Knot singularities in some Rational Map Parameter Spaces.

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ABSTRACT

The knot types are calculated for some singularities in a family V_n of varieties of quadratic rational maps. In particular, developing and using some theory, the simplest nontrivial knot types are identified and calculated, as is the simplest knot type which is not a torus knot.

RÉSUMÉ

Les genres des noeuds sont calculé s pour les singularités dans une famille V_n des variétés des fonctions rationelles quadratiques. En particulier, en développant et utilisant de la théorie, les genres des noeuds les plus simplessont calculés, et aussi le genre de noeud le plus simple qui n'est pas un noeud du tore.

In this note, we determine the knot types of some singularities of some algebraic curves which arise as parameters spaces of rational maps. We consider

$$f_{b,c,d}(z) = b + \frac{c}{z} + \frac{d}{z^2}$$

which, for $d \neq 0$, is a degree two map, with critical points 0 and $-2d/c$. We write $f_{c,d} = f_{1,d,c}$. The orbit of 0 under $f_{c,d}$ starts $0 \mapsto \infty \mapsto 1 \mapsto 1 + c + d$. We take V_n to be the set of $f_{c,d}$ - or equivalently the set of $(c, d) \in \mathbf{C}^2$ - for which 0 is of period n under $f_{c,d}$, for each integer $n \geq 3$. These curves were much studied in [**R**] and [**Sti**] (and indeed have been much studied by many authors, including A. Epstein [**E**]). We take R to be the quotient of the set of maps

$$\{f_{b,c,d} : (c, d) \neq (0, 0),\ (b^2 + 4c, d) \neq (0, 0)\}$$

by conjugations $z \mapsto \lambda z$ ($\lambda \neq 0$). These conjugations preserve the critical point 0 if $d \neq 0$. The space of maps $\{f_{c,d} : (c, d) \neq (0, 0),\ c \neq -\frac{1}{4}\}$ maps injectively into R. In fact, it maps onto the quotient of the set of maps with $b \neq 0$. The space R is a noncompact orbifold, with just two nonmanifold points, at the maps $z \mapsto z^{-1}$ and $z \mapsto z^{-2}$. Up to Möbius conjugation, all rational maps of degrees one or two are in R, apart from $z \mapsto z + t$ (any t) and $z \mapsto z^2$. Any degree two rational map f with critical point c is in R up to Möbius conjugacy mapping c to 0, except when c is fixed under f, that is, up to Möbius conjugation, f is a polynomial and $c = \infty$. The closure $\overline{V}_n$ of any V_n within R is compact. If c or $d \to \infty$ in V_n, then they both do, and in R, $f_{c,d}$ tends to $z \mapsto z^{-1}$. The curves V_n are nonsingular, with possible singularities at the points of $\overline{V}_n \setminus V_n$, that is, $d = 0$ or $d \to \infty$ (in which case $c \to \infty$ also). A proof of this is given in [**Sti**] Chapter 1. The proof uses standard techniques, and has also been proved by others, A. Epstein, for example. From now on we fix n and write $V = V_n$.

There are always finitely many points in $\overline{V} \setminus V$. These points can be precisely described [**Sti**]. The point $z \mapsto z^{-1}$ is always one of them. The others are the maps

$f_{c,0} : z \mapsto 1 + c/z$ for which this Möbius transformation has period $\leq n$ and ≥ 3. For such maps we necessarily have c real and $c < -\frac{1}{4}$. Up to Möbius conjugation which sends the fixed points of $f_{c,0}$ to 0, ∞, these maps are, of course, of the form $z \mapsto \zeta z$ for ζ a k'th root of unity, $2 \leq k \leq n$ and $\zeta = \lambda/|\lambda|$ where $\lambda = 1 \pm \sqrt{1+4c}$. Note that $z \mapsto \zeta z$ is Möbius conjugate to $z \mapsto \zeta^{-1} z$ under the conjugacy $z \mapsto z^{-1}$.

In order to study singularities, we change coordinates and consider the set of maps

$$S = \{h_{\zeta,\rho} : (\zeta, \rho) \in \mathbf{C}^* \times \mathbf{C}\},$$

where

$$\begin{aligned} h_{\zeta,\rho}(z) &= \zeta z \left(1 - \frac{2+\rho}{2(1+\rho)} z\right) \left(1 - \frac{2}{2+\rho} z\right)^{-1} \\ &= \zeta z \left(1 - \frac{\rho^2 z}{4(1+\rho)(1+\frac{1}{2}\rho - z)}\right) = \zeta z P_\rho(z). \end{aligned}$$

Stimson used slightly different coordinates. These particular coordinates were used in [**R**] 7.4. The fixed points are 0, ∞, with multipliers ζ, $\zeta^{-1}(1+\rho^2/4(1+\rho))^{-1}$, and a point near 1. The critical points are 1 and $1+\rho$. The map $h_{\zeta,\rho}$ is Möbius conjugate under $z \mapsto z^{-1}$ to $h_{\zeta^{-1},\rho'}$ where $\rho' = (1+\rho)^{-1} - 1$, and this is the only possible conjugacy fixing 1. We have $(\zeta, \rho) = (\zeta', \rho')$ if and only if $(\zeta, \rho) = (-1, 0)$ or $(1, 0)$. Although not every map in R is Möbius conjugate to a map in S under a conjugacy mapping 0 to 1, this is true near degree one maps in R, where the map thus locally defined from S to R is a local homeomorphism, except near $h_{-1,0}$, where it is two-to-one, and provides the manifold covering.

We let W be the lift of V (minus finitely many points - corresponding to polynomials) to S, which will, in fact, always be a connected double cover. The lift of $\overline{V}$ to S is closed, and is the closure, $\overline{W}$, of W. Using the (ζ, ρ)-coordinates, the points of $\overline{W} \setminus W$ are now of the form $(\zeta_0, 0)$ where ζ_0 is a k'th root of unity for some $2 \leq k \leq n$. Near a point x_0 of $\overline{W} \setminus W$, the continuous map π from $\overline{W}$ to $\overline{V}$ extending the map $W \to V$ is a local homeomorphism, unless $\pi(x_0) = f_{0,1,0} \in R$, in which case π is either a local homeomorphism or a degree two branched covering near x_0. A *branch* of W at x_0 is the equivalence class of a topological annulus (conformally a punctured disc) which is a component of intersection of W with a neighbourhood in S of x_0, whose closure is a topological disc D containing x_0. Two such annuli are equivalent and define the same branch if their intersection contains another such annulus. The branch is *smooth* if D is a smooth submanifold in S. If a branch is not smooth, we can study its *knot type*, its intersection with a three-sphere which is the boundary of a neighbourhood in S of x_0. The complement of this knot in the three-sphere is then uniquely determined up to homeomorphism. Classically, the knot is an iterated torus knot and one can ask what knot types occur.

Fix a branch U of W at $(\zeta_0, 0)$. Then $\pi \mid U$ is degree one unless $\zeta_0 = -1$, when it can also be degree two. A calculation in [**Sti**] (see below) shows that the degrees of ρ and ζ on U are the same. Classically, the knot type of U is trivial if and only if one of ζ, ρ can be used as a parameter on U: that is, since they both have the same degree, if and only if $\zeta \mid U$ has degree one. The method of Stimson's thesis [**Sti**] enables us to compute the local degree of ζ on U. (In [**Sti**] it was claimed that the local degree was always one, which later turned out to be false in general, although true for $n < 9$. But, as we shall see, in some sense $n = 7$ gives the first counterexample.) In [**Sti**], and in [**R**] 3.11, 7.4, it was shown how to calculate this degree - always a power of two - in terms of certain combinatorial data - the *Thurston equivalence class* of a *critically finite branched covering* associated to $\pi(U)$. The first example was given in 3.12 of [**R**] of such a branched covering for which the degree of $\zeta \circ \pi \mid U$ was degree > 1 for the associated branch U, but no knot types were calculated. (In fact, for that branch U, the degree of $\pi \mid U$ was two and the degree of $\zeta \mid U$ was one, so the knot type of U was trivial.) The purpose of this paper is to calculate some knot types, the main task being to identify the simplest nontrivial ones.

1. Some Expansions.

The following expansions will be important.

$$h_{\zeta,\rho}(1) = \zeta(1 - \rho^2/(4(1+\rho)(\rho/2))) = \zeta(1 - \tfrac{1}{2}\rho + \tfrac{1}{2}\rho^2 + O(\rho^3)), \tag{1.1}$$

$$h_{\zeta,\rho}(1+\rho) = \zeta(1 + \frac{3}{2}\rho + O(\rho^2)). \tag{1.2}$$

If $\zeta = \zeta_0 + o(1)$ where ζ_0 is a primitive k'th root of unity, then for $1 \le j \le k$, and $z = 1 + b\rho + c\rho^2 + o(\rho^2)$, $b \ne \frac{1}{2}$, c_j a function of b and c,

$$\begin{aligned} h^j_{\zeta,\rho}(z) &= \zeta^j \prod_{\ell=0}^{j-1} P_\rho(h^\ell_{\zeta,\rho}(z)) = \zeta^j z P_\rho(z)(1 + c_j\rho^2 + o(\rho^2)) \\ &= \zeta^j \left(1 + \rho\left(b + \frac{1}{2(2b-1)}\right) + d_j\rho^2 + o(\rho^2)\right). \end{aligned} \tag{1.3}$$

It follows that, restricted to a branch U, ζ^k must have an expansion $\zeta^k = 1 + a\rho + o(\rho)$, for a depending on the branch. It is explained in [**Sti**] (in slightly different coordinates), that $a \ne 0$. (The same argument is summarized in [**R**], 7.4.) So the degrees of $\rho \mid U$ and $\zeta \mid U$ are always the same. In particular, for $a \ne 0$,

$$h^k_{\zeta,\rho}(1 + b\rho) = 1 + \left(a + b + \frac{1}{2(2b-1)}\right)\rho + O(\rho^2). \tag{1.4}$$

If z is bounded and bounded from 1,

$$h'_{\zeta,\rho}(z) = \zeta + O(\rho^2). \tag{1.5}$$

$$h'_{\zeta,\rho}(1 + b\rho) = \frac{b(b-1)\zeta(4 + 4\rho + \rho^2)}{\rho(2b-1)^2},$$

$$h''_{\zeta,\rho}(1 + b\rho) = \frac{4\zeta}{\rho(2b-1)^3} + \frac{\zeta\rho}{(1+\rho)(2b-1)}.$$

So if $v = o(\rho)$, and b is bounded and bounded from $\frac{1}{2}$, then

$$\begin{aligned} h_{\zeta,\rho}(1 + b\rho + v) &= h_{\zeta,\rho}(1 + b\rho) + \frac{4b(b-1)\zeta v}{\rho(2b-1)^2} + \frac{2\zeta v^2}{\rho(2b-1)^3} \\ &+ O(\rho b(b-1)v) + O(\rho v^2) + O(v^3/\rho). \end{aligned} \tag{1.6}$$

$$h'(1+\tfrac{1}{2}\rho+c\rho^2) = -\frac{\zeta}{4c^2\rho^2}(1+O(\rho)).$$

So if $w=o(\rho^2)$, and c is bounded and bounded from 0, then

$$\begin{aligned} h_{\zeta,\rho}(1+\tfrac{1}{2}\rho+c\rho^2+w) &= h_{\zeta,\rho}(1+\tfrac{1}{2}\rho+c\rho^2) - \frac{\zeta w}{4c^2\rho^2} + O(w^2/\rho^4) + O(w/\rho) \\ &= \zeta\frac{1+4c}{4c} + o(1). \end{aligned} \tag{1.7}$$

2. Basic Combinatorial Data.

Sufficient details on critically finite branched coverings will be given to carry out simple calculations. For more details, see [**T**], [**Sti**], [**R**]. A *critically finite branched covering* f_0 of $\overline{\mathbf{C}}$ is a branched covering for which the union $X(f_0)$ of the forward orbits of numbered critical values is finite. Two such branched coverings f_0 and f_1 are *Thurston equivalent* if there is a homotopy f_t ($t\in[0,1]$) between them, such that each f_t is critically finite and $X(f_t)$ varies isotopically, with numbering of critical values preserved. (This condition about numbering being preserved is the convention of the current paper, but is not universal.) In the current work, the branched coverings are degree two, and there are always two critical points, and two critical values.

There is a particularly simple description of critically finite quadratic polynomials (up to Thurston equivalence) by Douady and Hubbard ([**D-H1**], [**D-H2**]) which was reinterpreted by Thurston [**T**] in terms of *minor leaves.* We restrict to those polynomials for which the finite critical point is periodic. Such Thurston equivalence classes (apart from that of $z\mapsto z^2$) are in natural one-to-one correspondence with *minor leaves* μ_p for p an odd denominator rational in $(0,1)$. The minor leaf μ_p is the chord in the unit circle with endpoints $e^{2\pi ip}$ and $e^{2\pi ip'}$, where p and p' have the same period under $x\mapsto 2x \bmod 1$, the interior of the chord μ_p is disjoint from all its forward images under $z\mapsto z^2$ (where the map on chords is determined by the map on endpoints) and μ_p is of shorter Euclidean length than any chord in its forward orbit (which is periodic). These properties uniquely determine p' from p. So the equivalence relation generated by $p\sim p'$ for all odd denominator rationals p in $(0,1)$ has exactly two elements in each equivalence class. All minor leaves μ_p for odd denominator rationals $p\in(0,1)$ are disjoint, with disjoint endpoints. A partial ordering is defined on such minor leaves by: $\mu_p<\mu_q$ if μ_p separates μ_q in the unit disc from 0. The minimal leaves in in this ordering are precisely those μ_p such that the forward orbit of μ_p bounds a single polygon in the unit disc. In the case of $\mu_{1/3}$, this polygon is degenerate: the forward orbit is just $\mu_{1/3}$ itself, with endpoints $e^{2\pi i/3}$, $e^{4\pi i/3}$. The other minimal minor leaf which will feature prominently in our examples, $\mu_{1/7}$, has forward orbit bounding a triangle, with vertices at $e^{2\pi i/7}$, $e^{4\pi i/7}$, $e^{8\pi i/7}$.

Any minor leaf μ_p generates an *invariant lamination* L_p in the unit disc, which is a closed set of leaves, with endpoints forward and backward invariant under $z\mapsto z^2$, and such that the preimages of μ_p (one of which is periodic, and also in the forward orbit of μ_p) are the two leaves in L_p of longest Euclidean length. There is a periodic *gap* $G_0(p)$ - a component of the complement of $\cup L_p$ - adjacent to μ_p, and separated from 0 by μ_p, and of the same period as μ_p. A degree two branched covering s_p

can be constructed which preserves the unit disc, L_p, gaps, is $z \mapsto z^2$ on the unit circle and with a critical value in $G_0(p)$ of the same period as μ_p. The map s_p is uniquely defined by μ_p up to Thurston equivalence, and every quadratic polynomial with periodic finite critical point is Thurston equivalent to s_p for exactly one μ_p. We define critically finite branched coverings $s_p \amalg s_q$ by

$$(s_p \amalg s_q)(z) = \begin{array}{l} s_p(z) \text{ if } |z| \leq 1, \\ (s_q(z^{-1}))^{-1} \text{ if } |z| \geq 1. \end{array}$$

It was shown in [**Sti**] that there is a Thurston equivalence class of critically finite branched covering $f_{\pi(U)}$ associated with $\pi(U)$ which is of the form $s_p \amalg s_q$ with μ_p minimal and $\mu_q \geq \mu_{1-p}$. (If μ_p is minimal then so is μ_{1-p}.) This construction was summarized in [**R**] (and will be summarized again below), and it was confirmed that $U \mapsto f_{\pi(U)}$ is a one-to-one correspondence. There may be several (finitely many) different ways of writing $f_{\pi(U)}$ in the form $s_p \amalg s_q$, but we can only have $s_p \amalg s_q \simeq s_{p'} \amalg s_{q'}$ if either $\mu_{p'} = \mu_p$ or $\mu_{p'} = \mu_{1-p}$. We have $\mu_p = \mu_{1-p}$ if and only if $p = 1/3$ or $p = 2/3 = 1 - 1/3$. The branch U is easily seen to be smooth if, for p and q such that $f_{\pi(U)} \simeq s_p \amalg s_q$, there is no q', $\mu_{q'} \neq \mu_q$, with $s_p \amalg s_q \simeq s_p \amalg s_{q'}$. In fact in [**Sti**] it was claimed that this is always true that therefore the branch is always smooth. This was later found [**R**] to be false, but in a constructive way. The number of different q' is always 2^m for some integer $m \geq 0$ (3.11 of [**R**]). The degree of $\zeta \mid U$ is 2^m if $\mu_p \neq \mu_{1/3}$ and $2^{m-1}\deg(\pi \mid U)$ if $\mu_p = \mu_{1/3}$, where $\deg(\pi \mid U) = 2$ if $m = 0$ and $\deg(\pi \mid U) = 1$ if $m \geq 1$ ([**R**] 7.4). So the branch U is knotted if and only if $\zeta \mid U$ is degree > 1, that is, if $m \geq 1$ in the case $\zeta_0 \neq -1$, or $m \geq 2$ if $\zeta_0 = -1$.

The number of distinct leaves $\mu_{q'}$ such that $s_p \amalg s_{q'} \simeq s_p \amalg s_q = f$ is then determined as follows. Let $X(f)$ denote (as usual) the union of the forward orbits of the critical points 0, ∞ under f. Write $\gamma_0 = \mu_p \cup \mu_{1-p}^{-1}$. Let Γ be the smallest loop set in $\overline{\mathbf{C}} \setminus X(s_p \amalg s_q)$ such that $\gamma_0 \in \Gamma$ and $f^{-1}(\Gamma) = \Gamma$ modulo trivial and peripheral loops and isotopy preserving $X(s_p \amalg s_q)$. The map f^{-1} on Γ gives a linear map $f^* : \mathbf{R}^\Gamma \mapsto \mathbf{R}^\Gamma$ ([**T**], [**R**] 2.4). Let A be the matrix of $f^* : \mathbf{R}^\Gamma \to \mathbf{R}^\Gamma$ with respect to the standard basis, which, in fact, has rational coefficients. This matrix has a unique eigenvector (up to scaling) with eigenvalue 1, with all entries equal in the coordinates corresponding to the forward orbit of $\mu_p \cup \mu_{1-p}^{-1}$. Let L be the least integer ≥ 1 such that an eigenvector with L in the entry corresponding to $\mu_p \cup \mu_{1-p}^{-1}$ has all entries integers. It was shown in 3.12 of [**R**] that $L = 2^m$ for m as above. We can write $m = m(f)$, because any Thurston equivalence between maps $s_p \amalg s_q$ and $s_{p'} \amalg s_{q'}$ will preserve γ_0 up to isotopy.

3. The construction of $\mathbf{f}_{\pi(\mathbf{U})}$.

We outline the construction of $f_{\pi(U)}$, following [**Sti**] and the outline from there given in 7.4 of [**R**]. Given $M > 0$ there is N such that $h_{\zeta,\rho}^{Nk}(1+\rho)$ is distance $\geq M\rho$ from 1 for $(\zeta, \rho) \in U$. As (ζ, ρ) traces a circle in U round the puncture of W at $(\zeta_0, 0)$, the straight line segment $\tau_\ell = \tau_\ell(\zeta, \rho)$ joining $h_{\zeta,\rho}^{(\ell+1)k}(1+\rho)$ and $h_{\zeta,\rho}^{\ell k}(1+\rho)$ does not intersect the periodic orbit of 1 under $h_{\zeta,\rho}$ for $\ell \geq N$. The multiplier of the fixed point 0 of $h_{\zeta,\rho}$ is ζ. So if $|\zeta| < 1$, 0 is an attractive fixed point. So in the case $|\zeta| < 1$, $h_{\zeta,\rho}$ lies in the hyperbolic component of a rational map which is Möbius conjugate to a polynomial. Choose an odd denominator rational $q = q(\zeta, \rho)$ so that this polynomial is Thurston equivalent to s_q. There are actually two choices for q given by the two endpoints of μ_q. Let $\varphi = \varphi_{\zeta,\rho}$ be the uniformising map so that

$$\varphi(0) = 0,\ \varphi(\zeta z) = h_{\zeta,\rho}(\varphi(z)),\ 1 + \rho = \varphi(\zeta).$$

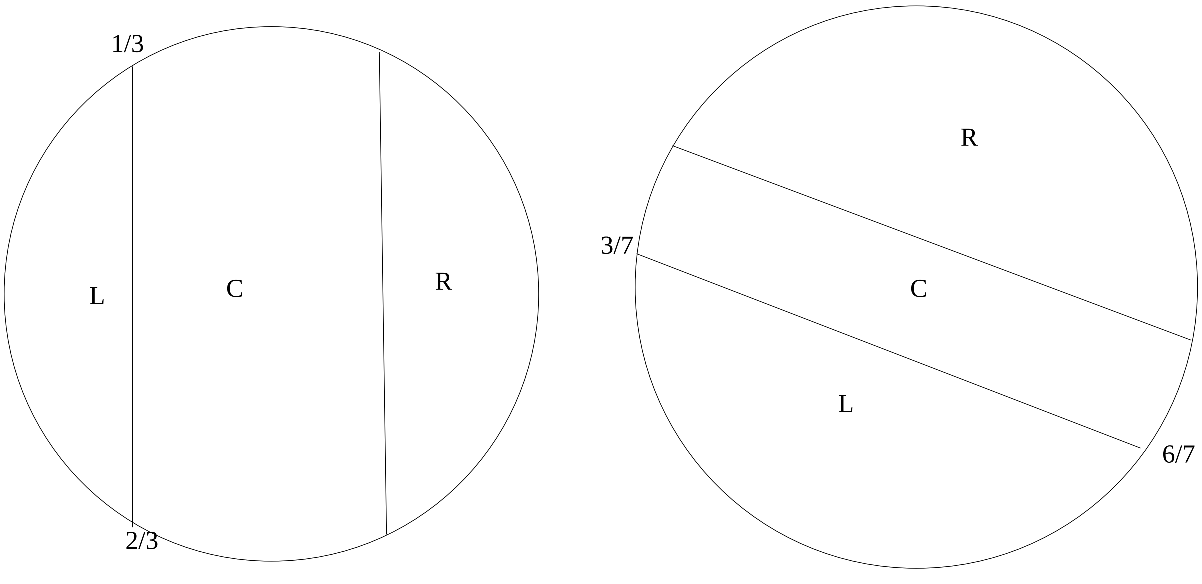
Now suppose, further, that $\zeta = \lambda\zeta_0$ for real λ, $0 < \lambda < 1$. Then for sufficiently large n, the image under φ of the straight line segment joining $\lambda^{(\ell+1)k}$ and $\lambda^{\ell k}$ is isotopic to τ_ℓ relative to the forward orbits of 1, $1+\rho$. It follows by induction that the same is true for all $\ell \geq N$, and that there is a preimage $\tau = \tau(\zeta, \rho)$ of τ_N under $h_{\zeta,\rho}^{Nk-1}$ which joins $h_{\zeta,\rho}(1+\rho)$ and $h_{\zeta,\rho}^{k+1}(1+\rho)$, and is disjoint from the periodic orbit of 1. Moreover, $\tau = \tau(\zeta, \rho)$ varies continuously with (ζ, ρ). Let σ_τ denote a homeomorphism which is the identity outside a small disc neighbourhood of the arc τ, and maps the initial endpoint $h_{\zeta,\rho}(1+\rho)$ of τ to the final endpoint $h_{\zeta,\rho}^{k+1}(1+\rho)$. All the critically finite maps

$$\sigma_{\tau(\zeta,\rho)}^{-1} \circ h_{\zeta,\rho}$$

are Thurston equivalent and are also equivalent to $s_p \amalg s_q$ for $p = p(\zeta_0)$ and $q = q(\zeta, \rho)$ for any (ζ, ρ) in a polynomial hyperbolic component in U with $|\zeta| < 1$.

4. Symbolic Dynamics.

If μ_p (and hence also μ_{1-p}) is minimal and q is an odd denominator rational with $\mu_q \geq \mu_{1-p}$, then, rather than using the numerical value q, it is probably more instructive to describe q in terms of symbolic dynamics with respect to a natural partition of the unit disc asssociated to μ_{1-p} - although this description will not determine $\mu_q \geq \mu_{1-p}$ uniquely. We use a coarsening of the much-used Yoccoz partition **[H]**. Given μ_{1-p}, there is a partition of the unit disc into three parts, using the two preimages under $x \mapsto 2x \bmod 1$ of μ_{1-p}. The partitions for $p = 1/3$ and $p = 1/7$ ($1 - p = 6/7$) are illustrated.



L, R, C

The justification for labelling the regions L, R, C is that the region labelled L contains the leftmost point on the unit circle, and the region labelled R contains the rightmost point. As is common, we associate to q the sequence $X_1 \cdots X_n$ with $\exp 2\pi i(2^{j-1}q) \in X_j \in \{L, C, R\}$ if q has period n under $x \mapsto 2x \bmod 1$. In the case $\mu_q = \mu_{1-p}$, when this does not define the sequence uniquely, we take the sequence $L^{n-1}C$. So then, in all cases, $X_1 = L$ and $X_n = C$. Any occurrence of C has to be followed by L, in fact, by $L^{k-1}(R \text{ or } C)$ if k is the period of p (and $1-p$) under $x \mapsto 2x \bmod 1$. Any occurrence of R has to be followed by $L^j(R \text{ or } C)$ for some $0 \le j \le k-2$. Any sequence obeying these rules does occur.

If $\mu_q \ge \mu_{1-p}$ for μ_p minimal, and $s_p \amalg s_q \simeq s_p \amalg s_{q'}$, then $\mu_{q'} \ge \mu_{1-p}$ also, and the symbolic sequences associated to q and q' must be the same. Given a branch U of W at a point $(\zeta_0, 0) \in S$, where $\zeta_0 = e^{2\pi i\alpha}$ is a root of unity, p is determined as being one of the two odd denominator rationals such that μ_p is minimal and the orders of the points $2^j p \bmod 1$ and $j\alpha \bmod 1$ on $(0,1)$ are the same. If p is one of the rationals, the other is r, where $e^{2\pi i r}$ is the other end of μ_p. Then the symbolic sequence for q can be determined from the sequence $h^j_{\zeta,\rho}(1)$ for $(\zeta, \rho) \in U$, $0 \le j < n$ (if $W = W_n$). For all j, we must have $h^j_{\zeta,\rho}(1) = \zeta_0^t + O(\rho)$ for some $0 \le t < k$, where ζ_0 is a k'th root of unity. If $1 \le t < k$, then the j'th letter of the symbolic sequence is L. If $h^j_{\zeta,\rho}(1) = 1 + \frac{1}{2}\rho + o(\rho)$ then the j'th letter is R. If $h^j_{\zeta,\rho}(1) = 1 + b\rho + o(\rho)$ for $b \ne \frac{1}{2}$ then the j'th letter is C. The justification for this is that the partition into L, R, C can be drawn directly on the dynamical plane of $h_{\zeta,\rho}$. For example, let

$$L' = \{z \in \overline{\mathbf{C}} : |z - 1| \ge |\rho|^{1/2}\},$$

$$R' = \{z \in \mathbf{C} : |z - 1 - \tfrac{1}{2}\rho| \le |\rho|^{3/2}\},$$

$$C' = \overline{\mathbf{C}} \setminus (L \cup R).$$

Let $f = \sigma^{-1}_{\tau(\zeta,\rho)} \circ h_{\zeta,\rho}$ as above. Then $\partial L'$ and $\partial R'$ are isotopic to components of $f^{-k}(\partial L)$ via an isotopy fixing $X(f)$, and are preimages under f of the same component γ_0' of $f^{-(k-1)}(\partial L)$ up to $X(f)$-preserving isotopy. Under the Thurston-equivalence between f and $s_p \amalg s_q$ (any p and q) γ_0' will be isotoped to $\mu_p \cup \mu_{1-p}^{-1} = \gamma_0$, and the $\{L', C', R'\}$ partition will be isotoped to $\{L, C, R\}$.

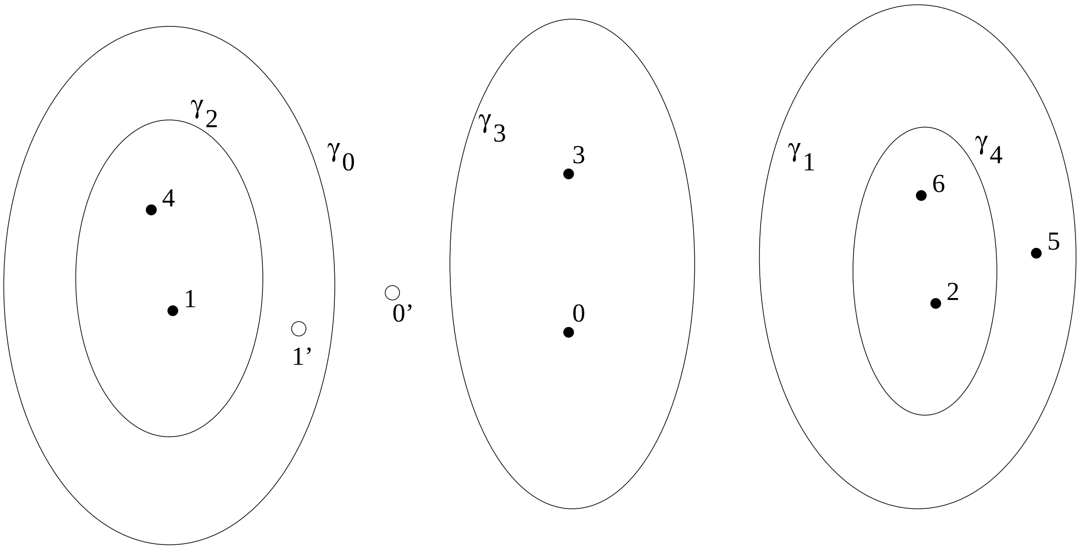
The same p and symbolic sequence X may be associated to more than one branch U. The number of different $\mu_q \ge \mu_{1-p}$ for which q has the symbolic sequence X is 2^ℓ, where $\ell + 1$ is the number of occurrences of C in X.

5. The first example with $\mathbf{m \ge 1}$.

5.1. Lemma. *Let μ_p be minimal and $\mu_q \ge \mu_p$, and let p have period k under $x \mapsto 2x \bmod 1$. Let $m = m(s_p \amalg s_q) \ge 1$. Then q has period $\ge (m+1)(k+1)+1$ under $x \mapsto 2x \bmod 1$.*

Proof: Let p have period k under $x \mapsto 2x \bmod 1$. As before, we write $\gamma_0 = \mu_p \cup \mu_{1-p}^{-1}$. By the definition of m in Section 2, the number 2^m is bounded by the maximum degree of $f^\ell \mid \gamma : \gamma \to \gamma_0$ ($\ell \geq 1$) where $\gamma \subset f^{-\ell}(\gamma_0)$ is nontrivial nonperipheral in $\overline{\mathbf{C}} \setminus X(f)$ and nonperiodic, that is, γ is not contained in $f^{-j}(\gamma)$ up to X-preserving isotopy for any $j > 0$. We can choose γ maximizing ℓ so that $f(\gamma)$ bounds a disc disjoint from the forward orbit of γ, containing $e^{2\pi iq}$ and $e^{2\pi it}$ where $t = 2^j q \bmod 1$ for some $j \geq k+1$, and the initial portion of the (L, C, R) sequence for $e^{2\pi iq}$ and $e^{2\pi it}$, up to and excluding the m'th occurrence of C in the sequence for $e^{2\pi iq}$, is the same. If sequence for $e^{2\pi iq}$ up to just before the first occurrence of C is X, then X is of the form $L^{k-1} \cdots (R \text{ or } L)$, the portion of the sequence which coincides for $e^{2\pi iq}$ and $e^{2\pi it}$ is $X(CX)^{m-1}$, and the sequence for $e^{2\pi iq}$ starts $X(CX)^m$. Of course, the sequence always has to end with C. But since γ is nonperiodic, the sequences of $e^{2\pi it}$ and $e^{\pi iq}$ should not be the same, and there must be an extra letter between the $(m+1)$'th occurrence of X and the final C. So the period of q under $x \mapsto 2x \bmod 1$ must be $\geq (m+1)(k+1)+1$. $\square$

It follows that if $m \geq 1$ then q has period ≥ 7, if $k \geq 3$ in addition then q has period ≥ 9, and if $m \geq 2$, and any $k \geq 2$, q has period ≥ 10. To have period 7 we must have $\mu_p = \mu_{1/3}$, $m = 1$, and the only possible symbolic sequence if $LRCLRRC$. This example was given in 3.12 of [**R**]. The possiblilities for q are $q = 55/127 \sim 56/127$, or $71/127 \sim 72/127$, which give the same branch. Writing $f = s_p \amalg s_q$, and letting $\gamma_0 = \mu_p \cup \mu_{1-p}^{-1}$, there is one nonperiodic component γ_1 of $f^{-1}(\gamma_0)$ and for $i \leq 3$ we can inductively find one nontrivial nonperipheral component of $f^{-1}(\gamma_i)$. If c and c' are the critical points of $f = s_p \amalg s_q$ in $\{z : |z| > 1\}$ and $\{z : |z| < 1$ respectively, then $f^j(c)$ is labelled j and $f^j(c')$ is labelled j'.



LRRCLRRC, $p = 1/3$.

Taking these five loops γ_i $(0 \leq i \leq 4)$ to index the rows and columns, the matrix of f^* is

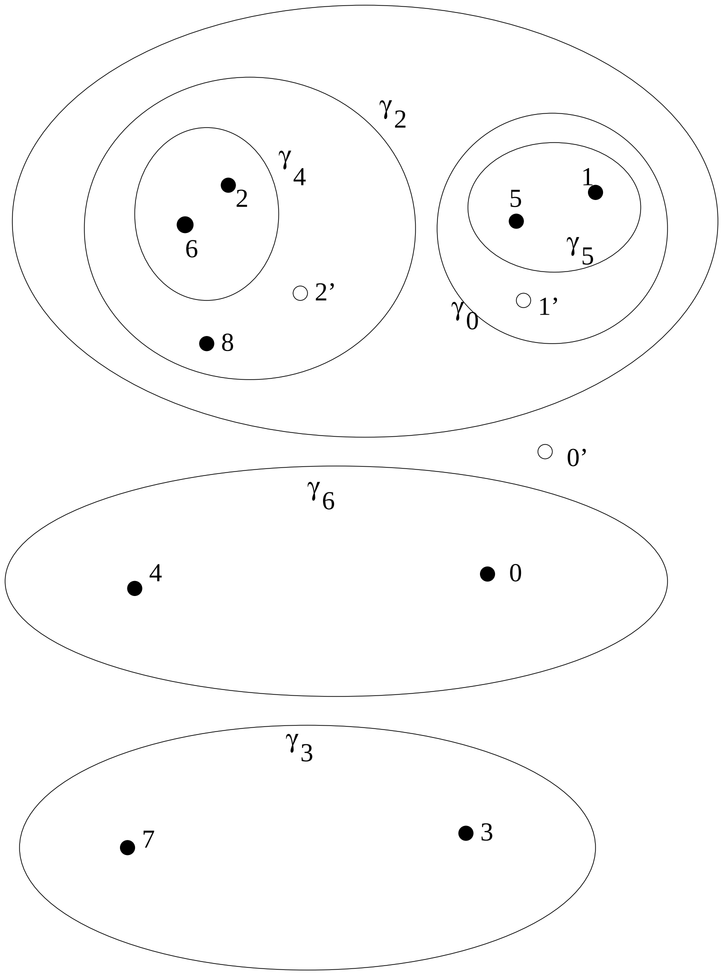
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The smallest eigenvector for eigenvalue 1 with integer entries, written as a row vector, is $(2,2,2,1,1)$, that is, $L = 2 = 2^m$ and $m = 1$. This example is unknotted, because $p = 1/3$ and the formula says that the degree of $\zeta + 1$ on the associated branch of W is $2^{m-1} = 1$. This can be verified directly.

6. Knotted Examples.

By Lemma 5.1, there are no knotted branches in the lift $W = W_n$ of $V = V_n$ for $n \leq 8$. This agrees with calculations in [**Sti**], where it was shown directly that there are no knots for $n \leq 7$.

Example 6.1. LLRCLLRLC, $p = 1/7$.



LLRCLLRLC, $p = 1/7$.

By the proof of Lemma 5.1, this is one of the two possible shortest sequences with $\mu_p \neq \mu_{1/3}$ which might give $m = 1$ (and hence a knot). The other possible sequence *LLRCLLRRC*, $p = 1/7$, gives $m = 0$, as can be computed (but this is omitted: this is the symbolic sequence for two different branches with $p = 1/7$). We again define $\gamma_0 = \mu_{1/7} \cup \mu_{6/7}^{-1}$ and let γ_2, γ_1 be the periodic components of $f^{-1}(\gamma_0)$, $f^{-1}(\gamma_1)$, $f = s_{1/7} \amalg s_q$. Then the nonperiodic components of $f^{-1}(\gamma_1)$, $f^{-1}(\gamma_2)$ are peripheral. Take γ_3 to be the nonperiodic component of $f^{-1}(\gamma_0)$ (as before) and let γ_{i+1} be the (unique) nontrivial nonperipheral component of $f^{-1}(\gamma_i)$ for $i = 3$, 4, 5. Then $f^{-1}(\gamma_5) = \gamma_6$ is connected. The lowest integer-value eigenvector with eigenvalue 1 is (written as a row vector) $(2,2,2,2,2,2,1)$. Therefore, ζ has a power series expansion in $\rho^{1/2}$.

Now we compute the knot type. For $(\zeta, \rho) \in U$, we must have

$$h^j_{\zeta,\rho}(1) = e^{2\pi i r/3}(1 + O(\rho)) \text{ for } r \in \{0,1,2\},\ j \geq 0.$$

Since the third letter is R, it follows from (1.1) and (1.4) that

$$\zeta = e^{2\pi i/3}(1 + \tfrac{1}{3}\rho + O(\rho^2)). \tag{6.1.1}$$

Now the third and seventh letters are both R, followed by different letters C and L. Then by (1.7),

$$h^7_{\zeta,\rho}(1) = h^3_{\zeta,\rho}(1) + b_1\rho^2 + o(\rho^2) = 1 + \tfrac{1}{2}\rho + O(\rho^2) \text{ for } b_1 \neq 0. \tag{6.1.2}$$

. Then by (1.5),

$$h^5_{\zeta,\rho}(1) = h_{\zeta,\rho}(1) + b_2\rho^2 + o(\rho^2) \text{ for } b_2 \neq 0. \tag{6.1.3}$$

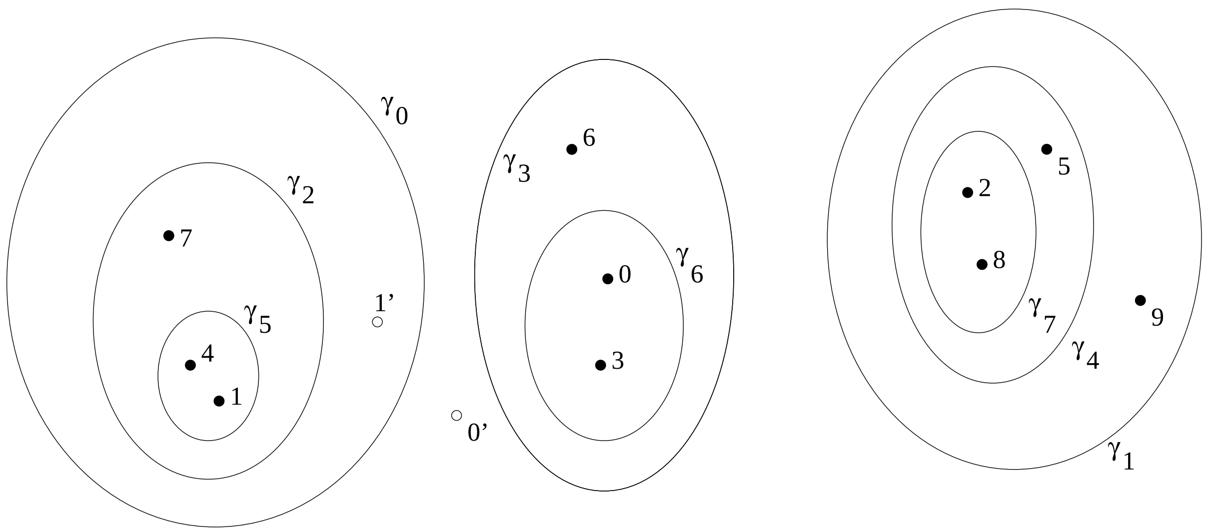
So by (1.6),

$$h^4_{\zeta,\rho}(1) = 1 + b_3\rho^{3/2} + o(\rho^{3/2}) \text{ for } b_3 \neq 0. \tag{6.1.4}$$

Let $\rho^{i/2}$ be the lowest noninteger power in the expansion of ζ. Then the lowest noninteger power in the expansion of $h^3_{\zeta,\rho}(1)$ is $\rho^{i/2}$ by (1.3), and the lowest noninteger power in $h^4_{\zeta,\rho}(1)$ is $\rho^{(i-4)/2}$ by (1.7), giving $i = 7$ and torus knot type $(7,2)$.

Example 6.2. LRCLRCLRRC, $p = 1/3$.

By the proof of 5.1, this is the lowest possible period example - and, in fact, the only one of period 10 - which gives $p = 1/3$ and $m \geq 2$.



LRCLRCLRRC, $p = 1/3$.

Fix some q such that $\mu_q > \mu_{1/3}$ and $s_q(0)$ has this sequence. (There are actually $4 = 2^2$ possibilities for μ_q.) Take $f = s_{1/3} \amalg s_q$. As usual take $\gamma_0 = \mu_{1/3} \cup \mu^{-1}_{1/3}$, and γ_1 to be the nonperiodic component of $f^{-1}(\gamma_0)$. Only one component, γ_2, of $f^{-1}(\gamma_1)$, is nonperipheral. It is separated from 0 by γ_0. Then $f^{-1}(\gamma_2) = \gamma_3$ is connected. Both components γ_4 and γ_5 of $f^{-1}(\gamma_3)$ are nontrivial nonperipheral. Take γ_4 to be the one separated from 0 by γ_2. Then $f^{-1}(\gamma_4) = \gamma_6$ is connected, nonperipheral and separates 0 from γ_3. Both components of $f^{-1}(\gamma_5)$ are trivial or peripheral, as are both components of $f^{-1}(\gamma_6)$. The smallest integer-entry eigenvector, written as a row vector, is then $(4,4,4,2,2,2,1)$, and $m=2$.

We shall show that, this time, the lowest noninteger power of ρ is $\rho^{9/2}$, giving knot type a torus knot $(9,2)$. Since $k=2$ and $m=2$, we know that ζ has an expansion in powers of $\rho^{1/2}$. By (1.7), if the j'th letter is R, then $h^j_{\zeta,\rho}(1)=1+\frac{1}{2}\rho-\frac{1}{8}\rho^2$. In analogy to (6.1.1), we have

$$\zeta=-1(1+\tfrac{1}{2}\rho+O(\rho^2)). \tag{6.2.1}$$

The j'th and $j+3$'rd letters agree for $j\leq 5$. Using this fact for $j=3$ - the third letter is C - we have

$$h^6_{\zeta,\rho}(1)=h^3_{\zeta,\rho}(1)+O(\rho). \tag{6.2.2}$$

Applying (1.7), (1.5), (1.6) respectively, we obtain

$$h^5_{\zeta,\rho}(1)=h^2_{\zeta,\rho}(1)+O(\rho^3), \tag{6.2.3}$$

$$h^4_{\zeta,\rho}(1)=h_{\zeta,\rho}(1)+O(\rho^3), \tag{6.2.4}$$

$$h^3_{\zeta,\rho}(1)=1+O(\rho^2). \tag{6.2.5}$$

In analogy to (6.1.2), (6.1.3) since fifth and eight letters are R, but the sixth and ninth letters are C and R, by (1.7):

$$h^8_{\zeta,\rho}(1)=h^5_{\zeta,\rho}(1)+b_1\rho^3+o(\rho^3)\ \text{ for } b_1\neq 0. \tag{6.2.6}$$

Now we pull this back, using (1.5), (1.6), (1.7), depending on whether the corresponding letter is L, C or R, and using (6.2.5) with (1.6) in (6.2.8).

$$h^7_{\zeta,\rho}(1)=h^4_{\zeta,\rho}(1)+b_2\rho^3+o(\rho^3)\ \text{ for } b_2\neq 0. \tag{6.2.7}$$

$$h^6_{\zeta,\rho}(1)=h^3_{\zeta,\rho}(1)+b_3\rho^2+o(\rho^2)\ \text{ for } b_3\neq 0, \tag{6.2.8}$$

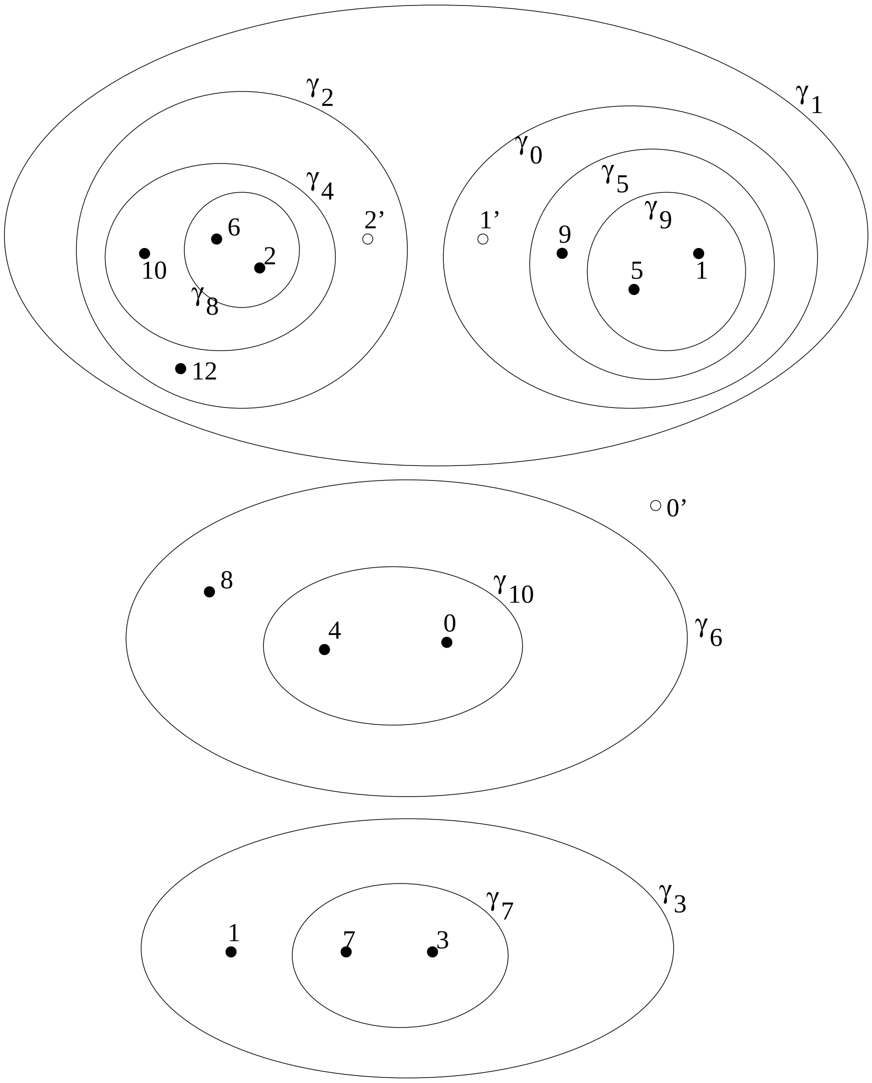
$$h^5_{\zeta,\rho}(1)=h^2_{\zeta,\rho}(1)+b_4\rho^4+o(\rho^4)\ \text{ for } b_4\neq 0, \tag{6.2.9}$$

$$h^4_{\zeta,\rho}(1)=h_{\zeta,\rho}(1)+b_5\rho^4+o(\rho^4)\ \text{ for } b_5\neq 0, \tag{6.2.10}$$

$$h^3_{\zeta,\rho}(1)=1+b_6\rho^{5/2}+o(\rho^{5/2})\ \text{ for } b_6\neq 0. \tag{6.2.11}$$

As in 6.1, we deduce that the lowest noninteger power of ρ in the expansion of ζ is $\rho^{2+5/2}=\rho^{9/2}$, and so the branch has torus knot type $(9,2)$.

Example 6.3. $LLRCLLRCLLRLC$, $p = 1/7$.



LLRCLLRCLLRLC, $p = 1/7$.

This diagram is an extension of the diagram for $LLRCLLRLC$, $p = 1/7$. This time, the nonperiodic component of $f^{-1}(\gamma_1)$ is not peripheral but is isotopic to one component of $f^{-1}(\gamma_6)$ - the other component of $f^{-1}(\gamma_6)$ is trivial - so we have labelled it γ_7. Then $f^{-1}(\gamma_9) = \gamma_{10}$ is connected, as is $f^{-1}(\gamma_5) = \gamma_6$ (as before). The lowest integer-valued eigenvector (written as a row vector) is $(4, 4, 4, 4, 4, 4, 2, 2, 2, 2, 2, 1)$. For this example, $m = 2$.

As in 6.1, we have (6.1.1). Working as in Example 6.2, (6.2.2-5), we deduce using (1.5)-(1.7) from

$$h^8_{\zeta,\rho}(1) = h^4_{\zeta,\rho}(1) + O(\rho) \tag{6.3.1}$$

that

$$h^4_{\zeta,\rho}(1) = 1 + O(\rho^2). \tag{6.3.2}$$

The 7'th and the 11'th letters are R, while the 8'th and 12'th are C and L. So as in 6.1, 6.2, again using (1.5), (1.6), (1.7) to deduce each line from the lines above:

$$h^{11}_{\zeta,\rho}(1) = h^7_{\zeta,\rho}(1) + b_1\rho^2 + o(\rho^2) \text{ for } b_1 \neq 0, \tag{6.3.3}$$

$$h^9_{\zeta,\rho}(1) = h^5_{\zeta,\rho}(1) + b_2\rho^2 + o(\rho^2) \text{ for } b_2 \neq 0, \tag{6.3.4}$$

$$h^8_{\zeta,\rho}(1) = h^4_{\zeta,\rho}(1) + b_3\rho^{3/2} + o(\rho^{3/2}) \text{ for } b_3 \neq 0, \tag{6.3.5}$$

$$h^7_{\zeta,\rho}(1) = h^3_{\zeta,\rho}(1) + b_4\rho^{7/2} + o(\rho^{7/2}) \text{ for } b_4 \neq 0, \tag{6.3.6}$$

$$h^5_{\zeta,\rho}(1) = h_{\zeta,\rho}(1) + b_5\rho^{7/2} + o(\rho^{7/2}) \text{ for } b_5 \neq 0, \tag{6.3.7}$$

$$h^4_{\zeta,\rho}(1) = 1 + b_6\rho^{9/4} + o(\rho^{9/4}) \text{ for } b_6 \neq 0. \tag{6.3.8}$$

As before, the lowest noninteger power of ρ in the expansion of ζ is $\rho^{2+9/4} = \rho^{17/4}$, giving torus knot type $(17,4)$, since we know that there is an expansion in powers of $\rho^{1/4}$.

7. Iterated torus knots.

It is possible to get iterated torus knots. The following may be the simplest example which is not a torus knot. Take sequence $X_2 = LLRCLLRLCLLRCLLRRC$, $p = 1/7$, for which $m = 2$, and corresponding branched covering f. This is the second in a sequence of examples, of which the first is $X_1 = LLRCLLRLC$. We can find the loop set generated by $\mu_{1/7} \cup \mu^{-1}_{6/7}$ under f by considering the loop set for X_1. Replace each of the points labelled i, $0 \leq i \leq 7$, in the diagram for X_1, by two points, labelled i and $i+9$. Also add an extra point labelled 17 inside the loop labelled γ_3. Insert an extra loop enclosing the point i and $i+9$ for $0 \leq i \leq 3$, labelled γ_{11-i}. Also insert a loop enclosing 3, 12, 17 (and enclosing γ_8 also) labelled γ_7. Then γ_7 is a nontrivial nonperipheral component of $f^{-1}(\gamma_1)$. The only nontrivial nonperipheral components of $f^{-1}(\gamma_7)$, $f^{-1}(\gamma_8)$ are both homotopic to γ_9. The only nontrivial nonperipheral component of $f^{-1}(\gamma_i)$ is γ_{i+1} for $i = 9$, 10. So the matrix for f^* is a 12×12 matrix. The equations for the eigenvector $(x_0, \cdots, x_{11})$ reduce to $x_i = x_0$ for $i \leq 5$, $x_6 = \frac{1}{2}x_5$, $x_7 = x_1$, $x_8 = x_6$, $x_9 = x_7 + x_8$, $x_{10} = x_9$ and $x_{11} = \frac{1}{2}x_{10}$. The lowest integer eigenvector is then (written as a row vector) $(4,4,4,4,4,4,2,4,2,6,6,3)$. So $m = 2$ and ζ has a power series expansion in $\rho^{1/4}$.

Steps (6.1.1)-(6.1.4) apply to this example, so that the first noninteger power of ρ in the expansion of ζ is $\rho^{7/2}$. But we can also look at the sixteenth and seventh letters, and argue as in Examples 6.1-3, using (1.5)-(1.7), and using (6.1.4) in (7.3):

$$h^{16}_{\zeta,\rho}(1) = h^7_{\zeta,\rho}(1) + c_1\rho^2 + o(\rho^2) \text{ for some } c_1 \neq 0, \tag{7.1}$$

$$h^{14}_{\zeta,\rho}(1) = h^5_{\zeta,\rho}(1) + c_2\rho^2 + o(\rho^2) \text{ for some } c_2 \neq 0, \tag{7.2}$$

$$h^{13}_{\zeta,\rho}(1) = h^4_{\zeta,\rho}(1) + c_3\rho^{3/2} + o(\rho^{3/2}) \text{ for some } c_3 \neq 0, \tag{7.3}$$

$$h^{12}_{\zeta,\rho}(1) = h^3_{\zeta,\rho}(1) + c_4\rho^{7/2} + o(\rho^{7/2}) \text{ for some } c_4 \neq 0, \tag{7.4}$$

$$h^9_{\zeta,\rho}(1) = 1 + c_5\rho^{9/4} + o(\rho^{9/4}) \text{ for some } c_5 \neq 0. \tag{7.5}$$

Now let $\rho^{i/4}$ be the lowest power in the expansion of ζ which has i and 4 coprime. Then by (1.3), the lowest such power in the expansion of $h^3_{\zeta,\rho}(1)$ is also $\rho^{i/4}$ and by (1.7), the lowest such power in the expansion of $h^4_{\zeta,\rho}(1)$ is $\rho^{(i-8)/4}$, which means that $i \geq 15$. Then the lowest such power in the expansion of $h^j_{\zeta,\rho}(1)$ is $\rho^{(i-6)/4}$ for

$j=5$ (using (1.6) and (6.1.4)), and for $j=6$, $j=7$ (using (1.5)) and $\rho^{(i-14)/4}$ for $j=8$ (using (1.7) and also for $j=9$ (using (1.5)). So $i=14+9=23$. In Brieskorn-Knörrer's notation [**B**] (page 405) the Puiseaux pairs are $(2,7)$, $(2,23)$, and this is an iterated torus knot.

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