This is an open book examination, so students can look up theory in their lecture notes and other sources. I don't think this makes the theoretical bits of questions completely trivial - there is some collating to do, and anyone who copies out large chunks of theory blindly will obviously waste quite a bit of time. Up to 6 hours will be allowed.

1.

(i) A *topological space* $(X, \mathcal{T})$ is a set X together with a collection $\mathcal{T}$ of subsets of X, (called *open*) such that a) -c) hold.

a) X, $\phi \in \mathcal{T}$

b) If $U_i \in \mathcal{T}$ for all $i \in I$, for any set I, then $\cup_{i \in I} U_i \in \mathcal{T}$.

c) If U_1, $U_2 \in \mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.

[5 marks]

For the standard topology on $\mathbf{R}^n$, the open sets are all sets U such that the following holds. For any $\underline{x} \in U$ there is $\varepsilon > 0$ such that

$$B(\underline{x}, \varepsilon) \subset U$$

where

$$B(\underline{x}, \varepsilon) = \{\underline{y} : \|\underline{y} - \underline{x}\| < \varepsilon\},$$

and, if $\underline{v} = (v_1 \cdots v_n)$ then

$$\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2}.$$

[2 marks]

Now suppose that U_1 and U_2 are open in $\mathbf{R}^n$. Take any $\underline{x} \in U_1 \cap U_2$. Then choose $\varepsilon_j > 0$ such that

$$B(\underline{x}, \varepsilon_j) \subset U_j$$

for $j = 1$, 2. Now take

$$\varepsilon = \mathrm{Min}(\varepsilon_1, \varepsilon_2).$$

Then

$$B(\underline{x}, \varepsilon) \subset B(\underline{x}, \varepsilon_1 \cap B(\underline{x}, \varepsilon_2) \subset U_1 \cap U_2.$$

So $U_1 \cap U_2$ is open also.

[4 marks]

(ii)

a) For any $\varepsilon > 0$, and any $x \in \mathbf{R}$, $B((x, 0), \varepsilon)$ is not contained in $\mathbf{R} \times \{0\}$. So $\mathbf{R} \times \{0\}$ is not open.

[1 mark]

b) For any $\varepsilon > 0$ and any $x_0 \in \mathbf{R}$, $B((x_0, 0), \varepsilon)$ contains the point $(x_0, -\frac{1}{2}\varepsilon)$. So

$$(x, y) \in \mathbf{R}^2 : y \geq 0)$$

is not open.

[2 marks]

c) Take any (x_0, y_0) with $x_0 - y_0 = \varepsilon > 0$. Then

$$\|(x_0, y_0) - (x, y)\| = \sqrt{(x_0 - x)^2 + (y_0 - y)^2} \geq \min(|x - x_0|, |y - y_0|.$$

So if $\|(x_0, y_0) - (x, y)\| < \varepsilon/2$ then

$$|(x - y) - (x_0 - y_0)| \leq |x - x_0| + |y_0 - y| < \varepsilon.$$

Then $|x - y - \varepsilon| < \varepsilon$ and $x - y > 0$. So

$$B((x_0, y_0), \varepsilon) \subset \{(x, y) : x > y\}$$

and so this set is open.

[4 marks]

(iii) To get a) $\Rightarrow$ b): Given V open in Y, for each $x \in f^{-1}(V)$ choose $U(x)$ open in X with $x \in U(x)$ and $f(U(x)) \subset V$. then $U(x) \subset f^{-1}(V)$. So

$$f^{-1}(V) = \cup\{U(x) : x \in f^{-1}(V)\}$$

is open in X.

[3 marks]

To get b) $\Leftrightarrow$ c): $V \subset Y$ is open $\Leftrightarrow$ $Y \setminus V$ is closed, $f^{-1}(Y \setminus V = X \setminus f^{-1}(V)$.

[2 marks]

To get b) $\Rightarrow$ a): given $V \subset Y$ and $x \in f^{-1}(V)$, take $U = f^{-1}(V)$.

[2 marks]

Of course different solutios will use different strategies: but 7 marks altogether.

$5 + 2 + 4 + 1 + 2 + 4 + 3 + 2 + 2 = 25$

First part of question is bookwork, second part of (i) and part (ii) similar to homework exercises. The equivalences in (iii) were mentioned in lectures but not proved - so this is unseen.

2. (i) $(X, \mathcal{T})$ is *compact* if for any collection $\{U_i : i \in I\}$ of open sets (that is, in $\mathcal{T}$) with $X \subset \cup_{i \in I} u_i$, there exists a finite $J \subset I$ with $X \subset \cup_{j \in J} U_j$.

[3 marks]

$(X, \mathcal{T})$ is *Hausdorff* if given any x, $y \in X$ with $y \neq x$, there are open U, V in X with $x \in U$, $y \in V$ and $U \cap V = \phi$.

[3 marks]

Let $[x]$ denote the equivalence class of x with respect to $\sim$. Then $X/\sim = \{[x] : x \in X\}$. We define $\pi : X \to X/\sim$ by $\pi(x) = [x]$. Then the *quotient topology* on $X/\sim$ is the collection of sets

$$U \subset X/\sim : \pi^{-1}(U) \in \mathcal{T}\},$$

where, as usual,

$$\pi^{-1}(U) = \{x : \pi(x) \in U\}.$$

[3 marks]

(ii) For reflexive: $x = x + 0$ so $x \sim x$. For symmetric: if $y = x + n$ then $x = y - n$, so $x \sim y$ implies $y \sim x$. For transitivity: if $y = x + m$ and $z = y + n$ then $z = x + m + n$, so if $x \sim y$ and $y \sim z$, $x \sim z$. We have $\pi([0, 1]) = \mathbf{R}/\sim$, π is continuous by the definition of the quotient topology and $[0, 1]$ is compact. Since the continuous image of a compact set is compact, $\mathbf{R}/\sim$ is compact also.

[3 marks]

If $[x] \neq [y]$ then $y \neq x+n$ for any $n \in \mathbf{Z}$. So there is $\varepsilon > 0$ such that $|x+n-y| > 2\varepsilon$ for all $n \in \mathbf{Z}$. Put $U_1 = \cup_{n\in\mathbf{Z}}(x+n-\varepsilon, x+n+\varepsilon)$ and $U_2 = \cup_{n\in\mathbf{Z}}(y+n-\varepsilon, y+n+\varepsilon)$. Then $U_1 \cap U_2 = \phi$, the U_j are open in $\mathbf{R}$ $\pi^{-1}\pi(U_j) = U_j$ for $j = 1,\ 2$. So $\pi(U_j)$ are open in $\mathbf{R}/\sim$, $\pi(U_1)\cap\pi(U_2) = \phi$ and $[x] \in \pi(U_1)$, $[y] \in \pi(U_2)$.

[3 marks]

(iii) If $[x_1] = [x_2]$ then $x_1\tilde{x}_2$ and $F(x_1) = F(x_2)$. So $[F]$ is well-defined. To show $[F]$ is continuous, it suffices to show that $[F]^{-1}(U)$ is open for all open U in Y. But $[F]^{-1}(U)$ is open in $X/\sim\ \Leftrightarrow\ \pi^{-1}([F]^{-1}(U)) = ([F]\circ\pi)^{-1}(U) = F^{-1}(U)$ is open - which is true, because F is continuous.

[4 marks]

(iv) Take $\lambda = 2\pi$. Then take

$$F(x) = (\cos 2\pi x, \sin 2\pi x).$$

Then F is continuous into $\mathbf{R}^2$ and maps $\mathbf{R}$ onto Y because $\cos^2 t + \sin^2 t = 1$ for all t. F maps onto Y because cos maps $[0,\pi]$ onto $[-1,1]$ (all with sin positive) and maps $[\pi, 2\pi]$ onto $[-1,1]$ (all with sin negative). $F(x) = F(x') \Leftrightarrow \cos 2\pi x = \sin 2\pi x'$ - which gives $x' = x+n$ or $x' = -x+n$ for some integer n - and $\sin 2\pi x = \sin 2\pi x'$ - which shows that $x' \neq -x+n$ for any integer n. So $F(x) = F(x') \Leftrightarrow x \sim x'$ and $[F]$ is well-defined, continuous and a bijection onto Y. So $\mathbf{R}/\sim$ and Y are homeomorphic.

[6 marks]

3+3+3+3+3+4+6=25

(i) is bookwork, (ii) is similar to homework exercises.

3.

An *orientable* C^1 *manifold* is Hausdorff topological space which is a countable union of compact spaces with an *atlas* Λ of *charts* (U,φ). For $(U,\varphi)\in\Lambda$, U is an open set, φ a homeomorphism of U onto some open subset of $\mathbf{R}^n$ for some n, and

$$M \subset \cup\{U : (U, \varphi \in \Lambda\}.$$

[3 marks]

In addition, if (U,φ) and $(V,\psi)\in\Lambda$ with $U\cap V \neq \phi$ (in which case U and V are necessarily homeomorphic to open subsets of $\mathbf{R}^n$ for the same n) then

$$\psi\circ\varphi^{-1} : \varphi(U\cap V) \to \psi(U\cap V)$$

is continuously differentiable with strictly positive Jacobian at all points. If $F = (F_1, \cdots F_n) : U(\subset \mathbf{R}^n) \to \mathbf{R}^n$ is differentiable, then the Jacobian $J(F)(\underline{x_0})$, for $\underline{x_0} \in U$, is the determinant of the matrix

$$\begin{pmatrix} \partial F_1/\partial x_1(\underline{x_0}) & \cdots & \partial F_1/\partial x_n(\underline{x_0}) \\ \cdot & \cdot & \cdot \\ \partial F_n/\partial x_1(\underline{x_0}) & \cdots & \partial F_n/\partial x_n(\underline{x_0}) \end{pmatrix}.$$

[3 marks]

(ii) $\varphi_2\circ\varphi_1^{-1}$ is defined on $(0,1)$ by

$$\varphi_2\circ\varphi_1^{-1}(t) = \varphi_2(t, +\sqrt{1-t^2}) = -\sqrt{1-t^2}$$

which is continuously differentiable with positive derivative

$$\frac{t}{+\sqrt{1-t^2}}$$

for $t \in (0,1)$. Similarly

$$\varphi_3 \circ \varphi_2^{-1}(t) = \varphi_3(+\sqrt{1-t^2}, -t) = -\sqrt{1-t^2},\ t \in (0,1),$$

has positive derivative $t/\sqrt{1-t^2}$,

$$\varphi_4 \circ \varphi_3^{-1}(t) = \varphi_3(-t, -\sqrt{1-t^2}) = -\sqrt{1-t^2},\ t \in (0,1)$$

and

$$\varphi_1 \circ \varphi_4^{-1}(t) = \varphi_1(-\sqrt{1-t^2}, t) = -\sqrt{1-t^2},\ t \in (0,1).$$

So in all cases the domain of the transition function, are the same as in the first case, so the transition funciton is C^1 and the derivative is positive everywhere.

[7 marks]

(iii) a) If

$$(0,1) + \lambda((x,y) - (0,1)) = (p_1(x,y), -1)$$

then

$$1 + \lambda(y-1) = -1,$$

So

$$\lambda = \frac{2}{1-y},\ p_1(x,y) = \frac{2x}{1-y}.$$

So p_1 is continuous on $X \setminus \{(0,1)\}$.To find the inverse function: if

$$t = \frac{2x}{1-y}$$

then $t - 2x = ty$. So $t^2(1-x^2) = t^2 - 4xt + 4x^2$. So if $x \neq 0$ we have $4t = x(4+t^2)$ and

$$x = \frac{4t}{4+t^2}.$$

This also holds when $x = 0$. Then

$$ty = t - \frac{8t}{4+t^2} = \frac{t(t^2-4)}{4+t^2}.$$

So $t = 0$ or

$$y = \frac{t^2-4}{4+t^2}.$$

This also holds at $t = 0$. So

$$p^{-1}(t) = \left(\frac{4t}{4+t^2}, \frac{t^2-4}{4+t^2}\right)$$

which is continuous. So p_1 is a homeomorphism.

[7 marks]

(iii)b)

$$p_1 \circ \varphi_1^{-1}(x) = \frac{2x}{1-\sqrt{1-x^2}}$$

for $0<x<1$ or $-1<x<1$, that is on $\varphi_1(U_1\setminus\{(0,1)\})$. This function is C^1.
[2 marks]
(iii)c) Similarly to the calculation of λ,

$$\mu=\frac{2}{1+y}\ p_2(x,y)=\frac{2x}{1+y}.$$

and so

$$p_1(x,y).p_2(x,y)=\frac{4x^2}{1-y^2}=\frac{4x^2}{x^2}=4$$

whenever $x\neq 0$. It follows that $p_2\circ p_1^{-1}(t)=4/t$ whenever $x\neq 0$, that is, whenever $t\neq 0$. The domain of the transition function is $\{t:t\neq 0\}$.
[3 marks]
$3+3+7+7+2+3=25$
(i) is bookwork. (ii) is an example similar to homework exercises. (iii) is some detail from an example in lectures which was not completely spelt out at the time, and with slightly different parameters. The calculation of the inverse function of p_1 is NOT in lecture notes, although the answer is given for a slightly different parametrisation.
4.(i) Let

$$A=\begin{pmatrix} a & \overline{c}\\ c & \overline{a}\end{pmatrix}$$

and define $\tau(z)=A.z$. Then

$$\tau(z)=\frac{az+\overline{c}}{cz+\overline{a}}\text{ and }\tau'(z)=\frac{a(cz+\overline{a})-c(az+\overline{c})}{(cz+\overline{a})^2}=\frac{1}{(cz+\overline{a})^2}.$$

Also

$$1-|\tau(z))|^2=\frac{|cz+\overline{a}|^2-|az+\overline{c}|^2}{|cz+\overline{a}|^2}$$

$$=\frac{|z|^2(|c|^2-|a|^2)+|a|^2-|c|^2+2\mathrm{Re}(cza)-2\mathrm{Re}(azc)}{|cz+\overline{a}|^2}$$

$$=\frac{1-|z|^2}{|cz+\overline{a}|^2}$$

So for any z,

$$\frac{|\tau'(z)|}{1-|\tau(z)|^2}=\frac{1}{1-|z|^2}.$$

[4 marks]
So

$$\ell_D(A.\gamma_1)=\int_0^1\frac{2|(\tau\circ\gamma_1)'(t)|}{1-|(\tau\circ\gamma_1)(t)|^2}dt=\int_0^1\frac{2|\tau'(\gamma_1(t))||\gamma_1'(t)|}{|1-|\tau(\gamma_1(t))|^2}dt$$

$$=\int_0^1\frac{|\gamma_1'(t)|}{1-|\gamma_1(t)|^2}dt=\ell_D(\gamma_1).$$

[2 marks]

So given any z_1, $z_2 \in D$,

$$d_D(z_1,z_2) = \inf\{\ell_D(\gamma) : \gamma \text{ is a path in } D \text{ from } z_1 \text{ to } z_2\}$$
$$= \inf\{\ell_D(A.\gamma) : \gamma \text{ is a path in } D \text{ from } z_1 \text{ to } z_2\} = d_D(A.z_1, A.z_2).$$

[2 marks]

(ii)

$$\gamma_2'(t) = r'(t)e^{i\theta(t)} + ir(t)\theta'(t)e^{i\theta(t)}.$$

So

$$|\gamma_2'(t)| = |r'(t) + ir(t)\theta'(t)| \geq |r'(t)|.$$

So

$$\ell_D(\gamma_2) = \int_0^1 \frac{2|\gamma_2'(t)|}{1-|\gamma_2(t)|^2}dt \geq \int_0^1 \frac{2|r'(t)|}{1-(r(t)^2}dt \geq \left|\int_0^1 \frac{2r'(t)}{1-(r(t))^2}dt\right|$$
$$= \int_{r(0)}^{r(1)} \frac{2dr}{1-r^2} = [\ln((1+r)/(1-r))]_0^{r(1)}$$
$$= \ln((1+r(1))/(1-r(1))).$$

[4 marks]

So the shortest path from 0 to z_1, for any $|z_1| = r_1 < 1$, is along the radius from 0 to z_1, and is $\ln((1+r_1)/(1-r_1))$. If this is s_1, then

$$e^{s_1} = \frac{1+r_1}{1-r_1}.$$

Then

$$r_1(e^{s_1}+1) = e^{s_1} - 1.$$

So

$$r_0 = \frac{e^{s_0}-1}{e^{s_0}+1}.$$

[4 marks]

(iii) Given any $z_0 \in D$, we want A such that $A.0 = z_0$. Take

$$A_0 = \frac{1}{\sqrt{1-|z|^2}}\begin{pmatrix} 1 & z_0 \\ \overline{z_0} & 1 \end{pmatrix}.$$

Let A_1 be similarly defined given any z_1. Then $(A_1A_0^{-1}).z_0 = z_1$.

[3 marks]

(iv) For any geodesic ℓ in D (for d_D) and $z_0 \in D$ with $z_0 \notin \ell$, we can find $A \in SU(1,1)$ such that $A.z_0 = 0$, and hence $0 \notin A.\ell$. Then the shortest distance between 0 and $A.\ell$ is the same as the shortest distance between z_0 and ℓ. By rotating we can assume that $A.\ell$ goes through points ζ and $\overline{\zeta}$ on the unit circle, and hence cuts the real axis at rightangles at a point r_0. The circle $|z| = r_0$ then meets $A.\ell$ only in the point r_0 and $\ln((1+r_0)/(1-r_0))$ is the minimum distance between 0 and $A.\ell$, acchieved at r_0 with the shortest path being along the radius, that is the along the unique geodesic segment from 0 to $A.\ell$ which meets $A.\ell$ at rightangles. Applying A^{-1}, the shortest distance between z_0 and ℓ is attained uniquely along the geodesic segment from z_0 to ℓ which meets ℓ at rightangles.

[6 marks]

$4+2+2+4+4+3+6=25$. This is essentially bookwork BUT the treatment is a little different from what I shall do in lectures, when I shall treat $\ell_H(A.\gamma)=\ell_H(\gamma)$ for $A\in SL(2,\mathbf{R})$ and then show $\ell_H(\gamma)=\ell_D(P.\gamma)$ for a suitable Möbius transformation mapping the upper half plane H to the disc D. So the easiest route for the candidates is to do some calculations similar to, but not identical to, what they have seen before. If they simply copy out lecture notes i shall give credit but it will take them much longer.

5. (i) Take any $w\in X$. Then $w=e^z$ for some $z\in U$, and

$$p^{-1}(w)=\{z+2\pi in : n\in\mathbf{Z}\}.$$

Then p is a homeomorphism restricted to $W_n=\{z' : |z'-(z+2\pi in)|<1\}\cap U$ with image $V=e^{z'} : z'\in U, |z'-z|<1\}$ and inverse function $w'\to z+2\pi in+\ln(w'/w)$. In particular, the image V is independent of n, and

$$p^{-1}(V)=\cup_{n\in\mathbf{Z}}W_n.$$

So p is a covering map.

[3 marks]

$\gamma_1(0)=5/4=\gamma_2(0)=\gamma_3(0)$ and $\gamma_1(1)=\gamma_2(1)=\gamma_3(1)=-7/4$.

[2 marks]

We have $5/4=e^{\ln(5/4)}$ and $5/4+t/2=e^{\ln((t/2)+(5/4))}$. So $\tilde{\gamma_j}$ is a lift of γ_j, where

$$\tilde{\gamma_1}(t)=\ln(5/4+t/2)+i\pi t,$$

$$\tilde{\gamma_2}(t)=\ln(5/4)+it\pi+t\ln(7/5),$$

$$\tilde{\gamma_3}(t)=\ln(5/4)-it\pi+t\ln(7/5).$$

[3 marks]

So we have $\tilde{\gamma_j}(0)=\ln(5/4)$ for $j=1,\ 2,\ 3$, and $\tilde{\gamma_1}(1)=\tilde{\gamma_2}(1)=\ln(7/4)+i\pi$, and $\tilde{\gamma_3}(1)=\ln(7/4)+i\pi$. So $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ have the same endpoints, but $\tilde{\gamma_3}$ has only the same first endpoint as the other two, and a different second endpoint. A homotopy between γ_1 and γ_2 is then given by

$$F(s,t)=e^{(1-s)\tilde{\gamma_1}(t)+s\tilde{\gamma_2}(t)},$$

which does satisfy $F(s,0)=\gamma_1(0)=\gamma_2(0)$ for all s and $F(s,1)=\gamma_1(1)=\gamma_2(1)$ for all s (as it must).

[3 marks]

However if two paths have lifts with the same first endpoints and diffenent second endpoints then they cannot be homotopic (because any homotopy would lift to the covering space - U in this case.)

[2 marks]

(ii). A topological space Y is *path-connected* if for any x, $y\in Y$ there is a continuous map $f:[0,1]\to Y$ with $f(0)=x$ and $f(1)=y$.

[1 mark]

Any points x and y in X can be represented in polar forms $r_0e^{i\theta_0}$ and $r_1e^{i\theta_1}$ for r_0, $r_1 \in (1,2)$ and θ_0, $\theta_1 \in [0,2\pi]$. Then write $r_0 = e^{s_0}$, $r_1 = e^{s_1}$ for s_0, $s_1 \in (0,\ln 2)$. Then define

$$f(t) = e^{(1-t)s_0+ts_1+i((1-t)\theta_0+t\theta_1)}.$$

Then $f(0) = x$, $f(1) = y$, $\ln|f(t)| = (1-t)s_0 + ts_1 \in (0,\ln 2)$ and so $f([0,1]) \subset X$, and $f : [0,1] \to X$ is continuous (because e is continuous on $\mathbf{C}$). So X is path-connected.

[3 marks]

(iii) A topological spce Y is *simply connected* if for any continuous map $f : [0,1] \to Y$ with $f(0) = f(1)$ (that is for any closed path f) there exists a continuous map $F : [0,1] \times [0,1] \to Y$ such that

$$F(0,t) = f(t) \text{ for all } t \in [0,1],$$
$$F(s,0) = F(s,1) = F(1,t) = f(0)(= f(1)) \text{ for all } t,\ s \in [0,1].$$

[3 marks]

If f is any closed path in U we can find such an F (a *homotopy to a constant path* by defining

$$F(s,t) = (1-s)f(t) + sf(0),$$

which works, because U is convex: the straight line segment between $f(0) \in U$ and $f(t) \in U$ is in U.

[2 marks].

(iv) Since $p(z + 2\pi in) = p(z)$ for all $n \in \mathbf{Z}$ and $z \in U$, and $p^{-1}p(z) = \{z + 2\pi in : n \in \mathbf{Z}\}$, the covering group of X is

$$\{z \mapsto z + 2\pi in : n \in \mathbf{Z}\}.$$

This group is isomorphic to $\mathbf{Z}$.

[2 marks]

A representative of the element of $\pi_1(X, 3/2)$ corresponding to n is given by

$$t \mapsto (3/2)e^{2n\pi it} : [0,1] \to X.$$

[1 mark]

$3 + 2 + 3 + 3 + 2 + 1 + 3 + 3 + 2 + 2 + 1 = 25$

(i) is similar to homework exercises. Part of (ii) and part of (iii) are bookwork. The rest is similar to homework exercises.

6a) We have

$$p^{-1}([\underline{x}]) = \{\underline{x} + (m,n) : m, n \in \mathbf{Z}\}$$

and $p(\underline{x} + (m,n)) = p(\underline{x})$ for all m, $n \in \mathbf{Z}$. So the covering group action is $(m,n).(x,y) = (x+m, y+n)$ for all $(m,n) \in \mathbf{Z}^2$, as claimed.

[1 mark]

For any $\underline{x_0}$, $p(\tilde{f}(x_0 + \underline{m}) = p(\tilde{f}(\underline{x_0})$ so given $\underline{x_0}$, $\underline{m}$, there is $\varphi(\underline{m}) \in \mathbf{Z}^2$ such that

$$\tilde{f}(\underline{x_0} + \underline{m}) = \tilde{f}(\underline{x_0}) + \varphi(\underline{m}).$$

[2 marks]

Since $\varphi(\underline{m}) \in \mathbf{Z}^2$ is continuous in $\underline{x_0}$, it is constant as a function of $\underline{x_0}$. So

$$\begin{aligned}\tilde{f}(\underline{x_0}+\underline{m}+\underline{n}) &= \tilde{f}(\underline{x_0}+\underline{m})+\varphi(\underline{n}) \\ &= \tilde{f}(\underline{x_0})+\varphi(\underline{m})+\varphi(\underline{n})\end{aligned}$$

Since we also have

$$\tilde{f}(\underline{x_0}+\underline{m}+\underline{n}) = \tilde{f}(\underline{x_0})+\varphi(\underline{m}+\underline{n})$$

we obtain

$$\varphi(\underline{m})+\varphi(\underline{n}) = \varphi(\underline{m}+\underline{n})$$

for all $\underline{m}$, $\underline{n} \in \mathbf{Z}^2$, that is, $\varphi : \mathbf{Z}^2 \to \mathbf{Z}^2$ is a group homomorphism
[4 marks]

Any other lift of f is given by $\underline{x} \mapsto \tilde{f}(\underline{x})+\underline{p}$ for some $\underline{p} \in \mathbf{Z}^2$. Then

$$\tilde{f}(\underline{x}+\underline{m})+\underline{p} = \tilde{f}(\underline{x})+\underline{p}+\varphi(\underline{m})$$

for all $\underline{x} \in \mathbf{R}^2$, $\underline{m}$, $\underline{p} \in \mathbf{Z}^2$. So φ is independent of the lift chosen.
[2 marks]
b) By induction on m, starting from $m=1$,

$$\psi(m,0) = (ma,mc) \tag{1}$$

for all integers $m \geq 1$. To pass from m to $m+1$ we use $\psi(m+1,0) = \psi(m,0)+\psi(1,0)$. By induction on n starting from $n=1$,

$$\psi(0,n) = (nb,nd) \tag{2}$$

for all integers $n \geq 1$. We then have $\psi(m,0)+\psi(-m,0) = \psi(0,0) = (0,0)$ and $\psi(0,n)+\varphi(0,-n) = (0,0)$, which extend (1) and (2) to all integers m and n. Then

$$\psi(m,n) = \psi(m,0)+\psi(0,n) = (ma,mc)+(nb,nd) = (ma+nb,mc+nd)$$

for all m, $n \in \mathbf{Z}$.
[3 marks]

Then take

$$f([x,y]) = [ax+by,cx+dy]$$

This is well-defined because

$$(a(x+m)+b(y+n),c(x+m)+d(y+n)) = (ax+by+(am+bn),cx+dy+(cm+dn))$$

and $am+bn$, $cm+dn \in \mathbf{Z}$ whenever m, $n \in \mathbf{Z}$. Then f is continuous because $\tilde{f}(x,y) = (ax+by,cx+dy)$ is a lift and continuous. (This uses the usual results about quotient topologies.). Then we see that

$$\tilde{f}(x+m,y+n) = \tilde{f}(x,y)+(am+bn,cm+dn)$$

and so $f_* = \varphi$.
[3 marks]
c)(i)

$$\tilde{f}\circ\tilde{f}(x,y) = \tilde{f}(x+\tfrac{1}{2},-y) = (x+1,y).$$

So $f\circ f$ is the identity.
[2 marks]

c)(ii) By definition $\approx$ is reflexive. We have $f^{-1} = f$, so $p_2 = f(p_1) \Leftrightarrow p_1 = f(p_2)$ and $\approx$ is symmetric. Since each equivalence class contains only two elements, $\approx$ is also transitive.
[2 marks]
c)(iii) We have

$$\tilde{f}.(m,n).(x,y) = \tilde{f}(x+m, y+n) = (x+m+\tfrac{1}{2}, -y-n) \tag{1}$$

and

$$(m,n).\tilde{f}(x,y) = ((x+m+\tfrac{1}{2} - y+n)$$

These are equal $\Leftrightarrow$ $n = 0$.
[2 marks]
Applying (1) twice, we have

$$\tilde{f}.(m,n).\tilde{f}.(m,n).(x,y) = (x+2m+1, y). \tag{2}$$

[2 marks]
The centre of G is $\{(m,0) : m \in \mathbf{Z}\}$ since any element $\tilde{f}.(p,q)$ does not compute with everything.
[2 marks]
$1+2+4+2+3+3+2+2+2+2+2 = 25$.
The torus example will be covered in lectures - in fact, it has already come up although we have not yet started covering space theory, and there has been a homework example with some relevance to this question. The Klein bottle will be mentioned briefly in lectures. So the first example shoudl be fairly familiar to them, the second one not so much so.
7.(i) $az + b = z \Leftrightarrow z(a-1) = b$. This has a solution if $a \neq 1$ or if $a = 1$ and $b = 0$.
[1 mark]
In a group of covering bijections, only the identity map is allowed to have fixed points. All holomorphic bijections of $\mathbf{C}$ are of the form $z \mapsto az + b$. So a covering group of holomorphic bijections of $\mathbf{C}$ must be contained in

$$\{z \mapsto z + c : c \in \mathbf{C}\}.$$

If the group is cyclic with generator $\sigma : z \mapsto z + b$, $b \neq 0$, then $\sigma^n(z) = z + nb$ and the group is

$$\{z \mapsto z + nb : n \in \mathbf{Z}\}.$$

[2 marks]
This is the covering group for

$$z \mapsto e^{2\pi i z/b}$$

[1 mark]
(ii) We have

$$\frac{az+b}{cz+d} = z \Leftrightarrow cz^2 + (d-a)z - b = 0.$$

If $c = 0$ then this equation has either one real root, or $a - d = 0$ and $b = 0$. Hence there are no fixed points of $z \mapsto A.z$ in H unless $ad = 1$ and $a = d$ and $b = c = 0$,

that is $A = \pm I$. If $c \neq 0$ then this equation has nonreal roots $\Leftrightarrow$ $(d-a)^2 + 4bc < 0$, that is, $\Leftrightarrow$ $d^2 + 2ad + a^2 - 4(ad - bc) < 0$, that is, $\Leftrightarrow$ $(d+a)^2 < 4$, that is $\Leftrightarrow$ $|a+d| < 2$. So there are no fixed points in H $\Leftrightarrow$ $A \neq \pm I$ and $|a+d| \geq 2$.
[3 marks]

The characteristic polynomial of A is

$$x^2 - (a+d)x + 1.$$

So the condition $|a+d| \geq 2$ is equivalent to both eigenvalues of A being real. Since the product of the eigenvalues is 1, the eigenvalues are $\lambda^{\pm 1}$ for some real $\lambda \neq 0$ and the eigenvalues are distinct unless they are both 1 or both -1. If $\lambda \neq \pm 1$ then A is diagonalisable and of the form

$$P \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} P^{-1}$$

for some real invertible matrix P. Interchanging the columns of P and interchanging λ and λ^{-1} if necessary, we can assume that P has positive determinant, and can then scale by a suitable constant and assume that P has determinant 1. If the eigenvalues of P are both 1, then taking the columns of P to be $\underline{v_1}$ and $\underline{v_2}$ where $\underline{v_1}$ is an eigenvector of A and $(A - I)\underline{v_2} = \pm \underline{v_1}$ we can ensure that P has positive determinant and

$$A = P \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} P^{-1}$$

By scaling $\underline{v_1}$ and $\underline{v_2}$ we can ensure that P has determinant 1. The case when both eigenvalues are -1 is similar.
[6 marks]
(iii)a) Define

$$p_2(z) = e^{2\pi i z}.$$

Then

$$p_2(B_2.z) = p_2(z \pm 1) = p_2(z)$$

and $p_2(z) = p_2(z') \leftrightarrow z' = z + n$ for some $n \in \mathbf{Z}$ $\Leftrightarrow$ $z' = B_2^n.z$ for some $n \in \mathbf{Z}$. Also

$$p_2(H) = \{e^{2\pi i z} : \mathrm{Im}(z) > 0\} = \{e^t : \mathrm{Re}(t) < 0\}$$
$$= \{z : 0 < |z| < 1\}.$$

[3 marks]
(iii)b) Define

$$p_{3,R}(z) = e^{-i(\ln R)z/\pi},$$

which can be defined for all $z \in \mathbf{C}$, and $0 < \mathrm{Im}(z) < \pi$ $\Leftrightarrow$ $\Leftrightarrow$ $0 < \mathrm{Re}(-i(\ln R)z/\pi) < \ln R$ $\Leftrightarrow$ $1 < |p_{3,R}(z)| < R$. So $p_{3,R} : U \to A(R)$.
[3 marks]

Also, $p_{3,R}(z) = p_{3,R}(z')$ $\Leftrightarrow$ $-i(\ln R)z/\pi + 2n\pi i = -i(\ln R)z'/\pi$ for some $n \in \mathbf{Z}$ $\Leftrightarrow$ $z' = z + 2n\pi^2/(\ln R)$ for some $n \in \mathbf{Z}$. So the covering group of $A(R)$ on U is

$$\{z \mapsto z + 2n\pi^2/(\ln R) : n \in \mathbf{Z}\}$$

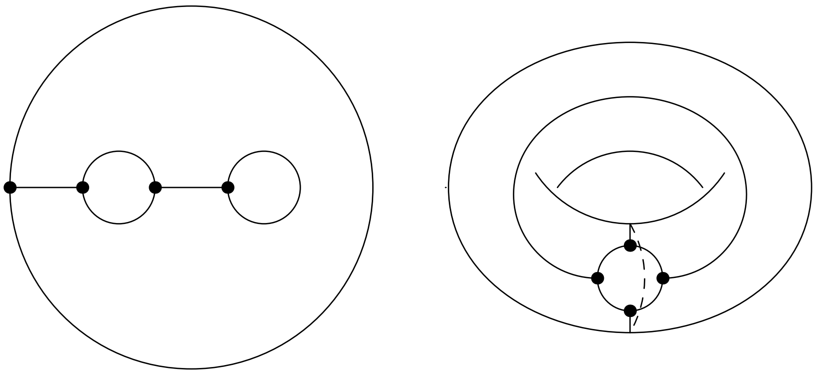
[3 marks]

Now $z \mapsto \log z$ maps H bijectively onto U. So we can take $p_{1,R}(z) = p_{3,R}(\ln z)$, and $p_{1,R} : H \to A_R$ is a covering.
[1 mark]

The covering group is

$$\{z \mapsto \lambda^{2n} z : n \in \mathbf{Z}\} = \{z \mapsto B_1^n(\lambda.z : n \in \mathbf{Z}\}$$

for $\lambda > 1$ such that $\ln \lambda = \pi^2/(\ln R)$.
[2 marks]
$1+2+1+3+6+3+3+3+1+2 = 25.$
Some aspects of this example will be covered in lectures and homeowrk - other similar examples in homework and lectures.
8(i) Let v denote the number of vertices of G, e the number of edges and f the number of components of $S \setminus G$ (which are all topolological discs, by the assumptions on G). Then the number $v - e + f$ is independent of the graph G chosen and is the Euler characteristic $\chi(S)$ of S
[2 marks]
(ii)



The pair of pants **The torus minus disc**

(ii)a) From the diagram we have $f = 1$, $e = 6$, $v = 4$. So $\chi(S) = 1 - 6 + 4 = -1$.
[2 marks]
(ii)b) From the diagram we have $v = 4$, $e = 6$, $f = 1$. So. again, $\chi(S) = -1$.
[2 marks]
(iii) Take a finite graph $G \subset S_1$ such that any component of $S_1 \setminus G$ is a topological open disc. Then any component γ of ∂S_1 is a union of alternate edges and vertices of G. So if $v(\gamma)$, $e(\gamma)$ are the numbers of vertices and edges of G in γ, we have $v(\gamma) = e(\gamma)$. Let G_2 be the graph in S_2 obtained by identifying the pairs of boundary components of S_1 which are identified in S_1. Then using G and G_2 to calculate $\chi(S_1)$, $\chi(S_2)$,

$$\chi(S_2) = \chi(S_1) - \sum_\gamma (v(\gamma) - e(\gamma)) = \chi(S_1)$$

where the sum is over one γ in each identified pair.
[4 marks]
(iv) $S \setminus A$ is a compact orientable surface-with-boundary. with either one or two components, depending on whether A does not disconnect, or does disconnect, S. If

$S\backslash A$ is connected, it has 2 boundary components, and te same Euler characteristic - 4 - as S. So in this case $S\backslash A$ is a two-holed torus minus 2 discs. Any compact connected orientable surface-with-boundary is uniquely determined, up to homeomorphism by its Euler characeristic and number of boundary components.

Now suppose that $S \setminus A$ has two components T_1 and T_2. Then each of T_1, T_2 is connected and has one boundary component. Neither of them is a disc, because A is homootpically nontrivial. So both T_1 and T_2 have negative odd Euleer chacteristic (since each has one boundary component). So one of them, say T_1, has Euler chacteristic -1 and is a torus minus one diec, while T_2 is a 2-holed torus minus a disc.

[5 marks]

(v) If $S \setminus (A_1 \cup A_2)$ is connected, it has four boundary components, and must have Euler characteristic -4, and must be a torus minus four discs.

If $S \setminus A_1$ is connected and $S \setminus (A_1 \cup A_2)$ is not then either both components of $S \setminus (A_1 \cup A_2)$ have two boundary components and Euler characteristic -2 - because if the Euler chacteristic of one component is 0 then it is an annulus and A_1 and A_2 are homotopic - or one component has three boundary components and the other has one. If two components have two boundary components and Euler characteristic 2 then they are both tori minus two discs. If one component has three boundary components and the other has one, then the Euler characteristics are -1 and -3 - either way round. The components of $S \setminus (A_1 \cup A_2)$ are either a two-holed torus minus one disc and a pair of pants, or a torus minus one disc and a torus minus 3 discs.

The possibilities are similar if $S \setminus A_2$ is connected and $S \setminus (A_1 \cup A_2)$ is not.

[5 marks]

Now suppose that both A_1 and A_2 disconnect S. Then A_2 disconnects one component T_1 or T_2 of $S \setminus A_1$ - otherswise $S \setminus A_2$ is connected. Now A_2 cannnot isconnect the one-holed torus T_1 because T_1 has Euler characteristic -1 and if $A_2' subset T_1$ and $T_1 \setminus A_2$ has two components, neither compoentn is allowed to have Euler characteristci 1 (since this give A_2 trivial) or 0 (since this gives A_2 homotopic to A_1. So $A_2 \subset T_2$ and the two components of $T_2 \setminus A_2$ have Euler characteristisc -1 and -2. One component is a torus minus tw discs and one a torus minus one disc.

[5 marks]

$2 + 2 + 2 + 4 + +5 + 5 + 5 = 25$

(i) and (iii) are bookwork. (ii), (iv) and (v) are similar to examples from lectures and homework, with the examples in (ii) being covered in lectures at soem point - but it is surely easier to teat from scratch.