MATH102
Solutions Week 9 (Set 8)

49a)

$$\nabla f = (yz)\mathbf{i} + (zx)\mathbf{j} + (xy)\mathbf{k} \quad \Rightarrow \quad \nabla f(P) = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

$$\nabla g = (2x)\mathbf{i} + 4y)\mathbf{j} + (6z)\mathbf{k} \quad \Rightarrow \quad \nabla g(P) = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}.$$

So

$$\nabla f(P) \times \nabla g(P) = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}.$$

This is the direction of the tanget of the curve at $P = (1, 1, 1)$. So the equation of the tangent line to the curve at P is

$$x = 1 + 2t, \quad y = 1 - 4t, \quad z = 1 + 2t,$$

or

$$\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$$

b)

$$\nabla f = (2x)\mathbf{i} + (2y)\mathbf{j} \quad \Rightarrow \quad \nabla f(P) = 2\sqrt{2}(\mathbf{i} + \mathbf{j}).$$

$$\nabla g = (2x)\mathbf{i} + 2y)\mathbf{j} - \mathbf{k} \quad \Rightarrow \quad \nabla g(P) = 2\sqrt{2}(\mathbf{i} + \mathbf{j}) - \mathbf{k}.$$

So

$$\nabla f(P) \times \nabla g(P) = 2\sqrt{2}(-\mathbf{i} + \mathbf{j}).$$

So the line is in the direction $-\mathbf{i} + \mathbf{j}$, and the lequation of the tangent line to the curve can be written as

$$x = \sqrt{2} - t, \quad y = \sqrt{2} + t, \quad z = 4.$$

50. Let $\nabla f(P) = a\mathbf{i} + b\mathbf{j}$. Then

$$\nabla f(P).\frac{(\mathbf{i} + \mathbf{j})}{\sqrt{2}} = 2\sqrt{2} \quad \Rightarrow \quad a + b = 4,$$

$$\nabla f(P).\frac{(-2\mathbf{j})}{2} = -3 \quad \Rightarrow \quad b = 3 \quad \Rightarrow \quad a = 1.$$

Hence $\nabla f(P) = \mathbf{i} + 3\mathbf{j}$. We have $|-\mathbf{i} - 2\mathbf{j}| = \sqrt{5}$. So the derivative of f in the direction of derivative in the direction of $-\mathbf{i} - 2\mathbf{j}$ is

$$(\mathbf{i} + 3\mathbf{j}).\frac{(-\mathbf{i} - 2\mathbf{j})}{\sqrt{5}} = -\frac{7}{\sqrt{5}}.$$

51a) $f_x = 2x + y + 3,\ f_y = x + 2y - 3.$

$f_x = f_y = 0 \quad \Rightarrow \quad x + 2(-2x - 3) - 3 = 0 \quad \Rightarrow \quad x = -3, y = 3.$

$f_{xx} = 2, f_{yy} = 2, f_{xy} = 1 \quad \Rightarrow \quad f_{xx} > 0, f_{xx}f_{yy} - f_{xy}^2 = 3 > 0.$
Hence there is a minimum at $(-3, 3)$ and $f(-3, 3) = -5$.

It is an absolute minimum, because f is a conic – that is, at most quadratic in x and y – and if a conic has a maximum or minimum it is always absolute. The easiest way to see this completely is to write f as a sum of squares minus a constant:

$$f(x, y) = \left(x + \frac{y}{2} + \frac{3}{2}\right)^2 + \frac{3}{4}(y-3)^2 - 5.$$

b) $f_x = 2x - 4y,\ f_y = -4x + 2y + 6$.
$f_x = f_y = 0 \quad \Rightarrow \quad -4x + x + 6 = 0 \quad \Rightarrow \quad x = 2, y = 1.$
$f_{xx} = 2, f_{yy} = 2, f_{xy} = -4 \quad \Rightarrow \quad f_{xx}f_{yy} - f_{xy}^2 = -12 < 0.$
Hence there is a saddle point at $(2, 1)$.

c) $f_x = 12x - 6x^2 + 6y, f_y = 6y + 6x$.
$f_x = f_y = 0 \quad \Rightarrow \quad 12x - 6x^2 - 6x = 0 \quad \Rightarrow \quad x = 0, y = 0 \ \text{ or } \ x = 1, y = -1.$
$f_{xx} = 12 - 12x, f_{yy} = 6, f_{xy} = 6.$
At $(0, 0) : f_{xx} = 12 > 0, f_{yy} = 6, f_{xy} = 6$
$\Rightarrow \quad f_{xx}f_{yy} - f_{xy}^2 = 36 > 0, f_{xx} > 0.$
Hence there is a local minimum at $(0, 0)$ and $f(0, 0) = 0$.
At $(1, -1) : f_{xx} = 0, f_{yy} = 6, fxy = 6 \quad \Rightarrow \quad f_{xx}f_{yy} - f_{xy}^2 = -36 < 0.$
Hence there is a saddle point at $(1, -1)$.
The point $(0, 0)$ is not an absolute minimum because, for example, $f(x, 0) \to -\infty$ as $x \to +\infty$, for example $f(4, 0) < 0 = f(0, 0)$.

d) $f_x = \dfrac{-2x}{(x^2 + y^2 - 1)^2}, f_y = \dfrac{-2y}{(x^2 + y^2 - 1)^2}.$
$f_x = f_y = 0 \quad \Rightarrow \quad x = 0, y = 0.$
$f_{xx} = \dfrac{-2}{(x^2 + y^2 - 1)^2} + \dfrac{8x^2}{(x^2 + y^2 - 1)^3}, f_{yy} = \dfrac{-2}{(x^2 + y^2 - 1)^2} + \dfrac{8y^2}{(x^2 + y^2 - 1)^3},$
$f_{xy} = \dfrac{8xy}{(x^2 + y^2 - 1)^3}.$
At $(0, 0) : f_{xx} = -2, f_{yy} = -2, f_{xy} = 0 \quad \Rightarrow \quad f_{xx}f_{yy} - f_{xy}^2 = 4 > 0, f_{xx} < 0.$
Hence there is a maximum at $(0, 0)$ and $f(0, 0) = -1$.
It is not an absolute maximum since $f \to +\infty$ as $x^2 + y^2 - 1 \to 0$. for $x^2 + y^2 > 1$.

e) $f_x = y + 2 - 2x^{-1}, f_y = x - y^{-1}$.
$f_x = f_y = 0 \quad \Rightarrow \quad y + 2 - 2y = 0 \quad \Rightarrow \quad x = 1/2, y = 2.$
$f_{xx} = 2x^{-2}, f_{yy} = y^{-2}, f_{xy} = 1.$
At $(1/2, 2) : f_{xx} = 8, f_{yy} = 1/4, f_{xy} = 1 \quad \Rightarrow \quad f_{xx}f_{yy} - f_{xy}^2 = 1 > 0, f_{xx} > 0.$
Hence there is a local minimum at $(1/2, 2)$ and $f(1/2, 2) = 2 + \ln 2$.
It is not absolute since $f(x, y) < 0$ when, for example, $x = -2, y = 1$.

f) $f_x = -x^{-2} + y, f_y = x - y^{-2}$.
$f_x = f_y = 0 \quad \Rightarrow \quad x = 1, y = 1.$
$f_{xx} = 2x^{-3}, f_{yy} = 2y^{-3}, f_{xy} = 1.$
At $(1, 1) : f_{xx} = 2, f_{yy} = 2, f_{xy} = 1 \quad \Rightarrow \quad f_{xx}f_{yy} - f_{xy}^2 = 3 > 0, f_{xx} > 0.$

Hence there is a local minimum at $(1, 1)$ and $f(1, 1) = 3$.
It is not absolute since $f(x, y) < 0$ when $x = -1, y = -1$.

52.
Interior: $T_x = 2x - 1, T_y = 4y$ and $T_x = T_y = 0 \quad \Rightarrow \quad x = 1/2, y = 0$.
$T_{xx} = 2, T_{yy} = 4, T_{xy} = 0 \quad \Rightarrow \quad T_{xx}T_{yy} - T_{xy}^2 = 8 > 0, f_{xx} > 0$.
Hence there is a minimum at $(1/2, 0)$ and $T(1/2, 0) = -0.25$.
Boundary: $T(x, y) = 2 - x^2 - x = f(x), f' = -2x - 1$ and $f'' = -2$.
Hence there is a maximum at the points $x = -1/2, y = \pm\sqrt{3}/2$.
At these points $T = 2.25$.
On the boundary $-1 \leq x \leq 1$. $f(1) = T(1, 0) = 0$, $f(-1) = T(-1, 0) = 2$.
Hence hottest points are at $x = -1/2, y = \pm\sqrt{3}/2$, when $T = 2.25$ and the coldest point is at $x = 1/2, y = 0$, when $T = -0.25$.