some notation; a pair of symmetric indices is written with ( ) and antisymmetric with [ ]. So, for example,

$$2X_{[a} X_{b]} = X_a X_b - X_b X_a \tag{A.0.44}$$

$$2X_{(a} X_{b)} = X_a X_b + X_b X_a \tag{A.0.45}$$

The Riemann (curvature) tensor is defined by

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma^e_{bc} \Gamma^a_{ed} \tag{A.0.46}$$

and for torsion free connections

$$\frac{1}{2}R^a{}_{bcd}X^b = \nabla_{[c}\nabla_{d]}X^a \,. \tag{A.0.47}$$

Properties of the Riemann tensor:

$$\begin{aligned} R^a{}_{bcd} &= -R^a{}_{bdc} && \text{(A.0.48)} \\ R_{abcd} &= R_{cdab} && \text{(A.0.49)} \\ R_{abcd} &= -R_{bacd} && \text{(A.0.50)} \\ R_{abcd} &= -R_{abdc} = -R_{bacd} = R_{cdab} && \text{(A.0.51)} \\ R^a{}_{bcd} &+ R^a{}_{dbc} + R^a{}_{cdb} \equiv 0 \,. && \text{(A.0.52)} \end{aligned}$$

The Ricci tensor is

$$R_{ab} = R^c{}_{acb} = g^{cd}R_{dacb} \,, \tag{A.0.53}$$

which is symmetic, and the Ricci scalar

$$R = g^{ab}R_{ab} \,. \tag{A.0.54}$$

These are used to build the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R \,, \tag{A.0.55}$$

which is also symmetric.

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