Type III hyperbolic components in V_3.

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Every quadratic rational map with a period 3 critical point is represented up to Möbius conjugacy by

$$h_a(z) = \frac{(z-a)(z-1)}{z^2}.$$

The critical points are 0 and $c_2(a) = \dfrac{2a}{a+1}$ and 0 is of period 3. In fact:

$$h_a(0) = \infty, \quad h_a(\infty) = 1, \quad h_a(1) = 0,$$

$$h_a(c_2(a)) = -\frac{(a-1)^2}{4a} = v_2(a), \quad h_a^{-1}(0) = \{1, a\}, \quad h_a^{-1}(1) = \left\{\infty, \frac{a}{a+1}\right\}.$$

Write

$$V_3 = \{h_a : a \in \mathbb{C},\ a \neq 0\}.$$

A ***hyperbolic component*** is a connected (open) set in which $\lim_{n\to\infty} h_a^n(c_2(a))$ is an attractive periodic orbit, which is either $\{0, 1, \infty\}$ (for types II and III) or a different periodic orbit (for type IV). Each hyperbolic component H in V_3 has a unique *centre* $a_c = a_c(H)$ such that the set $\{h_{a_c}^n(c_2(a_c) : n > 0\}$ is finite. Thus, $c_2(a_c)$ is either periodic under h_a (for types II and IV) or strictly preperiodic (for type III). There are only two type II hyperbolic components in V_3, with centres ± 1. There are infinitely many of types III and IV.

A type III hyperbolic component wiith centre a_* is *of preperiod m* if $c_2(a_c)$ is of preperiod m under h_a. This happens if and only if $a_c \neq 0$, ± 1 and a_c satisfies one of the equations:

$$h_a^m(c_2(a)) = a \text{ or } \frac{a}{a+1}.$$

This happens if and only if a_c is a nontrivial root ($a \neq 0, \pm 1$) of $p_m(a) - aq_m(a)$ or $(a+1)p_m(a) - aq_m(a)$, where

$$p_0(a) = 2a, \quad q_0(a) = a+1,$$

$$p_{m+1}(a) = (p_m(a))^2 - (a+1)q_m(a) + a(q_m(a))^2,$$

$$q_{m+1}(a) = (p_m(a))^2.$$

In both cases, the number of nontrivial roots is

$$2^m.\frac{23}{21} + O(1),$$

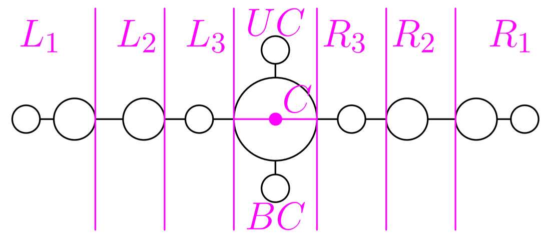
where the $O(1)$ term is period 6 in m.

Theorem *All roots are simple.* A full description of the dynamics in all type III hyperbolic components in V_3 can be found in [?]. It is enough to describe the dynamics of the centres of the hyperbolic components. As usual (or always) in dynamics, variation of dynamics with parameter is defined in terms of dynamics in particular dynamical planes. In this case, we use the three polynomials up to Möbius conjugacy in V_3, the two ***rabbit polynomials***, given up to Möbius conjgacy by $a = a_0$, $\overline{a_0}$ and the ***aeroplane polynomial*** given by $a = a_1$. Here, $a = a_0$, $\overline{a_0}$, $a = a_1$ are the roots of

$$c_2(a) = v_2(a)$$

A ***fundamental domain for*** V_3 is chosen to make the description of dynamics of each hyperbolic component unique. The description of dynamics of (the centre of) each type III hyperbolic component of preperiod $\leq m$ is then given by the following.

- $a_* = a_0$, $\overline{a_0}$ or a_1.
- a point x in $h_{a_*}^{-m}(\{0, 1, \infty\})$. In the case of the aeroplane polynomial this point has a symbolic coding by letters L_j, R_j ($j = 1$, 2 or 3) BC ,UC and C.



- A path from $c_2(a_*$ to x, which, apart from endpoints, does not intersect $Y_m(a_*) = h_{a_*}^{-m}(\{0, 1, \infty\}) \cup \{c_2(a_*)$ up to homotopy fixing $Y_m(a_*)$.

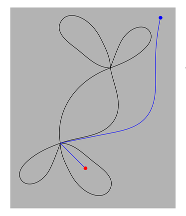
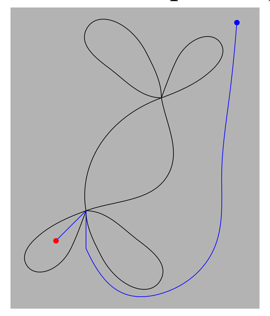
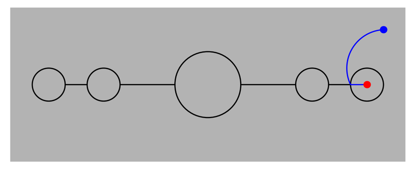
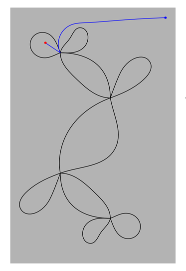
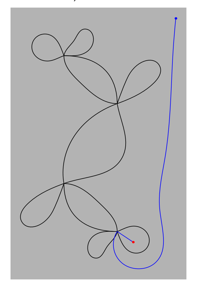
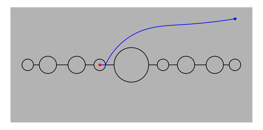
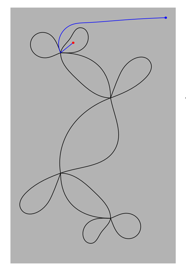
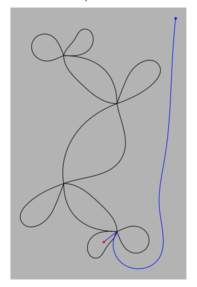
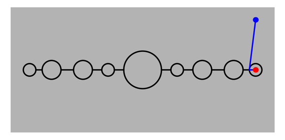
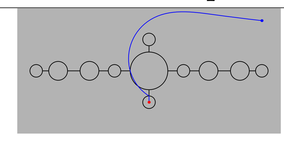
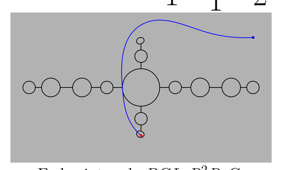
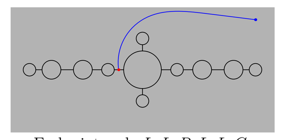
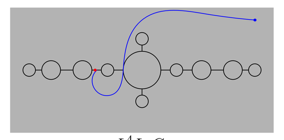
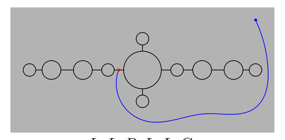
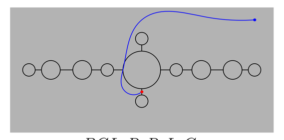
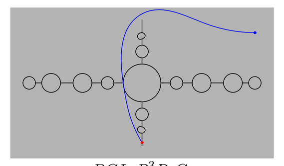
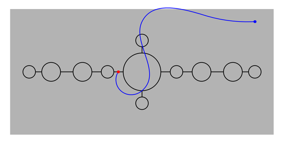
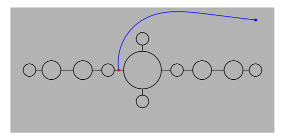
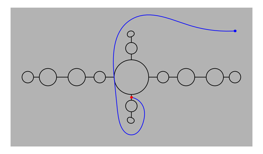
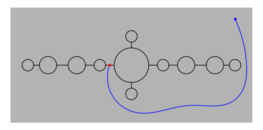
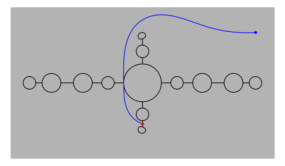
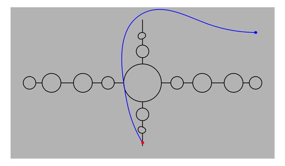
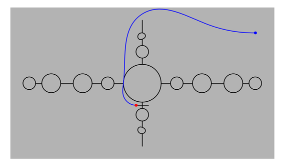
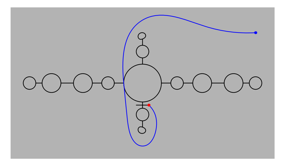
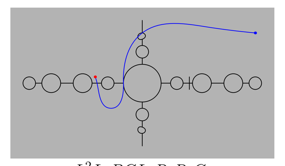
A path is called a *capture path* if it intersects the Julia set of h_{a_*} in exactly one point. A hyperbolic component is called a emcapture if it can be describe by a capture path.

Theorem *The number of capture type III hyperbolic components is* $\geq \lambda 2^m$ *and* $\leq \frac{22}{21}.2^m + O(1)$, *for some* $\lambda > 1$.

It can be considered a folklore-result that $\lambda > \frac{6}{7}$

References

[1] Rees, Mary: A Fundamental Domain for V_3. Preprint, May 2006.

m	nontrivial roots of $p_m(a) - aq_m(a)$	nontrivial roots of $(a+1)p_m(a) - aq_m(a)$
1	2, both rabbit captures 	$3 = 2 + 1$. 2 rabbit captures, one aeroplane capture.  
2	$5 = 4 + 1$. 4 rabbit captures, one aeroplane capture.   	$5 = 4 + 1$. 4 rabbit captures, one aeroplane capture.   
3	$7 = 6 + 1$. 6 rabbit captures, one aeroplane capture.	$8 = 6 + 1 + 1$. 6 rabbit captures, one aeroplane capture. *First noncapture*, using symbolic code $BCL_1R_1R_2C$
		
4	$18 = 14 + 4$ 14 rabbit captures, 4 aeroplane captures.	$18 = 14 + 3 + 1$. 14 rabbit captures, 3 aeroplane captures. Second noncapture. with code $BCL_1R_1^2R_2C$.  Endpoint code $BCL_1R_1^2R_2C$.
5	$35 = 28 + 6 + 1$ 28 rabbit captures, 6 aeroplane captures *up to equivalence*, 2 with same path endpoint, one of these represented by noncapture in fundamental domain. One other noncapture.  Endpoint code $L_3L_2R_3L_3L_2C$.  $L_3^4L_2C$. $\cong$  $L_3L_2R_3L_3L_2C$.  $BCL_1R_2R_3L_2C$.	$36 = 28 + 7 + 1$ 28 rabbit captures, 7 aeroplane captures, one noncapture.  $BCL_1R_1^3R_2C$.
6	$69 = 54 + 14 + 1$ 54 rabbit captures, 14 aeroplane captures, 2 with same path endpoint, both of these represented by noncaptures in the fundamental domain. One other noncapture.  $L_3L_2R_3L_3^2L_2C$. $\cong$  $L_3L_2R_3L_3^2L_2C$.  $BCL_1R_2R_3L_3L_2C$. $\cong$  $L_3L_2R_3L_3^2L_2C$.  $BCL_1R_1R_2R_3L_2C$.	$69 = 54 + 11 + 4$ 54 rabbit captures, 11 aeroplane captures, 4 noncaptures.  $BCL_1R_1^4R_2C$.  $BCL_1R_2BCL_1R_1R_2C$.  $BCL_1R_2UCL_1R_1R_2C$.  $L_3^2L_2BCL_1R_1R_2C$.