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Each question is worth 25 marks. Full marks may be obtained by answering four questions completely. This is an open book examination.

The following will be used in questions 4 and 7. The standard action of

$$GL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\ b,\ c,\ d \in \mathbb{C},\ ad - bc \neq 0 \right\}$$

on $\mathbb{C} \cup \{\infty\}$ is given by

$$A.z = \frac{az+b}{cz+d} \quad \text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Under this action, $SL(2,\mathbf{R})$ - the set of matrices with a, b, c, d all real and $ad - bc = 1$ - preserves H and $\partial H = \mathbf{R} \cup \{\infty\}$, where

$$H = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}.$$

The action of

$$SU(1,1) = \left\{ \begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix} : |a|^2 - |c|^2 = 1 \right\}$$

preserves D and ∂D where

$$D = \{z \in \mathbf{C} : |z| < 1\},\ \partial D = \{z \in \mathbf{C} : |z| = 1\}.$$

For piecewise C^1 paths $\gamma_1 : [s_1, s_2] \to H$ and $\gamma_2 : [t_1, t_2] \to D$ we define

$$\ell_H(\gamma_1) = \int_{s_1}^{s_2} \frac{|\gamma'(t)|}{\mathrm{Im}(\gamma(t))} dt, \qquad \ell_D(\gamma_2) = \int_{t_1}^{t_2} \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} dt.$$

We define a metric d_H on H by

$$d_H(z_1, z_2) = \inf\{\ell_H(\gamma) : \gamma \text{ has endpoints } z_1,\ z_2\}$$

and define d_D on D similarly using ℓ_D.

1.

subexno(i) Give the definition of a topological space. Define the open sets in the standard topology on $\mathbb{R}^n$, for any integer $n > 0$. Show that if U and V are open in the standard topology on $\mathbb{R}^n$, then $U \cap V$ is also.

(ii) Determine which of the following are open in the standard topology on $\mathbb{R}^2$, giving brief reasons.

a) $\{(x, 0) : x \in \mathbb{R}\}$.

b) $\{(x, y) : x, y \in \mathbb{R},\ y \geq 0\}$.

c) $\{(x, y) : x, y \in \mathbb{R},\ x > y\}$.

(iii) Let X and Y be topological spaces and let $f : X \to Y$. Show that the following three definitions of f being continuous are equivalent.

a) For all $x \in X$ and any open $V \subset Y$ with $f(x) \in V$, there is an open $U \subset X$ with $x \in U$ and $f(U) \subset V$.

b) For any open $W \subset Y$, $f^{-1}(W)$ is open.

c) For any closed $F \subset Y$, $f^{-1}(F)$ is closed.

[25 marks]

2.

(i) Given a topological space $(X, \mathcal{T})$, define what it means for $(X, \mathcal{T})$ to be *compact* and what it means for $(X, \mathcal{T})$ to be *Hausdorff*. If $\sim$ is an equivalence relation on X define the *quotient topology* on $X/\sim$ (with respect to the topology $\mathcal{T}$ on X).

(ii) For x, $y \in \mathbb{R}$, define $x \sim y \Leftrightarrow y = x + n$ for some $n \in \mathbf{Z}$. Check that $\sim$ is an equivalence relation. Show that $\mathbb{R}/\sim$ is compact and Hausdorff.

[*Hint:* You may assume without proof that the continuous image of a compact set is compact, and that a closed bounded interval in $\mathbf{R}$ is compact.]

(iii) Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $F : X \to Y$ be a continuous function with the property that $x_1 \sim x_2$ implies $F(x_1) = F(x_2)$. Show that $[F] : X/\sim \to Y : [x] \mapsto F(x)$ is well-defined and continuous.

(iv) Find the smallest $\lambda > 0$ such that $\cos(\lambda(x+n)) = \cos(\lambda x)$ and $\sin(\lambda(x+n)) = \sin(\lambda x)$ for all $x \in \mathbf{R}$ and $n \in \mathbf{Z}$. Now let

$$Y = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

By finding a continuous map $F : \mathbb{R} \to Y$ or otherwise, show that $\mathbb{R}/\sim$ and Y are homeomorphic. You are not required to prove continuity of the function F.

[*Hint:* You may assume without proof that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.]

[25 marks]

3.

(i) Define an *orientable C^1 manifold.*

(ii) Let
$$X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$
and let X have the subspace topology with respect to the standard topology on $\mathbb{R}^2$. Consider the set Λ of charts (U_j, φ_j), $j = 1, 2, 3, 4$ with
$$U_1 = \{(x, y) \in X : y > 0\}, \quad \varphi_1(x, y) = x,$$
$$U_2 = \{(x, y) \in X : x > 0\}, \quad \varphi_2(x, y) = -y,$$
$$U_3 = \{(x, y) \in X : y < 0\}, \quad \varphi_3(x, y) = -x,$$
$$U_4 = \{(x, y) \in X : x < 0\}, \quad \varphi_4(x, y) = y.$$

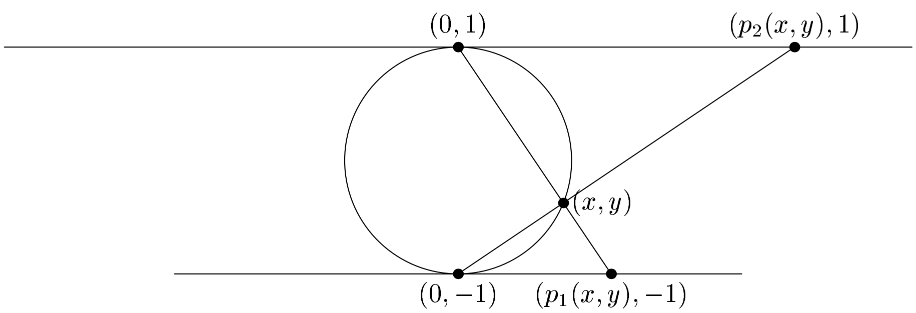
Show that these charts comprise an orientable C^1 atlas for X.
[*Hint.* You only need compute the four transition functions $\varphi_2 \circ \varphi_1^{-1}$, $\varphi_3 \circ \varphi_2^{-1}$, $\varphi_4 \circ \varphi_3^{-1}$, $\varphi_1 \circ \varphi_4^{-1}$, since the other four are inverses of these. You should get the same domain $(0, 1)$ *and the same transition function* in each case.]

(iii) Now consider the maps
$$p_1 : X \setminus \{(0, 1)\} \to \mathbf{R} \text{ and } p_2 : X \setminus \{(0, -1)\} \to \mathbf{R}$$
(which are known as *stereographic projections*) defined by the following diagram, that is, for a unique $\lambda > 0$, $\mu > 0$ depending on (x, y),
$$(0, 1) + \lambda((x, y) - (0, 1)) = (p_1(x, y), -1),$$
$$(0, -1) + \mu((x, y) - (0, -1)) = (p_2(x, y), 1).$$

a) Show that p_1 is a homeomorphism from $X \setminus \{(0, 1)\}$ to $\mathbf{R}$

b) Show that $p_1 \circ \varphi_1^{-1}$ is C^1 on $\varphi_1(U_1 \setminus \{(0, 1)\})$.

c) Compute $p_1(x, y).p_2(x, y)$ and hence compute the transition function $p_2 \circ p_1^{-1}$, giving its domain.



Stereographic Projection

[25 marks]

4. We use the notation established at the start of this paper.

(i) Show that if $A \in SU(1,1)$ and $\tau(z) = A.z$, then for all $z \in D$,

$$\frac{|\tau'(z)|}{1 - |\tau(z)|^2} = \frac{1}{1 - |z|^2}.$$

Hence, or otherwise, show that, for any piecewise C^1 path $\gamma_1 : [0,1] \to D$,

$$\ell_D(\tau \circ \gamma_1) = \ell_D(\gamma_1).$$

Hence or otherwise, show that for all z_1, $z_2 \in D$,

$$d_D(z_1, z_2) = d_D(A.z_1, A.z_2).$$

(ii) Let $\gamma_2 : [0,1] \to D$ satisfy $\gamma_2(0) = 0$ and

$$\gamma_2(t) = r(t)e^{i\theta(t)}$$

for piecewise C^1 functions $r : [0,1] \to [0,1)$ and $\theta : [0,1] \to \mathbf{R}$. Show that $|\gamma_2'(t)| \geq |r'(t)|$ and hence or otherwise show that

$$\ell_D(\gamma_2) \geq \int_0^1 \frac{2r'(t)}{1 - (r(t))^2} dr = \ln\left(\frac{1 + r(1)}{1 - r(1)}\right).$$

Hence or otherwise, show that for any $s_0 > 0$ there is $r_0 \in (0,1)$ depending on s_0 - which you should compute in terms of s_0 - such that

$$\{z \in D : d_D(0, z) < s_0\} = \{z : |z| < r_0\}.$$

(iii) Show that $SU(1,1)$ acts transitively on D.
Hint: It is enough to show that for any $z \in D$ there is $A \in SU(1,1)$ such that $A.0 = z$.

(iv) Show that for any geodesic ℓ in D and any $z_1 \in D \setminus \ell$, there is a unique point $z_2 \in \ell$ such that

$$d_D(z_1, z_2) = \text{Min}\{d_D(z_1, z) : z \in \ell\}$$

and that the (unique) geodesic segment between z_1 and z_2 is perpendicular to ℓ at z_2.

Hint. You may assume that the $GL(2, \mathbb{C})$ action preserves the set of circles and straight lines, and also preserves angles. You may also assume (although this virtually follows from an earlier part of the question) that the geodesics in D are the circle arcs and straight line segments which meet ∂D at right-angles. Of course, earlier parts of the question might be useful.

[25 marks]

5. Let
$$X = \{z \in \mathbb{C} : 1 < |z| < 2\}$$
and
$$U = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < \ln 2\}.$$
Let $p : U \to X$ be defined by $p(z) = e^z$.

(i) Show that p is a covering. Check that the following paths $\gamma_j : [0,1] \to X$ all have the same endpoints and find one lift to U of each of them.

$$\gamma_1(t) = \left(\frac{5}{4} + \frac{t}{2}\right) e^{i\pi t},$$
$$\gamma_2(t) = \frac{5}{4} e^{t(i\pi + \ln(7/5))},$$
$$\gamma_3(t) = \frac{5}{4} e^{t(-i\pi + \ln(7/5))}.$$

Hence, or otherwise, show that γ_1 and γ_2 are homotopic, and give a homotopy between them. Explain, quoting any necessary theory, why γ_3 is not homotopic to either γ_1 or γ_2.

(ii) Define what it means for a topological space to be path-connected. Show that X is path-connected.

(iii) Define what it means for a topological space to be simply-connected. Show that U is simply connected.

(iv) Determine the covering group of X. Hence, or otherwise, write down a representative of each homotopy class in $\pi_1(X, 3/2)$.

[25 marks]

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6. Let $X = \mathbb{R}^2/\sim$ where $(x, y) \sim (x', y') \Leftrightarrow x' = x + m$ and $y' = y + n$ for some $m,\ n \in \mathbf{Z}$. You may assume that $\sim$ is an equivalence relation. Let $p : \mathbf{R}^2 \to X$ be defined by $p(x, y) = [x, y]$, where $[x, y]$ denotes the equivalence class of (x, y) with respect to $\sim$. You may *assume* that p is a covering map.

a) Check that the covering group of X is $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ by

$$(m, n).(x, y) = (x + m, y + n).$$

Let $f : X \to X$ be any continuous map and $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ any lift of f to $\mathbb{R}^2$. For any $\underline{x_0} \in \mathbb{R}^2$ and $\underline{m} \in \mathbf{Z}^2$, show that there is $\varphi(\underline{m}) \in \mathbf{Z}^2$ such that

$$\tilde{f}(\underline{x_0} + \underline{m}) = \tilde{f}(\underline{x_0}) + \varphi(\underline{m}).$$

Explain, (quoting covering space theory if necessary), why $\varphi(\underline{m})$ is independent of $\underline{x_0}$ and why $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a group homomorphism. Show also that φ is independent of the choice of lift $\tilde{f}$ of f. For the rest of this question, write $\varphi = f_*$.

b) If $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a group homomorphism with $\psi(1, 0) = (a, c)$ and $\psi(0, 1) = (b, d)$ then compute $\psi(m, n)$ for all $(m, n) \in \mathbf{Z}^2$. Now for such a ψ, find a continuous $f : X \to X$ with $f_* = \psi$.

c) Now let $f : X \to X$ be the homeomorphism with lift $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\tilde{f}(x, y) = (x + \tfrac{1}{2}, -y).$$

(i) Show that $f \circ f$ is the identity map.

Let $Y = X/\approx$, where $z_1 \approx z_2 \Leftrightarrow z_1 = z_2$ or $z_2 = f(z_1)$.

(ii) Show that $\approx$ is an equivalence relation, using (i) or otherwise.

(iii) Now *assume* that the function which sends $z \in X$ to its $\approx$-equivalence class is a covering map, and that the covering group G of Y *on* $\mathbf{R}^2$ is generated by $\tilde{f}$ and the covering group $\mathbb{Z}^2$ of X.

Show that, for $(m, n) \in \mathbb{Z}^2$ and $(x, y) \in \mathbb{R}^2$,

$$\tilde{f}((m, n).(x, y)) = (m, n).\tilde{f}(x, y) \quad \Leftrightarrow \quad n = 0.$$

Show also that for all $(m, n) \in \mathbb{Z}^2$ and $(x, y) \in \mathbb{R}^2$,

$$\tilde{f}((m, n).\tilde{f}((m, n).(x, y))) = (x + 2m + 1, y).$$

Identify the *centre*

$$\{h \in G : gh = hg \text{ for all } g \in G\}$$

of G.

[25 marks]

7. We use the notation established at the start of the paper for part (ii) onwards.

(i) Let a, $b \in \mathbb{C}$ with $a \neq 0$. Show that the map

$$z \mapsto az + b : \mathbb{C} \to \mathbb{C}$$

has no fixed points $\Leftrightarrow a = 1$ and $b \neq 0$. Hence show that the only form an infinite cyclic covering group of holomorphic bijections of $\mathbb{C}$ can take is

$$\{z \mapsto z + nb : n \in \mathbb{Z}\}$$

for some $b \neq 0$.

[You may assume that every holomorphic bijection is of the form $z \mapsto az + b$ for a, $b \in \mathbb{C}$ with $a \neq 0$.]

Show, by finding an appropriate covering map, that any one of these groups can be realised as the covering group of $\mathbb{C} \setminus \{0\}$.

(ii) Let $A \in SL(2, \mathbb{R})$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that the map $z \mapsto A.z : H \to H$ has no fixed point in $H \Leftrightarrow |a + d| \geq 2$ and $A \neq \pm I$. Explain why this means that A is conjugate in $SL(2, \mathbb{R})$ to one of

$$B_1(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \ (\lambda \neq \pm 1) \ \text{ or } \ B_2 = \pm \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}.$$

(iii) Let $G_1(\lambda)$ and G_2 denote the groups of Möbius transformations of H generated by $B_1(\lambda)$ and B_2 respectively. (Any of the four choices of B_2 generates the same group G_2.)

a) Find a covering map

$$p_2 : H \to \{z \in \mathbf{C} : 0 < |z| < 1\} = A(+\infty)$$

for which the covering group is G_2.

b) Find a covering map

$$p_{3,R} : U = \{z : 0 < \mathrm{Im}(z) < \pi\} \to A_R = \{z : 1 < |z| < R\}$$

and the covering group of A_R acting on U. Hence, or otherwise, find a covering map $p_{1,R} : H \to A(R)$ with covering group $G_1(\lambda(R))$ for a suitable $\lambda(R)$, which you should specify.

[25 marks]

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8. (i) Explain briefly how the Euler characteristic $\chi(S)$ of a surface-with-boundary S is computed, using a finite graph $G \subset S$ such that all components of $S \setminus G$ are open discs, up to homeomorphism.

(ii) By drawing a suitable graph, compute $\chi(S)$ where S is

a) a pair of pants,

b) a torus minus an open disc.

(iii) Let S_1 be a (possibly disconnected) surface-with-boundary. Let S_2 be obtained from S_1 by identifying some pairs of boundary components of S_1. Show that

$$\chi(S_2) = \chi(S_1).$$

(iv) Let S be a 3-holed torus and let $A \subset S$ be an open annulus which is homotopically nontrivial, that is, not homotopic to a point. By using $\chi(S) = \chi(S \setminus A)$ and quoting any necessary results about Euler characteristic, find all possibilities, up to homeomorphism, for the surface-with-boundary $S \setminus A$.

Caution. A may, or may not, disconnect S.

(v) Now do the same if S is as in (iv) and $A_1 \cup A_2 \subset S$ where A_1 and A_2 are both homotopically nontrivial annuli, A_1 and A_2 are disjoint and not homotopic to each other.

[25 marks]