# Ph D thesis of Joan Cushen

# Normal varieties in the arithmetic theory of algebraic varieties

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The manuscript of Joan's Ph D thesis was found in a drawer in her flat in September 2013, after her death in August 2013. It was in an envelope posted to herself, Miss J.S. Cushen, Girton College, Girton, Cambs. She had at that time recently moved back to Cambridge, where she had been a Girton student, and where she started her Ph D, from a teaching post at Royal Holloway College. Her Ph D was awarded by the University of London. The postmark on the envelope is illegible. The stamps are of George VI, one 3d and the other 1/2d. The untitled and undated manuscript is written in blue ink, with a few comments written in, presumably by the typist, in red biro or grey pencil, with a one-page typed conclusion. Probably the manuscript was returned to Joan after the typist had finished work. The thesis is held, with the title and date given, in the closed access section of the library at Queen Mary University of London.

## Foreword

I should like to record here my sincere gratitude to Professor Hodge and Dr Pedoe for their constant help and encouragement in my work. Through their encouragement I have found an increasing interest in the arithmetic theory of algebraic varieties which has made this work worthwhile for me.

A large part of my thesis is concerned with a survey of the algebraic foundations of the arithmetic theory of algebraic varieties and with the results of other people who have developed the theory. The whole of chapter 1 and most of chapters 2 and 3 are concerned with this. The work in §§2.5 and 3.3 is my own. In §3.1 the results are, of course, not original, but in an endeavour to give something in the way of concise proofs for them I have used some of my own arguments. I could not, for instance, find an explicit statement of lemma 1 in this section and I found it a convenient result to use. In the next chapter, §§4.1 and 4.2 describe the results of Muhly and Gaeta. The remaining work in the thesis is original as far as I know.

## Symbols and Conventions

$S_n$ denotes a projective space of $n$ dimensions in which $(x_0, x_1, \cdots, x_n)$, or sometimes $(y_0, y_1, \cdots, y_n)$, are homogeneous coordinates. $A_n$ denotes an affine space of $n$ dimensions and $(x_1, \cdots, x_n)$ are non-homogeneous coordinates in $A_n$. Confusion through using $x$'s for both homogeneous and non-homogeneous coordinates is not likely to arise, but where necessary one set of coordinates will be distinguished by dashes.

$K$ is the ground-field in which the coefficients in the equations defining algebraic varieties are chosen. $K$ is assumed to be of zero characteristic but is not algebraically closed unless this is specifically stated.

$V_r^m$ will be used to denote an algebraic variety of dimension $r$ and order $m$. Either, or both, suffixes may be dropped when they are not relevant. Other letters which I have used to denote varieties are $U$ and $W$. Where there is no ambiguity I have often used "variety" or "algebraic variety" to mean "irreducible algebraic variety".

$K[x_1, \cdots, x_n]$ is the ring of polynomials in $x_1, \cdots, x_n$ with coefficients in $K$. $K(x_1, \cdots, x_n)$, the quotient field of $K[x_1, \cdots, x_n]$, is the field of rational functions in $x_1, \cdots, x_n$. $K\{x_1, \cdots, x_n\}$ denotes the ring of power series in $x_1, \cdots, x_n$ convergent in some neighbourhood of the origin.

Greek letters will usually be used for the coordinates of a generic point of the variety. If $(1, \xi_1, \cdots, \xi_n)$ are the coordinates of a normalised generic point of $V$, then $\sigma = K[\xi_1, \cdots, \xi_n]$, the ring of polynomials in the coordinates of this generic point with coefficients in $K$, is an integral domain associated with $V$. The quotient field of $\sigma$ is $\Sigma = K(\xi_1, \cdots, \xi_n)$. $\Sigma$ is the function field of $V$.

Other symbols will be explained as they occur.

I have divided my thesis into five chapters which are subdivided into numbered paragraphs. Wherever I have referred back to an equation or expression by a number in round brackets this will refer back to the equation or expression of that number in the same paragraph unless otherwise stated.

References to books and papers are given by numbers in square brackets in the text. The numbers are given in the list of references at the end of the thesis. If necessary, the exact page in the work concerned is given after the reference number.

A few footnotes occur and the sign "†" in the text alludes to these.

The sections into which I have divided my thesis are as follows : —

## Introduction

Modern algebraic geometry is concerned with problems in birational geometry. If $V_r$ is an algebraic locus in a projective space $S_n$ in which $(x_0, x_1, \cdots, x_n)$ are homogeneous coordinates, and if $V_r'$ is another algebraic locus in $S_m$ where $(y_0, y_1, \cdots, y_m)$ are homogeneous coordinates, then $V_r$ and $V_r'$ are said to be birationally equivalent if two sets of equations:

$$y_0 : y_1 : \cdots y_m = \varphi_0(x_0, x_1, \cdots x_n) : \varphi_1(x_0, x_1, \cdots x_n) : \cdots : \varphi_m(x_0, x_1, \cdots x_n)$$

$$x_0 : x_1 : \cdots x_n = \psi_0(y_0, y_1, \cdots y_m) : \psi_1(y_0, y_1, \cdots y_m) : \cdots : \psi_n(y_0, y_1, \cdots, y_m)$$

(where $\varphi_i$ $(i = 0, 1 \cdots m)$ are forms of the same degree in $x_0, \cdots x_n$ and $\psi_i$ $(i = 0, 1 \cdots n)$ are forms of the same degree in $y_0 \cdots y_m$) define a correspondence between the unexceptional points of $V_r$ and $V_r'$. $V_r'$ is called a projective model of $V_r$. The correspondence is $(1, 1)$ between unexceptional points of the two varieties but there may be points on either variety to which there corresponds more than one point on the other. Such points lie on what is often called the fundamental locus of the variety. † The fundamental locus on $V$ for example is the set of points on $V$ for which $\varphi_0 = \varphi_1 = \cdots = \varphi_m = 0$.

It often happens that results concerning birational correspondence between $V$ and $V'$ are true only for non-singular varieties. For example, for non-singular varieties it is true that the transform of the fundamental locus on $V$ is a pure $(r-1)$-dimensional subvariety of $V'$ [16], but this result is not necessarily true when $V$ is singular. For reasons of this nature the problem of the resolution of the singularities of an algebraic locus is one of the central problems in birational geometry.

† Zariski's definition of a fundamental locus is not quite this. [See [27] p515]

The most familiar solution of this problem in the case of algebraic curves depends on the theory of linear series of sets of points on an algebraic curve. In this theory the genus, the only numerical birational invariant of a curve, plays an important part. Many difficulties are encountered when the theory is generalised to cover the case of algebraic surfaces as Professor Baker pointed out in his Presidential Address to the London Mathematical Society in 1912 [2]. For surfaces three numerical invariants are well-known. They are the arithmetic, geometric and linear genera, and there may be others. The increased complications are well-illustrated in Zariski's work in "Algebraic Surfaces" [21]. The whole theory of linear series of equivalence is characteristic of the methods of the Italian school of geometry. In spite of many complicated difficulties, Severi has succceeded in developing a theory of linear series of equivalences for algebraic varieties of any dimension which may well prove to be the simplest approach to some types of problems in algebraic geometry.

In his recently published book "Fondamenti di Geometria Algebria" [13], Severi considers the theory for surfaces in detail and introduces the general theory. He works always with a surface in $S_3$ which he assumes to have only "ordinary singularities". This is the usual procedure and Zariski does the same thing in his book mentioned above. This approach presupposes the possibility of resolving the singularities of any given surface by transforming it birationally into a non-singular model in $S_5$. These "ordinary singularities" in $S_3$ are just those which cannot be avoided when a nonsingular surface in $S_5$ is projected from a general line into $S_3$. For three-dimensional varieties a non-singular projective model exists in $S_7$, and seven is the lowest dimension for which this result is true. Presumably, postulates Severi, a similar result holds for a variety $V_r$ of any dimension $r$ and a projective model can be found in $S_{2r+1}$ which is free from singularities. This is a conjecture and it is, in fact, only quite recently that* Zariski has shown it to be true in the case $r = 3$.

In the anniversary volume of "Courant" in 1948 under the title "The foundation of algebraic geometry. A very incomplete historical survey" [19] van der Waerden writes of the inspiration that he received from Emmy Noether when, in Göttingen in 1924, she taught him "that algebraic geometry ought to be based on Steimitz' algebraic theory of fields and on Dedekind's arithmetic theory of algebraic functions and ideals". "I at once saw that she was right", says van der Waerden. "I was enthused by her foundation of the general theory of ideals, based upon Hilbert's basis theorem, and by her theory of "ganz-abgeschlossene Ringe" ".

This admiration for Emmy Noether's work and its influences on his own reearches can be seen in van der Waerden's work. His well-known book "Moderne Algebra" was written, as its title page acknowledges, "unter Benutzung von Vorlesungen von E. Artin und E. Noether". In this work he devotes a good deal of space to the study of the algebraic theory of fields. A great many of the invariant properties of a class of birationally equivalent varieties can be derived from a study of their common function field and it is mainly for this reason that field theory is studied so extensively by geometers. The important idea of the generic point of an irreducible algebraic variety was, van der Waerden claims, implicit in Emmy Noether's work, although it was as a result of his own research that the idea first reached publication in Volume 96 of the Mathematische Annalen [14]. In this paper, he proves the important result that every irreducible algebraic variety has

* "that" suggested by the typist as a correction from "the"

a generic point. The long series of fifteen papers "Zur Algebraische Geometrie" was also published in the Mathematische Annalen between 1933 and 1938.

Throughout his work on algebraic geometry, van der Waerden has preferred to rely on purely algebraic methods with little recourse to the powerful methods of modern topology. This is because he maintains that these algebraic methods have a wider field of application than topological methods. By this constant appeal to results in modern algebra, he shows himself to be one of the pioneers of the modern school of geometry which seeks to apply arithmetic methods to birational geometry. In van der Waerden's earlier papers classical ideas involving intuition and continuity were still apparent, but gradually the arithmetic theory has asserted itself. It is certainly the arithmetic theory which has dominated the researches of his two eminent pupils Wei Liang Chow and André Weil. It is a far cry from the first of the Z.A.G. papers to Weil's book on "The Foundations of Algebraic Geometry" [20]. The same change of outlook is exhibited throughout Zariski's work if, for example, his "Algebraic Surfaces" [21] be compared with his recent solution of the resolution of the singularities of an algebraic surface [23, 25]. In his recent presidential address to the London Mathematical Society, Professor Hodge surveyed the achievements of present-day algebraic geometers [7]. His address was concerned mainly with contributions to the arithmetic theory.

The two main theories underlying this arithmetic approach are ideal theory and valuations theory. I should like to discuss both of these theories later on in my thesis and wish only to mention them here. Zariski seems to concentrate his interest on zero-dimensional valuations whereas van der Waerden has recently turned his attention to the $(r-1)$-dimensional valuations of a function field of degree of transcendence $r$ over the ground field. "I confess I do not like infinitely near points" he says, and he tries hard to dispose of them and the "odious complications" to which they give rise in the theory of linear series of equivalence [19]. Now, after twenty years of meditation on the problem, he believes that he can dispense with this awkward concept and use instead the $(r-1)$-dimensional valuations mentioned above. There is a brief explanatory note at the end of his essay in "Courant" and a detailed paper is published in a recent volume of Acta Salimanticensia.

It is the zero-dimensional valuations alone that Zariski has considered in resolving the singularities of an algebraic surface. One of the main steps in his work in this connection concerns normal varieties and the very important fact that an irreducible normal variety of dimension $r$ has no singular locus of dimension $r-1$. This is a property which is shared by the wider class of locally normal varieties. * The geometrical properties of these normal and locally normal varieties have formed the main subject of my study. In this I have tried to set my own investigations into a general survey of the arithmetical theory of algebraic varieties in an endeavour both to display the importance of normal varieties in algebraic geometry and also to explain the algebraic ideas which I have used in discussing the concept of normality.

*I have corrected this from "varies" in the manuscript. M.R.

# Chapter 1.

## Preliminary definitions

§1.1 Definition of an algebraic variety

Consider an affine space $A_n$ in which $(x_1, \cdots, x_n)$ are non-homogeneous coordinates. $f_i(x_1, \cdots, x_n) = 0$ where $f_i(x_1, \cdots, x_n)$ belong to $K[x_1, \cdots x_n]$, is a finite or infinite set of algebraic equations. The points in $A_n$ whose coordinates $(\alpha_1, \cdots, \alpha_n)$ satisfy all these equations, form an algebraic variety $V$ in $A_n$. The coordinates $\alpha_i$ $(i = 1, \cdots, n)$ belong to some extension of the groundfield $K$. The polynomials which vanish at every point of $V$ form an ideal in the ring $K[x_1, \cdots, x_n]$. As a consequence of Hilbert's Basis Theorem concerning ideals in a polynomial ring, we can find a finite set of equations $g_i(x_1, \cdots, x_n) = 0$ $(i = 1, \cdots, h)$ which are sufficient to determine the points of the variety $V$.

If we consider a set of equations $f_i(x_0, x_1 \cdots, x_n) = 0$ which are homogeneous in the indeterminates $x_0, x_1, \cdots x_n$, and if $(x_0, x_1, \cdots x_n)$ are regarded as homogeneous coordinates in a projective space $S_n$, the non-zero solutions only are of importance. A solution $(\alpha_0, \alpha_1, \cdots \alpha_n)$ is equivalent to $(\lambda\alpha_0, \lambda\alpha_1, \cdots, \lambda\alpha_n)$, $\lambda \neq 0$, since both sets of coordinates define the same point in $S_n$. The points whose coordinates form a solution for these homogeneous equations define an algebraic variety $V$ in the projective space $S_n$. This variety can be defined by a finite number of homogeneous equations $g_i(x_0, x_1, \cdots, x_n) = 0$ $(i = 1, \cdots, h)$, which form a base for the homogeneous ideal of polynomials in $K[x_0, x_1, \cdots, x_n]$ which vanish at every point of $V$.

Let $V$ be an algebraic variety defined in $S_n$ over the ground field $K$. The variety $V$ is said to be reducible over $K$ if we can find two varieties $V_1$ and $V_2$, both defined over $K$ and distinct from $V$, whose points are points of $V$ such that $V$, considered as a set of points, is the sum of the point sets $V_1$

and $V_2$. In this case we write

$$V = V_1 \dot{+} V_2.$$

If $V_1$ and $V_2$ cannot be found to satisfy these conditions, $V$ is irreducible. In fact every variety $V$ is the sum of a finite number of irreducible varieties $V_i$ $(i = 1, \cdots, s)$ and we can write

$$V = V_1 \dot{+} V_2 \dot{+} \cdots \dot{+} V_s.$$

The component varieties $V_i$ $(i = 1, \cdots, s)$ are unique provided that irrelevant components $V_i$, which are contained in some other $V_j$ $(i \neq j)$ are rejected.

A polynomial $\varphi(x_0, x_1, \cdots, x_n)$ is said to vanish over $V$ if it vanishes at every point of $V$. A necessary and sufficient condition for $V$ to be reducible over $K$ is the existence of two forms $\varphi(x_0, x_1, \cdots, x_n)$ and $\psi(x_0, x_1, \cdots, x_n)$ belonging to $K[x_0, x_1, \cdots, x_n]$ whose product vanishes over $V$ although neither $\varphi$ nor $\psi$ separately vanishes over $V$.

This definition of reducibility depends on the ground field $K$ and a variety which is irreducible over $K$ may become reducible when this ground field is extended. For example. the equation $x_0^2 + x_1^2 = 0$ defines an irreducible variety in $S_1$ over the real number field, but over the complex number field the same equation defines a reducible variety consisting of the two points $(1, i)$ and $(1, -i)$, where $i^2 = -1$.

§1.2 The function field of an irreducible variety and the idea of a generic point.

Associated with every irreducible variety $V$ in $S_n$ is a field known as the function field of $V$. It is usually denoted by $\Sigma$ and is defined as follows. We consider the quotients of forms of like degree

$$\frac{a}{b} = \frac{a(x_0, x_1, \cdots, x_n)}{b(x_0, x_1, \cdots, x_n)}$$

and divide these quotients into equivalence classes $\left\{\frac{a}{b}\right\}$ defining two quotients $\frac{a}{b}$ and $\frac{a'}{b'}$ as equivalent if $ab' - a'b$ vanishes over $V$. If $V$ is irreducible these classes can be shown to form a field $\Sigma$ which contains a subfield isomorphic to $K$. $\Sigma$ is the field of rational functions on $V$ and is called the function field of the variety.

One of the most important characteristic properties of an irreducible variety $V$ is the existence of a generic point. Let us suppose that $V$ is defined by the homogeneous equations

$$f_i(x_0, x_1, \cdots, x_n) = 0, \quad (i = 1, \cdots, h).$$

The point with coordinates $(\xi_0, \xi_1, \cdots, \xi_n)$, where $\xi_i$ $(i = 0, \cdots r)$ are in some extension of the ground field, is called a generic point of $V$ if it satisfies one of the following two conditions:

(i) $f_i(\xi_0, \xi_1, \cdots \xi_n) = 0$ $(i = 1, \cdots, h)$.

(ii) If $g(x_0, x_1, \cdots x_n)$ is a polynomial in $K[x_0, \cdots, x_n]$ such that $g(\xi_0, \xi_1, \cdots, \xi_n) = 0$ then $g(x_0, x_1, \cdots, x_n)$ vanishes over the variety $V$.

If the point $(\xi_0, \xi_1, \cdots, \xi_n)$ is a generic point of $V$ then obviously the point $(\lambda\xi_0, \lambda\xi_1, \cdots, \lambda\xi_n)$, where $\lambda$ is a non-zero quantity belonging to any extension of $K$, is also a generic point of $V$. If $\lambda$ is chosen to that the first nonvanishing coordinate is one, and, if necessary, rearranging the order of the coordinates, it is possible to write the coordinates of the generic point in the normalised form $(1, \xi_1', \cdots, \xi_n')$. This means that $V$ does not lie entirely in the prime $x_0 = 0$. Choosing this prime as the prime at infinity and $x_i' = \frac{x_i}{x_0}$, $(i = 1, \cdots, n)$, as non-homogeneous coordinates in the affine space $A_n$ obtained from $S_n$ by neglecting these points at infinity, then $(\xi_1', \cdots, \xi_n')$

are non-homogeneous coordinates of a generic point of $V$ considered as a variety defined in $A_n$ by the non-homogeneous equations

$$f_i(1, x'_1, \cdots, x'_n) = 0 \quad (i = 1, \cdots h).$$

The generic point of a variety is not unique. If $(\xi_0, \xi_1, \cdots, \xi_n)$ and $(\eta_0, \eta_1, \cdots, \eta_n)$ are two generic points of the same variety $V$ in $S_n$, then $\xi_0 \neq 0$ implies $\eta_0 \neq 0$ and therefore $(1, \xi'_1, \cdots \xi'_n)$ and $(1, \eta'_1, \cdots \eta'_n)$ can be regarded as the corresponding normalised generic points. It can be shown that the two rings $K[\xi'_1, \cdots, \xi'_n]$ and $K[\eta'_1, \cdots, \eta'_n]$ are isomorphic. It will appear later that these two rings are isomorphic also with $K[x'_1, \cdots, x'_n]/\mathfrak{p}$, the ring of residue classes of polynomials in $K[x'_1, \cdots, x'_n]$ modulo $\mathfrak{p}$, the prime ideal defined in $V$ by $K[x'_1, \cdots x'_n]$. These rings are in fact integral domains and $K[x'_1, \cdots, x'_n]/\mathfrak{p}$ is often called the integral domain of $V$. The quotient fields of these three rings are also isomorphic with each other and the quotient field of the integral domain of $V$ is $\Sigma$, the function field of $V$. Dropping now the dashes on the $\xi$'s, $\eta$'s and $x$'s so that $x_1, \cdots x_n$ refer to a non-homogeneous coordinate system we have therefore:

$$K[\xi_1, \cdots, \xi_n] \cong K[\eta_1, \cdots \eta_n] \cong K[x_1, \cdots x_n]/\mathfrak{p}$$

and

$$K(\xi_1, \cdots \xi_n) \cong K(\eta_1, \cdots, \eta_n) \cong \Sigma.$$

If $(1, \xi_1, \cdots, \xi_n)$ is a normalised generic point of $V$ it is quite often found convenient to refer to $K[\xi_1, \cdots, \xi_n]$ as the integral domain of $V$ and to $K(\xi_1, \cdots, \xi_n)$ as the function field of $V$.

If $V$ and $V'$ are birationally equivalent varieties with normalised generic points $(1, \xi_1, \cdots \xi_n)$ and $(1, \eta_1, \cdots, \eta_m)$ defined respectively in $S_n$ and $S_m$ over the same ground field $K$, then their function fields are isomorphic. If, more particularly the integral domains $K[\xi_1, \cdots, \xi_n]$ and $K[\eta_1, \cdots \eta_m]$ are isomorphic, then the correspondence between the finite points of $V$ and $V'$ is one-one without exception and $V$ and $V'$ are said to be integrally equivalent ([22], p.279).

If $(1, \xi_1, \cdots, \xi_n)$ is a normalised generic point of the irreducible variety $V$ in $S_n$ then the dimension of $V$ is defined to be the maximum number $r$ of the coordinates $\xi_i$ which are algebraically independent over $K$. It is always possible to choose a coordinate system so that the quantities $\xi_1, \cdots, \xi_r$ are algebraically independent over $K$ and so that $\xi_{r+1}, \cdots, \xi_n$ are each algebraic over the field $K(\xi_1, \cdots, \xi_r)$. This number $r$ is equal to the degree of transcendency of the function field of $V$, $\Sigma = K(\xi_1, \cdots, \xi_n)$, over $K$. In the general case of a reducible variety, its dimension is defined to be the same as that of the irreducible component varieties of highest dimension.

If $\xi_1, \cdots, \xi_n$ are any $n$ quantities belonging to some extension of the ground field $K$, then $(\xi_1, \cdots, \xi_n)$ can be regarded as the non-homogeneous coordinates of a generic point of some irreducible variety $V$ defined over $K$. If a given irreducible variety $V$ is of dimension $r$ and is defined in $A_n$, then any set of quantities $(\xi_1, \cdots, \xi_n)$ are the non-homogeneous coordinates of a generic point of $V$ provided that

(i) $(\xi_1, \cdots, \xi_n)$ satisfy the non-homogeneous equations for $V$,

(ii) $K(\xi_1, \cdots, \xi_n)$ is of degree of transcendency $r$ over $K$. ([20], p.73 Theorem 2).

Suppose that $(\xi_0, \xi_1, \cdots, \xi_n)$ are homogeneous coordinates of a generic point of $V_r$ in $S_n$. The non-zero set of quantities $(\zeta_0, \zeta_1, \cdots, \zeta_n)$ is called a non-zero specialisation of $(\xi_0, \xi_1, \cdots, \xi_n)$ if, given $f(x_0, x_1, \cdots x_n)$, a polynomial in $x_0, x_1, \cdots, x_n$ such that $f(\xi_0, \xi_1, \cdots, \xi_n) = 0$, then $f(\zeta_0, \zeta_1, \cdots, \zeta_n) = 0$.

This implies that $(\zeta_0, \zeta_1, \cdots \zeta_n)$ are the coordinates of a point on $V_r$. $(\lambda\zeta_0, \lambda\zeta_1, \cdots, \lambda\zeta_n)$ $(\lambda \neq 0)$, are the coordinates of the same point and, assuming $\xi_0 \neq 0$, it is possible to choose $\lambda$ so that this point has normalised coordinate $(1, \zeta_1', \cdots, \zeta_n')$. $\zeta_0 \neq 0$ implies $\xi_0 \neq 0$ so that we can normalise the coordinates of the above generic point of $V_r$ and write them in the form $(1, \xi_1, \cdots, \xi_n)$. If the field $K(\zeta_1', \cdots \zeta_n')$ is of degree of transcendency $s$ over $K$ then $s \leq r$ and $(1, \zeta_1', \cdots \zeta_n')$ are the coordinates of a generic point of an irreducible subvariety $W_s$ of $V_r$ of dimension $s$.

§1.3 Form of the elements of a function field

Let $(1, \xi_1, \cdots, \xi_n)$ be a normalised generic point of $V_r$, and let $\xi_1, \cdots, \xi_r$ be algebraically independent over the ground field $K$. Then any element of the function field $K(\xi_1, \cdots, \xi_n)$ can be written as a quotient of a polynomial in $K[\xi_1, \cdots, \xi_n]$ by a polynomial in $K[\xi_1, \cdots, \xi_r]$. This follows from the following * result. If we make a simple algebraic extension of the field $K$ using the irreducible equation $f(x) = 0$ where $f(x)$ is of degree $m$ in $x$, then, if $\xi$ is a root of this equation the extended field is isomorphic with $K(\xi)$ and $K(\xi) = K[\xi]$. We can in fact choose as elements of $K[\xi]$, polynomials in $\xi$ of degree less than $m$ ([8] p.100). The proof is as follows. The polynomials of the ring $K[x]$ are divided into equivalence classes modulo $f(x)$. These classes form a ring with the obvious definitions of addition and multiplication, the zero of the ring being the class $\{f(x)\}$ defined by $f(x)$. The ring is in fact a field. For suppose $\{g(x)\}$ is any non-zero class. Polynomials $r(x)$, $s(x)$ can be found in $K[x]$ such that

$$r(x)g(x) + s(x)f(x) = 1$$

* "following" has been inserted. M.R.

since $g(x)$ and $f(x)$ have highest common factor unity. Therefore

$$\{r(x)\}\{g(x)\} = \{1\}$$

showing that the class $\{g(x)\}$ possesses an inverse in the ring of residue classes which is consequently a field $K_1'$. The classes $\{a\}$ where $a$ belongs to $K$ form a subfield $K_1$ of $K_1'$ and $K_1$ is isomorphic with $K$. It follows that $K$ must possess an extension $K'$ isomorphic with $K_1'$. In fact if $\xi$ is the element of $K'$ corresponding to $\{x\}$, then $\xi$ is a root of the equation $f(x) = 0$. Also every element of $K'$ can be written as a polynomial in $\xi$ of degree less than $m$. Consider any element $\{g(x)\}$ of $K_1'$. In the class $\{g(x)\}$ there is a polynomial $g(x)$ of degree less than $m$ and the element of $K'$ which is isomorphic with $\{g(x)\}$ is $g(\xi)$.

This result extends simply to the case when $\xi_1, \cdots, \xi_n$ are a finite number of quantities each algebraic over $K$. In this case $K(\xi_1, \cdots, \xi_t) = K[\xi_1, \cdots, \xi_t]$. We assume the result for $t = s - 1$ and prove it for $t = s$.

$\xi_s$ is algebraic over $K$ and therefore, a fortiori over $K(\xi_1, \cdots, \xi_{s-1})$. By the induction hypothesis $K(\xi_1, \cdots, \xi_{s-1}) = K[\xi_1, \cdots, \xi_{s-1}]$. Using the result above, we find that

$$K(\xi_1, \cdots, \xi_{s-1}, \xi_s) = K(\xi_1, \cdots, \xi_{s-1})(\xi_s)$$

$$= K[\xi_1, \cdots, \xi_{s-1}](\xi_s)$$

$$= K[\xi_1, \cdots, \xi_{s-1}][\xi_s]$$

$$= K[\xi_1, \cdots, \xi_{s-1}, \xi_s]$$

The first result shows that the result is true for $s = 1$ and consequently for $s = 1, 2, \cdots, k$.

When we are considering the function field $K(\xi_1, \xi_2, \cdots, \xi_n)$ where $\xi_1, \cdots, \xi_r$ are algebraically independent and $\xi_{r+1} \cdots, \xi_n$ are quantities algebraically dependent on $K(\xi_1, \cdots, \xi_r)$ then it follows that the function field $\Sigma$ is given by

$$\Sigma = K(\xi_1, \cdots, \xi_n) = K(\xi_1, \cdots, \xi_r)[\xi_{r+1} \cdots, \xi_n].$$

# Chapter 2.

# Algebra involved in the arithmetic theory

§2.1 Ideal theory in a Noetherian ring.[†]

One of the most important algebraic theories used in modern algebraic geometry is the theory of ideals in a commutative ring.

An ideal $\mathfrak{a}$ in a commutative ring $\mathfrak{s}$ is a set of elements $a$, $b$,... of $\mathfrak{s}$ such that

(i) $a - b$ is in $\mathfrak{a}$ for all $a$, $b$ in $\mathfrak{a}$

(ii) $\lambda a$ is in $\mathfrak{a}$ for all $a$ in $\mathfrak{a}$ and $\lambda$ in $\mathfrak{s}$

$\mathfrak{s}$ is called a chain-condition ring or a Noetherian ring if it satisfies the three equivalent conditions

(i) Any chain of ideals $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_n \subset \mathfrak{a}_{n+1} \subset \cdots$ where $\mathfrak{a}_n$ is properly contained in $\mathfrak{a}_{n+1}$, necessarily contains only a finite number of ideals.

(ii) In any set of ideals in $\mathfrak{s}$ there is a maximal ideal; that is, an ideal which is not contained in any other ideal of the set.

(iii) Every ideal $\mathfrak{a}$ in $\mathfrak{s}$ has a finite basis $a_1, \cdots, a_h$ of elements in $\mathfrak{s}$; that is, if $a$ is in $\mathfrak{a}$ we can always write $a$ in the form $a = r_1 a_1 + \cdots + r_h a_h$ where $r_i$ is in $\mathfrak{s}$ $(i = 1, \cdots, h)$. We denote this by $\mathfrak{a} = (a_1, \cdots, a_h)$.

If the elements of an ideal $\mathfrak{a}$ all belong to some other ideal $\mathfrak{b}$ we say that $\mathfrak{b}$ divides $\mathfrak{a}$ or is a factor or divisor of $\mathfrak{a}$ and that $\mathfrak{a}$ is a multiple of $\mathfrak{b}$. This relationship between $\mathfrak{a}$ and $\mathfrak{b}$ can be written in the equivalent forms

$$\mathfrak{a} \subseteq \mathfrak{b}$$

$$\mathfrak{b} \supseteq \mathfrak{a}$$

[†]See [17] vol. II chap. 12

$$\mathfrak{a} \equiv 0 \pmod{\mathfrak{b}}$$

The "H.C.F.", $(\mathfrak{a}, \mathfrak{b})$ of $\mathfrak{a}$ and $\mathfrak{b}$ is the smallest ideal containing $\mathfrak{a}$ and $\mathfrak{b}$. If $\mathfrak{a} = (a_1, \cdots, a_h)$ and $\mathfrak{b} = (b_1, \cdots, b_k)$ then $(\mathfrak{a}, \mathfrak{b}) = (a_1, \cdots a_h, b_1, \cdots b_k)$. The "L.C.M.", $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{a} \cap \mathfrak{b}$, of $\mathfrak{a}$ and $\mathfrak{b}$ is the set of elements common to $\mathfrak{a}$ and $\mathfrak{b}$ and is itself an ideal. The product $\mathfrak{a}\mathfrak{b}$ (or $\mathfrak{a} \cdot \mathfrak{b}$) is the ideal with basis $(\cdots, a_i b_j, \cdots)$ and the ideal quotient $\mathfrak{a} : \mathfrak{b}$ is an ideal which consists of the set of elements $\gamma$ in $\mathfrak{s}$ such that $\gamma b$ is in $\mathfrak{a}$ for all $b$ in $\mathfrak{b}$.

An ideal $\mathfrak{m}$ is said to be reducible if we can write $\mathfrak{m} = \mathfrak{a} \cap \mathfrak{b}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are proper divisors of $\mathfrak{m}$: that is $\mathfrak{m} \equiv 0 \pmod{\mathfrak{a}}$, $\mathfrak{m} \equiv 0 \pmod{\mathfrak{b}}$ but $\mathfrak{a} \neq \mathfrak{m}$ and $\mathfrak{b} \neq \mathfrak{m}$. Otherwise $\mathfrak{m}$ is irreducible.

I should like to sketch the method used by van der Waerden to obtain the normal representation of an ideal in a Noetherian ring.

Using an induction proof he shows that every ideal $\mathfrak{m}$ can be written in the form

$$\mathfrak{m} = [\mathfrak{i}_1, \mathfrak{i}_2, \cdots, \mathfrak{i}_s] = \mathfrak{i}_1 \cap \mathfrak{i}_2 \cap \cdots \cap \mathfrak{i}_s \tag{1}$$

where $\mathfrak{i}_1, \cdots, \mathfrak{i}_s$ are irreducible ideals.

An ideal $\mathfrak{q}$ is called primary if $\mathfrak{a}\mathfrak{b} \equiv 0 \pmod{\mathfrak{q}}$ and $\mathfrak{a} \not\equiv 0 \pmod{\mathfrak{q}}$ implies $\mathfrak{b}^p \equiv 0 \pmod{\mathfrak{q}}$ for some positive integer $p$. The set of primary ideals includes the prime ideals. These are ideals $\mathfrak{p}$ such that $\mathfrak{a}\mathfrak{b} \equiv 0 \pmod{\mathfrak{p}}$ and $\mathfrak{a} \not\equiv 0 \pmod{\mathfrak{p}}$ implies $\mathfrak{b} \equiv 0$ $(\mathfrak{p})$. Now every irredicible ideal can be shown to be primary and so we obtain a representation of $\mathfrak{m}$ as

$$\mathfrak{m} = [\mathfrak{q}_1, \cdots, \mathfrak{q}_s] \tag{2}$$

where $\mathfrak{q}_i$ are primary and in which we shall omit any component $\mathfrak{q}_i$ which divides any other $\mathfrak{q}_j$ $(i \neq j)$.

With every primary ideal $\mathfrak{q}$ is associated a prime ideal $\mathfrak{p}$ where $\mathfrak{p}$ is the set of elements $p$ such that $p^\rho \equiv 0 \pmod{\mathfrak{q}}$ for some positive integer $\rho$. There exists a smallest integer $\kappa$ for which $\mathfrak{p}^\kappa \equiv 0 \ \ (\mathfrak{q})$ and $\kappa$ is called the exponent of $\mathfrak{q}$. It can be shown that the intersection of primary ideals having the same associated prime ideal $\mathfrak{p}$ is a primary ideal which also has $\mathfrak{p}$ as its associated prime. On account of this fact we may assume that in the representation (2), the prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ associated with $\mathfrak{q}_1, \cdots, \mathfrak{q}_s$ are all distinct. We call (2) a normal representation of the ideal $\mathfrak{m}$.

There are two uniqueness properties of normal representations. If $\mathfrak{m} = [\mathfrak{q}_1, \cdots, \mathfrak{q}_s]$ and $\mathfrak{m} = [\mathfrak{q}'_1, \cdots, \mathfrak{q}'_{s'}]$ are two normal representations of the same ideal $\mathfrak{m}$ then $s = s'$ and the prime ideals associated with $\mathfrak{q}_1, \cdots, \mathfrak{q}_s$ are the same as the prime ideals associated with $\mathfrak{q}'_1, \cdots, \mathfrak{q}'_s$. In the representation (2) if $\mathfrak{p}_i$ is the prime associated with $\mathfrak{q}_i$ $(i = 1, \cdots, s)$ we shall call $\mathfrak{p}_i$ an isolated prime if it does not divide any other $\mathfrak{p}_j$ $(i \neq j)$.

Example:

If $\mathfrak{s} = K[x, y]$, the ring of polynomials in two indeterminates with ground field $K$, and if $\mathfrak{m} = (xy, x^2)$, then $\mathfrak{m} = [\mathfrak{q}_1, \mathfrak{q}_2] = [\mathfrak{q}_1, \mathfrak{q}_3]$ where $\mathfrak{q}_1 = (x)$, $\mathfrak{q}_2 = (x^2, xy, y^2)$ and $\mathfrak{q}_3 = (x^2, y)$ are two normal representations of $\mathfrak{m}$. In this case $\mathfrak{p}_1 = (x)$, $\mathfrak{p}_2 = \mathfrak{p}_3 = (x.y)$ and $\mathfrak{p}_1$ is an isolated prime.

## §2.2 Geometrical interpretation of the normal representation of an ideal.

I should like to explain the geometrical significance of this normal representation in the case when $\mathfrak{s} = K[x_1, \cdots, x_n]$ and $x_1, \cdots, x_n$ are independent indeterminates over $K$. $\mathfrak{s}$ is a Noetherian rring and any ideal in $\mathfrak{s}$ therefore has a normal representation.

We may think of $(x_1, \cdots, x_n)$ as non-homogeneous coordinates of a point in afficne space $A_n$. With any ideal $\mathfrak{a}$ in $\mathfrak{s}$ we can then associate a variety $V(\mathfrak{a})$ in $A_n$ which consists of points $(\alpha_1, \cdots \alpha_n)$ which are zeros of all the polynomials in $\mathfrak{a}$. If

$$\mathfrak{a} = (f_1(x_1, \cdots, x_n), \cdots, f_k(x_1, \cdots, x_n))$$

this simply means that $V(\mathfrak{a})$ is the variety defined in $A_n$ by the equations

$$f_i(x_1, \cdots, x_n) = 0 \quad (i = 1, \cdots, k).$$

Conversely if we consider a variety $U$ in $A_n$ then the set of all polynomials vanishing over $U$ is an ideal in $\mathfrak{s}$ associated with $U$. We may denote it by $I(U)$. Here we are thinking of $U$ as a point set with no idea of multiple component varieties. It is evident that

$$V(I(U)) = U$$

but it is not always true that $I(V(\mathfrak{a})) = \mathfrak{a}$. In fact

$$I(V(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$$

where the ideal on the right hand side, called the radical of $\mathfrak{a}$, is the set of all elements $a$ in $\mathfrak{s}$ such that $a^p$ belongs to $\mathfrak{a}$ for some positive integer $p$. In particular the radical of a primary ideal $\mathfrak{q}$ is the associated prime ideal $\mathfrak{p}$.

If $U$ is an irreducible variety in $A_n$ then $I(U)$ is a prime ideal $\mathfrak{p}$ in $\mathfrak{a}$. $V(\mathfrak{a})$ is an irreducible variety in $A_n$ if and only if $\mathfrak{a}$ is a primary ideal, or, in particular, a prime ideal. Suppose that the variety $U$ is made up of irreducible components $U_1, U_2, \cdots, U_s$ so that

$$U = U_1 \dot{+} U_2 \dot{+} \cdots \dot{+} U_s.$$

then

$$I(U) = I(U_1) \cap I(U_2) \cap \cdots \cap I(U_s)$$

$$= \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_s$$

where

$$\mathfrak{p}_i = I(U_i) \quad (i = 1, \cdots, s).$$

On the other hand suppose that

$$\mathfrak{a} = [\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_s] = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_s$$

is a normal representation of an ideal $\mathfrak{a}$ in $\mathfrak{s}$ and that $\mathfrak{p}_i$ is the prime associated with $\mathfrak{q}_i$ $(i = 1, \cdots, s)$. Then

$$V(\mathfrak{a}) = V(\mathfrak{q}_1) \dot{+} V(\mathfrak{q}_2) \dot{+} \cdots \dot{+} V(\mathfrak{q}_s)$$

$$= U_1 \dot{+} U_2 \dot{+} \cdots \dot{+} U_s.$$

$U_i$ is an irreducible variety and

$$I(U_i) = \mathfrak{p}_i \quad (i = 1, \cdots, s).$$

It is easily seen that $I(V(\mathfrak{a})) = \mathfrak{a}$ if and only if $\mathfrak{a}$ is the intersection of a finite number of prime ideals, $\mathfrak{a} = [\mathfrak{p}_1.\mathfrak{p}_2, \cdots, \mathfrak{p}_s]$.

In the general case

$$I(V(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$$

where if

$$\mathfrak{a} = [\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_s]$$

then

$$\mathrm{rad}(\mathfrak{a}) = [\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s]$$

$\mathfrak{p}_i$ being the prime ideal corresponding to the primary ideal $\mathfrak{q}_i$ $(i = 1, \cdots, s)$.

Since

$$V(\mathrm{rad}(\mathfrak{a}, \mathfrak{b})) = V(\mathfrak{a}, \mathfrak{b})$$

we find

$$V(\mathfrak{a}, \mathfrak{b}) = V(\mathfrak{a}) \cap V(\mathfrak{b}).$$

When varieties in the projective space $S_n$ are considered it is necessary to introduce the idea of homogeneous ideals, or $H$-ideals. Suppose $(x_0, x_1, \cdots, x_n)$ are the homogeneous coordinates of a point in $S_n$. The points which do not lie in the prime $x_0 = 0$ have coordinates which can be written in the normalised form $(1, x_1, \cdots, x_n)$. These points form an affine space $A_n$ in which $x_1, \cdots, x_n$ can be regarded as non-homogeneous coordinates.

Suppose that $\mathfrak{a}$ is an ideal in $K[x_1, \cdots, x_n]$. To an element $f(x_1, \cdots, x_n)$ of $\mathfrak{a}$ there corresponds a form $f^*(x_0, x_1, \cdots, x_n)$ which is obtained from $f(x_1, \cdots, x_n)$ by writing $x_0$ instead of 1 in order to make $f$ homogeneous. These forms $f^*(x_0, x_1, \cdots, x_n)$ form an $H$-ideal $\mathfrak{a}^*$ in $K[x_0, x_1, \cdots x_n]$, $\mathfrak{a}^*$ being the smallest ideal in $K[x_0, \cdots, x_n]$ which contains all these forms. $\mathfrak{a}^*$ has a basis consisting of forms. This is the characteristic property of an $H$-ideal. If $f^*(x_0, x_1, \cdots, x_n)$ is a form in $K[x_0, x_1, \cdots, x_n]$ then $f^*(\xi_0, \xi_1, \cdots \xi_n) = 0$ implies that $f^*(\lambda\xi_0, \lambda\xi_1, \cdots, \lambda\xi_n) = 0$. It follows that the homogeneous ideal $\mathfrak{a}^*$ in $K[x_0, x_1, \cdots, x_n]$ determines an algebraic variety $V^* = V(\mathfrak{a}^*)$ in $S_n$. If $V = V(\mathfrak{a})$ is the variety determined by $\mathfrak{a}$ in $A_n$, then $V^*$ is the smallest variety in $S_n$ which contains $V$. An $H$-ideal in $K[x_0, x_1, \cdots, x_n]$ always defines a variety in $S_n$.

§2.3 Contraction and extension of ideals.

Important geometrical operations on varieties correspond to the algebraic processes of contraction and extension of ideals. Suppose $\sigma$ is a subring of the ring $\sigma'$. For simplicity we shall assume that $\sigma$ and $\sigma'$ have a unit element and are Noetherian, which conditions will be satisfied in any rings we shall consider from the point of view of application of this theory to geometry. Let $\mathfrak{a}$ be an ideal in $\sigma$. The smallest ideal $\mathfrak{a}'$ in $\sigma'$ which contains all the elements of $\mathfrak{a}$ is called the extension of $\mathfrak{a}$ in $\sigma'$ and we write $\mathfrak{a}' = \sigma'\mathfrak{a}$. If $\mathfrak{a} = (a_1, \cdots, a_n)$ then $\mathfrak{a}'$ has the same basis $a_1, \cdots, a_n$ in $\sigma'$.

Now suppose that $\mathfrak{b}'$ is an ideal in $\mathfrak{a}'$. Then $\mathfrak{b} = \sigma \cap \mathfrak{b}'$, the set of elements of $\mathfrak{b}'$ which belong also to $\sigma$, is an ideal in $\sigma$ which is known as the contraction of $\mathfrak{b}'$ in $\sigma$. The following stated results show the relations existing between ideals $\mathfrak{a}$, $\mathfrak{b}$ in $\sigma$ and their extensions in $\sigma'$, and between ideals $\mathfrak{a}'$, $\mathfrak{b}'$ in $\sigma'$ and their contractions in $\sigma$.

(i) If $\mathfrak{a} \subseteq \mathfrak{b}$ then $\sigma'\mathfrak{a} \subseteq \sigma'\mathfrak{b}$

(ii) $\sigma'(\mathfrak{a}, \mathfrak{b}) = (\sigma'\mathfrak{a}, \sigma'\mathfrak{b})$

(iii) $\sigma'(\mathfrak{a} \cdot \mathfrak{b}) = (\sigma'\mathfrak{a}) \cdot (\sigma'\mathfrak{b})$

(iv) $\sigma'(\mathfrak{a} \cap \mathfrak{b}) = \sigma'\mathfrak{a} \cap \sigma'\mathfrak{b}$

(i)' If $\mathfrak{a}' \subseteq \mathfrak{b}'$ then $\sigma \cap \mathfrak{a}' \subseteq \sigma \cap \mathfrak{b}'$

(ii)' $\sigma \cap (\mathfrak{a}', \mathfrak{b}') \supseteq (\sigma \cap \mathfrak{a}', \sigma \cap \mathfrak{b}')$

(iii)' $\sigma \cap (\mathfrak{a}' \cdot \mathfrak{b}') \supseteq (\sigma \cap \mathfrak{a}') \cdot (\sigma \cap \mathfrak{b}')$

(iv)' $\sigma \cap (\mathfrak{a}' \cap \mathfrak{b}') = (\sigma \cap \mathfrak{a}') \cap (\sigma \cap \mathfrak{b}')$

Two other important results relating to the extension of ideals are as follows.

(v)' The contraction of a prime ideal in $\sigma'$ is a prime ideal in $\mathfrak{s}$.

(vi)' The contraction of a primary ideal $\mathfrak{q}'$ in $\sigma'$ with associated prime $\mathfrak{p}'$ is a primary ideal $\mathfrak{q}$ in $\sigma$ with associated prime $\mathfrak{p}$, where $\mathfrak{p}$ is the contraction of $\mathfrak{p}'$ in $\sigma$.

If $\sigma' = K[x_1, \cdots, x_n]$ and $\sigma = K[x_1, \cdots, x_{n-1}]$ then the contraction of an ideal corresponds to the idea of orthogonal projection of a variety in

$A_n$ where $(x_1, \cdots, x_n)$ are non-homogeneous coordinates. For suppose $\mathfrak{a}'$ in $\mathfrak{s}'$ defines a variety $V'$ in $A_n$. $f$ is in $\mathfrak{a}$, the contraction of $\mathfrak{a}'$ in $\mathfrak{s}$, if $f$ belongs to $\mathfrak{a}'$ and also to $\sigma$. Therefore $f$ is a polynomial in $x_1, \cdots, x_{n-1}$ and $f = f(x_1, \cdots, x_{n-1}) = 0$ is the equation of a primal in $A_n$ which is a cylinder with generators parallel to the $x_n$-axis and this cylinder contains $V'$. $f(x_1, \cdots, x_{n-1}) = 0$ represents in $A_{n-1}$, the prime of $A_n$ with equation $x_n = 0$, a primal through the orthogonal projection $V$ of $V'$ onto $x_n = 0$. Moreover if $g(x_1, \cdots x_{n-1}) = 0$ represents any primal in $A_{n-1}$ through $V$, then $g(x_1, \cdots, x_{n-1})$ belongs to $\mathfrak{a}'$ and therefore to $\mathfrak{a} = \sigma \cap \mathfrak{a}'$. The result (v)' tells us that the orthogonal projection of an irredicible variety is itself irreducible.

If $\sigma' = K[x_0, x_1 \cdots, x_n]$ and $\sigma = K[x_0, x_1, \cdots, x_{n-1}]$ and if $(x_0, x_1, \cdots x_n)$ are regarded as homogeneous coordinates in $S_n$, then the contraction of an $H$-ideal $\mathfrak{a}'$ in $\sigma$ is an $H$-ideal $\mathfrak{a}$ which determines a variety $V$ in the $S_{n-1}$ given by the equation $x_n = 0$. If $V'$ is the variety determined by $\mathfrak{a}'$ in $S_n$, then $V$ is the projection of $V'$ from the point $(0, \cdots, 0, 1)$ onto the opposite face of the simplex of reference.

Another case important from the geometrical point of view arises when $\sigma = K[x_1, \cdots, x_n]$ and $\sigma' = K\{x_1, \cdots, x_n\}$, the ring of convergent power series in $x_1, \cdots, x_n$. The ring $K\{x_1, \cdots, x_n\}$ is Noetherian ([20], p.47) and has to be considered when local properties of the variety are to be investigated. The theory of ideals in this ring is developed in a paper by Rückert [11].

Suppose we wish to investigate the local properties of an irreducible variety $V$ in $S_n$ at a generic point $P$ with non-homogeneous coordinates $(a_0, a_1, \cdots, a_n)$. If $a_0 \neq 0$, $P$ is at a finite distance with respect to non-homogeneous coordinates $x'_i = x_i/x_0$ $(i = 1, \cdots, n)$. A non-singular transformation will produce a coordinate system with $P$ at the origin. We may assume that $(x_1, \cdots, x_n)$ are non-homogeneous coordinates referred to $P$ as origin. $V$ determines a prime ideal $\mathfrak{p}$in $K[x_1, \cdots, x_n]$. Form the extended ideal $K\{x_1, \cdots, x_n\} \cdot \mathfrak{p}$. This need no longer be a prime ideal in $K\{x_1, \cdots, x_n\}$ and will have a normal representation as an intersection of primary ideals $\mathfrak{q}_i$ $(i = 1, \cdots s)$ in $K\{x_1, \cdots, x_n\}$. Each of these primary ideals defines an irreducible analytic branch of $V$ through $P$. In fact Zariski has shown that $K\{x_1, \cdots x_n\} \cdot \mathfrak{p}$ is an intersection of prime ideals implying that an irreducible algebraic variety $V$ can have only simple analytic branches in the neighbourhood of a point $P$ [29].

Another application of ideal theory to geometry arises when we consider ideals in the integral domain of a given irreducible variety $V$ in $S_n$. Let $(1, \xi_1, \cdots \xi_n)$ be a normalised generic point of $V$ and consider the ring $\sigma = K[\xi_1, \cdots \xi_n]$. This ring is isomorphic with the ring $K[x_1, \cdots x_n]/\mathfrak{p}$ of residue classes of $K[x_1, \cdots, x_n]$ modulo $\mathfrak{p}$ where $\mathfrak{p} = I(V)$ is a prime ideal in $K[x_1, \cdots, x_n]$. Since $\mathfrak{p}$ is prime $\sigma$ is an integral domain and its quotient field $\Sigma = K(\xi_1, \cdots, \xi_n)$ is of degree of transcendency $r$ over $K$ where $r$ is the dimension of $V$. Also $\sigma$ is Noetherian ([17] vol II p. 21). Any ideal $\mathfrak{a} \neq (0)$ in $\sigma$ defines a subvariety on $V$ of dimension less than $r$ which is at a finite distance with respect to the chosen coordinate system.

§2.4 Integral dependence and integral closure

A very important idea which features in modern algebra and which Zariski has used with considerable success in algebraic geometry, is that of integral dependence and of the integral closure of a commutative ring $R$ in some larger commutative ring $R'$ containing $R$. For simplicity I shall assume that $R$ has a unit element.

An element $\xi$ of $R'$ is said to be integrally dependent on $R$ if all positive powers of $\xi$ belong to a finite $R$-module $(a_1, \cdots a_h)$ in $R'$, that is, if there exist elements $a_1, \cdots, a_h$ in $R'$ so that, for any positive integer $p$, $\xi^p$ can be written in the form

$$\xi^p = r_1 a_1 + \cdots + r_h a_h, \quad (r_i \text{ belongs to } R, \quad i = 1, \cdots h). \tag{1}$$

Suppose now that $R$ is Noetherian. Van der Waerden has shown that for any finite $R$-module $\mathfrak{m}$ in $R$ there is a chain condition for the submodules of $\mathfrak{m}$. That is, if $\mathfrak{m}_i$ are submodules of $\mathfrak{m}$ then a chain $\mathfrak{m}_1 \subset \mathfrak{m}_2 \subset \cdots$ is necessarily finite, where $\mathfrak{m}_i \subset \mathfrak{m}_{i+1}$ means that $\mathfrak{m}_i$ is strictly contained in $\mathfrak{m}_{i+1}$. As in the case of the ideals in a Noetherian ring $R$, this is equivalent to the condition that every submodule of $\mathfrak{m}$ has a finite basis ([17] vol II p. 77). Consider the chain of modules

$$(1) \subset (1, \xi) \subset (1, \xi, \xi^2) \subset (1, \xi, \xi^2, \xi^3) \subset \cdots$$

This chain is necessarily finite, and therefore, for some integer $m$,

$$(1, \xi, \xi^2, \cdots, \xi^{m-1}) = (1, \xi, \xi^2, \cdots, \xi^{m-1}, \xi^m).$$

This implies that

$$\xi^m = \alpha_1 \xi^{m-1} + \alpha_2 \xi^{m-2} + \cdots + \alpha_{m-1} \xi + \alpha_m \quad (\alpha_i \text{ is in } R, \ i = 1, \cdots, m). \tag{2}$$

Conversely suppose that $\xi$ is an element of $R'$ satisfying an equation of the form (2). On account of this relation every power of $\xi$ can be written in the form

$$\beta_1 \xi^{m-1} + \beta_2 \xi^{m-2} + \cdots + \beta_{m-1} \xi + \beta_m \quad (\beta \text{ is in } R, \ i = 1, \cdots, m) \tag{3}$$

so that any power of $\xi$ is in the finite $R$-module $(1, \xi, \xi^2, \cdots, \xi^{m-1})$. Consequently $\xi$ is integral over $R$ according to the original definition. This means that we may give an alternative definition of integral dependence and define

$\xi$, belonging to $R'$, to be integral over $R$ if it satisfies an equation of the form (2), or, as this equation is more usually written,

$$\xi^m + \alpha_1 \xi^{m-1} + \cdots + \alpha_{m_1} \xi + \alpha_m = 0 \quad (\alpha_i \text{ is in } R,\ i = 1, \cdots m). \qquad (4)$$

The set of elements of $R'$ which are integral over $R$ form a ring $R^*$ which is called the integral closure of $R$ in $R'$. For suppose that all positive powers of $\xi$ belong to the $R$-module $(a_1, \cdots, a_h)$ and that all positive powers of $\eta$ belong to the $R$-module $(b_1, \cdots, b_k)$. Evidently all powers of $\xi \pm \eta$ or $\xi\eta$ belong to the $R$-module

$$(a_1, \cdots, a_h, b_1, \cdots, b_k, a_1 b_1, \cdots a_1 b_k, a_2 b_1, \cdots a_h b_k).$$

It follows that $R^*$ is a ring. Obviously $R \subseteq R^* \subseteq R'$. If $R$ coincides with $R^*$ we say that $R$ is integrally closed in $R'$. In tbis case a relation of the form (4) where $\xi$ is in $R'$ implies that $\xi$ is in $R$.

If $R$ is a subring of a larger ring $R_1$ and if every element of $R_1$ is integral over $R$, we say that the ring $R_1$ is itself integral over $R$. An important property of this relation of integral dependence is its transitivity. Given three rings $R \subset R_1 \subset R_2$, such that $R_1$ is integral over $R$ and $R_2$ is integral over $R_1$, then $R_2$ is integral over $R$.

§2.5 Various results concerning elements integral over a given ring

From the geometrical point of view the most important case arises when $R$ is an integral domain and $R'$ is its quotient field. I should like, therefore, to give three results concerning the integral closure of an integral domain in its function field.

**Theorem 1** *If $K$ is a field and $x$ an indeterminate over $K$, then $K[x]$ is integrally closed in $K(x)$, its quotient field.*

For if $\xi = \dfrac{a}{b}$, where $a$ and $b$ belong to $K[x]$, and if $\xi$ is integrally dependent on $K[x]$, we have a relation

$$\xi^m + \alpha_1 \xi^{m-1} + \cdots + \alpha_{m-1}\xi + \alpha_m = 0 \ (\alpha \text{ is in } K[x],\ i = 1, \cdots, m).$$

This implies

$$a^m + \alpha_1 a^{m-1} b + \cdots + \alpha_{m-1} a b^{m-1} + \alpha_m b^m = 0.$$

This means that $b$ is a factor of $a^m$ and since in the original representation of $\xi$ as the quotient of $a$ by $b$ we may assume $a$ and $b$ to be free of common factors, this implies that $b$ is in $K$, and hence that $\xi$ is in $K[x]$.

An exactly similar proof will suffice to show that any integral domain which is a unique factorisation domain is integrally closed in its quotient field ([17] vol II p. 78). The property of being a unique factorisation domain is however not a necessary condition for an integral domain to be integrally closed in its quotient field.

Let us suppose that $R$ is an integral domain which is integrally closed in its quotient field $K$ and consider an extension of the integral domain $R$ in $R^* = K[\zeta]$ where $\zeta$ is a root of the irreducible equation:

$$f(z) = z^m + A_1 z^{m-1} + \cdots + A_{m-1} z + A_m = 0,\ (A_i \text{ is in } R,\ i = 1, \cdots .m).$$

Van der Waerden considers the elements of the quotient field $K^*$ of $R^*$ which are integrally dependent on $R^*$, and he shows that such an element, $\eta$, can be written in the form

$$\eta = \frac{\sum_{j=0}^{m-1} r_j \zeta^j}{D}, \quad (r_j \text{ is in } R,\ j = 0, 1, \ldots, m-1) \qquad (5)$$

where $D$ is the $z$-discriminant of $f(z)$ and is in $R$ ([17] vol II p.80).

I shall use the main ideas of the proof of this result later on. I should like here to show how to derive an alternative form for $\eta$.

**Theorem 2** *If $R$ is an integral domain which is integrally closed in its quotient field $K$, and if $R^*$ is the integral domain $K[\zeta]$, where $\zeta$ is a root of the irreducible equation:*

$$f(z) = z^m + A_1 z^{m-1} + \cdots + A_{m-1} z + A_m = 0, \ (A_i \text{ is in } R, \ i = 1, \cdots .m) \quad (6)$$

*then any element, $\eta$, of the quotient field $K^*$ of $R^*$, which is integrally dependent on $R^*$, can be written in the form:*

$$\eta = \frac{\sum_{j=0}^{m-1} r_j \zeta^j}{\frac{df}{d\zeta}}, \quad (r_j \text{ is in } R, \ j = 0, 1, \cdots m-1) \quad (7)$$

*where*

$$\frac{df}{d\zeta} = \left(\frac{df}{dz}\right)_{z=\zeta}.$$

$K^*$ is a simple algebraic extension of $K$ by the equation (6). Any element $\eta$ of $K^*$ is of the form

$$\eta = \sum_{j=0}^{m-1} \rho_j \zeta^j, \ (\rho_j \text{ is in } K, \ j = 0, 1, \cdots m-1) \ \ ([8] \ p.100). \quad (8)$$

Consider the conjugate quantities $\zeta = \zeta_1, \zeta_2, \cdots, \zeta_m$ which are the roots of $f(z) = 0$, and define

$$\eta_i = \sum_{j=0}^{m-1} \rho_j \zeta_i^j \quad (i = 1, 2, \cdots, m) \quad (9)$$

Now if we set

$$\frac{f(z)}{z - \zeta_1} = B_1 z^{m-1} + B_1 z^{m-2} + \cdots + B_{m-1} \quad (10)$$

we find, equating the coefficients of powers if $z$ in the identity

$$f(z) = (B_0 z^{m-1} + B_1 z^{m-2} + \cdots + B_{m-1})(z - \zeta_1) \quad (11)$$

that the following relations hold.

$$
\begin{aligned}
&1 = B + 0 \\
&A_1 = -\zeta_1 B_0 + B_1 \\
&A_2 = -\zeta_1 B_1 + B_2 \\
&. \\
&. \\
&A_{m-1} = -\zeta_1 B_{m-2} + B_{m-1} \\
&A_m = -\zeta_1 B_{m-1}
\end{aligned} \tag{12}
$$

Therefore

$$
\begin{aligned}
&B_0 = 1 \\
&B_1 = \zeta_1 + A_1 \\
&B_2 = \zeta_1^2 + A_1\zeta_1 + A_2 \\
&. \\
&. \\
&B_{m-1} = \zeta_1^{m-1} + A_1\zeta_1^{m-2} + \cdots + A_{m-1}
\end{aligned} \tag{13}
$$

so that $B_k$ is in $R[\zeta_1]$ $(k = 0, 1, \cdots m-1)$.

We know that $\dfrac{f(z)}{z-\zeta_1}$ vanishes when $z = \zeta_2, \cdots \zeta_m$ and has the value

$$\left(\frac{df}{dz}\right)_{z=\zeta_1} = f_z(\zeta_1)$$

when $z = \zeta_1$. Therefore

$$B_0\zeta_i^{m-1} + B_1\zeta_i^{m-2} + \cdots + B_{m-1} = f_z(\zeta_1)\delta_{i1} \quad (i = 1, 2 \cdots m). \tag{14}$$

Let

$$\eta = \sum_{j=0}^{m-1} \rho_j \zeta^j = \varphi(\zeta)$$

and consider

$$P_k = \zeta_1^k \varphi(\zeta_1) + \cdots + \zeta_m^k \varphi(\zeta_m) \quad (k = 0, 1, \cdots m-1) \tag{15}$$

$$= \zeta_1^k \eta_1 + \zeta_2^k \eta_2 + \cdots + \zeta_m^k \eta_m \quad (k = 0, 1, \cdots m-1). \tag{16}$$

$P_k$ is symmetric in the conjugates $\zeta_1, \cdots, \zeta_m$ and $\eta_1, \cdots, \eta_m$ over $K$, and therefore $P_k$ is in $K$. Also since

$$P_k = \zeta_1^k \eta_1 + \zeta_2^k \eta_2 + \cdots + \zeta_m^k \eta_m,$$

$P_k$ is integral over $R$ and therefore $P_k$ is in $R$ since $R$ is integrally closed in $K$.

We therefore have relations

$$\begin{aligned}
&\varphi(\zeta_1) + \varphi(\zeta_2) + \cdots + \varphi(\zeta_m) = P_0 \\
&\zeta_1\varphi(\zeta_1) + \zeta_2\varphi(\zeta_2) + \cdots + \zeta_m\varphi(\zeta_m) + P_1 \\
&\cdots \\
&\zeta_1^{m-1}\varphi(\zeta_1) + \zeta_2^{m-1}\varphi(\zeta_2) + \cdots + \zeta_m^{m-1}\varphi(\zeta_m) = P_{m-1}
\end{aligned} \tag{17}$$

where $P_0, P_1, \cdots, P_{m-1}$ are all in the ring $R$.

Multiply these equations by $B_{m-1}, B_{m-2}, \cdots, B_0$ respectively and add them together. Using equation (14) we obtain

$$f_z(\zeta_1)\varphi(\zeta_1) = P_0 B_{m-1} + P_1 B_{m-2} + \cdots + P_{m-1} B_0. \tag{18}$$

The expression on the right hand side of this equation is a polynomial in the ring $R[\zeta_1]$ and consequently can be written in the form

$$\sum_{j=0}^{m-1} r_j \zeta_1^j \quad (r_j \text{ is in } R,\ j = 0, 1, \cdots m-1).$$

Therefore, dropping the suffix $j$,

$$\eta = \varphi(\zeta) = \frac{\sum_{j=0}^{m-1} r_j \zeta^j}{f_z(\zeta)} \quad (r_j \text{ is in } R,\ j = 0, 1, \cdots, m-1). \tag{19}$$

Van der Waerden's form for $\eta$ follows as a corollary from this result. The $z$-discriminant of $f(z)$ is the $z$-resultant of $f(z)$ and $\dfrac{df}{dz}$. It is known that if $f(z)$ and $g(z)$ are two polynomials in $R[z]$ then their $z$-resultant, $S$, is of the form

$$S = Af + Bg$$

where $A$ ad $B$ are polynomials in $R[z]$ ([17] vol I p. 91).

Therefore the $z$-discriminant of $f(z)$, $D$, can be written in the form

$$D = Af + B\frac{df}{dz} \tag{20}$$

where $A$, $B$ are in $K[z]$. Substituting $z = \zeta$ in this relation we find

$$D = Bf_z(\zeta) \tag{21}$$

Consequently

$$\begin{aligned}
\eta &= \frac{\sum_{j=0}^{m-1} r_j\zeta^j}{f_z(\zeta)} \qquad (r_j \text{ is in } R,\ j = 0, 1, \cdots m-1) \\
&= \frac{B\sum_{j=0}^{m-1} r_j\zeta^j}{Bf_z(\zeta)} \qquad (r_j \text{ is in } R,\ j = 0, 1, \cdots m-1) \qquad (22) \\
&= \frac{\sum_{j=0}^{m-1} s_j\zeta^j}{D} \qquad (r_j \text{ is in } R,\ j = 0, 1, \cdots m-1).
\end{aligned}$$

Two common ways of extending an integral domain $R$ involve respectively the addition of an indeterminate or a quantity algebraic over $K$, the quotient field of $R$. We have just investigated the elements of $K(\zeta)$ integrally dependent on $K[\zeta]$ where $\zeta$ is algebraic over $K$. I should like now to consider the integral closure of the ring $R[x]$ in $K(x)$ where $x$ is an indeterminate over $K$.

When $R$ is a unique factorisation domain $R[x]$ is also a unique factorisation domain ([1], p. 36) and is, in consequence, integrally closed in $K(x)$. One of the most common types of integral domain occurring in algebraic geometry is the ring $R = K_0[\zeta]$ where $\zeta$ is a quantity algebraic over the field $K_0$. $R$ is in this case an integral domain but is not, in general, a unique factorisation domain. We can prove in general the following result.

**Theorem 3** *If $K_0(\zeta)$ is a simple algebraic extension of the ground field $K_0$, and if $x$ is an indeterminate over $K_0$, and therefore also over $K_0(\zeta)$, then, denoting by $(K_0[\zeta])^*$ and $(K_0[\zeta, x])^*$ the integral closures of $K_0[\zeta]$ and $K_0[\zeta, x]$ in their respective quotient fields,*

$$(K_)[\zeta])^*[x] = (K_0[\zeta, x])^*.$$

Suppose that $u$ is an element of $(K_0[\zeta, x])^*$. $u$ satisfies an equation of integral dependence

$$u^m + \alpha_1 u^{m-1} + \cdots + \alpha_{m-1} u + \alpha_m = 0 \ (\alpha_i \text{ is in } K_0[\zeta, x], i = 1, \cdots, m). \quad (23)$$

In this relation the coefficients $\alpha_i$ $(i = 1, \cdots, m)$ belong a fortiori to the ring $K'[x] = K_0(\zeta)[x]$ where $K' = K_0(\zeta)$ is a field. $K'[x]$ is a unique factorisation domain and is therefore integrally closed in the quotient field $K'(x) = K_0(\zeta, x)$. Since $u$ belongs to $(K'[x])^*$ we deduce that $u$ is in $K_0(\zeta)[x]$ and consequently can be written in the form

$$u = a_0 x^m n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \ \ (a_i \text{ is in } K_0(\zeta) \ \ i = 0, 1, \cdots, n). \quad (24)$$

In this relation we substitute the values $1, 2, \cdots (n+1)$ for $x$ in turn. We obtain a set of $n+1$ quantities $p_1, p_2, \cdots, p_{n+1}$ given by

$$p_k = a_0 k^n + a_1 k^{n-1} + \cdots + a_{n-1} k + a_n \ (k = 1, 2, \cdots, (n+1)). \qquad (25)$$

and, substituting these same values for $x$ in the equation (23) we find that $p_k$ $(k = 1, 2, \cdots, (n+1))$ are elements of the ring $(K_0[\zeta])^*$.

If we solve the equations (25) for $a_i$ $(i = 0, 1, \cdots, n)$ we see that the quantities $(n+1)!a_i$ $(i = 0, 1, \cdots, n)$ are linear combinations of the quantities $p_k$ with coefficients in $K_0$. Therefore $(n+1)!a_i$ is in $(K_0[\zeta])^*$ $(i = 0, 1, \cdots, n)$. It follows that $a_i$ is in $(K_0[\zeta])^*$ since it is assumed that $K_0$ is of characteristic zero and therefore must contain the field of rational numbers as a subfield. Therefore

$$(K_0[\zeta, x])^* \subseteq (K_0[\zeta])^*[x] \tag{26}$$

Now suppose that $u$ is in $(K_0[\zeta])^*[x]$. Then

$$u = b_0 x^p + b_1 x^{p-1} + \cdots + b_{p-1} x + b_p \tag{27}$$

where $b_i$ is in $K_0(\zeta)$ $(i = 0, 1, \cdots, p)$ and satisfies an equation of integral dependence over $K_0[\zeta]$:

$$b_i^s + \beta_{1,i} b_i^{s-1} + \cdots + \beta_{s-1,i} b_i + \beta_{s,i} = 0 \quad (i = 0, 1, \cdots p) \tag{28}$$

where the $\beta$'s are in $K_0[\zeta]$. This relation can be multiplied by $x^{(p-i)s}$ for each value of $i = 0, 1 \cdots p$ to give

$$(b_i x^{p-i})^s + \beta_{1,i} x^{p-i} (b_i x^{p-i})^{s-1} + \cdots + \beta_{s-1,i} x^{(s-1)(p-i)} (b_i x^{p-i}) + \beta_{s,i} x^{s(p-i)} = 0. \tag{29}$$

Therefore $b_i x^{p-i}$ satisfies an equation of integral dependence on $K_0[\zeta, x]$ and so $u$ belongs to $(K_0[\zeta, x])^*$. This implies that

$$(K_0[\zeta])^*[x] \subseteq (K_0[\zeta, x])^*. \tag{30}$$

The relations (26) and (30) give the result

$$(K_0[\zeta])^*[x] = (K_0[\zeta, x])^* \tag{31}$$

**Corollary** *If $K_0[\zeta]$ is integrally closed then so also is $K_0[\zeta, x]$*

For then

$$\begin{aligned}(K_0[\zeta, x])^* &= (K_0[\zeta])^*[x] \\ &= K_0[\zeta][x] \\ &= K_0[\zeta, x].\end{aligned}$$

§2.6 Quasi-equal ideals

One of the reasons for the importance of the notion of integral closure arises from a consideration of the ideal theory in an integrally closed integral domain $\mathfrak{s}$, that is, an integral domain which is integrally closed in its quotient field.

The classical ideal theory seeks to express every ideal $\mathfrak{a}$ in a ring $\sigma$ uniquely as a product of prime ideals in the form

$$\mathfrak{a} = \mathfrak{p}_1^{p_1} \mathfrak{p}_2^{p_2} \cdots \mathfrak{p}_r^{p_r}$$

Necessary and sufficient conditions on $\sigma$ for this to be possible are:

(i) $\sigma$ is an integral domain which is Noetherian.

(ii) Every non-null prime ideal $\mathfrak{p}$ is divisorless, that is $\sigma$ is the only proper divisor of $\mathfrak{p}$.

(iii) $\sigma$ is integrally closed in its quotient field.

The second condition is very restrictive and in algebraic geometry the integral domains which have to be studied rarely satisfy it. One of the simplest examples of an integral domain satisfying all three conditions is given by $\sigma = K[x]$. The integral domain of any irreducible algebraic curve which is normal in the affine sense as defined later on provides a second example.

To overcome the restriction on $\sigma$ and yet to obtain a representation of every ideal as a product of prime ideals, the concept of quasi-equality, or Artin-equality, is introduced to replace strict equality. Artin's form of the definition for quasi-equality ([17] vol. II p. 93) does not immediately display the geometrical significance that appears with Van der Waerden's later definition [15] which is an exactly equivalent definition.

Van der Waerden considers the theory of ideals in an integral domain $\sigma$ satisfying the conditions (i) and (iii) above. He divides the ideals in $\sigma$ into two classes of higher and lower ideals by the definition:

a) A higher prime ideal has no prime multiples other than (0).

b) A higher ideal is divisible by at least one higher prime ideal.

c) All other ideals, and in particular $\mathfrak{s}$, are lower ideals.

Various criteria are available for deciding to which class a given ideal belongs. Given an ideal $\mathfrak{a}$ with normal representation $\mathfrak{a} = [\mathfrak{q}_1, \cdots, \mathfrak{q}_r]$, where $\mathfrak{p}_i$ is the prime associated with $\mathfrak{q}_i$ $(i = 1, \cdots, r)$, then $\mathfrak{a}$ is a higher ideal if at least one of the $\mathfrak{p}_i$ is a higher ideal and $\mathfrak{a}$ is a lower ideal if all the $\mathfrak{p}_i$ are lower ideals. A product of lower ideals is a lower ideal, and a product with at least one higher ideal as a factor is a higher ideal.

Van der Waerden then defines quasi-equality. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are quasi-equal if there exist lower ideals $\mathfrak{c}_1$ and $\mathfrak{c}_2$ such that

$$\mathfrak{c}_1\mathfrak{a} \equiv 0(\mathfrak{b})$$

$$\mathfrak{c}_2\mathfrak{b} \equiv 0(\mathfrak{a}).$$

We write $\mathfrak{a} \sim \mathfrak{b}$ and it is easy to see that this is an equivalence relation.

Among the set of ideals quasi-equal to $\mathfrak{a}$ is a maximal ideal $\mathfrak{a}^*$ which is unique and contains every ideal quasi-equal to $\mathfrak{a}$. Suppose in the normal representation $\mathfrak{a} = [\mathfrak{q}_1, \cdots, \mathfrak{q}_s]$ of a higher ideal $\mathfrak{a}$, that $\mathfrak{q}_{r+1}, \cdots, \mathfrak{q}_s$ are lower ideals and $\mathfrak{q}_1, \cdots, \mathfrak{q}_r$ are higher ideals. Then

$$\mathfrak{a} \sim [\mathfrak{q}_1, \cdots, \mathfrak{q}_r] = \mathfrak{a}^* \tag{1}$$

In this representation (1) each $\mathfrak{q}_i$ is such that the H.C.F. of $\mathfrak{q}_i$ and the intersection of the remaining $\mathfrak{q}_j$ $(i \neq j)$ is a lower ideal. Now in the case of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ whose H.C.F. is a lower ideal,

$$[\mathfrak{a}, \mathfrak{b}] \sim \mathfrak{a}\mathfrak{b},$$

and therefore it follows

$$\mathfrak{a} \sim [\mathfrak{q}_1, \cdots, \mathfrak{q}_r] \sim \mathfrak{q}_1 \cdots \mathfrak{q}_r. \tag{2}$$

The representation of any ideal in $\sigma$ as a product of primary ideals depends only on the fact that $\sigma$ satisfies condition (1) above. At this point Van der Waerden introduces the fractional ideals of $\sigma$. These are finite $\sigma$-modules whose base belongs to $\Sigma$, the quotient field of $\sigma$. He extends his definition of quasi-equality to cover the case of these fractional ideals.

If $\mathfrak{p}$ is a higher prime ideal then the module quotient $\mathfrak{p}^{-1} = \sigma : \mathfrak{p}$ (that is, the set of elements $\gamma$ belonging to $\Sigma$ such that $\gamma p$ is in $\sigma$ for all $p$ in $\mathfrak{p}$) is a fractional ideal. Obviously $\mathfrak{p}^{-1} \supseteq \sigma$ and it can be shown that $\mathfrak{p}^{-1}$ contains elements of $\Sigma$ which do not belong to $\sigma$. Using the property of the integral closure of $\sigma$ in $\Sigma$, Van der Waerden shows

$$\mathfrak{p}\mathfrak{p}^{-1} \sim \sigma.$$

Finally he shows that every higher primary ideal of $\mathfrak{q}$ is quasi-equal to a power of $\mathfrak{p}$, its associated prime ideal and hence obtains a unique representation

$$\mathfrak{a} \sim \mathfrak{p}_1^{p_1} \mathfrak{p}_2^{p_2} \cdots \mathfrak{p}_r^{p_r}. \tag{3}$$

Van der Waerden gives the following example to show that ordinary equality cannot, in general, replace quasi-equality. Suppose $\zeta$ is an algebraic function of the indeterminate $x$ and $uy$ given by $x^2-y^2+\zeta^2=0$. Let $\sigma$ be the integral domain $K[x, y, \zeta]$, where $K$ is the field of complex numbers. $\sigma$ satisfies the conditions (i) and (iii) above. The ideal $(x)$ in $\sigma$ has no prime multipliers and is the L.C.H. of the prime ideals $\mathfrak{p}_1 = (x, y-\zeta)$ and $\mathfrak{p}_2 = (x, y+\zeta)$.

$$\mathfrak{p}_1\mathfrak{p}_2 = (x^2, xy, x\zeta)$$

$$= (x) \cdot (x, y, \zeta)$$

and since $(x, y, \zeta)$ is a lower ideal it follows that

$$\mathfrak{p}_1\mathfrak{p}_2 \sim (x).$$

On the other hand $\mathfrak{p}_1\mathfrak{p}_2 \neq (x)$.

We may interpret this example geometrically. The equation $x^2 - y^2 + z^2 = 0$ represents a cone in the affine three-dimensional space in which $x$, $y$, $z$ are non-homogeneous coordinates. This cone has its vertex at the origin of coordinates. Non-homogeneous coordinates of a generic point of the locus are $(x, y, \zeta)$ where $\zeta$ is the algebraic function of $x$ and $y$ previously defined. As will be clear later on the cone is normal in the affine sense which means that $K[x, y, \zeta]$ is integrally closed in $K(x, y, \zeta)$. The ideals in $\sigma$ define subvarieties on this cone. The subvariety defined by $(x)$ consists of two generators lying in the plane $x = 0$. $\mathfrak{p}_1$ defines one of these generators and $\mathfrak{p}_2$ defines the other. The lower ideal $(x, y, \zeta)$ defines the vertex of the cone which is the point of intersection of these two generators.

In general suppose that $V$ is an irreducible $r$-dimensional variety with a generic point having non-homogeneous coordinates $(\xi_1, \cdots, \xi_n)$ which is such that its integral domain $K[\xi_1, \cdots, \xi_n]$ is integrally closed in the field

$K(\xi_1, \cdots, \xi_n)$. Any ideal $\mathfrak{a}$ in $\sigma$ defines a subvariety $W$ of $V$ and if $\mathfrak{a}$ is a higher ideal this subvariety is of dimension $r-1$. In $\sigma$, $\mathfrak{a}$ is quasi-equal to a product of higher prime ideals

$$\mathfrak{a} \sim \mathfrak{p}_1^{p_1} \cdots \mathfrak{p}_s^{p_s} \qquad (\mathfrak{p}_i \text{ is a higher prime ideal, } i-1, \cdots, s).$$

These higher prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ define on $V$ the $(r-1)$-dimensional irreducible components of $W$.

The theory is simplified in the case when $\sigma = K[x_1, \cdots, x_n]$ where the $x_i$ $(i = 1, \cdots, n)$ are all indeterminates - so that $(x_1, \cdots, x_n)$ is a nonhomogeneous generic point of $A_n$. Every higher prime ideal of $\sigma$ is a principal ideal, $(p(x_1, \cdots, x_n))$, where $p(x_1, \cdots, x_n)$ is an irreducible polynomial, and therefore defines an irreducible primal in $A_n$. Any higher ideal $\mathfrak{a}$ is contained in some higher prime ideal $(p(x_1, \cdots, x_n))$ and therefore a basis of $\mathfrak{a}$ is contained in $(p(x_1, \cdots, x_n))$. Consequently, writing $p(x)$ for $p(x_1, \cdots, x_n)$ etc. for brevity,

$$\mathfrak{a} = (p(x)\varphi_1(x), p(x)\varphi_2(x), \cdots, p(x)\varphi_s(x))$$

and $\mathfrak{a}$ represents a variety having the primal given by $p(x) = 0$ as an irreducible component. Also any higher prime ideal $\mathfrak{p}' = (p'(x))$ which divides $\mathfrak{a}$ is such that the basis of $\mathfrak{a}$ is divisible by $p'(x)$. Suppose

$$\mathfrak{a} = (f(x)\psi_1(x), f(x)\psi_2(x), \cdots, f(x)\psi_s(x))$$

where the H.C.F. of the polynomials $\psi_i(x)$ $(i = 1, \cdots s)$ is 1. Then

$$\mathfrak{a} \sim (f(x)) = \mathfrak{a}^*$$

and $\mathfrak{a}^*$ defines a bunch of primals in $A_n$.

§2.7 Birational correspondence

Let us consider a correspondence between points $(x_0, x_1, \cdots, x_n)$ in $S_n$ and points $(y_0, y_1, \cdots, y_m)$ in $S'_m$ which is such that, given a point $P$ in $S_n$, the corresponding point $P'$ in $S'_m$ is defined by the equations

$$\rho y_i = \varphi_i(x_0, x_1, \cdots, x_n) \qquad (i = 0, 1, \cdots, m) \tag{4}$$

where the $\varphi_i$'s are forms of the same degree in $x_0, x_1, \cdots, x_n$ and $\rho$ is a factor of proportionality. If $P$ describes an algebraic locus, say $V$ a variety of $r$ dimensions, then $P'$ also traces out a locus $V'$ in $S'_m$. It may happen that a general point $P'$ on $V'$ arises from only one point $P$ on $V$. In this case there is a one-one algebraic relationship between $P$ and $P'$ and another set of equations exists giving the conditions of $P$ in terms of those of $P'$,

$$\tau x_j = \psi_j(y_0, y_1, \cdots, y_m) \qquad (j = 0, 1, \cdots, n) \tag{5}$$

where the $\psi_j$'s are forms of the same degree in $y_0, y_1, \cdots y_m$ and $\tau$ is a factor of proportionality. The varieties $V$ and $V'$ are in birational correspondence and are both of the same dimension. Although the correspondence between general points of $V$ and $V'$ is one-one, there may well be special points at which the correspondence is not one-one.

In general the equations (4) and (5) do not define a correspondence which is birational in character for the whole of the two spaces $S_n$ and $S'_m$. If, however, this is the case, $n = m$ and we have a Cremona transformation.

Consider now an irreducible variety $V_r$ in $S_n$ not lying entirely in the prime $x_0 = 0$. If $x_0 = 0$ is chosen as the prime at infinity and if $x_i$ replaces $x_i/x_0$, $(x_1, \cdots, x_n)$ may be taken as the non-homogeneous coordinates of those points of $S_n$ which do not lie in this prime. Relative to this coordinate system $V_r$ has a generic point with non-homogeneous coordinates

$(\xi_1, \cdots, \xi_n)$. Similarly choosing non-homogeneous coordinates $(y_1, \cdots, y_m)$ in $S'_m$ by writing $y_i$ for $y_i/y_0$ $(i = 1, \cdots, m)$ assuming that $V'$, birationally related to $V$ by equations (4) and (5), does not lie entirely in the prime $y_0 = 0$, $V'$ has a generic point $(\eta_1, \cdots, \eta_m)$. Define $\eta_i^*$ $(i = 1, \cdots, m)$ by the equations

$$\eta_i^* = \frac{\varphi_i(1, \xi_1, \cdots, \xi_n)}{\varphi_0(1, \xi_1, \cdots, \xi_n)} \qquad (i = 1, \cdots, m). \tag{6}$$

$(\eta_1^*, \cdots, \eta_m^*)$ is a generic point of $V'$ and the $\xi_j$ $(j = 1, \cdots, m)$ can be expressed in terms of these coordinates by using equations (5)

$$\xi_j = \frac{\psi_j(1, \eta_1^*, \cdots, \eta_m^*)}{\psi_0(1, \eta_1^*, \cdots, \eta_m^*)} \qquad (j = 1, \cdots, n). \tag{7}$$

Obviously the function fields $K(\xi_1, \cdots, \xi_m)$ and $K(\eta_1^*, \cdots, \eta_m^*)$ coincide and since $K(\eta_1^*, \cdots, \eta_m^*)$ and $K(\eta_1, \cdots, \eta_m)$ are isomorphic we obtain the relation

$$K(\xi_, \cdots, \xi_n) \cong K(\eta_1, \cdots, \eta m)$$

so that the function fields of $V$ and $V'$ are isomorphic. This is in fact the characteristic property of two birationally related varieties defined over the same ground field.

In the original treatment of a birational correspondence as developed by the Italian school of geometry, the points on $V'$ corresponding to a special point $P$ on $V$ at which the one-one nature of the correspondence breaks down, are determined by limiting processes involving intuitive ideas of continuity. A point $P$ might possibly correspond to a finite set of points as, for example, the double point $P$ of a rational plane cubic obtained by projecting a twisted cubic from a general point in space. In other cases $P$ might correspond to an algebraic subvariety of dimension greater than zero. This case occurs in the stereographic projection of a quadric into a plane when the centre of projection, $P$, is transformed into a line $p'$ in a plane. A birational correspondence is thought of as a point-point correspondence, and, in this last example, every point $P'$ of $p'$ corresponds to the same point $P$ of the quadric.

§2.8 Valuation Theory

One of the aims of modern algebraic geometry is to eliminate processes which depend ultimately on intuitive reasoning. With this objective in view, valuation theory has been introduced to define the birational correspondence for points at which the one-one character breaks down. The correspondence is given directly between subvarieties of birationally equivalent varieties and these subvarieties are no longer thought of as being the aggregates of corresponding points.

A valuation of the field $\Sigma$ is a homomorphic mapping $v$ if the multiplicative group consisting of the non-zero elements of $\Sigma$ onto an ordered additive Abelian group $\Gamma$ such that the following axioms are satisfied.

(i) $v(w_1 w_2) = v(w_1) + v(w_2)$

(ii) $v(w_1 \pm w_2) \geq \min\{v(w_1), v(w_2)\}$

Conventionally the third condition is included

(iii) $v(0) = +\infty$

Here $v(w)$ denotes the map of the element $w$ of $\Sigma$ in $\Gamma$. When $\Sigma$ is a function field, which is the important case in geometry, a fourth axiom is added, namely

(iv) $v(a) = 0$ for all elements $a \neq 0$ of the ground field $K$.

Those element $w$ of $\Sigma$ for which $v(w) \geq 0$ form a ring $R$ in $\Sigma$ called the valuation ring of $v$. The elements $w$ such that $v(w) > 0$ form a prime ideal $\mathfrak{p}$ in $R$. $\mathfrak{p}$ is called the prime ideal of the valuation $v$.

Let $V$ be an algebraic variety in $S_n$ with a generic point $(\xi_0^*, \xi_1^*, \cdots, \xi_m^*)$ and function field $\Sigma$. We may multiply these quantities $\xi_i^*$ $(i = 0, 1, \cdots, n)$ by any non-zero factor belonging to any extension of $\Sigma$ and in this way we can arrange for the $\xi_i^*$ to belong to $\Sigma$. For a given valuation $v$ of $\Sigma$ consider the set of values $v(\xi_i^*)$, $(i = 0, 1, \cdots, n)$ and suppose that $v(\xi_0^*)$ is the least, or one of the least. Then we may normalise the generic point so that its coordinates are $(1, \xi_1, \cdots, \xi_n)$ where $\xi_i = \xi_i^*/\xi_0^*$, $(i = 1, \cdots, n)$, and then $v(\xi_i) \geq 0$ for $i = 1, \cdots, n$. This preliminary step ensures that $\sigma = K[\xi_1, \cdots, \xi_n]$, the integral domain of $V$, is contained in the valuation ring of $v$.

Let $\mathfrak{p}$ be the prime ideal of the valuation $v$. Assuming that $V$ does not lie entirely in the prime $x_i =$ const, $v(\xi_i) \neq 0$. Those polynomials in $\xi_1, \cdots, \xi_n$ which have strictly positive values for a prime ideal $\overline{\mathfrak{p}}$ in $\sigma$ where $\overline{\mathfrak{p}} = \sigma \cap \mathfrak{p}$ is the contraction of $\mathfrak{p}$ in $\sigma$.

This ideal $\overline{\mathfrak{p}}$ in $\sigma$ defines an irreducible subvariety $W$ of $V$ which is called the centre of $v$ in $V$. It is evident that $\overline{\mathfrak{p}} \neq \sigma$ and therefore $W$ has dimension zero at least.

The birational correspondence between two algebraic varieties $V$ and $V'$ with a common function field $\Sigma$ is then defined as follows.

Two subvarieties $W$, $W'$ of $V$, $V'$ respectively correspond to each other if there exists a valuation of $\Sigma$ whose centre on $V$ is $W$ and on $V'$ is $W'$.

Example.

The most obvious example concerns a valuation of the function field of an irreducible plane curve given in a plane where $(x, y)$ are non-homogeneous coordinates by an equation $f(x, y) = 0$. Suppose there is a branch of this curve passing through the origin of coordinates and that in the neighbourhood of the origin the points of this branch have coordinates given as convergent power series in some parameter $t$

$$\begin{aligned} x &= t^{\sigma}(a_0 + a_1 t + \cdots) \\ y &= t^{\rho}(b_0 + b_1 t + \cdots) \end{aligned} \quad \text{where } a_0 \neq 0,\ b_0 \neq 0.$$

If $w$ belongs to the function field of this curve then $w$ is a rational function $R(x, y)$ of the coordinates $x$, $y$ and

$$w = R(x, y) = t^{\tau}(c_0 + c_1 t + \cdots) \qquad \text{where } c_0 \neq 0.$$

If we define $v(w) = \tau$, then we shall find that $v$ is in fact a mapping which satisfies the valuation axioms stated above. The centre of this valuation is the origin. In fact every valuation of this function field arises from a branch of the curve and the corresponding value $v(w)$ is the order of $w$ at the branch. $\Gamma$, the valuation group, in this case is the set of integers.

This definition of a birational correspondence by means of valuation theory is that on which Zariski builds his theory. "As one advances into the general theory of algebraic varieties one . . . reaches the conclusion that there does not exist a general theory of birational correpondences" he writes at the beginning of one paper [26]. Shortly after this in a paper dealing with the fundamental theory of birational correspondences he develops his theory for algebraic varieties defined over arbitrary ground fields [27]. Characteristic of his work in this connection is his use of the normal varieties and a good many of his results for birational correspondences hold only for these normal varieties.

In view of their importance in his connection I shall now discuss the idea of normal and locally normal varieties.

# Chapter 3.

## Normal varieties and their application in algebraic geometry

§3.1.Definitions and results relating affine and local normality

Suppose $V_r$ is an irreducible $r$-dimensional algebraic variety in an affine space of $n$ dimensions in which $x_1, \cdots, x_n$ are non-homogeneous coordinates. Let $(\xi_1, \cdots, \xi_n)$ be non-homogeneous coordinates of a generic point of $V_r$. Using a sufficiently general coordinate system we may assume that the first $r$ of these coordinates $\xi_1, \cdots \xi_r$ are algebraically independent and that $\xi_{r+1}, \cdots, \xi_n$ are integrally dependent on the ring $K[\xi_1, \cdots, \xi_r]$.

Definition 1 $V_r$ is said to be normal in the affine sense if $K[\xi_1, \cdots \xi_r]$ is integrally closed in $K(\xi_1, \cdots \xi_r)$.

Now suppose that $V_r$ is defined in a projective space of $n$ dimensions and consider an irreducible variety $W$ of $V$. Choose a non-homogeneous coordinate system with respect to which $W$ lies at a finite distance and form the quotient ring, $Q(W)$, of $W$ on $V$. This is a ring consisting of the elements of $K(\xi_1, \cdots, \xi_n)$ which can be written in the form $\alpha/\beta$ where $\alpha$ and $\beta$ belong to $K[\xi_1, \cdots, \xi_n]$ and $\beta$ does not vanish over $W$.

Definition 2 $V_r$ is locally normal at $W$ if the quotient ring $Q(W)$ is integrally closed in $K(\xi_1, \cdots, \xi_n)$.

Definition 3 $V_r$ is locally normal if it is locally normal at every irreducible subvariety $W$.

In regard to the last definition we can actually assert the following result.

**Theorem 1** *A necessary and sufficient condition for $V$ to be locally normal is that $V$ is locally normal at every point.*

The necessity of this condition is a consequence of the definition. The sufficiency can be deduced from the following two lemmas. Let the non-homogeneous coordinates of a generic point of $V$ be $(\xi_1, \cdots, \xi_n)$ and consider the local normality of $V$ at a point $P$ at a finite distance. The integral domain of $V$ is $\sigma = K[\xi_1, \cdots \xi_n]$ and $\sigma^*$ denotes the integral closure of $\sigma$ in $K(\xi_1, \cdots, \xi_n)$.

**Lemma 1** *A necessary and sufficient condition for $V$ to be locally normal at $P$ is that $\sigma^* \subseteq Q(P)$, where $Q(P)$ is the quotient ring of $P$ on $V$.*

Suppose that $V$ is locally normal at $P$ and let $\zeta$ belong to $\sigma^*$. $\zeta$ satisfies a relation

$$\zeta^m + \alpha_1 \zeta^{m-1} + \cdots + \alpha_{m-1}\zeta + \alpha_m = 0 \quad (\alpha_i \text{ is is } \sigma, \quad i = 1, \cdots, m). \tag{1}$$

Since the $\alpha_i$ belong also to $Q(P)$, this equation further asserts that $\zeta$ is integrally dependent on $Q(P)$, and therefore, by our hypothesis, $\zeta$ is in $Q(P)$. Hence $\sigma^* \subseteq Q(P)$.

On the other hand suppose $\sigma^* \subseteq Q(P)$ and let $\zeta$ be an element of $K(\xi_1, \cdots, \xi_n)$ which is integrally dependent on $Q(P)$. $\zeta$ satisfies a relation

$$\beta_0\zeta^m + \beta_1\zeta^{m-1} + \cdots + \beta_{m-1}\zeta + \beta_m = 0 \quad \begin{array}{l}(\beta_i \text{ is in } \sigma,\ i = 0, 1, \cdots, m, \\ \beta_0 \neq 0 \text{ at } P).\end{array} \tag{2}$$

This equation may be multiplied by $\beta_0^{m-1}$ and the following relation is then obtained.

$$(\beta_0\zeta)^m + \beta_1(\beta_0\zeta)^{m-1} + \cdots + \beta_{m-1}\beta_0^{m-2}(\beta_0\zeta) + \beta_m\beta_0^{m-1} = 0. \tag{3}$$

$(\beta_0\zeta)$ therefore belongs to $\sigma^*$ and since by hypothesis $\sigma^* \subseteq Q(P)$, $\beta_0\zeta$ is in $Q(P)$. Therefore

$$\beta_0\zeta = \frac{a}{b} \quad (a \text{ and } b \text{ are in } \sigma, \quad b \neq 0 \text{ at } P).$$

Hence

$$\zeta = \frac{a}{\beta_0 b}.$$

$\beta_0 b$ is an element of $\sigma$ and $\beta_0 b \neq 0$ at $P$ since neither $\beta_0$ nor $b$ vanishes at $P$. Consequently $\zeta$ is an element of $Q(P)$ and this implies that $V$ is locally normal at $P$.

The second lemma expresses a generalisation of this result and can be proved in the same way.

**Lemma 2** *A necessary and sufficient condition for $V$ to be locally normal at an irreducible subvariety $W$ is that $\sigma^* \subseteq Q(W)$ where $Q(W)$ is the quotient ring of $W$ on $V$.*

The sufficiency of the condition in Theorem 1 can now be proved simply. Consider any irreducible subvariety $W$ of $V$ and $P$, a point on $W$. We assume that $V$ is locally normal at $P$ and therefore, using Lemma 1, $\sigma^* \subseteq Q(P)$. Since $P$ is a point on $W$ it follows that $Q(P) \subseteq Q(W)$. Consequently $\sigma^* \subseteq Q(W)$ and the result of Theorem 1 follows on account of Lemma 2.

Two theorems demonstrate the relationship between affine and local normality.

**Theorem 2** *If a variety $V$ in $A_n$ is locally normal at every finite point then $V$ is affinely normal and conversely ([22] p 294 Theorem 16).*

**Theorem 3** *If a variety $V$ in $S_n$ is locally normal then it is affinely normal for every choice of the prime at infinity. Conversely, if $V$ is affinely normal for every choice of the prime at infinity then $V$ is locally normal.*

Theorem 3 is obviously a consequence of Theorem 2 and the converse of Theorem 3 can evidently be modified in the following way. If $V$ is affinely normal when each of $n+1$ linearly independent primes is chosen as the prime at infinity, then $V$ is locally normal.

The converse of Theorem 2 is immediate. Let $P$ be a finite point of $V$ and assume that $V$ is affinely normal. Continuing with the notation already used this means $\sigma^* = \sigma$. Obviously, $\sigma \subseteq Q(P)$ and therefore $\sigma^* \subseteq Q(P)$. On account of Lemma 1 $V$ is locally normal at $P$.

In order to define those points at which a given variety is not locally normal, Zariski introduces the idea of the conductor of a ring $\sigma$ with respect to a larger ring $\sigma^*$ containing $\sigma$ [22]. This conductor is defined as the largest ideal in $\sigma$ which is also an ideal in $\sigma^*$ and is denoted by $\mathfrak{c} = \mathfrak{c}(\sigma, \sigma^*)$. Obviously $\mathfrak{c}$ is the set of all elements $\zeta$ in $\sigma$ such that $\zeta\sigma^* \subseteq \sigma$. Therefore, $\sigma^* \subseteq \sigma$.

When $V$ is affinely normal, $\sigma = \sigma^*$ and $\mathfrak{c} = \sigma$. Otherwise $(0) \subset \mathfrak{c} \subset \sigma$, where $(0) \neq \mathfrak{c} \neq \sigma$, and then the ideal $\mathfrak{c}$ in $\sigma$ defines a proper algebraic subvariety $C$ of $V$. This subvariety consists of just those points at which $V$ is not locally normal and which are at a finite distance with respect to the non-homogeneous coordinates chosen.

As an example consider the plane cubic curve with non-homogeneous generic point $(\theta^2, \theta^3)$, where $\theta$ is an indeterminate,

$$\sigma = K[\theta^2, \theta^3], \quad \sigma^* = K[\theta[] \text{ and } \mathfrak{c}(\sigma, \sigma^*) = \sigma \cdot (\theta^2, \theta^3) = \mathfrak{p}_0.$$

$\mathfrak{c}$ is a zero-dimensional ideal which is a prime ideal $\mathfrak{p}_0$ in $\sigma$ defining the cusp of this cubic curve. Regarded as an ideal in $\sigma^*$, $\mathfrak{c}$ is a primary ideal. We have in fact

$$\sigma^*\mathfrak{p}_0 = \sigma(\theta^2) = \mathfrak{p}_0^{*2}$$

where $\mathfrak{p}_0^* = \sigma^*(\theta)$ and $\mathfrak{p}_0^*$ is a prime ideal in $\sigma^*$. $\mathfrak{p}_0^{*2}$ is a primary ideal in $\sigma^*$.

Using this idea of the conductor, Theorem 2 can be proved quite simply. Still using the same notation and assuming that $V$ is locally normal at every finite point it is required to show that $\sigma = \sigma^*$. Suppose this is not the case and that $\sigma$ is properly contained in $\sigma^*$. We have

$$(0) \subset \mathfrak{c} \subset \sigma \text{ where } (0) \neq \mathfrak{c} \text{ and } \mathfrak{c} \neq \sigma.$$

For if $\sigma = \mathfrak{c}$ then, since $\mathfrak{c}\sigma^* \subseteq \sigma$ it would follow that $\sigma^* \subseteq \sigma$ and therefore $\sigma^* = \sigma$ contrary to our hypothesis. Therefore $\mathfrak{c}$ defines a proper subvariety of $V$ at a finite distance and hence there is at least one finite point at which $V$ is not locally normal in contradiction to the original hypothesis that $V$ is locally normal at every finite point. It follows that $\sigma = \sigma^*$.

§3.2 Homogeneous generic point and projective normality

Let $V_r$ now be defined in the projective space $S_n$ in which $x_0, x_1, \cdots x_n$ are homogeneous coordinates. Then we may use the quantities $x_i' = x_i/x_0$ $(i = 1, \cdots, n)$ as the non-homogeneous coordinates of a point in the affine space $A_n$ which consists of those points in $S_n$ which do not lie in the prime $x_0 = 0$. The points of $V_r$ not in this prime form a variety in $A_n$ which has a generic point with non-homogeneous coordinates $(\xi_1, \cdots, \xi_n)$. Homogeneous coordinates of this point are $(1, \xi_1, \cdots, \xi_n)$ or $(\lambda, \lambda\xi_1, \cdots, \lambda\xi_n)$ where $\lambda \neq 0$. If $\lambda$ is an indeterminate over $K(\xi_1, \cdots, \xi n)$ the coordinates $(\lambda, \lambda\xi_1, \cdots, \lambda\xi_n)$ are the coordinates of a "homogeneous generic point" of $V_r$ in $S_n$. Zariski ([22], p284) defines this homogeneous generic point by introducing a cone $W_{r+1}$ of dimension $r + 1$ where the equations of this cone are the same as those in homogeneous coordiates for $V_r$, but interpreted as if $(x_0, x_1, \cdots, x_n)$ are non-homogeneous coordinates in an affine space $A_{n+1}$ of dimension $n+1$. This cone is therefore obtained by joining $V_r$ to an external point. Consequently if $(\eta_0, \eta_1, \cdots, \eta_n)$ are the coordinates of a homogeneous generic point of $V_r$ the same coordinates are non-homogeneous coordinates of a generic point of $W_{r+1}$ in $A_{n+1}$. It follows that $(\eta_0, \eta_1, \cdots, \eta_n)$ are the coordinates of a homogeneous generic point of a variety $V_r$ in $S_n$ if and only if

(i) $(\eta_0, \eta_1, \cdots, \eta_r)$ lies in $V_r$

(ii) $K(\eta_0, \eta_1, \cdots, \eta_n)$ is of degree of transcendency $r + 1$ over $K$.

We can now define the concept of projective normality. If $(\eta_0, \eta_1, \cdots \eta_n)$ are coordinates of a homogeneous generic point of $V_r$, then $V_r$ is defined to be projectively normal, or, more briefly, normal, if $K[\eta_0, \eta_1, \cdots, \eta_n)$ is integrally closed in $K(\eta_0, \eta_1, \cdots, \eta_n)$.

The properties of a variety being normal in the affine sense, locally normal and projectively normal are increasingly more restrictive. A normal variety is necessarily locally normal and a locally normal variety is necessarily affinely normal for every choice of the prime at infinity. It is to be noted that a variety is normal if and only if the cone projecting it from an external point is affinely normal for a coordinate system for which the vertex of the cone is a finite point. This condition in fact implies that the cone is locally normal. A locally normal variety $V$ is normal if it can be shown that the cone projecting the variety from a point 0 external to the ambient space, is locally normal at its vertex 0. If $(\eta_0, \eta_1, \cdots, \eta_n)$ are coordinates of a homogeneous generic point of $V$ this means, in the terminology of ideal theory, the variety $V$ is locally normal if the conductor of the ring $\sigma' = K[\eta_0, \eta_1, \cdots, \eta_n]$ with respect to the integral closure of this ring in its quotient field is an irrelevant ideal, that is, if it is a primary ideal with the ideal $\sigma' \cdot (\eta_0, \eta_1, \cdots, \eta_n)$ as its associated prime ideal.

§3.3 Regular birational correspondence

Consider a birational correspondence between two varieties $V$ in $S_n$ and $V'$ in $S'_m$. Let the homogeneous coordinates in $S_n$, $S'_m$ be $(x_0, x_1, \cdots, x_n)$ and $(y_0, y_1, \cdots, y_m)$ respectively. Suppose the equations defining the correspondences are

$$x_0 : x_1 : \cdots : x_n = \varphi_0(y_0, y_1, \cdots y_m) : \varphi_1(y_0, y_1, \cdots, y_m) : \cdots \varphi_n(y_0, y_1, \cdots, y_m)$$

$$y_0 : \cdots : y_m = \psi_0(x_0, x_1, \cdots, x_n) : \psi_1(x_0, x_1, \cdots, x_n) : \cdots : \psi_m(x_0, x_1, \cdots, x_m)$$

where the $\varphi$'s are forms of the same degree, $k$, in $y_0, y_1, \cdots y_m$ and the $\psi$'s are forms of the same degree, $\ell$, in $x_0, x_1, \cdots, x_n$.

I want to consider two particular points $P$ of $V$ and $P'$ of $V'$ such that the $\psi$'s do not vanish at $P$ and the $\psi$'s do not vanish at $P'$. In this case it can be shown that the quotient ring of $P$ on $V$ is the same as the quotient ring of $P'$ on $V'$. I shall denote these two quotient rings by $Q_V(P)$ and $Q_{V'}(P')$ respectively.

We may assume preliminary non-singular linear transformations in both spaces so that $P$ is the point with coordinates $x_0 = 1, x_1 = \cdots = x_n = 0$ and $P'$ has coordinates $y_0 = 1$, $y_1 = \cdots = y_m = 0$. It is then permissible to choose $P$ as the origin of non-homogeneous coordinates $x'_1 = x_1/x_0, \cdots, x'_n = x_n/x_0$ on $S_n$ and $P'$ as the origin of non-homogeneous coordinates $y'_0 = y_1/y_0, \cdots, y'_m = y_m/y_0$ in $S'_m$. Then if $(\xi_1, \cdots, \xi_n)$ are non-homogeneous coordinates of a generic point of $V$, $(\eta_1, \cdots, \eta_m)$ are non-homogeneous coordinates of a generic point of $V'$ where

$$\xi_i = \frac{\varphi_i(1, \eta_1, \cdots, \eta_m)}{\varphi_0(1, \eta_1, \cdots, \eta_m)} \qquad (i = 1, \cdots, n)$$

$$\eta_j = \frac{\psi_j(1, \xi_1, \cdots, \xi_m)}{\psi_0(1, \xi_1, \cdots, \xi_m)} \qquad (j = 1, \cdots, m).$$

By hypothesis

$$\psi_0 \neq 0 \text{ and } \psi_1 = \cdots = \psi_m = 0 \text{ at } P,$$

$$\varphi_0 \neq 0 \text{ and } \varphi_1 = \cdots = \varphi_n = 0 \text{ at } P'.$$

Suppose that $\omega$ is an element of the common function field, $K(\xi_1, \cdots \xi_n) = K(\eta_1, \cdots, \eta_m)$, of $V$ and $V'$ which belongs to $Q_V(P)$. Then there is a relation

$$\alpha_0\omega + \alpha_1 = 0 \tag{4}$$

where $\alpha_0$, $\alpha_1$ belong to $K[\xi_1, \cdots, \xi_n]$ and $\alpha_0 \neq 0$ at $P$. $\alpha_0 = \alpha_0(\xi_1, \cdots, \xi_n)$ and $\alpha_1 = \alpha_1(\xi_1, \cdots, \xi_n)$ can be written in the following forms

$$\begin{aligned} \alpha_0(\xi_1, \cdots, \xi_n) &= \alpha_0\left(\frac{\varphi_1(1, \eta_1, \cdots, \eta_m)}{\varphi_0(1, \eta_1, \cdots, \eta_m)}, \cdots \frac{\varphi_n(1, \eta_1, \cdots, \eta_m)}{\varphi_0(1, \eta_1, \cdots, \eta_m)}\right) \\ &= \frac{\beta_0(\eta_1, \cdots, \eta_m)}{(\varphi_0(1, \eta_1, \cdots, \eta_m))^{\rho_0}} \end{aligned} \tag{5}$$

where $\beta_0(\eta_1, \cdots, \eta_m)$, $\varphi_0(1, \eta_1, \cdots, \eta_m)$ belong to $K[\eta_1, \cdots, \eta_m]$ and $\rho_0$ is a positive integer,

$$\alpha_1(\xi_1, \cdots, \xi_n) = \frac{\beta_1(\eta_1, \cdots, \eta_m)}{(\varphi_0(1, \eta_1, \cdots, \eta_m))^{\rho_1}} \tag{6}$$

where $\beta_1(\eta_1, \cdots, \eta_m)$ and $\rho_1$ is a positive integer. Multiplying (4) by $(\varphi_0(1, \eta_1, \cdots, \eta_m))^{\rho}$, where $\rho = \max(\rho_0, \rho_1)$, we find that $\omega$ satisfies an equation

$$\beta_0(\eta_1, \cdots, \eta_m)(\varphi_0(1, \eta_1, \cdots, \eta_m))^{\rho-\rho_O}\omega + \beta_1(\eta_1, \cdots, \eta_m)(\varphi_0(1, \eta_1, \cdots, \eta_m))^{\rho-\rho_1} = 0 \tag{7}$$

In this equation the quantities $\beta_0\varphi^{\rho-\rho_0}$ and $\beta_1\varphi^{\rho-\rho_1}$ belong to $K[\eta_1, \cdots, \eta_m]$ and $\beta_0\varphi_0^{\rho-\rho_0} \neq 0$ at $P'$, since $\varphi_0 \neq 0$ at $P'$ and also $\beta_0 \neq 0$ at $P'$. This last remark follow from the following consideration. When $(\eta_1, \cdots, \eta_m)$ specialise to the values $(0, \cdots, 0)$, the corresponding values of $(\xi_1, \ldots, \xi_n)$ specialise to the values $(0, \cdots, 0)$.

$$(\varphi_0(1, \eta_1, \cdots, \eta_m))^{\rho_0} \alpha_0(\xi_1, \cdots, \xi n) = \beta_0(\eta_1, \cdots, \eta_m)$$

and therefore

$$(\varphi_0(1, 0, \cdots, 0))^{\rho_0} \alpha_0(0, \cdots, 0) = \beta_0(0, \cdots, 0).$$

$\varphi_0(1, 0, \cdots, 0) \neq 0$ and therefore $\beta_0(0, \cdots, 0) = 0$ implies $\alpha_0(0, \cdots, 0) = 0$ contrary to our assumption that $\alpha_0 \neq 0$ at $P$. Therefore $\beta_0 \neq 0$ at $P'$. Equation (7) then shows that $\omega$ belongs to the quotient ring $Q_{V'}(P')$. We deduce that $Q_V(P) \subseteq Q_{V'}(P')$. Similarly it can be proved that $Q_{V'}(P') \subseteq Q_V(P)$. From these two relations we deduce that $Q_V(P) = Q_{V'}(P')$.

**Corollary 1** *If $V$ is locally normal at $P$ then $V'$ is locally normal at $P'$ and conversely.*

**Corollary 2** *If the $\psi$'s do not vanish simultaneously at any point of $V$ and the $\varphi$'s do not vanish siuylataneously at any point of $V'$ then if $V$ is locally normal, $V'$ also is locally normal and conversely.*

This result follows directly since if $V$ is locally normal it is locally normal at every point and conversely.

This birational correspondence is an example of a correspondence which Zariski calls a regular correspondence ([27] p513). It is, namely, a correspondence in which quotient rings are preserved. If $W$ is any irreducible subvariety on $V$ and $W'$ is the corresponding subvariety on $V'$, then the quotient ring of $W$ on $V$ is the same as the quotient ring of $W'$ on $V'$, that is, extending the notation already used in an obvious fashion

$$Q_V(W) = Q_{V'}(W').$$

If for two corresponding irreducible subvarieties $W$, $W'$ on$V$, $V'$ respectively we have

$$Q_V(W) = Q_{V'}(W'),$$

Zariski defines the correspondence to be regular at $W$ and $W'$.

Zariski has shown that a birational correspondence which is regular in his sense, is one-one without exception. The converse is not true as is evident if we consider the the correspondence between the twisted cubic curve $\gamma$ defined in $S_3$ by the parametric equations

$$x_0 : x_1 : x_2 : x_3 = 1 : \theta^2 : \theta^3 : \theta$$

and its projection $\gamma'$ from the point $(0, 0, 0, 1)$onto the plane $x_3 = 0$ which has equations

$$x_0 : x_1 : x_2 = 1 : \theta^2 : \theta^3,$$

$$x_3 = 0.$$

The point $O = (1, 0, 0, 0)$ on $\gamma$ corresponds to the point $O' = (1, 0, 0, 0)$ on $\gamma'$. The element $\theta$ in $K(\theta)$ is an element of the quotient ring of $O$ on $\gamma$ but $\theta$ does not belong to the quotient ring of $O'$ on $\gamma'$. Consequently, although the correspondence between $\gamma$ and $\gamma'$ is one-one without exception, it is not regular at 0 and therefore is not a regular correspondence.

§3.4 Birational correspondences and derived normal varieties

One of the reasons for the importance of locally normal varieties lies in the use made of such varieties by Zariski in his work concerning birational correspondences.

Van der Waerden in his paper, "Algebraische Korrespondenzen und rationale Abbildungen" [16], proves two theorems about a birational correspondence between a non-singluar $r$-dimensional algebraic variety $V$ and another variety $V'$. They are as follows.

**Theorem 1** *If $W$ is an irreducible $s$-dimensional variety contained in the fundamental locus of $V$, then $W$ corresponds to an algebraic subvariety of $V'$ whose irreducible components are all of dimension greater than $s$.*

**Theorem 2** *The fundamental locus on $V'$ corresponds to a pure $(r-1)$-dimensional subvariety of $V'$.*

In a paper [26] in which he discusses normal varieties in relation to birational correspondences, Zariski gives examples showing that both theorems are untrue if $V$ is allowed to have singular points. He suggests in this paper that for a general theory of birational correspondence we should consider only varieties which are locally irreducible. In fact he finds that the varieties to which he wishes to confine his attention are those satisfying the more restrictive condition

- If a point $P$ on $V$ corresponds to a unique point $P'$ on $V'$, then the birational correspondence, regarded as an analytical transformation, is regular at $P$.

This restricted class of varieties is the class of locally normal varieties. The first theorem stated above is true for locally normal varieties but the second holds only when $V$ is non-singular.

In a later paper Zariski examines the properties of a birational correspondence between two locally normal varieties [27]. In the same paper he lays the foundations for a general theory of birational correspondence defined in terms of valuation theory. In an earlier paper Zariski establishes the existence of locally normal varieties birationally equivalent to any variety $V$

([22] §20 p 290). These he calls derived normal varieties of $V$. They are obtained by using the algebraic operation of integral closure. A derived normal variety of a given variety $V$ is defined uniquely to within regular birational transformations. Consequently when the properties of a birational correspondence between a variety $V$ and one of its dervied normal varieties are known, the properties of a general birational correspondence between any two varieties $V$ and $V'$ can be examined by splitting up the correspondence into three stages: the correspondence between $V$ and a derived normal variety $V^*$, that between $V^*$ and $V'^*$, a derived normal variety of $V'$, and that between $V'^*$ and $V'$.

I should like to describe briefly the method by which a derived normal variety is obtained.

Let $(\xi_1, \cdots, \xi_n)$ be non-homogeneous coordinates of a generic point $V$ in $S_n$. Let $\sigma$ denote the ring $K[\xi_1, \cdots, \xi_n]$ and $\Sigma$ the function field $K(\xi_1, \cdots, \xi_n)$ of $V$. If $\eta_0$ is transcendental over $\Sigma$ we may regard $(\eta_0, \eta_1 = \eta_0\xi_1, \cdots, \eta_n = \eta_0\xi_n)$ as a homogeneous generic point of $V$. Let $\sigma'$ denote $K[\eta_0, \eta_1, \cdots, \eta_n]$ and $\Sigma'$ the quotient field of $\sigma'$ so that

$$\Sigma' = K(\eta_0, \eta_1, \cdots, \eta_n) = \Sigma(\eta_0).$$

Suppose $\sigma'^*$ is the integral closure of $\sigma'$ in $\Sigma'$. The transformation $\eta_i \mapsto t\eta_i$, $t$ in $K$, defines an automorphism $\overline{\omega}$ of $\sigma'^*$. we say that $\zeta$, an element of $\sigma'^*$, is homogeneous of degree $\rho$ if $\overline{\omega}\zeta = t^\rho\zeta$. Now the homogeneous elements in $\sigma'^*$ of a given degree $\rho$ possess a finite linearly independent base over $K$. Let $\eta_0^*, \cdots, \eta_m^*$ be such a base. It can be proved that if $\rho$ is sufficiently high, then the integral domain $K[\eta_0^*, \eta_1^*, \cdots, \eta_m^*]$ is integrally closed in its quotient field. The quotients $\eta_i^*/\eta_0^*$ $(i = 1, \cdots, m)$ belong to $\Sigma$. Therefore the quantities $(\eta_0^*, \eta_1^*, \cdots, \eta_m^*)$ can be regarded as the coordinates of a homogeneous generic point of a normal variety $V^*$ which is birationally equivalent to $V$. $V^*$ is called the derived normal variety for $V$ relative to the character of homogeneity $\rho$.

Without loss of generality we may always assume that this character of homogeneity is unity. For consider the birational correspondence between $V$, with a homogeneous generic point $(\eta_0, \eta_1, \cdots, \eta_n)$ and $V^*$, a derived normal variety of $V$ with a homogeneous generic point $(\eta_0^*, \eta_1^*, \cdots, \eta_m^*)$ relative to the character of homogeneity $\rho$. The quantities $\eta_0^*, \eta_1^*, \cdots, \eta_m^*$ form a linear base for the elements of $K(\eta_0, \eta_1, \cdots, \eta_n)$ which are integrally dependent on $K[\eta_0, \eta_1, \cdots, \eta_n]$ and are homogeneous of degree $\rho$. $K[\eta_0^*, \eta_1^*, \cdots, \eta_m^*]$ is integrally closed in $K(\eta_0^*, \eta_1^*, \cdots, \eta_m^*)$. Now consider the variety $\overline{V}$ whose homogeneous generic point is $(\overline{\eta}_0, \overline{\eta}_1, \cdots \overline{\eta}_s)$ where $\overline{\eta}_0, \overline{\eta}_1, \cdots, \overline{\eta}_s$ is a minimal basis for the forms of degree $\rho$ in $\eta_0, \eta_1, \cdots, \eta_n$. It can be shown that the birational correspondence between $V$ and $\overline{V}$ is one-one without exception and that any two corresponding irreducible subvarieties of $V$ and $\overline{V}$ have the same dimension and quotient ring. It is therefore allowable to replace $V$ by $\overline{V}$ for purposes of birational correspondence since they are related by a regular birational transformation.

Consider now the correspondence between $V^*$ and $\overline{V}$. Every homogeneous element in $K(\eta_0, \eta_1, \cdots, \eta_n)$ which is of degree $\rho$, and consequently every $\eta_i^*$ $(i = 0, 1, \cdots, m)$ can be written as a quotient of two forms in the quantities $\eta_i$ $(i = 0, 1, \cdots, n)$ whose degrees are multiples of $\rho$. Any such element is a quotient of two forms in $\overline{\eta}_i$ $(i = 0, 1, \cdots s)$ and therefore $\eta_i^*$ belongs to the field $K(\overline{\eta}_0, \overline{\eta}_1, \cdots, \overline{\eta}_s)$ $(i = 0, 1, \cdots, m)$. On the other hand, $\overline{\eta}_i$ is linear in $\eta_0^*, \eta_1^*, \cdots, \eta_m^*$ $(i - 0, 1, \cdots, s)$ and hence we deduce $K(\overline{\eta}_0, \overline{\eta}_1, \cdots, \overline{\eta}_s) = K(\eta_0^*, \eta_1^*, \cdots, \eta_m^*)$. The elements $\overline{\eta}_i$, $(i = 0, 1, \cdots, s)$, which, considered as elements of the field $K(\eta_0, \eta_1, \cdots, \eta_n)$ are homogeneous of degree $\rho$, when considered as elements of the field $K(\overline{\eta}_0, \overline{\eta}_1, \cdots, \overline{\eta}_s)$ are homogeneous of degree 1. A similar argument holds for $\eta_i^*$ $(i = 0, 1, \cdots, m)$. The elements $\eta_0^*, \eta_1^*, \cdots, \eta_m^*$ constitute a linear base for those elements of the field $K(\overline{\eta}_0, \overline{\eta}_1, \cdots, \overline{\eta}_s)$ which are homogeneous of degree one and which are integrally dependent on $K[\overline{\eta}_0, \overline{\eta}_1, \cdots, \overline{\eta}_s]$. Therefore $V^*$ is a derived normal variety of $\overline{V}$ relative to the character of homoegneity 1. In fact, since the $\overline{\eta}_i$ $(i = 0, 1, \cdots, s)$ are linear combinations of rhe $\eta_i^*$ $(i - 0, 1 \cdots, m)$, it follows that $\overline{V}$ is a projection of the normal variety $V^*$.

From the definition it is evident that the coordinates of a homogeneous generic point of a derived normal variety are linear forms in the coordinates of a homgeneous generic point of any other derived normal variety relative to the same character of homogeneity. It follows that two derived normal varieties relative to the same character of homogeneity are projectively related.

As an example consider the following special cases of derived normal varieties.

1. Cubic curve in $A_2$ with a cusp.

Choosing the origin of non-homogeneous coordinates at the cusp, axes can be chosen so that the non-homogeneous coordinates of a generic point are $(\theta^2, \theta^3)$ where $\theta$ is an indeterminate. A homogeneous generic point is $(\omega, \omega\theta^2, \omega\theta^3)$ where $\omega$ is a second indeterminate. A homogeneous generic point is $(\omega, \omega\theta^2, \omega\theta^3)$ where $\omega$ is second indeterminate. Let $\sigma' = K[\omega, \omega\theta^2, \omega\theta^3]$.

$\omega\theta$ is in the field $K(\omega, \omega\theta^2, \omega\theta^3) = K(\omega, \theta)$ and is integrally dependent on $\sigma'$. $\omega, \omega\theta, \omega\theta^2, \omega\theta^3$ provide a base for the elements of $\sigma'^*$ which are of degree 1, $\sigma'^*$ being the integral closure of $\sigma'$ in its quotient field. In fact $\sigma'^* = K[\omega, \omega\theta, \omega\theta^2, \omega\theta^3]$ and $(\omega, \omega\theta, \omega\theta^2, \omega\theta^3)$ is a homogeneous generic point of a twisted cubic curve in $S_3$. It is easy to verify that this integral domain

$K[\omega, \omega\theta, \omega\theta^2, \omega\theta^3]$ is in fact integrally closed.

2. Derived normal varieties of a straight line

A homogeneous generic point of a straight line is $(\omega, \theta)$ where $\omega$, $\theta$ are independent indeterminates over $K$.

$\sigma' = K[\omega, \theta]$ is integrally closed in its quotient field. A basis for elements of degree $k$ in this ring is $\omega^k, \omega^{k-1}\theta, \omega^{k-2}\theta^2, \cdots, \omega\theta^{k-1}, \theta^k$. Interpreting these elements as defining a homogeneous generic point of a variety we have a rational curve of order $k$ in $S_k$.

3. Derived normal varieties of a plane.

A homogeneous generic point of a plane is $(\theta, \varphi, \psi)$ where $\theta$, $\varphi$, $\psi$ are independent indeterminates over $K$.

$\sigma' = K[\theta, \varphi, \psi]$ is integrally closed in its quotient field. The elements $\theta^t\varphi^u\psi^v$, where $t + u + v = k$, form a basis for the elements of degree $k$ in $\sigma'$.

When $k = 2$ $(\theta^2, \varphi^2, \psi^2, \theta\varphi, \varphi\psi, \theta\psi)$ is a homogeneous generic point of a Veronesal surface, $V_2^4$, in $S_5$.

When $k = 3$, $(\theta^3, \theta^2\varphi, \cdots, \psi^3)$ is a homogeneous generic point of a del Pezzo surface $V_2^9$ in $S_9$.

When $k = 4$ the corresponding derived normal variety is a $V_2^{16}$ in $S_{14}$ and in general the derived normal variety corresponding to character of homogeneity $k$ is a $V_2^{k^2}$ in $S_\nu$ where $\nu = \frac{1}{2}(k+1)(k+2) - 1$.

§3.5 Resolution of the singularities of an algebraic curve

A very important fact about a locally normal variety of dimension $r$ is that the dimension of the singular locus is less than or equal to $r - 2$. In particular a locally normal curve is non-singular. This fact is used in Muhly's and Zariski's paper on the resolution of the singularities of an algebraic curce [10]. In this paper the following results are proved.

(i) Every algebraic curve is birationally equivalent to a curve in higher space $S_n$ which is free from singularities both at a finite distance and at infinity.

(ii) Any curve free from singularities in $S_n$ can be projected into $S_3$ into a curve which is free from singularities.

(iii) Every algebraic curve can be birationally transformed into a curve in $S_2$ whose only singularities are ordinary double points.

These results are, of course, well known and can be proved in a rigorous and aesthetically satisfying fashion using the theory of linear series of points on a curve and the classical methods of Italian geometry. Muhly and Zariski do not present us with an essentially more concise or more beautiful method, but this alternative method of solving a classical problem is certainly justified since it can be generalised and used to provide a solution for the problem of the resolution of the singularities of an algebraic surface. A solution of this problem was published by Zariski in 1939 [23] and a simplified version of the same work appeared later [25]. The methods used by Zariski involve valuation theory extensively.

§3.6 Resolution of the singulatrities of an algebraic surface

There are two main theorems employed. The first one is called by Zariski the Local Uniformisation Theorem:

- Given a variety $V$ with function field $\Sigma$ and a valuation $v$ of $\Sigma$, then there exists abother variety $V'$, birationally equivalent to $V$ and such that the centre of $v$ on $V'$ is a simple subvariety of $V'$.

In fact Zariski finds it sufficient to confine his attention to valuations of zero dimension and the second main theorem is concerned with establishing the existence of what he calls a “finite resolving system” for these valuations.

- Consider the set of birationally equivalent varieties all having $\Sigma$ as function field and also the set of all zero-dimensional valuations of $\Sigma$. Then, if $V^{(i)}$ is the subset of these varieties, not necessarily denumerable , which are such that the centre of every zero-dimensional valuation is a simpler point on at least one $V^{(i)}$, there exists a finite set of the $V^{(i)}$ satisfying this same condition. This finite set is called a resolving system of the set of zero-dimensional valuations.

The problem of the reduction of the singularities of an algebraic surface, once the existence of a finite resolving system is established, is then equivalent to showing that there exists a resolving system consisting of only one surface.

Zariski shows that the transition from any finite resolving system to a system containing just one variety depends on the fundamental theorem which he enunciates as follows.

- If $\mathcal{V}$ is an arbitrary subset of $\mathcal{W}$, the set of zero-dimensional valuations of a given function field $\Sigma$, and if there exists a resolving system of $\mathcal{V}$ consisting of two varieties, then there also exists a resolving system consisting of just one variety.

Suppose $V_1 \cdots, V_n$* is a resolving system for $\mathcal{W}$ and let $\mathcal{V}$ be the subset of those valuations which have a singular centre on each of $V_1, \cdots, V_n$.

*The manuscript contains lower suffixes. A comment in the margin is “use upper suffixes”. This has been ignored because of the awkwardness of attaching an upper suffix to $V'$.

Then, in this case, $V_{n-1}$ and $V_n$ are a resolving system for $\mathcal{V}$. Assuming the fundamental theorem stated above, there exists a resolving system for $\mathcal{V}$ consisting of a single variety $V'_{n-1}$. Then $V_1, V_2, \cdots V_{n-2}, V'_{n-1}$ are $n-1$ varieties forming a resolving system for $\mathcal{W}$, and having reduced our system to one consisting of $n-1$ varieties instead of $n$, we can proceed similarly until a resolving system is obtained which consists of one model only.

In his paper [25] Zariski demonstrates the existence of resolving systems for the function field of an algebraic surface and proves his fundamental theorem. It is in the proof of this last theorem that the operation of integral closure is used to eliminate any multiple curves which may appear on the surface at each stage of the reduction which is effected by a sequence of quadratic transformations followed by transformations to derived normal varieties.

A normal surface $F$ with homogeneous generic point $(\xi_0, \xi_1, \cdots, \xi_n)$ has a finite number of singular points. Let one of them be $P$ and consider the quadratic forms in $\xi_0, \xi_1, \cdots, \xi_n$ which vanish at $P$. If $\omega_0, \omega_1, \cdots, \omega_m$ is a linearly independent base for these forms we may regard these quantities $\omega_0, \omega_1, \cdots, \omega_m$ as coordinates of a homogeneous generic point of a surface $F'$ birationally equivalent to $F$. This is a quadratic transformation of $F$ with centre $P$. The sections of $F$ by quadrics through $P$ correspond to prime sections of $F'$. The point $P$ corresponds to a pure one-dimensional subvariety of $F'$ and to any other point of $F$ corresponds a unique point of $F'$. There are no fundamental points for this correspondence on $F'$.

To prove the fundamental theorem quoted above for surfaces, Zariski then considers two surfaces $F$ and $F'$ forming a resolving system for a given set $\mathcal{V}$ of zero-dimensional valuations * of the function field $\Sigma$. A sequence of quadratic transformations with centres at the fundamental points on $F$, followed at each stage by transformations to the derived normal surface, leads to a surface $\overline{F}$ such that for the birational correspondence between $\overline{F}$ and $F'$ no fundamental points exist on $\overline{F}$. The next step is to eliminate those fundamental points of $F'$ which are simple points on $F'$ and obtain the surface $\overline{F'}$. Zariski then shows that the join $F^*$ of $\overline{F}$ and $\overline{F'}$ is a surface which forms a resolving system for $\mathcal{V}$, thus proving his fundamental theorem quoted above, and hence resolving the whole problem of the reduction of the singularities of an algebraic surface. All this is explained in his paper [25].

* "valuation" in the manuscript

# Chapter 4.

## Geometrical properties of normal varieties

§4.1 Muhly's characterisation of normal varieties.

In considering the normality of primals and curves we find an illustration of the general fact that the subvariety consisting of the singular points of a normal variety $V$ of dimension $r$ is of dimension less than or equal to $r-2$ ([22] p. 280 Theorem 11). For primals and curves the converse is true also, namely, non-singular irreducible curves and irreducible primals in $S_n$ having no multiple subvarieties of dimension $n-2$, are locally normal. In the case of primals local normality further implies projective normality.

It is of interest to find geometrical criteria by which we may investigate the normality, both local and projective, of given varieties. Regarding such criteria, the most important theorem concerning projectively normal varieties is due to Muhly. In his paper [9] he proves that a necessary and sufficient condition for an $r$-dimensional variety $V_r$ to be normal in its ambient projective space $S_n$ is that for every integer $p$, the linear system cut out on $V_r$ by the primals of order $p$ in $S_n$ be complete.

The necessity of this condition follows, as Muhly points out, immediately from Zariski's work regarding characters of homogeneity ([22] p. 290 §20). For suppose $(\eta_0, \eta_1, \cdots, \eta_n)$ are coordinates of a homogeneous generic point of $V_r$ and assume that $K[\eta_0, \eta_1, \cdots, \eta_n]$ is integrally closed in $K(\eta_0, \eta_1, \cdots, \eta_n)$. Let $\omega_0, \omega_1, \cdots, \omega_m$ be a linearly independent base for the forms of degree $p$ in $\eta_0, \eta_1, \cdots, \eta_n$. Then $(\omega_0, \omega_1, \cdots, \omega_m)$ can be regarded as coordinates of a homogeneous generic point of a variety $V_r'$ which is birationally equivalent to $V_r$. $V_r'$ is projectively normal and consequently geometrically normal ([22] p. 288 Theorem 14). Therefore the prime sections of $V_r'$ form a complete linear system on $V_r'$. Since these prime sections correspond to the sections of $V_r$ by the primals of order $p$ in $S_n$, these primals must cut out a complete series on $V_r$. Muhly's proof of the sufficiency of the condition uses valuation theory.

A similar theorem holds for locally normal varieties ([26] p. 409). In this case a necessary and sufficient condition that $V_r$ be locally normal is that there exists a positive integer $p_0$ such that for every integer $p$ with $p \geq p_0$, the linear system cut out on $V_r$ by the primals of order $p$ in the ambient space of $V_r$ is complete.

§4.2 Gaeta's work on the geometrical properties.

Using Muhly's theorem Gaeta has been able to establish some very interesting results regarding the normality of varieties [4, 5]. In the first of his two papers on this subject Gaeta begins by proving the following theorem. Suppose $C$ is an irreducible normal curve of $S_n$ and that $n-1$ primals of orders $p_1, p_2, \cdots, p_{n-1}$ pass simply through $C$ and cut residually in a simple, irreducible, non-singular curve $C'$. Then $C'$ also is normal.

Gaeta then considers curves of finite residue as defined by Severi. An irreducible non-singular curve in $S_3$ which is the complete intersection of two surfaces is said to be of residue zero. Curves of finite residue are defined inductively. An irreducible non-singular curve $C$ in $S_3$ is of residue $p$ when it is possible to pass through $C$ two surfaces which cut residually in a simple, irreducible and non-singular curve $C'$ of residue $p-1$, and where no curve $C'$ can be defined in the same way which is of residue less than $p-1$. Curves of finite residue in $S_n$ $(n > 3)$ are defined similarly when the curves of zero residue in $S_n$ have been defined as simple, complete, non-singular and irreducible intersections of $n-1$ primals. An irreducible non-singular curve $C$ in $S_n$ is of residue $p$ if $n-1$ primals passing simply through $C$ meet residually and simply in a curve $C'$ of residue $p-1$, where again $p$ has the smallest possible value.

Severi has shown that any curve of zero residue is projectively normal.[†] Gaeta's preceding theorem shows that every curve in $S_n$ which is of finite residue is projectively normal. The proof of this first theorem depends directly on Muhly's theorem and makes use of Severi's postulation formulae for complete intersections and Noether's formula for the genus of a reducible curve. Gaeta continues his paper with a study of the case $n = 3$, and proves conversely that, in this particular case, all normal curves in $S_3$ are necessarily of finite residue. He points out that this converse is not true when $n$ is greater than three and gives as an example the rational normal quartic curve in $S_4$. Such a curve is normal but not of finite residue.

The paper concludes with a generalisation of this work for surfaces in $S_n$. Suppose $F$ and $F'$ are two non-singular irreducible surfaces in $S_n$ $(n \geq 4)$ which make up the complete, simple intersection of $n-2$ primals. Then, if $F$ is regular and normal, $F'$ also is regular and normal. Surfaces of finite residue are defined in a fashion analogous to the corresponding definition for curves. A surface $F$ of zero residue is a simple, non-singular, complete intersection of $n-2$ primals in $S_n$, and an irreducible, non-singular surface $F$ is of residue $p$ if $n-3$ primals passing simply through $F$ meet residually and simply in a surface $F'$ of residue $p-1$ where here again $p$ has the smallest possible value. Severi has proved that an irreducible surface of $S_n$ without singular points which is the simple intersection of $n-2$ primals is regular and normal. It follows that every surface of finite residue in $S_n$ is regular and arithmetically normal.

In his next paper [5] Gaeta goes on to show that in the preceding theorem concerning the two non-singular irreducible surfaces $F$ and $F'$ which together

[†]For $n = 3$ see [13] §§20, 21.

make up the complete simple intersection of $n-2$ primals, regularity is not only a sufficient condition for the theorem but is also necessary. He shows that in fact a necessary and sufficient condition for $F'$ to be regular is that primals of any order $p$ ($p > 0$) cut a complete system on $F$. Therefore if $F$ is normal it follows that $F'$ is regular. If $F$ is irregular then $F'$ is not normal.

The next part of this paper is devoted to a characterisation of the surfaces of $S_4$ which are the simple complete intersection of two primals and leads to the following result. An irreducible surface $F$ of $S_4$ which is regular, subcanonical and arithmetically normal is the simple complete intersection of two primals and conversely.

The remainder of this paper is concerned with varieties of any dimension $d$. The author considers the intersection of two varieties $V_d$ and $W_d$ both of dimension $d$, which are together the complete intersection of $n-d$ primals in $S_n$. The intersection $V_d \cap W_d$ is $Z_{d-1}$, a variety of dimension $d-1$. A result analogous to those already found for curves and surfaces is obtained. Namely, if $V_d$ and $W_d$ are normal and if $Z_{d-1}$ is non-singular then $Z_{d-1}$ is normal. For $d \geq 2$ the converse theorem is true.

§4.3 Zero-dimensional varieties

A zero-dimensional variety need not necessarily consist of a single point. If the ground field $K$ is not algebraically closed a zero-dmensional variety consists of a finite number, $g$, of conjugate points. $g$ is the order of the variety. General results regarding varieties of dimension $r$ should be investigated separately for the case $r = 0$. Two illustrations should show the need for such precaution.

Consider first, problems concerning the normality of varieties. Such problems are closely connected with the series cut out on a variety by the primals of a given order in the ambient space. These series do not exist when the variety concerned is of zero dimension.

For a second example, consider the generic prime section of an irreducible variety $V_r$ of dimension $r$ in an ambient space $S_n$ in which the homogeneous coordinates of a general point are $(x_0, x_1, \cdots, x_n)$. A generic section is the section by a prime with equation

$$u_0x_0 + u_1x_1 + \cdots + u_nx_n = 0$$

where $u_0, u_1, \cdots, u_n$ are indeterminates over the ground field $K$. This section is defined over the field $K(u_0, u_1, \cdots, u_n)$ and is irreducible. If we choose non-special values in $K$ for the $u_i$ we shall obtain a "general" section which, in the case $r > 1$, is irreducible over $K$. When $r = 1$, the generic prime section is still irreducible over $K(u_0, u_1, \cdots, u_n)$ but the "general" section becomes reducible if $K$ is algebraically closed.

We can show that any zero-dimensional variety is normal in the affine sense for any choice of the non-homogeneous coordinates and therefore locally normal, whereas no zero-dimensional variety is projectively normal unless its order $g$ is equal to unity.

Consider a set of conjugate points in $S_n$. Choosing non-homogeneous coordinates $x_1, \cdots, x_n$ sufficiently generally a generic point of this zero-dimensional vriety $V_0^g$ has coordinates of the form $(\alpha, \theta_2(\alpha), \cdots, \theta_n(\alpha))$ where $\alpha$ is a root of an irreducible algebraic equation $R(x_1) = 0$, and $\theta_i(\alpha)$ belongs to $K[\alpha]$ $(i = 2, \cdots, n)$.

$$K[\alpha, \theta_2(\alpha), \cdots, \theta_n(\alpha)] = K[\alpha] = K(\alpha).$$

$K[\alpha, \theta_2(\alpha), \cdots, \theta_n(\alpha)]$, the integral domain of the variety, is trivially closed in $K(\alpha)$ its quotient field, so that the set of conjugate points is an affinely-normal variety.

Consider now projective normality. With suitably chosen homogeneous coordinates in $S_n$ a normalised generic point of the variety is $(1, \alpha, \theta_2(\alpha), \cdots, \theta_n(\alpha))$ and a homogeneous generic point is $(\lambda, \lambda\alpha, \lambda\theta_2(\alpha), \cdots, \lambda\theta_n(\alpha))$ where $\lambda$ is an indeterminate over $K(\alpha)$.

$K[\lambda, \lambda\alpha, \lambda\theta_2(\alpha), \cdots, \lambda\theta_n(\alpha)]$ is not integrally closed in $K(\lambda, \alpha)$ its quotient field. For $R(x_1) = 0$ is an equation with coefficients in $K$ and can be written in the form

$$x_1^g + a_1 z_1^{g-1} + \cdots + a_{g-1}x_1 + a_g = 0 \quad (a_i \text{ is in } K,\ i = 1, \cdots, g).$$

The equation

$$\alpha^g + a_1\alpha^{g-1} + \cdots + a_{g-1}\alpha + a_g = 0$$

shows that $\alpha$ is integrally dependent on $K[\lambda, \lambda\alpha, \lambda\theta_2(\alpha), \cdots, \lambda\theta_n(\alpha)]$, but if $g > 1$ $\alpha$ does not belong to this integral domain although belonging to its quotient field.

Geometrically these results are obvious since a zero-dimensional variety is non-singular and consequently locally normal. On the other hand if we form a cone by joining the conjugate points to another point outside the ambient $S_n$ this cone is irreducible, of dimension one, and has a singular point at which it is not locally normal. Consequently, the set of conjugate points cannot be projectively normal.

§4.4 <u>Example of a locally normal cubic surface</u>

The cubic surface $F$ in $S_3$ which is given by the homogeneous equation in $(x_0, x_1, x_2, x_3)$, homogeneous coordinates in $S_3$

$$x_0x_1x_3 + x_0x_2^2 + x_1x_2^2 = 0 \qquad (1)$$

where the ground field $K$ is the complex number field, is locally normal.

This surface is trinodal with nodes at the points $(1,0,0,0)$, $(0,1,0,0)$ and $(0,0,0,1)$.

If $x_0 = 0$ is chosen as the plane at infinity and $y_i = x_i/x_0$ $(i = 1,2,3)$ are non-homogeneous coordinates, the equation of $F$ is

$$y_1 y_3 + y_2^2 + y_1 y_2^2 = 0 \tag{2}$$

and a generic point of this rational surface is $\left(u, v, \dfrac{-v^2 - uv^2}{u}\right)$ where $u$, $v$ are independent indeterminates over $K$. $F$ is affinely normal for this choice of the plane at infinity if $K\left[u, v, \dfrac{-v^2 - uv^2}{u}\right]$ is integrally closed in $K\left(u, v, \dfrac{-v^2 - uv^2}{u}\right)$. Now

$$K\left[u, v, \frac{-v^2 - uv^2}{u}\right] = K\left[u, v, \frac{v^2}{u}\right]$$

and

$$K\left(u, v, \frac{-v^2 - uv^2}{u}\right) = K(u, v).$$

Therefore it is necessary to show that $K\left[u, v, \dfrac{v^2}{u}\right]$ is integrally closed in $K(u,v)$. I shall solve here the more general problem and show that the ring $K\left[u, v, \dfrac{v^s}{u}\right]$ is integrally closed in $K(u,v)$ where $s$ is any positive integer.

Any power product occurring in a polynomial of the ring $K\left[u, v, \dfrac{v^s}{u}\right]$ is of the form $u^a v^b \left(\dfrac{v^s}{u}\right)^c$, $a$, $b$, $c \geq 0$, and $a$, $b$, $c$ and any other indieces occurring in this paragraph are integers. Now

$$u^a v^b \left(\frac{v^s}{u}\right)^c = u^{a-c} v^{b+sc} = u^i v^j$$

is in $K\left[u, v, \dfrac{v^s}{u}\right]$.

(i) If $i \geq 0$ then obviously $u^i v^j$ is in $K\left[u, v, \dfrac{v^s}{u}\right]$.

(ii) Suppose $i < 0$ and $i = -k$ with $k > 0$.

Then $j - sk \geq 0$ and

$$u^i v^j = \left(\frac{v^s}{u}\right)^k v^{j-sk}$$

which is in $K\left[u, v, \dfrac{v^s}{u}\right]$ since $j - sk \geq 0$. We conclude that the elements of $K\left[u, v, \dfrac{v^s}{u}\right]$ are the finite sums $\sum_{i,j} \alpha_{i,j} u^i v^j$ where the coefficients $\alpha_{i,j}$ belong to $K$ and $j \geq 0$, $si + j \geq 0$.

Now suppose that $\zeta$, an element of $K(u, v)$, is integrally dependent on $K\left[u, v, \dfrac{v^s}{u}\right]$. $\zeta$ satisfies an equation

$$\zeta^n + a_1\zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n = 0 \quad (a_i \text{ is in } K\left[u, v, \frac{v^s}{u}\right],\ i = 1, \cdots, n). \quad (3)$$

Therefore we can multiply this equation by $u^{\rho n}$, where $\rho$ is some suitably chosen positive integer, and obtain the equation

$$(u^\rho\zeta)^n + b_1(u^\rho\zeta)^{n-1} + \cdots + b_{n-1}(u^\rho\zeta) + b_n = 0 \quad (b_i \text{ is in } K[u, v],\ i = 1, \cdots, n). \quad (4)$$

Since $K[u, v]$ is integrally closed in $K(u, v)$ (§2.5) $u^\rho\zeta$ must be in $K[u, v]$. Therefore

$$\zeta = \sum_{i,j} -\beta_{i,j} u^i v^j,$$

where the coefficients $\beta_{i,j}$ belong to $K$ and $j \geq 0$ in each term of this finite sum.

Writing $u = \lambda^s$, $v = \lambda\mu$ where $\lambda$, $\mu$ are two independent indeterminates over $K$, we find

$$\zeta = \sum_{i,j} \beta_{i,j} u^i v^j = \sum_{i,j} \beta_{i,j} \lambda^{si+j} \mu^j. \quad (5)$$

Now, since $\dfrac{v^s}{u} = \mu^s$, the $a_i$ $(i = 1, \cdots, n)$ occurring in equation (3) are in the ring $K[\lambda, \mu]$. This ring is integrally closed in $K(\lambda, \mu)$ and therefore $\zeta$ must belong to the ring $K[\lambda, \mu]$. But, according to expression (5), $\zeta = \sum_{i,j} \beta_{i,j} \lambda^{si+j} \mu^j$, and is therefore a polynomial in $\lambda$, $\mu$ only if $j \geq 0$, $si+j \geq 0$ in each term of the sum. Therefore $\zeta$ is in the ring $K\left[u, v, \dfrac{v^s}{u}\right]$.

In the particular case with which we are concerned, $s = 2$, and hence the surface $F$ defined by equation (2) is affinely normal.

We can show that the surface defined by equation (1) is locally normal by choosing new non-homogeneous coordinate systems with respect to which the other two nodes $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$ are at a finite distance.

When, for example $x_1 = 0$ is chosen as the plane at infinity and $z_i = x_i/x_1$ $(i = 0, 1, 2, 3)$ are non-homogeneous coordinates, $F$ has an equation

$$z_0 z_3 + z_0 z_2^2 + z_2^2 = 0 \tag{6}$$

and a generic point is again given by $\left(u, v, \dfrac{-uv^2 - v^2}{u}\right)$, as in the first case.

A slightly different situation arises when we deal with the third node. Choosing $t_i = x_i/x_3$ $(i = 0, 1, 2)$ as non-homogeneous coordinates, the surface has an equation

$$t_0 t_1 + t_0 t_2^2 + t_1 t_2^2 = 0 \tag{7}$$

and a generic point is $\left(\dfrac{-uv^2}{u + v^2}, u, v\right)$ where $u$, $v$ are independent indeterminates over $K$. It is necessary to show that $K\left[u, v, \dfrac{-uv^2}{u + v^2}\right]$ is integrally closed in $K\left(u, v, \dfrac{-uv^2}{u + v^2}\right)$. Consider the integral domain $K\left[u, v, \dfrac{-uv^2}{u + v^2}\right]$ and introduce a new quantity $w = u + v^2$. then

$$\begin{aligned} K\left[u, v, \frac{-uv^2}{u + v^2}\right] &= K\left[w - v^2, v, \frac{-wv + v^4}{w}\right] \\ &= K\left[w, v\frac{v^4}{w}\right]. \end{aligned}$$

Since $K\left[w, v, \dfrac{v^4}{w}\right]$ is integrally closed in $K(w, v)$, the case when $s = 4$ in the first part of this example, it follows that $K\left[u, v, \dfrac{-uv^2}{u+v^2}\right]$ is integrally closed in $K(u, v)$.

$F$ is locally normal at its three nodes, the only singular points, and therefore $F$ is a locally normal surface.

In the last part of this example, the surface as defined by equation (7) is evidently equivalent to the surface with generic point $\left(w, v, \dfrac{v^4}{w}\right)$. This is a rational surface with equation obtained by eliminating $w$ and $v$ from the parametric equations

$$t_0 = w, \quad t_1 = v, \quad t_2 = \frac{v^4}{w}.$$

The equation of the new surface is therefore

$$t_1^4 = t_0 t_2$$

where $(t_0, t_1, t_2)$ are non-homogeneous coordinates as before.

As an example of a surface in $A_3$ which is not affinely normal consider that given by the equation $x_2^3 - x_1^2 = 0$, a cylinder with axis parallel to the $x_3$-axis, $x_1$, $x_2$, $x_3$ being non-homogeneous coordinates. This cylinder has the $x_3$-axis as cuspidal generator. A generic point is $(\theta^3, \theta^2, \varphi)$ where $\theta$ and $\varphi$ are independent indeterminates over the ground field $K$. $K[\theta^3, \theta^2, \varphi]$ is not integrally closed in $K(\theta^3, \theta^2, \varphi) = K(\theta, \varphi)$ because for example $\theta$ is not in $K[\theta^3, \theta^2, \varphi]$ but the equation $z^2 - \theta^2 = 0$ is satisfied by $z = \theta$ showing that $\theta$ is integral over $K[\theta^3, \theta^2, \varphi]$.

For all but the simplest cases this direct investigation of affine normality of a variety is of little use. General theorems are needed to tell us whether or not a given variety is normal in any particular sense. I have tried, in the next section of my work, to find general criteria for normality in the case of a surface in ordinary space of three dimensions.

§4.5 An investigation into the normality of a surface in three dimensions.

Let $(x, y, z)$ be the non-homogeneous coordinates of a point in ordinary space. Any irreducible algebraic surface in space has an equation of the form $f(x, y, z) = 0$ where $f(x, y, z)$ is an irreducible polynomial in $(x, y, z)$ with coefficients in some ground field $K$. If the surface is of order $n$ then, for a sufficiently general choice of coordinate system, the coefficient of $z^n$ may be assumed to be unity. We shall think of the polynomial $f(x, y, z)$ as a polynomial in $z$ with coefficient in the ring $K[x, y]$.

**Theorem 1** *If $R^* = R[\zeta]$ is a simple algebraic extension of the integral domain $R$ using the equation $F(z) = 0$ where $F(z)$ is an irreducible polynomial in the ring $R[z]$ with leading coefficient unity, and if $R$ is a unique factorisation domain which is integrally closed in its quotient field $K$, then, if the discriminant of $F(z)$ with respect to $z$ has no repeated factors in $R$, $R^*$ is closed in its quotient field.*

Suppose $F(z)$ is of degree $n$ in $z$ and let $\zeta$ be a root of the equation $F(z) = 0$. $\zeta$ is then integral over $R$. Let $K^*$ denote the quotient field of $R^*$. Then $K^* = K(\zeta) = K[\zeta]$ ([8], p. 100). Suppose that

$$\eta = \sum_{i=1}^{n} p_i \zeta^{i-1} \quad (p_i \text{ is in } K,\ i = 1, \cdots, n)$$

is any element of $K^*$ which is integrally dependent on $R^*$. Now $R^*$ is integral over $R$ since $\zeta$ is integral over $R$. On account of the transitive property of integral dependence, $\eta$ is integral over $R$. Let $\zeta_1 = \zeta, \zeta_2, \cdots, \zeta_n$ be the conjugates of $\zeta$ over $K$ and define

$$\eta_j = \sum_{i=1}^{n} p_i \zeta_j^{i-1} \quad (j = 1, \cdots, n). \tag{1}$$

then these quantities $\eta_j$ $(j = 1, \cdots, n)$ are integral over $R$. Consequently

$$\begin{vmatrix} 1 & \zeta_1 & \cdots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \cdots & \zeta_2^{n-1} \\ \vdots & & & \\ 1 & \zeta_n & \cdots & \zeta_n^{n-1} \end{vmatrix} \cdot p_i \text{ is integral over } R, \quad (i = 1, \cdots n).$$

For

$$\begin{aligned} \eta_1 &= p_1 + p_2\zeta_1 + \cdots + p + n\zeta_1^{n-1} \\ \eta_2 &= p_1 + p_2\zeta_2 + \cdots + p + n\zeta_2^{n-1} \\ \vdots & \\ \eta_n &= p_1 + p_2\zeta_n + \cdots + p + n\zeta_n^{n-1} \end{aligned}$$

and therefore

$$\begin{vmatrix} 1 & \zeta_1 & \cdots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \cdots & \zeta_2^{n-1} \\ \vdots & & & \\ 1 & \zeta_n & \cdots & \zeta_n^{n-1} \end{vmatrix} \cdot p_i = c_{i,1}\eta_1 + \cdots + c_{i,n}\eta_n \quad (i = 1, \cdots, n) \tag{2}$$

where the coefficients $c_{i,j}$ $(i = 1, \cdots n, \quad j = 1, \cdots, n)$ are polynomials in $\zeta_1, \cdots, \zeta_n$ with the natural integers as coefficients, that is with coefficients in $R$, assuming that unity belongs to $R$, which is here the case.

We have

$$p_i = \frac{\sum_{j=1}^{n} c_{i,j}\eta_j}{D^{1/2}} \qquad (i = 1, \cdots n) \tag{3}$$

where $D$ is the discriminant of $F(z)$ with respect to $z$ and is in $R$. Hence

$$Dp_i^2 = \left( \sum_{j=1}^{n} c_{i,j}\eta_j \right)^2 \qquad (i = 1, \cdots, n). \tag{4}$$

Now $Dp_i^2$ belongs to $K$ $(i = 1, \cdots, n)$, and the expression on the right hand side of this equation is integral over $R$. Therefore $Dp_i^2$ belongs to $K$ and is integral over $R$, and since by hypothesis $R$ is integrally closed in $K$,

$$Dp_i^2 \;=\; A_i \text{ with } A_i \text{ in } R \quad (i = 1, \cdots, n). \tag{5}$$

Write $p_i$ in the form $p_i = \dfrac{\alpha_i}{\beta_i}$ $(i = 1, \cdots, n)$ where $\alpha_i$ and $\beta_i$ belong to $R$ and have no common factor in $R$ other than the units of $R$. We have

$$D\alpha_i^2 = A_i\beta_i^2 \quad (i = 1, \cdots, n), \tag{6}$$

and therefore if $D$ has no repeated factors in $R$, every factor of $\beta_i$ divides $\alpha_i$ and consequently $\beta_i$ is a unit of $R$. Hence $p_i$ belongs to $R$ and since this is true for $i = 1, \cdots, n$, it follows that $\eta$ belongs to $R^*$.

We deduce that $R^*$ is integrally closed in $K^*$.

With the help of this theorem we can show that a necessary and sufficient condition for a surface in three dimensions to be affinely normal is that it contains no multiple curves. For if an irreducible surface of order $n$ is given by the equation $f(x, y, z) = 0$ with the coefficient of $z^n$ unity, then to show that the surface is affinely normal it is sufficient to show that the coordinate axes can be chosen so that $D(x, y)$, the discriminant of $f(x, y, z)$ with respect to $z$, has no repeated factors. For then $K[x, y, \zeta]$, the algebraic extension of $K[x, y]$ defined by $f(x, y, z) = 0$, is integrally closed in its quotient field, and since $(x, y, \zeta)$ are non-homogeneous coordinates of a generic point of the surface, this will imply normality.

We shall see that such a choice of axes is possible if the surface does not possess a multiple curve.

Suppose $f(x, y, z)$, an element of the ring $K[x, y, z]$, is an irreducible polynomial of degree $n$ and that the coefficient of $z^n$ is unity. $D(x, y)$, the

$z$-discriminant of $f(x,y,z)$, is a polynomial in the ring $K[x,y]$. $f(x,y,z)=0$ is the equation of a surface $F$ in $A_3$ where $x$, $y$, $z$ are non-homogeneous coordinates, and $D(x,y)=0$ represents a cylinder with generators parallel to the $z$-axis. The generators of this cylinder meet the surface in two coincident points.

For if

$$f(x,y,z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \quad (a_i \text{ is in } K[x,y],\ i=1,\cdots,n)$$
$$= \prod_{i=1}^{n}(z-\zeta_i)$$

then

$$D(x,y) = \prod_{n\geq i>j\geq 1}(\zeta_i-\zeta_j)^2 \qquad ([17]\text{vol 1 p.87}).$$

Therefore if $\zeta_i=\zeta_j$ for some $i\neq j$, $D(x,y)=0$, but if all the $\zeta_i$ are distinct, $D(x,y)\neq 0$.

The generators of the cylinder may be tangents to the surface or they may be lines through multiple points of the surface. We have to determine the circumstances in which such a generator is a double line on the cylinder.

All generators which are ordinary tangents at just one simple point of $F$ are simple lines of the cylinder. For suppose the line $x=y=0$ is such a generator which touches $F$ at the point $(0,0,z)$. Using an extension of the Weierstrass Preparation Theorem ([6] pp 94-96), the equation of $F$ in the neighbourhood of $(0,0,z)$ can be written in the following form:

$$f(x,y,z) \equiv F(x,y,z'), \qquad \text{substituting } z=z'+z$$
$$\equiv (z'^2 + a_1(x,y)z' + a_2(x,y))\cdot U(x,y,z') = 0,$$

where $a_1$,$a_2$ are in the ring $K[x,y]$ and vanish when $x=y=0$. Also $a_2$ contains non-vanishing linear terms since we are assuming that $(0,0,z_1)$ is a simple point in the surface. $U$ is in $K\{x, y, z'\}$ and does not vanish when $x = y = z' = 0$.

The discriminant $D(x, y)$ of $f(x, y, z)$ with respect to $z$ is the same as the discriminant of $F(x, y, z')$ with respect to $z'$ and is a polynomial in $K[x, y]$. In the ring $K\{x, y\}$, $D(x, y)$ has a representation

$$D(x, y) = (a_1^2 - 4a_2)V(x, y)$$

where $V(x, y)$ is a unit in $K\{x, y\}$. Since $a_2$ has non-vanishing linear terms it follows that $D(x, y)$ has a simple zero when $x = y = 0$.

The only possible lines therefore which can be multiple generators of the cylinder $D(x, y) = 0$ are among

1. inflexional tangents to the surface $f(x, y, z) = 0$ parallel to the $z$-axis,

2. double tangents parallel to the $z$-axis,

3. lines parallel to the $z$-axis through multiple points of the surface.

We can show, in fact, that all such lines are multiple generators.

Suppose first that the line $x = y = 0$ is an inflexional tangent to the sirface at the point $(0, 0, z_1)$. Replacing $z$ by $z' + z_1$, the equation of the surface in the neighbourhood of this point is given by

$$f(x, y, z) = F(x, y, z') = (z'^3 + 3a_1 z'^2 + 3a_2 z' + a_3)U(x, y, z')$$

where $a_1$, $a_2$, $a_3$ belong to $K\{x, y\}$ and vanish when $x = y = 0$ and $U(x, y, z)$ is a unit of the ring $K\{x, y, z'\}$. The $z$-discriminant of $f(x, y, z)$ is a polynomial in $K[x, y]$. The polynomial has a factor in $K\{x, y\}$ of the form

$$(a_3 - a_1 a_2)^2 - 4(a_2 - a_1^2)(a_1 a_3 - a_2^2).$$

This power series has no constant or linear terms in $x$, $y$ and consequently $x = y = 0$ is a double generator on the cylinder $D(x, y) = 0$.

Secondly if $x = y = 0$ is a double tangent to $F$ at two ordinary points $(0, 0, z_1)$, $(0, 0, z_2)$ then, by considering parallel tangents near to these two points, it is intuitively evident that this line will be a double generator of the cylinder. Near to the point $(0, 0, z_1)$ the points * of the surface $F$ lie on the analytic sheet given by

$$z'^2 + a_1(x, y)z' + a_2(x, y) = 0 \qquad \text{where } z = z' + z_1,$$

and near to $(0, 0, z_2)$, the surface is given by

$$z''^2 + b_1(x, y)z'' + b_2(x, y) = 0 \qquad \text{where } z = z'' + z_2,$$

where $a_1$,$a_2$, $b_1$, $b_2$ are in the ring $K\{x, y\}$ and vanish when $x = y = 0$, and $a_2$, $b_2$ contain linear terms. The two sheets of the cylinder passing through the generator $x = y = 0$ are given by

$$a_1^2 - 4a_2 = 0$$
$$\text{and} \quad b_1^2 - 4b_2 = 0.$$

Finally suppose $F$ has a double point at $(x_1, y_1, z_1)$. Writing $x' = x - x_1$, $y' = y - y_1$, $z' = z - z_1$, the equation of $F$ is given by

$$f(x, y, z) = F(x', y', z')$$

$$= z'^n + b_1 z'^{n-1} + \cdots + b_{n-2}z'^2 + a_1 z' + a_2 = 0$$

where $b_1, \cdots b_{n-2}, a_1, a_2$ belong to $K[x', y']$ and $a_i$ is a polynomial in $x'$, $y'$ with no terms of degree less than $i$ ($i = 1$, $2$). The discriminant, $\Delta(x', y')$ of $F$ with respect to $z'$ is obtained by eliminating $z'$ between the equations

$$F = z'^n + b_1 z'^{n-1} + \cdots + b_{n-2}z'^2 + a_1 z' + a_2 = 0$$

* "point" in the ms

$$\frac{\partial F}{\partial z'} = nz'^{n-1} + \cdots + 2b_{n-2}z' + a_1 = 0.$$

Therefore

$$\Lambda(x', y') = \begin{vmatrix}
1 & \cdot & \cdot & \cdot & & b_{n-2} & a_1 & a_2 & 0 & \cdot & \cdot & \cdot & & 0 \\
& \cdot & & & & & & & \cdot & & & & & \\
& & \cdot & & & & & & & \cdot & & & & \\
& & & \cdot & & & & & & & \cdot & & & \\
& & & & \cdot & & & & & & & \cdot & & \\
& & & & & & & & & & & & a_2 & \\
& & & & & 1 & b_1 & & & & & b_{n-1} & a_1 & a_2 \\
n & & & & & 2b_{n-2} & a_1 & & & & & & & \\
& \cdot & & & & & & \cdot & & & & & & \\
& & \cdot & & & & & & \cdot & & & & & \\
& & & \cdot & & & & & & \cdot & & & & \\
& & & & \cdot & & & & & & \cdot & & & \\
& & & & & \cdot & & & & & & \cdot & & \\
& & & & & & & & & & & & a_1 & \\
& & & & & & n & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_1
\end{vmatrix}$$

Each of the non-zero elements in the last two columns of this determinant is of degree greater than or equal to one in $x'$, $y'$ and therefore $\Delta$ is a polynomial in $x'$ and $y'$ containing no constant or linear terms. Substituting $x' = x - x_1$, $y' = y - y_1$ in $\Delta(x', y')$ we shall find $\Delta(x', y') = D(x, y)$ where, as before, $D(x, y)$ is the $z$-discriminant of $f(x, y, z)$. It follows that the line $x = x_1$, $y = y_1$ is a multiple generator of the cylinder $D(x, y) = 0$.

A line which is a combination of these three types, for example, a tangent through a double point which is parallel to the $z$-axis, will evidently be a generator of $D(x, y) = 0$ of multiplicity greater than or equal to two.

If an algebraic surface contains no multiple curves then it contains only a finite number of multiple points and $D(x, y)$ will have a repeated factor if and only if there is an infinity of double or inflexional tangents to the surface from the point at infinity on the $z$-axis. Using the principle of counting constants we can prove that from a general point, not on the surface, only a finite number of inflexional and double tangents can be drawn ([12] p. 283). If therefore the surface does not contain a multiple curve, we may choose general axes so that the $z$-discriminant of $f(x, y, z)$ does not contain a repeated factor in $K[x, y]$, and in this case the surface is affinely normal since $K[x, y]$ is a unique factorisation domain integrally closed in its quotient field (Theorem 1).

For a general choice of coordinate we may say that a surface given by

$$f(x, y, z) = z^n + a_1 z^{n-1} + \cdots + a_n = 0 \quad (a_i \text{ is in } K[x, y] \quad i = 1, \cdots, n)$$

is affinely normal if and only if the discriminant of $f(x, y, z)$ with respect to $z$ contains no repeated factors. For a special choice of axes however this need not be true. For example, the Kummer surface contains no multiple curves and is therefore affinely normal. But if for the point at infinity on the $z$-axis we choose a point on one of the sixteen tropes, not on the surface, then we have an infinity of double tangents parallel to the $z$-axis, giving rise to a repeated linear factor in the $z$-discriminant.

In view of Zariski's result that an affinely normal variety of dimension $r$ contains a singular variety of dimension $r - 2$ at most ([22] p.280 Theorem 11) we can conclude as follows.

**Theorem 2** *A necessary and sufficient condition for a surface in $A_3$ to be affinely normal is that it contains no multiple curves not lying entirely at* *infinity.*

§4.6 Normality of a primal in $S_n$

This result for surfaces in $S_3$ generalises on the case of a primal $V_{n-1}$ in $A_n$.

**Theorem 3** *A necessary and sufficient condition for a primal $V_{n-1}$ in affine space of $n$ dimensions to be affinely normal is that it contains no multiple subvarieties of dimension $n-2$ not lying entirely at infinity.*

This result can be used to deduce a necessary and sufficient condition for the projective normality of a primal in $S_n$. It has already been remarked that a variety $V$ in an ambient space $S_n$ is projectively normal if the cone obtained by joining $V$ to a point $O$ outside $S_n$ is locally normal in its ambient space $S_{n+1}$ obtained by joining $O$ to the original space $S_n$. Since a necessary and sufficient condition for a variety to be locally normal is that it is affinely normal for all choices of the prime at infinity (§3.1 Theorem 1), it follows that a primal $V_{n-1}$ in $S_n$ is locally normal if it contains no multiple subvarieties of dimension $n-2$ either at a finite distance or at infinity for any particular choice of non-homogeneous coordinates. Now if a primal $V_{n-1}$ in $S_n$ contains no multiple subvarieties of dimension $n-2$, the cone $C_n$ obtained by joining $V_{n-1}$ to a point $O$ outside $S_n$ is a primal in $S_{n+1}$ and contains no multiple subvarieties of dimension $n-1$. On the other hand if $V_{n-1}$ has a multiple subvariety of dimension $n-2$, then $C_n$ has a multiple subvariety of dimension $n-1$. The required result then follows.

**Theorem 4** *A primal $V_{n-1}$ in $S_n$ which is locally normal is also projectively normal, the necessary and sufficient condition for normality being that no multiple subvarieties of dimension $n-2$ exist on the primal.*

§4.7 Local normality of a surface in $S_3$ at a point

In a very interesting paper [11] concerning the ideals in the ring $K\{x_1, \cdots, x_n\}$ of convergent power series in $x_1, \cdots, x_n$, Rückert gives an ideal-theoretic proof of Weierstrass's theorem that the null points of a set of analytic equations

$$P_i(x_1, \cdots, x_n) = 0 \quad (i = 1, \cdots, n),$$

where $P_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$ are power series in the ring $K\{x_1, \cdots, x_n\}$, which are convergent in the neighbourhood of one particular null-point $O$, lie on a finite number of irreducible analytic varieties passing through the point $O$. When, in particular the $P_i(x_1, \cdots, x_n)$ are polynomials in $x_1, \cdots, x_n$ in the ring $K[x_1, \cdots, x_n]$, the null points lie on an algebraic variety $V$. When $V$ is irreducible any point of $V$ may be taken as the null point $O$, and then the finite number of analytic varieties in the statement of the theorem are the sheets of $V$ which pass through $O$. In the subsequent work I shall assume a coordinate system chosen so that $O$ is the origin of coordinates.

Let $\mathfrak{J}_j$ denote the ring $K\{x_1, x_2, \cdots, x_j\}$. The units of $\mathfrak{J}_j$ are those power series of $\mathfrak{J}_j$ which do not vanish when $x_1 = x_2 = \cdots = x_j = 0$. A power series $Q$ of the ring $\mathfrak{J}_n$ is defined to be regular with respect to $x_k$ if at least one term $cx_k^p$ occurs where $c$ is in $K$, $c \neq 0$.

If $Q$ is a power series in $\mathfrak{J}_n$ which is regular with respect to $x_n$, and if $m$ is the smallest integer for which $cx_n^m$ ($c$ in $K$, $c \neq 0$) occurs in $Q$, then there exists a uniquely determined unit $F$ in $\mathfrak{J}_n$ such that

$$QF = x_n^m + A_1 x_n^{m-1} + \cdots + A_{m-1} x_n + A_m$$

where $A_i$ is in $\mathfrak{J}_{n-1}$ and is not a unit ($i = 1, \cdots, m$). The expression on the right hand side of this equation is a polynomial in $x_n$ in the ring $\mathfrak{J}_{n-1}[x_n]$. It has the property that the coefficient of the highest power of $x_n$ is unity and the other coefficients of the various powers of $x_n$ are non-units in $\mathfrak{J}_{n-1}$. Rückert calls such a polynomial "ausgezeichnet" which I shall translate as "fundamental".

Divisibility in $\mathfrak{J}_n$ is defined in the usual way, and two power series are regarded as equivalent if they differ only by a factor which is a unit in $\mathfrak{J}_n$. A power series is irreducible if it cannot be written as a product of two power series neither of which is a unit in $\mathfrak{J}_n$. With this understanding, factorisation of $P$, a power series in $\mathfrak{J}_n$, is unique and $P$ can be expressed as the product of a finite number of irreducible non-units of $\mathfrak{J}_n$. For fundamental polynomials of the ring $\mathfrak{J}_{n-1}[x_n]$, Rückert shows that their non-unit factors are also fundamental polynomials and that factorisation into fundamental factors is unique to within multiplication by a unit.

Consider the equation of an irreducible primal in $S_n$ given, in terms of non-homogeneous coordinates, by an algebraic equation

$$f(x_1, \cdots, x_n) = 0.$$

Suppose further that the origin lies on this locus and that the coordinate axes are chosen sufficiently generally for $f(x_1, \cdots, x_n)$ to be regular in $x_n$. $f(x_1, \cdots, x_n)$ is an irreducible polynomial in the ring $K[x_1, \cdots, x_n]$ but, regarded as belonging to the ring $K\{x_1, \cdots, x_n\}$ in which it is a non-unit, $f(x_1, \cdots, x_n)$ may be reducible and have a representation as a product of irreducible factors in $K\{x_1, \cdots, x_n\}$

$$f = f_1 f_2 \cdots f_n \quad (f_i \text{ is in } K\{x_1, \cdots, x_n\},\ i = 1, \cdots, n). \tag{1}$$

Now there exists a unit $E$ in $K\{x_1, \cdots, x_n\}$ such that

$$Ef = x_n^m + A_1 x_n^{m-1} + \cdots + A_{m-1} X_n + A_m$$

where the right hand side of this equation is a fundamental polynomial in $\mathfrak{J}_{n-1}[x_n]$. Consequently $E.f$ factorises uniquely as a product of irreducible fundamental factors $G_i$ in $\mathfrak{J}_{n-1}[x_n]$, $(i = 1, \cdots, k)$, and

$$Ef = G_1 G_2 \cdots G_k. \tag{2}$$

Now if $f$ is regular in $X_n$ so also are each of the factors $f_i$ $(i = 1, \cdots, h)$. Therefore there exist units $E'_i$ in $\mathfrak{J}_n$ such that

$$E'_i f_i = G'_i$$

where $G'_i$ is a fundamental polynomial of $\mathfrak{J}_{n-1}[x_n]$. Since $f_i$ is irreducible in $\mathfrak{J}_n$, $G'_i$ must be an irreducible fundamental polynomial. This is true for each $i = 1, \cdots, h$. Therefore

$$E' f = G'_1 G'_2 \cdots G'_h \tag{3}$$

where

$$E' = E'_1 E'_2 \cdots E'_h.$$

Equations (2) and (3) show that

$$G'_1 G'_2 \cdots G'_h = E'' G_1 G_2 \cdots G_k$$

where $E'' = E' E^{-1}$ is a unit in $\mathfrak{J}_n$. We deduce that $h = k$ and the set $G'_i$ are the same as the set $G_i$, apart from unit factors.

Incidentally the uniqueness of irreducible factors of an element of $\mathfrak{J}_n$ follows immediately from the uniqueness property of the fundamental factors of a fundamental polynomial in $\mathfrak{J}_{n-1}[x_n]$. In this case, for example, the factors $f_1, \cdots, f_h$ must be equivalent to the unique set $G_1, \cdots, G_h$.

The analytic equations

$$f_i = 0 \quad (i = 1, 2 \cdots, h)$$

define the irreducible sheets of the primal through 0, these sheets being equally well-defined by the equations

$$G_i = 0 \quad (i = 1, 2 \cdots, h).$$

Since the original polynomial $f(x_1, \cdots, x_n)$ is assumed to be irreducible in $K[x_1, \cdots, x_n]$ it follows that the factors $f_1, f_2, \cdots, f_h$ in $\mathfrak{J}_n$ are all distinct ([29] p.352).

Since $\mathfrak{J}_n$ is a unique factorisation domain it is also an integral domain which is integrally closed in its quotient field.

Before investigating the local normality of a surface in $S_3$ at a given point I require the following two results.

**Lemma 1** *Let $f(z)$ be a polynomial in $z$ of degree $n$ with coefficients in the unique factorisation domain $R$. Suppose that $f(z)$ is reducible in $R[z]$ and has the factorisation*

$$f(z) = f_1(z) f_2(z) \cdots f_h(z).$$

*If $D$ is the $z$-discrminant of $f(z)$, $D_i$ is the $z$-discriminant of $f_i(z)$ $(i = 1, \cdots, h)$ and $R_{k_1,k_2}$ is the discriminant of $f_{k_1}(z)$ and $f_{k_2}(z)$ with respect to $z$, then*

$$D = \prod_{k=1,\cdots,h} D_k \left\{ \prod_{k_1,k_2=1,\cdots,h,k_1>k_2} R_{k_1,k_2} \right\}^2. \tag{4}$$

For suppose the roots of the equation $f(z) = 0$ are $\zeta_1, \cdots, \zeta_n$, these roots belonging to some extension of the quotient field of $R$. then

$$\begin{aligned} f &= (z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_n) \\ f_1 &= (z-\zeta_1)\cdots(z-\zeta_{\alpha_1}) \\ f_2 &= (z-\zeta_{\alpha_1+1})\cdots(z-\zeta_{\alpha_2}) \\ &. \\ &. \\ &. \\ f_h &= (z-\zeta_{\alpha_{h-1}+1})\cdots(z-\zeta_n). \end{aligned}$$

Also

$$D = \prod_{i,j=1,\cdots,n,i\neq j} (\zeta_i - \zeta_j)$$

$$D_k = \prod_{\alpha_{k-1}<i\neq j\leq\alpha_k} (\zeta_i - \zeta_j) \quad (k = 1, \cdots, h)$$

$$R_{k_1,k_2} = \prod_{\alpha_{k_1-1}<i\leq\alpha_{k_1},\ \alpha_{k_2-1}<j\leq\alpha_{k_2}} (\zeta_i - \zeta_j) \quad (k_1 \neq k_2 \ \ k_1, k_2 = 1, \cdots, h).$$

The result stated follows by considering these last three equations.

**Lemma 2** *If $(\xi_1, \cdots, \xi_n)$ are non-homogeneous coordinates of a generic point of a variety $V$ in $S_n$ and if $P$ is a point of $V$ at which $V$ is locally normal, then $Q(P)$, the quotient ring of $P$ in $V$, is given by*

$$Q(P) = K(\xi_1, \cdots, \xi_n) \cap K\{\xi_1, \cdots, \xi_n\}$$

*where $K\{\xi_1, \cdots \xi_n\}$ is the ring of power series convergent in the neighbourhood of $P$.*

I would like to give two short proofs based on results proved by André Weil. I shall assume non-homogeneous coordinates chosen so that $P$ is the origin of coordinates.

*First proof.* Obviously

$$Q(P) \subseteq K(\xi_1, \cdots \xi_n) \cap K\{\xi_1, \cdots, \xi_n\}$$

since $Q(P) \subset K(\xi_1, \cdots, \xi_n)$ and every element of $Q(P)$ is expressible in the form $a/b$ where $a$, $b$ belong to $K[\xi_1, \cdots, \xi_n]$ and $b$ is a unit of $K\{\xi_1, \cdots, \xi_n\}$.

If $u$ is in $K(\xi_1, \cdots, \xi_n) \cap K\{\xi_1, \cdots, \xi_n\}$ then $u$ has a unique specialisation as $\xi_i \to 0$ $(i = 1, \cdots, n)$. Therefore $u$ is integrally dependent on $Q(P)$ ([20] p41 Prop. 22) and consequently is in $Q(P)$ since, by hypothesis the quotient ring of $P$ is integrally closed.

*Second proof.*

$$Q(P) \subseteq K(\xi_1, \cdots \xi_n) \cap K\{\xi_1, \cdots, \xi_n\}$$

as before. Suppose $u$ is integrally closed in $K(\xi_1, \cdots \xi_n) \cap K\{\xi_1, \cdots, \xi_n\}$. $u$ is in $K(\xi_1, \cdots, \xi_n)$ and $Q(P)$ is integrally closed. Therefore either $u$ or $\dfrac{1}{u}$ is in $Q(P)$ ([20] p.268 Prop. 1). Let us assume that $u$ is not in $Q(P)$ and therefore that $u = a/b$ where $a$ and $b$ are in $K[\xi_1, \cdots, \xi_n]$ with $a \neq 0$ at $P$ and $b = o$ at $P$. Also $a/b = \alpha$ where $\alpha$ is in $K\{\xi_1, \cdots, \xi_n\}$. Therefore

$$a(\xi_1, \cdots, \xi_n) = b(\xi_1, \cdots, \xi_n)\alpha(\xi_1, \cdots, \xi_n).$$

When $\xi_i \to 0$ $(i = 0, \cdots, n)$ the expression on the left hand side of this equation does not become zero whereas the expression on the right hand side does become zero and hence a contradiction arises. Therefore $u$ must belong to $Q(P)$.

In fact I require to use this lemma in the specialised form of the corollary.

**Corollary** *If $x_1, \cdots, x_n$ are $n$ independent determinates over $K$, then the elements of the intersection $K(x_1, \cdots, x_n) \cap K\{x_1, \cdots, x_n\}$ can be written in the form $\dfrac{a(x_1, \cdots, x_n)}{b(x_1, \cdots, x_n)}$ where $a(x_1, \cdots, x_n)$ and $b(x_1, \cdots, x_n)$ belong to $K[x_1, \cdots, x_n]$ and $b$ does not vanish when $x_1 = \cdots = x_n = 0$.*

This follows if we consider $x_1, \cdots, x_n$ as the coordinates of a non-homogeneous generic point of a space of dimension $n$.

If $f(x, y, z)$ is an irreducible polynomial in the ring $K[x, y, z]$, the equation $f(x, y, z) = 0$ represents an irreducible algebraic surface in $A_3$ where $x$, $y$, $z$ are non-homogeneous coordinates. Suppose that $O$, the origin of coordinates, lies on this surface. It may happen that $f(x, y, z)$ is reducible in the ring $K\{x, y, z\}$. Suppose that its factors are given in the form of equation (2) as

$$f(x, y, z) = f_1(x, y; z) f_2(x, y; z) \cdots f_h(x, y; z) E,$$

where $E$ is a unit of $K\{x, y, z\}$ and $f_i(x, y; z)$ is in $K\{x, y\}[z]$ $(i = 1, \cdots, h)$. $D$, the $z$-discriminant of $f(x, y, z)$ is a polynomial in $K[x, y]$ and has for its factors in $K\{x, y\}$ the $z$-discriminants $D_i$ of $f_i(x, y; z)$ $(i = 1, \cdots, h)$ and the $z$-resultants $R_{k_1,k_2}$ of $f_{k_1}(x, y; z)$ and $f_{k_2}(x, y; z)$ $(k_1 \neq k_2;\ k_1, k_2 = 1, \cdots, h)$. This follows on account of Lemma 1. Using again Zariski's result ([29] p.352) we deduce that $D$ has a repeated factor in $K[x, y]$ vanishing when $x = y = 0$ if and only if $D$ has a repeated factor in $K\{x, y\}$. This implies either that $h$ is greater than one, or that one of the $D_i$ has a repeated factor in $K\{x, y\}$. From this the obvious remark follows, that a multiple curve passes through a point $O$ on the surface if and only if two analytic sheets pass through $O$ and intersect in a branch of this multiple curve, or a multiple curve branch through $O$ lies on one of the analytic sheets.

Consider now the local normality of a surface in $S_3$ at a given point. A surface which is affinely normal is locally normal at every finite point (§3.1 Theorem 2). If therefore a surface contains a multiple curve there must be some points at which it is not locally normal. Such points are those on the multiple curve $C$. The surface is certainly not locally normal along the multiple curve ([20] p 269 Prop. 2). Consequently it cannot be locally normal at any point of $C$. †

On the other hand an irreducible surface $F$ in $S_3$ is locally normal at every point which does not lie on a multiple curve.

For consider such a point $O$ on the surface $F$ in $S_3$. Choosing $O$ as the origin of sufficiently general non-homogeneous coordinates $x$, $y$, $z$ the equation of $F$ can be written in the form

$$f(x,y,z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \quad (a_i \text{ is in } K[x,y],\ i = 1, \cdots, n)$$

where $D(x,y)$, the $z$-discriminant of $f(x,y,z)$ has no repeated factor in $K[x,y]$ which vanishes when $x = y = 0$. Also since $O$ does not lie on a multiple curve of $F$, $f(x,y,z)$ is irreducible in $K\{x,y,z\}$.

$(x,y,\zeta)$, where $\zeta$ is algebraic over $K(x,y)$ and is determined by the equation $f(x,y,z) = 0$, is a non-homogeneous generic point of $F$. Consider an element $u$ of $K(x,y,\zeta)$ which is integrally dependent on $Q(O)$, the quotient

†In general if a variety $V$ is not locally normal along any subvariety $W_1$, it is not locally normal along any subvariety $W_2$ contained in $W_1$. Choose a non-homogeneous coordinate system for which $W_2$, and therefore $W_1$, is at a finite distance. Let $\sigma$ denote the integral domain of $V$, $\Sigma$ its function field and $\sigma^*$ the integral closure of $\sigma$ in $\Sigma$. $W_2$ is contained in $W_1$ and therefore $Q(W_2)$, the quotient ring of $W_2$ on $V$, is contained in $Q(W_1)$, the quotient ring of $W_1$ on $V$. If we suppose $V$ is locally normal along $W_2$ we should have

$$\sigma^* \subseteq Q(W_2) \quad (\S 3.1 \text{ Lemma } 2)$$

and since $Q(W_2) \subseteq Q(W_1)$ this would imply

$$\sigma^* \subseteq Q(W_1)$$

and therefore, on account of Lemma 2,§3.1, $V$ would be locally normal along $W_1$ in contradiction to the original hypothesis.

ring of $O$ on $F$. $u$ satisfies an equation

$b_0u^m+b_1u^{m-1}+\cdots+b_{m-1}u+b_m=0$ ($b_i$ is in $K[x,y,\zeta]$, $i=0,1,\cdots,m, b_0\neq 0$ at 0).

Multiplying this equation by $b_0^{m-1}$ we obtain

$$(b_0u)^m+b_1(b_0u)^{m-1}+\cdots+b_{m-1}(b_0u)+b_mb_0^{m-1}=0. \qquad (5)$$

Now $b_0u$ is an element in the quotient field of $K\{x,y\}[\zeta]$ and the coefficient $b_1,\cdots,b_{m-1}b_0^{m-2},b_mb_0^{m-1}$ of the various powers of $b_0u$ in this equation (5) are in the ring $K\{x,y\}[\zeta]$. The $z$-discriminant of $f(x,y,z)$ has no repeated factor in $K\{x,y\}$. Otherwise, on account of Zariski' result ([29] p. 352) this same discriminant would have a factor in $K[x,y]$ vanishing when $x=y=0$. If we now apply the result of Theorem 1 in §4.5 with $K\{x,y\}$ in place of $R$ we find that $b_0u$ is an element of $K\{x,y\}[\zeta]$ since equation (5) shows that $b_0u$ is integrally dependent on $K\{x,y\}[\zeta]$.

Therefore

$$b_0u=\alpha_1\zeta^{n-1}+\cdots+\alpha_{n-1}\zeta+\alpha_n \qquad (\alpha_i \text{ is in } K\{x,y\},\ i=1,\cdots,n)$$

and also $b_0u$ is an element of $K(x,y,\zeta)$ and can therefore be written in the form

$$b_0u=\frac{c_1}{c}\zeta^{n-1}+\cdots+\frac{c_{n-1}}{c}\zeta+\frac{c_n}{c} \qquad (c,\ c_1,\cdots,c_n \text{ are in } K[x,y]).$$

It follows that $\zeta$ satisfies the equation

$$\left(\alpha_1-\frac{c_1}{c}\right)z^{n-1}+\cdots+\left(\alpha_{n-1}-\frac{c_{n-1}}{c}\right)z+\left(\alpha_n-\frac{c_n}{c}\right)=0.$$

The coefficients in this algebraic equation for $z$ are in the quotient field of the integral domain $K\{x,y\}$. But $f(x,y,z)=0$ is the irreducible equation for $z$ with coefficients in this field of lowest possible order.* It follows that

$$\alpha_i - \frac{c_i}{c} = 0 \quad (i = 1, \cdots, n)$$

or

$$\alpha_i = \frac{c_i}{c} \quad (i = 1, \cdots, n).$$

$\alpha_i$ is an element of $K(x, y)$ and also of $K\{x, y\}$ and so as a consequence of the corollary of Lemma 2 above,

$$\alpha_i = \frac{d_i}{e_i},$$

$d_i$ and $e_i$ belong to $K[x, y]$ and $e_i$ does not vanish when$x = y = 0$. This is true for $i = 1, \cdots, n$. Therefore

$$b_0 u = \sum_{i=1}^{n} \frac{d_i}{e_i} \zeta^{n-i}$$

and if $e = e_1 e_2 \cdots e_n$,

$$b_0 u = \frac{\sum_{i=1}^{n} d_i' \zeta^{n-i}}{e} \quad \text{where } d_i' \text{ is in } K[x, y] \quad (i = 1, \cdots, n)$$

and $e$ does not vanish when $x = y = 0$.

It follows that

$$u = \frac{\sum_{i=1}^{n} d_i' \zeta^{n-i}}{b_0 e}.$$

The numerator in the expression on the right hand side is in the ring $K[x, y, \zeta]$ and the denominator is in $K[x, y, \zeta]$ and does not vanish when $(x, y, \zeta)$ specialise to the values $(0, 0, 0)$. It follows that $u$ belongs to the quotient ring of $O$ on $F$ which is consequently integrally closed.

These results may be stated as follows.

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*Margin note says "Doesn't work: degree of (undecipherable word) eqn over $K\{x, y\}[\zeta]$ need not be $n$."

**Theorem 1** *A necessary and sufficient condition for an irreducible surface $F$ in $S_3$ to be locally normal at a point $O$ is that no multiple curve of $F$ passes through $O$.*

A similar result is true for primals in $S_n$.

**Theorem 2** *A necessary and sufficient condition for an irreducible primal $V_{n-1}$ in $S_n$ to be locally normal at a point $O$ is that no multiple subvariety of $F$ of dimension $n-2$ passes through $O$.*

§4.8.Normality of curves

From Zariski's general result that the singular manifold of a locally normal variety of dimension $r$ is of dimension less than or equal to $r-2$ it follows that no curve with a singular point can be locally normal. On the other hand a variety is locally normal at any simple point. It follows that a necessary and sufficient condition for a curve to be locally normal is that the curve is nonsingular. The transformation from any irreducible algebraic curve to one of its derived normal models provides us with a birational transformation which will resolve the singularities of the curve.

An algebraic curve in a plane is a special case of a primal and if it is nonsingular then the curve is not only locally, but also projectively, normal. This is not the case for the skew curve in ordinary space. Consider, for example, the rational quartic curve in $S_3$ given parametrically by the equation $x_0 : x_1 : x_2 : x_3 = 1 : \theta : \theta^2 : \theta^4$. Taking $x_0 = 0$ as the plane at infinity we may consider $(\theta, \theta^2, \theta^4)$ as the non-homogeneous coordinates of a generic point. $K[\theta, \theta^2, \theta^4] = K[\theta]$ is integrally closed in $K(\theta)$. Consequently the curve is affinely normal for this choice of the plane at infinity. In fact the curve is nonsingular and therefore locally normal. A homogeneous generic point for the curve is $(\varphi, \theta\varphi, \theta^2\varphi, \theta^4\varphi)$ where $\theta$, $\varphi$ are independent indeterminates over $K$.

$K[\varphi, \theta\varphi, \theta^2\varphi, \theta^4\varphi]$ is not integrally closed in $K(\theta, \varphi)$ its quotient field. For consider the element $\varphi\theta^3$ of $K(\theta, \varphi)$. This element is integrally dependent on $K[\varphi, \theta\varphi, \theta^2\varphi, \theta^4\varphi]$ since $z = \varphi\theta^3$ satisfies the equation $z^2 - \varphi\theta^2 \cdot \varphi\theta^4 = 0$. But $\varphi\theta^3$ does not belong to $K[\varphi, \theta\varphi, \theta^2\varphi, \theta^4\varphi]$ and therefore this rational quartic curve is not projectively normal. It can be obtained by projection from the rational quartic curve in $S_4$ with normalised generic point $(1, \theta, \theta^2, \theta^3, \theta^4)$, a curve which is projectively normal.

§4.9. A non-singular curve in $S_3$ defined by the complete intersecion of two cones.

One of the simplest types of curves in $S_3$ is the curve of residue zero. This is a non-singular curve which is a complete simple intersection of two surfaces. A very special case of such a curve is one which is the complete intersection of two cones. I shall give an algebraic proof showing that if two cones in $S_3$ intersect simply and completely in an irreducible, nonsingular curve, then this curve is projectively normal.

I shall begin by proving the following theorem.

**Theorem 1** *$R$ is an integral domain which is a unique factorisation domain. $K$ is the quotient field of $R$. $\alpha$, $\beta$ are quantities which are algebraic over $K$ and integrally dependent on $R$ satisfying irreducible equations*

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

$$g(z) = z^m + b_1 z^{m-1} + \cdots + b_m = 0 \ \ (a_i,\ i = 1, \cdots, n;\ b_j,\ j = 1, \cdots, m \text{ are in } R)$$

*respectively. $D_{(f)}$, $D_{(g)}$ denote the $z$-discriminants of $f(z)$, $g(z)$ respectively. If $R[\alpha]$ and $R[\beta]$ are both integrally closed in their quotient fields $K(\alpha)$ and $K(\beta)$, and if $D_{(f)}$ and $D_{(g)}$ have no factors in $R$ in common, other than the units of $R$, then $R[\alpha, \beta]$ is integrally closed in $K(\alpha, \beta)$, its quotient field.*

If $\xi$ is any element of $K(\alpha, \beta)$, then $\xi = s/t$ where $s$ belongs to $R[\alpha, \beta]$ and $t$ to $R$. For, as I have shown earlier, we can certainly write $\xi = s/t$ with $s$ in $K[\alpha, \beta]$ and $t$ in $K$. The required form is obtained from this since every element of $K$ is of the form $a/b$ with $a$, $b$ in $R$ and $b \neq 0$.

Suppose $\xi$ is any element of $K(\alpha, \beta)$ which is integrally dependent in $R[\alpha, \beta]$. Since $\alpha$ and $\beta$ are integral over $R$, the ring $R[\alpha, \beta]$ is integral over $R$ and on account of the transitive property of integral dependence, $\xi$ is integral over $R$.

We may write

$$\xi = \sum_{i,j} \rho_{i,j} \alpha^i \beta^j \qquad \left( \rho_{i,j} \text{ is in } K, \quad \begin{array}{l} i = 0, 1, \cdots, (n-1) \\ j = 0, 1, \cdots (m-1) \end{array} \right) \tag{1}$$

Let $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_n$ be the $n$ conjugate values satisfying $f(z) = 0$, and let $\beta = \beta_1, \beta_2, \cdots, \beta_m$ be the $m$ conjugate values satisfying $g(z) = 0$. Define the quantities $\xi_{1,\nu}$ $(\nu = 1, \cdots, m)$ by the equations

$$\xi_{1,\nu} = \sum_{i,j} \rho_{i,j} \alpha_1^i \beta_\nu^j \qquad (\nu = 1, \cdots m) \tag{2}$$

Solving these equations for the $m$ quantities $\sum_{i=0}^{n-1} \rho_{i,j} \alpha_1^i$ $(j = 0, 1, \cdots (m-1))$ we find

$$\sum_{i=0}^{n-1} \rho_{i,j} \alpha_1^i = \frac{\sum_{\nu=1}^{m} A_{\nu,j} \xi_{1,\nu}}{D_{(g)}^{1/2}} \tag{3}$$

where $A_{\nu,j}$ $(\nu = 1, \cdots, m;\ j = 0, 1, \cdots (m-1))$ are polynomials in $\beta_\nu$ with coefficients in $R$ and therefore they are integral over $R$. Multiplying equation (3) by $D_{(g)}$ we obtain

$$\sum_{i=0}^{n-1} D_{(g)} \rho_{i,j} \alpha_1^i = \sum_{\nu=1}^{m} D_{(g)}^{1/2} A_{\nu,j} \xi_{1,\nu} \qquad (j = 0, 1, \cdots, (m-1)) \tag{4}$$

The left hand expression in this equation belongs to the field $K(\alpha)$ and the right hand expression is integral over $R$ and therefore a fortiori over

$R[\alpha]$. Therefore, since $R[\alpha]$ is, by hypothesis, integrally closed in $K(\alpha)$, $\sum_{i=0}^{n-1} D_{(g)}\rho_{i,j}\alpha^i$ is in $R[\alpha]$.

Suppose

$$D_{(g)}\rho_{i,j} = \frac{a_{i,j}}{b_j} \text{ where } a_{i,j}, b_j \text{ are in } R \quad \begin{pmatrix} i = 0, 1, \cdots, (n-1) \\ j = 0, 1, \cdots, (m-1) \end{pmatrix} \tag{5}$$

Any element of $R[\alpha]$ can be written as a polynomial in $\alpha$ of degree less than or equal to $n-1$ with coefficients in $R$, since any term $\alpha^p$ with $p$ greater than or equal to $n$ can always be reduced to a sum of terms of lower degree by the substitution

$$\alpha^n = -a_1\alpha^{n-1} - a_2\alpha^{n-2} - \cdots - a_n.$$

We can therefore write

$$\sum_{i=0}^{n-1} \frac{a_{i,j}\alpha^i}{b_j} = \sum_{k=0}^{n-1} c_k\alpha^k \text{ with } c_k \text{ in } R,\ k = 0, 1, \cdots (n-1)\ (j = 0, 1, \cdots, (m-1)) \tag{6}$$

or

$$\sum_{i=0}^{n-1} a_{i,j}\alpha^i = \sum_{k=0}^{n-1} b_j c_k \alpha^k \qquad (j = 0, 1, \cdots, (m-1)). \tag{7}$$

In this relation we must have $a_{i,j} = b_j c_i$ $(i = 0, 1, \cdots (m-1),\ j = 0, 1, \cdots, (n-1))$, otherwise we should have an equation with coefficients in $R$ of degree less than $n$ which is satisfied by $\alpha$. Therefore

$$D_{(g)}\rho_{i,j} \text{ is in } R \text{ for } i = 0, 1, \cdots (n-1);\ j = 0, 1, \cdots (m-1).$$

Similarly

$$D_{(f)}\rho_{i,j} \text{ is in } R \text{ for } i = 0, 1, \cdots (n-1);\ j = 0, 1, \cdots (m-1).$$

Let us write $\rho_{i,j} = \frac{\lambda_{i,j}}{\mu_{i,j}}$ where $\lambda_{i,j}$, $\mu_{i,j}$ are in $R$ for each $i$, $j$ and assume that for a given pair $i$, $j$, $\lambda_{i,j}$ and $\mu_{i,j}$ have no factors in $R$ in common.

We see that for each $i$, $j$, $\mu_{i,j}$ must divide both $D_{(g)}$ and $D_{(f)}$, and since we have assumed that $D_{(g)}$ and $D_{(f)}$ have no factors in common in $R$, $\mu_{i,j}$ must be a unit of $R$ for each $i$, $j$. Consequently $\rho_{i,j}$ belongs to $R$ ($i = 0, 1, \cdots (n-1)$; $j = 0, 1, \cdots (m-1)$) and therefore $\xi$ belongs to $R[\alpha, \beta]$ and the theorem is proved.

An obvious generalisation can be made as follows. If $R$ and $K$ are as above, and if $\alpha_1, \cdots, \alpha_s$ are quantities algebraic over $K$ and integrally dependent on $R$ satisfying relations $f_i(z) = 0$ ($i = 1, \cdots, s$) respectively, where the leading coefficient in each equation is unity, then, if $R[\alpha_i]$ is integrally closed in $K(\alpha_i)$ for each $i = 1, \cdots, s$, and if $D_{(f_i)}$ ($i = 1, \cdots, s$) have no common factors in $R$, then $R[\alpha_1, \cdots, \alpha_s]$ is integrally closed in $K(\alpha_1, \cdots \alpha_s)$.

From this result we can deduce the following.

**Theorem 2** *If two cones in $S_3$ intersect simply and completely in a non-singular irreducible curve $\gamma$, then the curve is projectively normal.*

Let $(x, y, z, t)$ be homogeneous coordinates in $S_3$ and choose the vertices of the two cones as vertices $Z$ and $T$ of the tetrahedron of reference. the equations of the two cones can then be written in the form

$$\begin{aligned} f(x,y,z) &\equiv a_0 z^n + a_1(x,y) z^{n-1} + \cdots + a_n(x,y) = 0 \\ g(x,y,t) &\equiv b_0 t^m + b_1(x,y) t^{m-1} + \cdots + b_m(x,y) = 0 \end{aligned} \tag{8}$$

where $a_0$, $b_0$ are constants in $K$, the ground field of complex numbers, and $a_i(x, y)$, $b_j(x, y)$ are forms of order $i$, $j$ respectively in $x$, $y$; $i = 1, \cdots, n$; $j = 1, \cdots m$. We shall assume that $\gamma$ is not a plane curve.

Neither $a_0$ nor $b_0$ can vanish. For if $a_0 = 0$ then the cone $f(x, y, z) = 0$ passes through $Z$, the vertex of the other cone, and since $Z$ is a singular point of order $m$ for this second cone, this would mean that $\gamma$ would have an $m$-ple point at $z$ and since $m$ is greater than one this is untrue since $\gamma$ is assumed to be non-singular.

Since $a_0 \neq 0$ and $b_0 \neq 0$ it is permissible to assume that the equations of the two cones are in the form given above (8) with $a_0 = b_0 = 1$.

A homogeneous generic point of the first cone is given by $(x, y, \alpha, t)$ where $\alpha$ is algebraic over $K(x, y)$ and is determined by $f(x, y, \alpha) = 0$, and a homogeneous generic point of the second cone is $(x, y, z, \beta)$ where $\beta$ is algebraic over $K(x, y)$ and is determined by $g(x, y, \beta) = 0$. The coordinates $(x, y, \alpha, \beta)$ define a homogeneous generic point of the curve $\gamma$.

The two cones are both free from multiple curves and so the curve of intersection of the cone $f(x, y, z) = 0$ with the plane $t = 0$ is nonsingular and, since it is a primal, it is therefore normal according to a previous result (Theorem 4, §4.6). In the plane $t = 0$, a homogeneous generic point of this curve is $(x, y, \alpha)$ and so $K[x, y, \alpha]$ is integrally closed in $K(x, y, \alpha)$. Similarly we can show that $K[x, y, \beta]$ is integrally closed in $K(x, y, \beta)$.

To show that $\gamma$ is normal we have only to show that $D_{(f)}$ and $D_{(g)}$, the $z$-discriminant of $f(x, y, z)$ and the $t$-discriminant of $g(x, y, t)$ respectively, have no factors in $K[x, y]$ in common and we may apply the preceding theorem writing $K[x, y]$ for $R$ and $K(x, y)$ for $K$.

$D_{(f)}$ and $D_{(g)}$ are both homogeneous polynomials in $x$, $y$ with coefficients in $K$. Each of these forms factorises into a product of linear forms $ax + by$ with $a$, $b$ in $K$ since $K$ is here the field of complex numbers and is algebraically closed. Then the equation $D_{(f)} = 0$ represents a pencil of planes through the side $ZT$ of the fundamental tetrahedron and each of these planes touches the cone $f(x, y, z) = 0$ along one of its generators.

If $D_{(f)}$ and $D_{(g)}$ have a linear factor in common, this means that there is a plane touching each cone along a generator. These two generators intersect in a point $P$ at which this plane is a tangent plane to each cone. Therefore the two cones touch at $P$ and $P$ is a multiple point of their curve of intersection. Since, by hypothesis we assume the intersection $\gamma$ to be without multiple points it follows that $D_{(f)}$ and $D_{(g)}$ have no common factors in $K[x, y]$ and hence $\gamma$ is normal.

This result generalises immediately to give the following result.

**Theorem 3** *In $S_n$ consider the two cones each of dimension $n-1$ whose simple complete intersection is an irreducible variety $V_{n-2}$ containing no multiple subvarieties of dimension $n-3$. Then this $V_{n-2}$ is normal.*

# Chapter 5.

# Ideas for further investigation

§5.1 Three possible methods

As far as I can discover from the existing literature, general theorems completely describing geometrical properties which characterise locally normal and normal varieties are restricted to Muhly's theorem [9] for locally normal varieties*. When one is confronted with a general variety, the completeness or incompleteness of a linear series of subvarieties on it is not at all immediately evident, and it seems that alternative criteria should be investigated.

There are three possible methods which suggest themselves to me.

1. By extending the ground field over which a given variety $V$ is defined, a new variety $V'$ of lower dimension is determined. The relation between this variety and the original variety with regard to normality should be investigated. Possible information about the normality of $V$ might be deduced by considering the case, for example, when $V'$ is an algebraic curve.

2. By considering the projection of a variety $V_r$ in $S_n$ into a lower primal $V'_r$ in $S_{r+1}$, results might possibly be deduced from known properties of primals.

3. By considering prime sections it might be possible to derive information about the normality of $V_r$ in $S_n$. More generally, sections by any generic flat space could be investigated and, in particular, sections by a generic $S_{n-r+1}$, which are algebraic curves, might form a useful study.

*Probably, "varies" in the ms is a misprint.

§5.2 Extension of the ground field of a variety

**Theorem 1** *A variety $V_r$ of dimension $r$ in $S_n$ is given by a set of equations $f_i(x_1, \cdots, x_n) = 0$ $(i = 1, \cdots, h)$ where the $f_i(x_1, \cdots x_n)$ $(i = 1, \cdots, h)$ are polynomials in the ring $K[x_1, \cdots, x_n]$, and $(x_1, \cdots x_n)$ are non-homogeneous coordinates of a point in $S_n$. If $V_r$ is irreducible over the ground field $K$, then $V'$, the variety defined by the same set of equations but regarded as being defined over the ground field $K(x_1, \cdots, x_d)$ $(d < r)$ is irreducible and of dimension $r - d$.*

Let $(\xi_1, \cdots, \xi_n)$ be the non-homogeneous coordinates of a generic point of $V_r$, where $\xi_1, \cdots, \xi_r$ are algebraically independent over $K$. Then

$$f_i(\xi_1, \cdots, \xi_d, x_{d+1}, \cdots, x_n) = 0$$

regarded as an equation in $x_{d+1}, \cdots x_n$ is satisfied by the values $\xi_{d+1}, \cdots \xi_n$ $(i = 1, \cdots, h)$. On the other hand if $F(x_{d+1}, \cdots, x_n)$ is in $K(\xi_1, \cdots, \xi_d)[x_{d+1}, \cdots, x_n]$ then

$$F(x_{d+1}, \cdots, x_n) = \frac{\Phi(\xi_1, \cdots, \xi_d, x_{d+1}, \cdots, x_n)}{\Psi(\xi_1, \cdots, \xi_d)}$$

where $\Phi(\xi_1, \cdots, \xi_d, x_{d+1}, \cdots, x_n)$ is in $K[\xi_1, \cdots, \xi_d, x_{d+1}, \cdots, x_n]$ and $\Psi(\xi_1, \cdots, \xi_d)$ is in $K[\xi_1, \cdots, \xi_d]$. If

$$F(\xi_{d+1}, \cdots, \xi_n) = 0$$

then

$$\Phi(\xi_1, \cdots, \xi_d, \xi_{d+1}, \cdots, \xi_n) = 0$$

showing that $\Phi(x_1, \cdots, x_n)$ vanishes over $V_r$. Therefore, applying the result of Hilbert's Zero Theorem (see [17] Vol II p 6 for one proof)

$$\{\Phi(x_1, \cdots x_n)\}^\rho = \sum_{i=1}^{h} A_i(x_1, \cdots, x_n) f_i(x_1, \cdots, x_n) \ \ (A_i \text{ is in } K[x_1, \cdots, x_n],\ i = 1, \cdots, h)$$

and therefore

$$\{F(x_{d+1}, \cdots, x_n\}^\rho = \sum_{i=1}^{h} B_i(x_{d+1}, \cdots, x_n) f_i(\xi_1, \cdots, \xi_d, x_{d+1}, \cdots, x_n) \quad (1)$$

where

$$B_i(x_{d+1}, \cdots, x_n) = \frac{A_i(\xi_1, \cdots, \xi_d, x_{d+1}, \cdots x_n)}{\{\Psi(\xi_1, \cdots, \xi_d)\}^\rho}.$$

The right hand side of equation (1) is in $K(\xi_1, \cdots, \xi_d)[x_{d+1}, \cdots, x_n]$ ($i = 1, \ldots, h$) provided that $\xi_1, \cdots, \xi_d$ are algebraically independent over $K$ which is assured since $\xi_1, \cdots, \xi_r$ are assumed to be algebraically independent and $d$ is less than $r$.

Therefore $F(x_{d+1}, \cdots, x_n)$ vanishes over $V'$ showing that $(\xi_{d+1}, \cdots, \xi_n)$ is a non-homogeneous generic point of $V'$ which must consequently be irreducible over $K(x_1, \cdots, x_d)$.

The dimension of $V'$ is the degree of transcendency of $K(\xi_1, \cdots, \xi_d, \xi_{d+1}, \cdots, \xi_n)$ over $K(\xi_1, \cdots, \xi_d)$. We know that the degree of transcendency of $K(\xi_1, \cdots, \xi_d)$ over $K$ is $d$ and also that the degree of transcendency of $K(\xi_1, \cdots, \xi_n)$ over $K$ is $r$. Therefore the dimension of $V'$ is equal to $r - d$ ([17] vol I p 212).

**Theorem 2** *If $V_r$ is affinely normal in $S_n$ then $V'$ is affinely normal in the $S_{n-d}$ in which $(x_{d+1}, \cdots, x_n)$ are non-homogeneous coordinates of a general point and $S_{n-d}$ is defined over $K(x_1, \cdots, x_d)$.*

Choose the non-homogeneous coordinates of a generic point of $V$ so that the first $r$ are algebraically independent. Let such a generic point be $(x_1, \cdots, x_d, x_{d+1}, \cdots, x_r, \xi_{r+1}, \cdots, \xi_n)$ where $\xi_{r+1}, \cdots \xi_n$ are algebraically dependent on $K(x_1, \cdots, x_r)$. Then $V$ and $V'$ have a common function field $\Sigma$ and

$$\Sigma = K(x_1, \cdots, x_r, \xi_{r+1}, \cdots, \xi_n) = K(x_1, \cdots .x_r)[\xi_{r+1}, \cdots, \xi_n].$$

The integral domain of $V$ is $\sigma = K[x_1, \cdots, x_r, \xi_{r+1}, \cdots, \xi_n]$ and the integral domain of $V'$ is $\sigma' = K(x_1, \cdots, x_d)[x_{d+1}, \cdots, x_r, \xi_{r+1}, \cdots, \xi_n]$.

Suppose $u$, an element of the function field of $V'$, is integrally dependent on $\sigma'$. Then

$$u = \frac{a(x_1, \cdots, x_r, \xi_{r+1}, \cdots, \xi_n)}{b(x_1, \cdots, x_r)}$$

where $a$ is in $K[x_1, \cdots, x_r, \xi_{r+1}, \cdots, \xi_n]$ and $b$ is in $K[x_1, \cdots, x_r]$, and $u$ satisfies an equation

$$u^m + \alpha_1 u^{m-1} + \cdots + \alpha_{m-1} u + \alpha_m = 0 \quad (\alpha_i \text{ is in } \sigma', \quad i = 1, \cdots, m). \tag{2}$$

$\alpha_i$ can be written in the form $\beta_i/\beta$ where $\beta_i$ is in $\sigma$, and $\beta$ is in $K[x_1, \cdots, x_d]$, $(i = 1, \cdots, m)$.

Multiplying equation (2) by $\beta^m$ we obtain the equation

$$(\beta u)^m + \beta_1(\beta u)^{m-1} + \beta_2 \beta (\beta u)^{m-2} + \cdots + \beta_m \beta^{m-1} = 0. \tag{3}$$

In this equation the coefficients of the powers of $\beta u$ are in $\sigma$ and we see that $\beta u$ depends integrally on $\sigma$. Since $\beta u$ depends on $\Sigma$, the function field of $V$, and $V$ is assumed to be affinely normal, this implies that $\beta u$ belongs to $\sigma$.

Therefore $\beta \frac{a}{b}$ is in $K[x_1, \cdots x_r, \xi_{r+1}, \cdots, \xi_n]$, so that $\frac{a}{b}$ is in

$$K(x_1, \cdots, x_d)[x_{d+1}, \cdots, x_r, \xi_{r+1}, \cdots, \xi_n] = \sigma'.$$

Hence $V'$ is affinely normal.

**Corollary** *It follows that if $V$ is locally normal then $V'$ also is locally normal since a variety is locally normal if it is affinely normal for every choice of the prime at infinity.*

In defining a point $P$ of a variety $V_r$ over an abstract field $K$, Zariski does not immediately confer geometric reality to the term "point" ([24] p.198). He considers a field $\Sigma = K(\xi_1, \cdots, \xi_n)$ of degree of transcendency $r$ over $K$

so that if $A_n$ is an $n$-dimensional affine space defined over $K$, then the $n$ quantities $(\xi_1, \cdots, \xi_n)$ are non-homogeneous coordinates of a generic point of some $r$-dimensional variety. He defines a point $P$ on $V$ to be associated with a prime zero-dimensional ideal $\mathcal{P}_0$ in the ring $\sigma = K[\xi_1, \cdots, \xi_n]$. In general, when $K$ is not algebraically closed, there will not exist elements $c_1, \cdots, c_n$ in $K$ such that $\xi_i \equiv c_i \ (\mathcal{P}_0) \ (i = 1, \cdots, n)$. The variety in $A_n$ corresponding to $\mathcal{P}_0$ is a zero-dimensional variety consisting of a set of conjugate points. If we extend the ground-field over which the space $A_n$ is defined the "point" $P$ may be represented by a finite set of points.

Zariski's definition of a simple variety is as follows.

- An $s$-dimensional subvariety $W_s$ of the $r$-dimensional variety $V_r$ is simple on $V_r$ if and only if the ideal of non-units in the quotient ring of $W_s$ has a basis of $r - s$ elements.

In the case of a ground field of characteristic zero, this definition is equivalent to the classical one.

- Suppose $V_r$ is defined by the non-homogeneous equations $f_i(x_1, \cdots, x_n) = 0 \ (i = 1, \cdots, h)$ and let $(\eta_1, \cdots, \eta_n)$ be a non-homogeneous generic point of the subvariety $W_s$ of $V_r$. $W_s$ is simple on $V_r$ if the matrix
$$\left[\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}\right]_{x_1=\eta_1, \cdots, x_n=\eta_n}$$
is of rank $n - r$.

In his paper "Algebraic varieties over ground fields of characteristic zero" [24], Zariski begins by pointing out that a subvariety $W_s$ of $V_r$ can be treated as a point $P'$ of a variety $V'_{r-s}$ by a suitable transcendental extension of the ground field $K$ to $K'$ in $\Sigma$, the function field of $V_r$. The original equations defining $V_r$ then define $V'_{r-s}$ over $K'$. It follows almost immediately from either definition that $W_s$ is simple for $V_r$ if and only if $P'$ is simple for $V'_{r-s}$.

Let us consider as an example the special case when by an extension of the ground field $K$ a variety $V$ is considered as a curve $V'$. Suppose in addition that $V_r$ has an irreducible multiple subvariety $W_{r-1}$. Let $(x_1, \cdots, x_r, \xi_{r+1}, \cdots, \xi_n)$ be a non-homogeneous generic point of $V_r$ and let $(x_1, \cdots, x_{r-1}, \eta_r, \cdots, \eta_n)$ be a generic point of $W_{r-1}$ where we assume $x_1, \cdots, x_r$ are algebraically independent. The point $(\eta) = (x_1, \cdots, x_{r-1}, \eta_r, \cdots, \eta_n)$ is a specialisation of the point $(\xi) = (x_1, \cdots, x_r, \xi_{r+1}, \cdots, \xi_n)$.

Suppose $f_i(x_1, \cdots, x_n) = 0$ $(i = 1, \cdots, h)$ are the defining equations of $V$. Then, since $W$ is multiple on $V$ the $(h \times n)$-matrix $\dfrac{\partial(f_1, \cdots, f_h)}{\partial(x_1, \cdots, x_{r-1}, \eta_r, \cdots, \eta_n)}$ is of rank less than $n - r$.

Now let these same equations define an irreducible curve over $K(x_1, \cdots, x_{r-1})$ in $S_{n-(r-1)}$. A generic point of this curve is $(x_r, \xi_{r+1}, \cdots, \xi_n)$ and, since $(\eta_r, \eta_{r+1}, \cdots, \eta_n)$ is a specialisation of this generic point it represents a point on the curve. This point is multiple on the curve since the rank of the $(h \times (n - r + 1))$ matrix $\dfrac{\partial(f_1, \cdots, f_h)}{\partial(\eta_r, \cdots, \eta_n)}$ is less than $n - r = n - (r - 1) - 1$. If, therefore we assume that a curve with a multiple point is known to be not locally normal the well-known result follows that any $V_r$ containing a multiple sub-locus of dimension $r - 1$ cannot be locally normal.

There appears to be no reason why one should exclude the extreme case when $d = r$. If $V$ is an irreducible variety of dimension $r$ then $V'$ is a set of conjugate points defined over the ground field $K(x_1, \cdots, x_r)$. Such a set of conjugate points is always affinely normal and consequently the converse statement that the affine normality of $V$ would follow from the affine normality of $V'$ cannot be true.

§5.3 Projection of a variety

In his tract on Modular Systems, Macaulay shows that the equations of a $V_r$ in an affine space $A_n$ in which $x_1, \cdots, x_n$ are non-homogeneous coordinates can be written in the form

$$f(x_1, \cdots, x_{r+1}) = 0,$$

$$\frac{\partial f}{\partial x_{r+1}} \cdot x_{r+i} = \varphi_i(x_1, \cdots, x_{r+1}) \qquad (i = 2, \cdots, (n-r)),$$

provided that the choice of axes is sufficiently general. Van der Waerden derives this parametric representation of a variety in a slightly modified form ([15] p.114 §20). He considers a normalised generic point $(1, \xi_1, \cdots, \xi_n)$ of $V_r$ in $S_n$ in which $\xi_1, \cdots, \xi_r$ are algebraically independent and $\xi_{r+1}, \cdots \xi_n$ algebraically dependent on $\xi_1, \cdots, \xi_r$. If $\Sigma = K(\xi_1, \cdots, \xi_n)$ and $P = K(\xi_1, \cdots, \xi_r)$ then $\Sigma = P(\xi_{r+1}, \cdots \xi_n)$. There exists a primitive element $\xi'_{r+1}$ where $\xi'_{r+1} = \xi_{r+1} + \alpha_{r+2}\xi_{r+2} + \cdots + \alpha_n \xi_n$ ($\alpha_i$ is in $P$, $i = r+1, \cdots, n$) such that $\Sigma = P(\xi'_{r+1})$. Making a non-singular linear transformation of coordinates we may write $\xi_{r+1}$ instead of $\xi'_{r+1}$ and then

$$\Sigma = P(\xi_{r+1}, \cdots, \xi_n) = P(\xi_{r+1}).$$

$\xi_{r+1}$ satisfies an algebraic equation

$$f^{(1)}(\xi_1, \cdots, \xi_r, \xi_{r+1}) = 0$$

and making this equation homogeneous by introducing $\xi_0$,

$$f(\xi_0, \xi_1, \cdots, \xi_{r+1}) = 0. \tag{1}$$

Now $\xi_{r+2}, \cdots, \xi_n$ are rational functions of $\xi_1, \cdots, \xi_{r+1}$ and consequently

$$\xi_{r+i} = \frac{\varphi_i^{(1)}(\xi_1, \cdots, \xi_{r+1})}{\psi^{(1)}(\xi_1, \cdots, \xi_{r+1})} \qquad (i = 2, \cdots, n-r).$$

Making this equation homogeneous and multiplying by $\psi_i$ we obtain

$$\psi_i(\xi_0, \cdots, \xi_{r+1}\xi_{r+i}) - \varphi_i(\xi_0, \cdots, \xi_{r+1}) = 0 \quad (i = 1, \cdots, n-r). \qquad (2)$$

The $n-r$ equations

$$\begin{aligned} &f(x_0, \cdots, x_{r+1}) = 0 \\ &\psi_i(x_0, x_1, \cdots, x_{r+1})x_{r+i} - \varphi_i(x_0, x_1, \cdots, x_{r+1}) = 0 \quad (i = 2, \cdots, n-r) \end{aligned} \qquad (3)$$

define an algebraic variety $D$ in $S_n$ of which $V$ is an irreducible component. In fact van der Warerden shows that if we write $D$ as a sum of two varieties $U$ and $V$

$$D = U \dot{+} V$$

then $U$ is the variety defined by those points of $D$ for which $\psi(x_0, x_1, \cdots, x_{r+1}) = 0$ where $\psi$ is the least common mutiple of the forms $\psi_i$ $(i = r+1, \cdots, n)$.

The equation $f(x_0, x_1, \cdots, x_{r+1}) = 0$ defines a cone with the space $S_{n-r-2}$ given by $x_0 = x_1 = \cdots = x_{r+1} = 0$ for vertex. If $O$ is a point in this space $S_{n-r-2}$ for which none of the coordinates $x_{r+1}, \cdots, x_n$ is zero then every other equation in the set (3) defines a primal which is such that a general line through $O$ meets it in only one further point. For consider the point $(0, 0, \cdots, 0, y_{r+1}, \cdots, y_n)$ and join it to the point $(z_0, z_1, \cdots, z_n)$. A general point on this line has coordinates $(z_0, \cdots, z_{r+1}, z_{r+2} + \lambda y_{r+2}, \cdots, z_n + \lambda y_n)$ which lies on the primal given by

$$\psi_i(x_0, \cdots, x_{r+1})x_{r+i} - \varphi_i(x_0, \cdots, x_{r+1}) = 0 \quad (2 \leq i \leq n-r)$$

if

$$\psi_i(z_0, \cdots, z_{r+1})(z_{r+i} + \lambda y_{r+i}) - \varphi_i z_0, \cdots, z_{r+1}) = 0.$$

This last equation for given values of the $y$'s and $z$'s determines just one value for $\lambda$. Such a primal is called a monoid.

As an example of this parametric representation consider a twisted cubic curve in $S_3$ with a generic point $(1, \theta^2, \theta^3, \theta)$. Its equations can be written in the form (3) as

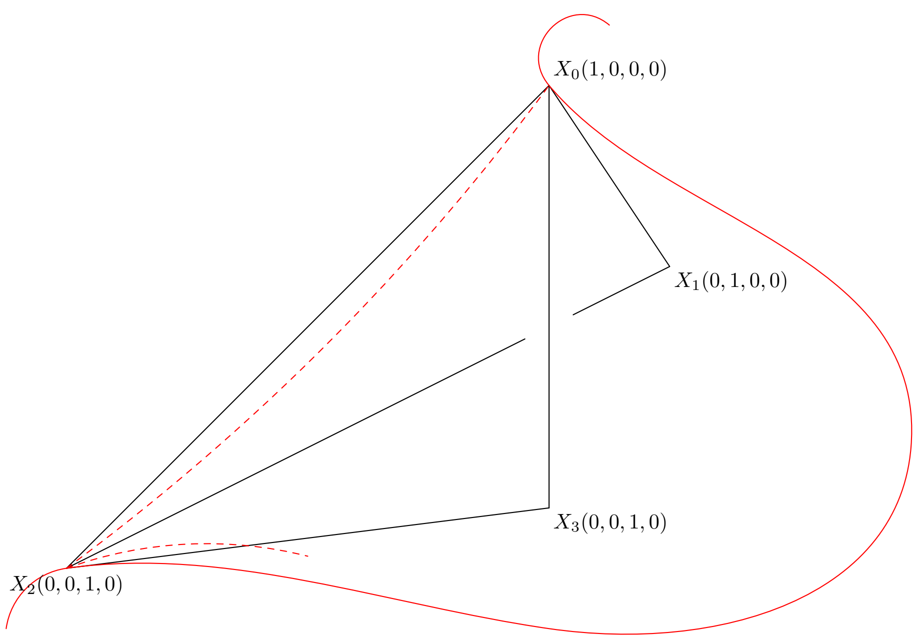
$$\begin{aligned} x_1^3 - x_0 x_2^2 &= 0 \\ x_1 x_3 - x_2 x_0 &= 0. \end{aligned}$$

The variety $V$ in this case is given by

$$\begin{aligned} x_1^3 - x_0 x_2^2 &= 0 \\ x_1 &= 0 \\ x_2 x_0 &= 0 \end{aligned}$$

and consists of the two lines joining the point $(0,0,0,1)$ to each of the points $(1,0,0,0)$ and $(0,0,1,0)$.

The equation $x_1^3 - x_0 x_2^2 = 0$ represents a cubic cone with vertex at the point $(0,0,0,1)$ and having the line joining this vertex to the point $(0,0,1,0)$ as a cuspidal generator.



The section of this cone by the plane $x_3 = 0$ is a cubic curve with a cusp and is shown in the diagram by a dotted curve. The cuspidal tangent to this curve is the line $X_2X_1$. This cuspidal cubic curve is the projection of the twisted cubic onto the plane $x_3 = 0$ from the point $X_3$ which lies on the tangent at $X_2$ to the twisted cubic.

Given any irreducible variety $V_r$ in $A_n$, an affine space of $n$ dimensions, the equation of $V_r$ can be written in the form

$$\begin{aligned} f(x_1, \cdots, x_{r+1}) &= 0, \\ \psi_i(x_1, \cdots, x_{r+1}1)x_{r+i} &= \varphi_i(x_1, \cdots, x_{r+1}) \quad i = 2, \cdots, (n-r). \end{aligned} \tag{4}$$

If $(\xi_1, \cdots \xi_n)$ is a non-homogeneous generic point of $V_r$ then, with a sufficiently general choice of coordinate axes $\xi_{r+2}, \cdots, \xi_n$ * and hence the whole ring $K[\xi_1, \cdots, \xi_n]$ is integrally dependent on $K[\xi_1, \cdots, \xi_{r+1}]$.

The primal $U_r$ given by $f(x_1, \cdots, x_{r+1}) = 0$ in $A_{r+1}$, where the coordinate of a general point are $(x_1, \cdots, x_{r+1})$, is birationally equivalent to $V_r$ and $(\xi_1, \cdots, \xi_{r+1})$ is a generic point of $U_r$.

To find the elements of $\Sigma = K(\xi_1, \cdots, \xi_n) = K(\xi_1, \cdots, \xi_{r+1})$ which are integrally dependent on $\sigma_{(1)} = K[\xi_1, \cdots, \xi_n]$ we have to find the elements of $K(\xi_1, \cdots, \xi_n)$ which are integrally dependent on $\sigma_{(2)} = K[\xi_1, \cdots, \xi_{r+1}]$. If we denote by $\sigma^*_{(j)}$ the integral closure of $\sigma_{(j)}$ in $\Sigma$ $(j = 1, 2)$, this last remark implies

$$\sigma^*_{(1)} = \sigma^*_{(2)}.$$

If we now suppose that $U_r$ is normal in the affine sense,

$$\sigma_{(2)} = \sigma^*_{(2)}.$$

*Something is missing here: these coordinates are independent?

It is obvious that $\sigma_{(2)} \subseteq \sigma_{(1)}$ and consequently

$$\sigma^*_{(1)} = \sigma^*_{(2)} = \sigma_{(2)} \subseteq \sigma_{(1)},$$

showing that

$$\sigma^*_{(1)} \subseteq \sigma_{(1)}$$

whence, since $\sigma_{(1)} \subseteq \sigma^*_{(1)}$, $\sigma_{(1)} = \sigma^*_{(1)}$, so that the affine normality of $U_r$ implies the affine normality of $V_r$.

In both of these cases $U_r$ is a quite general projection of $V_r$ into a space $S_{r=1}$ of $r+1$ dimensions.

Therefore we may state the above results in the form

**Theorem 1** *If the general projection $U_r$ of $V_r$ in $S_n$ into a space of $r+1$ dimensions is affinely normal, then $V_r$ itself is affinely normal, and if $U_r$ is projectively normal then $V_r$ also is projectively normal.*

This result is true more generally if we consider projection into a space of $r+i$ dimensions for $i = 1, 2 \cdots, n-r$.

In fact since projective normality implies geometric normality it is not possible to project a normal variety in $S_n$ into a normal variety of the same order in a space of dimension less than $n$, so the above result is of no interest in the case of projective normality.

The converse statement, that the general projection of a variety $V_r$ which is normal in the affine or projective sense into a space of $r+1$ dimensions is normal in the same sense, is not true. For consider a twisted cubic curve in $S_3$. Such a curve is projectively normal. A general projection of this curve into $S_2$ is a rational plane cubic with a double point. This plane curve is not affinely normal for any choice of the line at infinity which leaves this double point at a finite distance.

Using the same notation as above let us assume that $U_r$ and therefore $V_r$ are both affinely normal. Then

$$\sigma_{(2)} = \sigma^*_{(2)} = \sigma^*_{(1)} = \sigma_{(1)}$$

so that

$$K[\xi_1, \cdots, \xi_{r+1}] = K[\xi_1, \cdots, \xi_n]$$

and $U_r$ and $V_r$ are integrally equivalent varieties in Zariski's sense ([22] p279).

From this we find that $\xi_{r+i}$ is in $K[\xi_1, \cdots, \xi_{r+1}]$ $(i = 2, \cdots, n-r)$. The parametric equations of $V_r$ are therefore

$$\begin{aligned} f(x_1, \cdots, x_r) &= 0 \\ x_{r+i} &= \varphi_i(x_1, \cdots, x_{r+1}) \quad (i = 2, \cdots, (n-r)) \end{aligned} \tag{5}$$

and these $n-r$ equations define $V_r$ as their complete intersection considering only points at a finite distance, since, in the equations (3) all points for which the least common multiple of the $\psi_i$ does not vanish lie on $V_r$ and in this case if we made our equations homogeneous by introducing a new indeterminate $x_0$ this least common multiple would be some power of $x_0$.

It is perhaps just worthwhile noticing what happens if similar reasoning is applied to the case of projective normality. By an analogous process we should find that if $U_r$ is projectively normal then the equations for $V_r$ are the homogeneous equations

$$\begin{aligned} f(x_0, x_1, \cdots, x_r) &= 0 \\ x_{r+i} &= \varphi_i(x_0, x_1, \cdots, x_{r+1}) \quad (i = 2, \cdots n-r) \end{aligned} \tag{6}$$

The homogeneity of the last $n-r-1$ equations implies that $\varphi_i(x_0, x_1, \cdots, x_{r+1})$ $(i = 2, \cdots, n-r)$ are linear forms in $x_0, x_1, \cdots, x_{r+1}$. This means that the ambient space of the original $V_r$ is of $(r+1)$ dimensions, a result which was to be expected in view of the remark above that projective normality implies geometric normality.

Since a variety in $S_n$ which is affinely normal for every choice of the prime at infinity is locally normal it follows as a corollary of the result above for affine normality that, if $U_r$ is locally normal, then $V_r$ also is locally normal. Alternatively we can deduce this from the following result.

**Theorem 2** *If $P$ is a point of $V_r$ and $P'$ is the projection of $P$ on $U_r$ then if $U_r$ is locally normal at $P'$, $V_r$ is locally normal at $P$.*

Let $P$ be the point with coordinates $(\alpha_1, \cdots, \alpha_n)$. Its projection $P'$ is the point $(\alpha_1, \cdots, \alpha_{r+1}, 0, \cdots, 0)$. Let $Q_V(P)$, $Q_U(P')$ denote the quotient rings of $P$, $P'$ on $V_r$, $U_r$ respectively. Then

$$Q_U(P') \subseteq Q_V(P).$$

For let $\zeta'$ be an element of $Q_U(P')$.

$$\zeta' = \frac{a'(\xi_1, \cdots, \xi_{r+1})}{b'(\xi_1, \cdots, \xi_{r+1})}, \quad a', b' \text{ belong to } K[\xi_1, \cdots, \xi_{r+1}]$$

and

$$b'(\alpha_1, \cdots, \alpha_{r+1}) \neq 0.$$

Obviously $\zeta'$ belongs to $Q_U(P)$.

Therefore

$$\sigma^*_{(1)} = \sigma^*_{(2)} \subseteq Q_U(P') \subseteq Q_V(P)$$

whence

$$\sigma^*_{(1)} \subseteq Q_V(P)$$

showing that $V_r$ is locally normal at $P$.

This result is true if we consider a general projection into a space of $r+i$ dimensions ($1 \leq i \leq n-r$). Here again the converse is not true. For consider the surface in affine four-dimensional space $A_4$ given by the vanishing of the $2 \times 2$ minors in the matrix $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$.

A non-homogeneous generic point of this surface is $(\lambda, \lambda\theta, \lambda\theta^2, \lambda\theta^3)$ where $\lambda$, $\theta$ are independent indeterminates. We can show that $K[\lambda, \lambda\theta, \lambda\theta^2, \lambda\theta^3]$ is integrally closed in $K(\lambda, \theta)$. The surface is obtained by joining a twisted cubic curve in $A_3$ to a point external to $A_3$, in this case the origin of coordinates 0. Now project the surface onto a general subspace $A_3$. This $A_3$ meets the surface in a twisted cubic curve $\gamma'$ and $O$ projects onto a point $O'$ not on $\gamma'$ for a general projection. The projected surface is a rational cubic cone with vertex at $O'$ and containing a double generator through $O'$. This projected surface is not locally normal at $O'$ but the original surface is locally normal at $O$.

A slight generalisation of the previous result would be:

**Theorem 3** *If $W$ is an irreducible subvariety of $V_r$ and $W'$ its projection on $U_r$, then if $U_r$ is locally normal at $W'$, $V_r$ is locally normal at $W$.*

§5.4 Normality of prime sections

In the course of his second paper on arithmetic normality [5], Gaeta remarks in a footnote without any reference that if an irreducible prime section of the irreducible variety $V_r$ is arithmetically normal then $V_r$ is arithmetically normal. This seems to be an important result but I have unfortunately been unable to trace its existence in the Italian literature where it probably exists. The exact converse is not true as can be seen by considering an irreducible section of a cubic surface in three dimensions by a tangent plane. A result of a similar nature that the generic prime section of a normal variety is itself normal has recently been published by one of Zariski's disciples. In a slightly generalised form the first result becomes:

**Theorem 1** *If an irreducible section by $S_{n-i}$ of the irreducible variety $V_r$ in $S_n$ is arithmetically normal. $i$ can have any of the values $1, 2, \cdots, r-1$.*

I think it is of some interest to point out that this result cannot be true if we replace the words "arithmetically normal" by "locally normal". For if this were so any $r$-dimensional variety $V_r$ in $S_n$ which does not possess a multiple sub-variety of dimension $r-1$ would be locally normal, since a general section of $V_r$ by a space $S_{n-r+1}$ is a non-singular irreducible curve and such a curve is locally normal. There is an obvious fallacy here. We have only to consider the cone $\Gamma^{4*}$ in $S_4$ obtained by joining a rational quartic curve $\gamma^4$ in $S_3$ to an external point. $\gamma^4$ is locally normal and is an irreducible prime section of $\Gamma^4$. But $\Gamma^4$ is not locally normal at its vertex.

A result concerning generic normality analogous to Theorem 1 is as follows.

**Theorem 2** *If an irreducible prime section of $V_r$ ($r \geq 2$) is geometrically normal then $V_r$ is geometrically normal.*

A very simple argument can be used to prove this. Suppose that $V_r^m$ in an ambient space $S_n$ has been obtained by projection from $V_r'^m$, a variety of the same order $m$ lying in an ambient space $S_{n'}$ where $n'$ is greater than or equal to $n$. To prove the theorem we have to show that, assuming an irreducible prime section of $V_r$ to be geometrically normal, then $n' = n$.

*The ms actually says $\Gamma_2^4$ here, but $\Gamma_4$ for subsequent instances

We can choose a coordinate system so that, if $x_0, x_1, \cdots x_{n'}$ are homogeneous coordinates in $S_{n'}$, $S_n$ is the subspace given by $x_{n+1} = \cdots = x_{n'} = 0$ and the vertex of projection is the space defined by $x_0 = x_1 = \cdots = x_n = 0$. Consider $U^m_{r-1}$, the section of $V^m_r$ by a general prime with equation $u_0x_0 + \cdots + u_nx_n = 0$ in $S_n$. $U^m_{r-1}$ is of order $m$ and lies in an ambient space $S_{n-1}$ ([3] chap IX §6). Then $U^m_{r-1}$ is the projection of $U'^m_{r-1}$, the section of $V'^m_r$ by the prime in $S_{n'}$ with equation $u_0x_0 + \cdots + u_nx_n = 0$. This prime passes through the vertex of projection. $U'^m_{r-1}$ is of order $m$ and dimension $r-1$ and lies in an ambient space $S_{n'-1}$ of dimension $n'-1$.

If we assume $n' > n$ then $n'-1 > n-1$. But $n'-1 > n-1$ implies that $U^m_{r-1}$ is not geometrically normal contrary to hypothesis. Therefore $n' = n$ and $V^m_r$ must be geometrically normal.

Severi gives a geometrical proof that if two surfaces in $S_3$ intersect simply and completely in a nonsingular irreducible curve then this curve is projectively normal ([13] §21). Gaeta states that Severi has also proved the more general result that if $n-1$ primals in $S_n$ intersect simply and completely in a non-singular irreducible curve then this curve is normal [4]. Assuming these results, the following result can be deduced immediately.

**Theorem 3** *If $n-r$ primals $F^{(1)},\ F^{(2)}, \cdots, F^{(n-r)}$ in $S_n$ intersect simply and completely in an irreducible variety $V_r$ of dimension $r$ which has on it no singular subvariety of dimension greater than $r-2$, then $V_r$ is normal.*

For consider the section of $V_r$ by a general $S_{n-r+1}$ in $S_n$. This section is an irreducible non-singular curve $C$. The $S_{n-r+1}$ meets the primal $F^{(i)}$ in an irreducible variety of dimension $n-r$ which is a primal $F'^{(i)}$ in $S_{n-r+1}$, $(i = 1, \cdots, n-r)$. The primals $F'^{(i)}$, $i = 1, \cdots, n-r$ in $S_{n-r+1}$ meet simply and completely in the non-singular curve $C$ which is therefore normal. In consequence of the first result stated in this section $V_r$ is normal.

## Conclusion

The last section of my thesis is so sketchy and inconclusive that I feel it needs some apology. I include it here in order to indicate the various avenues that I have explored in an attempt to characterise geometrically the general normal variety of dimension $r$ in $S_n$. I am still hoping to find an arithmetic proof of the last theorem that I have enunciated (Theorem 3 §5.4) but I have come up against many difficulties in this task. Even when this problem is solved only sufficient conditions for normality will have been found. This is evident from Gaeta's result that a necessary and sufficient condition for a non-singular curve in $S_3$ to be normal is that it is of finite residue.

I am inclined to think that the methods of the Italian geometry are more suited to the solution of this particular problem. The results of the Italian school used by Gaeta in his two quoted papers are well-authenticated. Justification for arithmetic proofs of his theorems could be made only if greater rigour, elegance or simplicity could thereby be obtained, or if such methods yielded more readily to further generalisation. This does not seem very likely at the moment.

The most important use which Zariski makes of the projectively normal variety is in the resolution of the singularities of 2 and 3-dimensional varieties by birational transformations. The property made use of in this connection is common also to the locally normal varieties, namely the absence of multiple curves on a locally normal surface and of multiple surfaces on a locally normal 3-dimensional variety. In much of Zariski's work it appears quite often to be the concept of local normality rather than projective normality which is of fundamental importance. This is the case in his work on the foundations for a general theory of birational correspondence. Even local normality is not always required and the less restrictive property of analytic irreducibility may prove to be a more fundamental concept.

## References


[1] A.A. Albert. Modern Higher Algebra.

[2] H.F.Baker. Some recent developments in the theory of algebraic surfaces. Proceedings of the London Mathematical Society. vol 12(1913) pp 1-40.

[3] E. Bertini. Geometria Proiettiva degli Iperspazi.

[4] F. Gaeta, Sobre las curvas y las superficies del $S_r$ aritmeticamente normales. Revista Matematica Hispano-Americana. Series 4. vol. 7 (1947) pp 255-268.

[5] F. Gaeta. On the arithmetically normal surfaces and varieties of $S_r$. Revista Matematica Hispano-Americana. Series 4. vol. 8 (1948) pp 72-82.

[6] J. Harkness and F. Morley. A Treatise on the Theory of Functions.

[7] W.V.D. Hodge. Some recent developments in the theory of algebraic varieties. Journal of the London Mathematical Society. vol. 25 (1950) pp 148-157.

[8] W.V.D. Hodge and D. Pedoe. Methods of Algebraic Geometry. vol I.

[9] H.T.Muhly. A remark on normal varieties. Annals of Mathematics. Series 2 vol.42 (1941) pp 921-925.

[10] H.T. Muhly and O. Zariski. Resolution of Singularities of an algebraic curve. American Journal of Mathematics vol 61 (1939) pp107-114.

[11] W. Rückert. Zum Eliminationsproblem der Potenzreihenideale. Mathematische Annalen. vol. 107 (1933) pp259-281.

[12] G. Salmon. Three-dimensional Geometry. Vol. I.

[13] F. Severi. Fondamenti di Geometria Algebraica.

[14] B.L. van der Waerden. Zur Nullstellentheorie der Polynomideale. Mathematische Annalen. vol. 96 (1927) pp183-208.

[15] B.L. van der Waerden. Zur Produktzerlegung der Ideale in ganz-abgeschlossenen Ringen. Mathematische Annalen. vol. 101 (1929) pp 293-308.

[16] B.L. van der Waerden. Zur Algebraischen Geometrie VI. Algebraische Korrespondenzen und rationale Abbildungen. Mathematische Annalen. vol. 110 (1935) pp134-160.

[17] B.L. van der Waerden. Moderne Algebra (second edition).

[18] B.L. van der Waerden. Einführung in die Algebraische Geometrie.

[19] B.L. van der Waerden. The foundations of algebraic geometry. A very incomplete historical survey. Courant. Anniversary colume (1948) pp437-449.

[20] A. Weil. Foundations of Algebraic Geometry.

[21] O. Zariski. Algebraic Surfaces.

[22] O. Zariski. Some results in the arithmetic theoery of algebraic varieties. American Journal of Mathematics. vol. 61 (1939) pp 249-294.

[23] O. Zariski. The reduction of singularities of an algebraic surface. Annals of Mathematics series 2. vol. 40 (1939) pp 639-689.

[24] O. Zariski. Algebraic varieties and ground-fields of characteristic zero. American Journal of Mathematics vol. 62 (1940) pp 187-221.

[25] O. Zariski. A simplified proof for the resolution of singularities of an algebraic surface. Annals of Mathematics series 2. vol. 43 (1942) pp 583-593.

[26] O. Zariski. Normal varieties and birational correspondence. Bulletin of the American Mathematical Society vol. 45 (1942) pp 402-413.

[27] O. Zariski. Foundations of a general theory of birational correspondence. Transactions of the American Mathematical Society. vol. 53 (1943) pp 490-542.

[28] O. Zariski. Concept of a simple point of an abstract algebraic variety. Transactions of the American Mathematical Society. vol. 62 (1947) pp 1-52.

[29] O. Zariski. Analytic Irreducibility of normal varieties. Annals of Mathematics series 2 vol. 49 (1948) pp 352-361.