Open problems submitted to the Workshop on Matings in Toulouse, June 2011

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Problem: the over-count of matings

Every mating of critically periodic quadratic polynomials which is a rational map is the centre of a *type IV* hyperbolic component (also called type D). The rational maps in such a hyperbolic component have two disjoint periodic orbits of attractive fixed points, of some periods m and n. It is possible to count the number of such hyperbolic components [1]. It is also possible to count the number of matings of critically periodic polynomials $f_1(z) = z^2 + c_1$ and $f_2(z) = z^2 + c_2$ such that 0 is of period m under f_1 and of period n under f_2.

Surprisingly the number of matings exceeds the number of hyperbolic components for all $n \geq 5$ and sufficiently large m. In fact there is an excess even if one discounts the shared matings arising from the *Wittner flip*, those arising from *period one clusters*, when a mating with a star like polynomial can be realised in another way, and a slight generalisation of this: those arising from *period two cluster* when a mating with a critically periodic polynomial for which the periodic Fatou components accumulate on a period two repelling orbit can be realised in three other ways. It was, in fact, the overcount which led to the discovery of the shared matings arising from period two clusters and removed the over-count in the case of $n = 4$. There must be many more shared matings to account for the over-count for all $n \geq 5$. It is possible that some of these are simply described.

Here is a brief description of the numerics. We write $\eta'_{IV}(m,n)$ of type IV hyperbolic components with disjoint orbits of attractive periodic points of periods ≥ 3 and dividing n and m. For a strictly positive integer q, let $\varphi(q)$ be the number of integers between 0 and q which are coprime to q (the usual Euler phi-function). Let $\nu_q(n)$ be the number of minor leaves of period dividing n in a limb of period q, that is,

$$\nu_q(n) = \begin{cases} \left\lfloor \dfrac{2^{n-1}}{2^q - 1} \right\rfloor \text{ if} q \not| n, \\ \\ \left\lfloor \dfrac{2^{n-1}}{2^q - 1} \right\rfloor + 1 \text{ if } q | n. \end{cases}$$

Then $\eta'_{IV}(m,n)$ is

$$\left(\frac{7}{18}(2^{n-1}-1)-\frac{1}{4}\sum\nolimits_{3\le q\le n}\frac{\varphi(q)\nu_q(n)}{2^q-1}\right)\cdot 2^m+o(2^m)\text{ if }n\text{ is odd,}$$

$$\left(\frac{7}{9}(2^{n-2}-1)-\frac{1}{4}\sum\nolimits_{3\le q\le n}\frac{\varphi(q)\nu_q(n)}{2^q-1}\right)\cdot 2^m+o(2^m)\text{ if }n\text{ is even.}$$

if n is even.

Meanwhile the number of critically periodic matings with critical points of periods ≥ 3 and dividing m and n respectively is

$$\left(\frac{4}{9}(2^{n-1}-1)-\frac{1}{2}\sum\nolimits_{3\le q\le n}\frac{\varphi(q)\nu_q(n)}{2^q-1}\right)\cdot 2^m+o(2^m)\text{ if }n\text{ is odd,}$$

$$\left(\frac{8}{9}(2^{n-2}-1)-\frac{1}{2}\sum\nolimits_{3\le q\le n}\frac{\varphi(q)\nu_q(n)}{2^q-1}\right)\cdot 2^m+o(2^m)\text{ if }n\text{ is even.}$$

if n is even. The shared matings coming from the Wittner flip and period two clusters reduce this by, respectively,

$$\frac{(n-2)\varphi(n)}{2(2^n-1)}\cdot 2^m+o(2^m)$$

and

$$\frac{2n\varphi(n/2)}{2^n-1}\cdot 2^m+o(2^m).$$

So if $\theta(n,m)$ is the number of matings, discounting known sharings,

$$\eta'_{IV}(n,m)-\theta(n,m)=-\left(\frac{1}{18}(2^{n-1}-1)-\frac{1}{4}\sum_{3\le q\le n}\frac{\varphi(q)\nu_q(n)}{2^q-1}\right)\cdot 2^m$$
$$+\frac{(n-2)\varphi(n)}{2(2^n-1)}2^m+o(2^m)$$

if n is odd and

$$\eta'_{IV}(n,m)-\theta(n,m)=-\left(\frac{1}{9}(2^{n-2}-1)-\frac{1}{4}\sum_{3\le q\le n}\frac{\varphi(q)\nu_q(n)}{2^q-1}\right)\cdot 2^m$$
$$+\left(\frac{(n-2)\varphi(n)}{2(2^n-1)}+\frac{2n\varphi(n/2)}{2^n-1}\right)\cdot 2^m$$

if n is even. For $n=3$, 4 and 5 this is respectively

$$\frac{1}{21}2^m+o(2^m),\quad \frac{6}{35}2^m+o(2^m),\quad -\frac{156}{1085}\cdot 2^m+o(2^m)$$

But for $n\geq 6$

$$\eta'_{IV}(n,m)-\theta(n,m)\le-\left(\frac{1}{18}-\frac{1}{4}\sum_{3\le q\le n}\frac{q-1}{(2^q-1)^2}\right)\cdot 2^{m+n-1}$$

$$+2^m \cdot \left(\frac{1}{9} + \frac{1}{8} \sum_{3 \le q \le n} \frac{q-1}{2^q - 1} + \frac{(n-2)(3n-1)}{2(2^n - 1)} \right)$$
$$< -\frac{1}{27} \cdot 2^{m+n-1} + 2^m < 0$$

Problem: Type II hyperbolic components, periodic Wittner captures and matings, type IV components adjoining the boundary

Questions: Are there type II hyperbolic components of quadratic rational maps whose centres are not periodic Wittner captures? Is there a critically finite quadratic rational map with Fatou components in a single periodic orbit of clusters, which is not Thurston equivalent to a mating?

The second question is slightly weaker than the first, and can be considered without understanding the definition of periodic Wittner capture. A type II hyperbolic component is one for which both critical points of maps in the component lie in a single periodic cycle of Fatou domains of some period $n > 1$. In the boundary of such a hyperbolic component, if $n \geq 3$, there are at most three, and at least two, critically periodic rational maps, distinct up to Möbius conjugacy, with two distinct cycles of critical points of period n. If there are less than three, then the hyperbolic component is unbounded. In all the examples I know, all the boundary maps are Thurston equivalent to matings. Are there any which are not? The representation as a mating is not usually unique, if it does exist, which, of course, raises other questions.

Any missing boundary points are certainly represented by inadmissible matings. If at least one of these boundary rational maps is equivalent to a mating, even an admissible mating, then the centre of the hyperbolic component is equivalent to a periodic Witttner capture.

For any type II hyperbolic component of quadratic rational maps with Fatou comoponent of period $n \geq 3$, and any $q \geq 2$, there are $6(2^{q-1} - 1)$ type IV hyperbolic components with periodic Fatou components of periods $n$and nq_1, for $q_1 > 1$ dividing q, such that the closures of the type II and type IV components intersect. The intersection contains a rational map with one critical point of period n and an orbit of parabolic basins of period dividing nq. Much of this is proved in [2], although there is no count there: and the count given here does need checking.

Question: How many of the centres of these type IV hyperbolic components can be represented as matings?

I believe that such examples occur even for $n = 3$ and $q = 2$, but have not checked this properly. There are two type II components of period 3, and hence there are twelve type IV hyperbolic components to consider for $n = 3$ and $q = 2$. Of these, ten are represented by matings, eight of them in a unique way and two in two different ways. The last two of the twelve appear not to be represented by matings. This situation is probably reproduced for $n = 3$ and any $q > 2$, but, again, I have not checked this properly

References

[1] Kiwi, J. and Rees, M., Counting hyperbolic components, submitted to the London Mathematical Society

[2] Rees, M., Components of degree two hyperbolic rational maps. Invent. Math., 100 (1990), 357-382.