MATH553. Topology and Geometry of Surfaces
Revision Problems

It would certainly be worth going back through the problem sheets and tackling problems that you have not previously had time to do. Problem Sheet 4, for example, had a large number of questions, many of which, I think, are quite short. If you want me to look at and mark any solutions that you attempt, I shall be happy to do so. The 2002 exam, on VITAL, and also accessible from my own webpage for MATH553, does look to me like a perfectly reasonable mock exam, so it is a good idea to try questions from there also.

Anyway, here are some additional exercises.

1. Give the definitions of open and closed sets in the standard topology on $\mathbb{R}^n$. Determine which of the following is a) open, b) closed, c) neither, all in the standard topology on $\mathbb{R}^2$ (or $\mathbb{C}$), giving brief reasons.
(i) $[0,1] \times [0,1]$.
(ii) $\{z \in \mathbb{C} : |z| > 2\}$.
(iii) $\{z \in \mathbb{C} : -\frac{1}{4} < |z|(1-|z|) < \frac{1}{4}\}$.
(iv) $\{2^n : n \in \mathbb{Z}, n \geq 0\}$.
(v) $\{2^n : n \in \mathbb{Z}, n \leq 0\}$.

2a). Show that if U and V are open sets in the satndard topology on $\mathbb{R}$, then $U \times V$ is open in the standard topology on $\mathbb{R}^2$.
b) Define what it means for a map $f : X \to Y$ between topological spaces X and Y to be *continuous*, and what to means for s to be a *homeomorphism*.
c) Show that $x \mapsto (x,0) : \mathbb{R} \to \mathbb{R} \times \{0\}$ is a homeomorphism, if $\mathbb{R}$ and $\mathbb{R}^2$ are given the standard topology, and $\mathbb{R} \times \{0\}$ is given the subspace topology as a subspace of $\mathbb{R}^2$.

3. Give the definition of compact. Determine which of the sets in 1 is compact in the standard topology on $\mathbb{R}^2$ (or $\mathbb{C}$). In the cases where the set is noncompact, give an example of an open cover with no finite subcover.

4. Prove, from the definitions of compact and continuous, that if and X and Y are topological spaces, X is compact and $f : X \to Y$ is continuous onto, then Y is compact.

5a) Give the definition of Hausdorff.

Now let $\sim_1$ and $\sim_2$ be two equivalence relations on the set $X = [0,1] \times \{1,2,3\}$. Let the nontrivial equivalence classes for $\sim_1$ be $\{(1,1),(0,2)\}$ and $\{(1,2),(0,3)\}$. Let the only nontrivial equivalence classe for $\sim_2$ be $\{(0,1),(0,2),(0,3)\}$. Let X be given the standard topology, as a subset of $\mathbb{R}^2$. Let $[x]_i$ denote the equivakence class of x with respect to $\sim_i$.
b) Show that $X/\sim_1$ is homeomorphic to $[0,1]$, where $[0,1]$ is given the standard topology, that is, the subspace topology as a subset of $\mathbb{R}$, where $\mathbb{R}$ has the standard topology. You may assume that if $F : X \to [0,1]$ is continuous and $[F]([x]_i) = F(x)$ is well defined then $[F]$ is continuous. State any other result that you use.

c) Show that if $x \in [0,1]$ then $[0,x)$ and $(x,1]$ are path-connected. (Of course, $[0,0)$ and $(1,1]$ are empty.) Hence, or otherwise, show that $X/\sim_1$ and $X/\sim_2$ are not homeomorphic, possibly by removing a suitable point from $X/\sim_2$.

6a). Define what it means for a topological space $(X, \mathcal{T})$ to be a topological manifold, and what it means for $(X, \mathcal{T})$ to be an n-dimensional manifold. Also, define a *transition function*.

b) Let X be a one-dimensional manifold and let (U_1, φ_1) and (U_2, φ_2) be charts $\varphi_i(U_i) = (0,1)$ for $i = 1$, 2 and with $U_1 \cap U_2 \neq \emptyset$. Show that if $(a,b) \subset \varphi_1(U_1 \cap U_2)$ with $0 < a < b \leq 1$ and $a \notin \varphi_1(U_1 \cap U_2)$ then

$$\lim_{x \to a, x > a} \varphi_2 \circ \varphi_1^{-1}(x) = 0 \text{ or } 1.$$

Hint If not so, let $c \in (0,1)$ be a limit of a sequence $\varphi_2 \circ \varphi_1^{-1}(x_n)$ where $\lim_{n \to \infty} x_n = a$ and show that $\varphi_2^{-1}(c) \neq \varphi_1^{-1}(a)$ and that there are no disjoint open sets in X with one containing $\varphi_2^{-1}(c)$ and the other containing $\varphi_1^{-1}(a)$.

This is probably a bit hard. It is the basis of a proof that every one-dimensional manifold is homeomorphic to either $\mathbb{R}$ or S^1.

7a) Define what it means for a 2-dimensional manifold X to be a *complex manifold* (also called a *Riemann surface*) and to be an *orientable complex manifold* (also called an orientable Riemann surface) and what it means for X to be a *hyperbolic manifold* . Explain why every two-dimensional hyperbolic manifold is an orientable complex manifold.

b) Show that $\mathbb{C}/\mathbb{Z} = \mathbb{C}/(z \sim z + n, n \in \mathbb{Z})$ is a complex manifold.

c) Explain why $\mathbb{C}/\mathbb{Q} = \mathbb{C}/(z \sim z + p, p \in \mathbb{Q})$ is not a complex manifold.

d) Let the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be given the one-point compactification topology. Show that the map from $\{\infty\} \cup \{z : |z| > 1\} \to \{z : |z| < 1\}$ given by $\infty \mapsto 0$, $z \mapsto z^{-1}$ if $z \in \mathbb{C}$, $|z| > 1$, is a homeomorphism. Hence, or otherwise, show that $\overline{\mathbb{C}}$ is a (orientable) complex manifold.

8a) Let X_1 and X_2 be orientable Riemann surfaces. Define what it means for $f : X_1 \to X_2$ to be holomorphic.

b) Show that $f : z \mapsto z^3 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is holomorphic, defining $f(\infty) = \infty$. *Hint*: for any $z \in \overline{\mathbb{C}}$, show that there is at least one chart $(U_1, \varphi 1)$ with $z \in U_1$, and at least one chart (U_2, φ_2) with $f(z) \in U_2$, such that $\varphi_2 \circ f \circ \varphi_1^{-1}$ is holomorphic on $\varphi_1(U_1 \cap f^{-1}(U_2))$.

c) Now let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map, that is, of the form p/q, where p and q are polynomials, extended to $\overline{\mathbb{C}}$ in the obvious way. Show that f is holomorphic, possibly by consideringthe functions f, $z \mapsto 1/f(z)$, $z \mapsto f(1/z)$ and $z \mapsto 1/f(1/z)$.

9. a) Show that $z \mapsto \frac{z-i}{z+i}$ maps $H = \{z : \text{Im}(z) > 0\}$ onto $D = \{z : |z| < 1\}$.

b) Define Poincaré length of a piecewise C^1 path $\gamma : [a,b] \to D$. Compute the length $\ell_D(\gamma)$ of a horizontal line segment γ from 0 to r, for any $0 < r < 1$. Show that this is the shortest path from 0 to r.

Hint: look at the corresponding proof for vertical segments in H.

c) Show that if

$$\tau(z) = \frac{az+c}{\overline{c}z+\overline{a}}$$

and $|a|^2 - |c|^2 = 1$ then $\tau(D) = D$ and for any $z \in D$,

$$|tau'(z)| = 1 - |tau(z)|^2$$

and hence or otherwise, for any piecewise C^1 path $\gamma : [a,b] \to D$, $\ell_D(\gamma) = \ell_D(\tau \circ \gamma)$.

d) Define the Poincaré metric d_D on D in terms of Poincaré length. Show that the set of points $\{z : d_D(0,z) < s\}$ is a Euclidean disc centred on 0, for any $s > 0$.

Hint: Show that the shortest path from 0 to z is a straight line segment, possibly by finding a length-preserving Möbius transformation mapping it to the line segment $[0, |z|]$.

10. This question is back to Poincaré distance in H.
a) Find the intersection point between Freddie's geodesic — which goes through the points i and $3+i$ — and Bian Ce's geodesic — which goes through the points $-1+i$ and $2+2i$.
b) Find the shortest distance between Paul's geodesic — though the points $1+2i$, $2+i$ — and Joel's geodesic — though the points $-5+i$ and $3+i$.
11a). Give the definitions of path-connected, connected and simply-connected.
b) Determine which of the sets in question 1 is path-connected, with brief reasons.
c) One of the sets in question one has exactly two connected components. Which one? Show that one of the components is simply connected and that one is not, possibly by producing a simply-connected covering space and a closed loop which lifts to a non-closed path in the simply-connected covering space.

12. Show that the set

$$X = \{(x,0) : x \in \mathbb{R}\} \cup \{(0,y) : y \in \mathbb{R}\} \subset \mathbb{R}^2$$

is path-connected.
Hint: think how to find paths from all points in X to $(0,0)$.
13. Let $\mathbb{R}P^2 = S^2/\sim$, where

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\},$$

$$\underline{x} \sim \underline{y} \Leftrightarrow \underline{x} = \pm\underline{y}.$$

a) Given that S^2 is simply connected, find the covering group G of $\mathbb{R}P^2$ and its action on S^2.
Hint: G is a (small!) finite group.
b) Find all homomorphisms of G to itself.
c) For each homomorphism φ of G to itself, find a map $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ with $f_* = \varphi$.

14. Let $(\mathbb{R} \times (-1,1))/\mathbb{Z}$ be as on Sheet 5. Show that $f_2 : (\mathbb{R} \times (-1,1))/\mathbb{Z} \to (\mathbb{R} \times (-1,1))/\mathbb{Z}$ is well-defined, where $f_2([\underline{x}]) = [-\underline{x}]$. (*Hint:* it is probably best to write $\underline{x} = (x_1, x_2)$.)
15. Let $f_1 : (\mathbb{R} \times (-1,1))/\mathbb{Z} \to (\mathbb{R} \times (-1,1))/\mathbb{Z}$ denote the identity map. Let f_2 be as in question 14. Then find lifts $\tilde{f}_1$, $\tilde{f}_2 : \mathbb{R} \times (-1,1) \to \mathbb{R} \times (-1,1)$ of f_1, f_2. Then find $(\tilde{f}_1)_* : \mathbb{Z} \to \mathbb{Z}$ and $(\tilde{f}_2)_* : \mathbb{Z} \to \mathbb{Z}$. [Since $\mathbb{Z}$ is abelian, we can write $(\tilde{f}_1)_* = (f_1)_*$ and $(\tilde{f}_2)_* = (f_2)_*$.]
16. Let $f : (\mathbb{R} \times (-1,1))/\mathbb{Z} \to (\mathbb{R} \times (-1,1))/\mathbb{Z}$ be continuous. Find a homotopy between f and f_1 and f_2, depending on whether $f_* = (f_1)_*$ or $(f_2)_*$.
Hint: it suffices to find a homotopy $\tilde{F}(\underline{x}, t)$ between lifts $\tilde{f}$ and $\tilde{f}_j$, $j = 1$ or 2, such that

$$\tilde{F}(x+n, (-1)^n y, t) = (f_j)_*(n).\tilde{F}(x, y, t) \mathrm{forall} (x, y) \in \mathbb{R}\times(-1,1),\ t \in [0,1],\ n \in \mathbb{Z}.$$

17. Let X and Y be path-connected topological spaces. A continuous map $f : X \to Y$ is said to be a *homotopy equivalence* if there is a continuous map $g : Y \to X$ such that $g \circ f$ is homotopic to the identity on X and $f \circ g$ is homotopic to the identity on Y. (Two continuous maps h_0, $h_1 : Z \to W$ are *homotopic* if there is a continuous map $H : Z \times [0,1] \to W$ such that $H(z, 0) = h_0(z)$ for all $z \in Z$ and $H(z, 1) = h_1(z)$ for all $z \in Z$. H is then said to be a *homotopy* between h_0 and h_1.)
a) Let X and Y be path-connected topological spaces and $f : X \to Y$ be continuous and $g : Y \to X$ be continuous. Show that if α_0, $\alpha_1 : [0,1] \to X$ are homotopic paths, then $f \circ \alpha_0$, $f \circ \alpha_1 : [0,1] \to Y$ are homotopic and if β_0, $\beta_1 : [0,1] \to Y$ are homotopic paths, then $g \circ \beta_0$, $g \circ \beta_1 : [0,1] \to X$ are homotopic
b) Now show that if $f : X \to Y$ is a homotopy equivalence, with g as in the definition, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.
Hint You may assume that $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic for any x_1, $x_0 \in X$, consider this for $x_1 = g \circ f(x_0)$.
18. Let $X = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ be the torus $\mathbb{C}/\sim$ where $z \sim w \Leftrightarrow w = m + ni$ for m, $n \in \mathbb{Z}$ (as usual). Fix a, b, c, $d \in mathbbZ$.
a) Show that $f : \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is well-defined by

$$f(\mathbb{Z} + i\mathbb{Z} + x + iy) = \mathbb{Z} + i\mathbb{Z} + ax + cy + i(bx + dy)$$

and find a lift $\tilde{f} : \mathbb{C} \to \mathbb{C}$.
b) Show that f is holomorphic if and only if $a = d$ and $b = -c$, that is, if and only if, for all x and y, $ax + cy + i(bx + dy) = (a - ic)(x + iy)$.
Hint: f is holomorphic if and only if $\tilde{f}$ is, and the Cauchy Riemann equations might come in useful.
c) Show that f is a homeomorphism of $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ if and only if $ad - bc = 1$.
Hint: If g is an inverse of f then $\tilde{f}^{-1}$ must be a lift of g and $(\tilde{f})_*$ must map $\mathbb{Z} + i\mathbb{Z}$ onto $\mathbb{Z} + i\mathbb{Z}$.
19. Let $X = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ be the torus $\mathbb{C}/\sim$ where $z \sim w \Leftrightarrow w = m + ni$ for m, $n \in \mathbb{Z}$ (as usual). Let $f : X \to X$ be continuous with lift $\tilde{f} : \mathbb{C} \to \mathbb{C}$.
a) Write down the formula for a general lift of f, in terms of $\tilde{f}$.

b) Let $(\tilde{f})_* : \mathbb{Z} + i\mathbb{Z}\mathbb{Z} + i\mathbb{Z}$ be defined by: $\tilde{f}(z+m+ni) = \tilde{f} + (\tilde{f})_*(m+ni)$ for any $z \in \mathbb{C}$. Show that $(\tilde{f})_*$ is independent of the choice of lift, and can therefore be denoted by f_*. Show also that there are a, b, c, $d \in \mathbb{C}$ such that

$$f_*(m+ni) = m(a+bi) + n(c+di).$$

c) Show that if $g_1(x+iy) = \mathrm{Re}('\tilde{f}(x+iy) - (a-ic)(x+iy))$ then $g_1(x+iy) = g_1(x+iy+m+in)$ for all m, $n \in \mathbb{Z}$. Deduce that $g_1(z)$ and $e^{\tilde{f}(z)-(a-ic)z}$ are bounded.

d) Deduce that if f is holomorphic, $f(\mathbb{Z} + i\mathbb{Z} + z) = \mathbb{Z} + i\mathbb{Z} + (a-ic)z + \beta$ for some constant $\beta \in \mathbb{C}$.

Hint: f is holomorphic if and only if $\tilde{f}$ is. What do you know about bounded holomorphic functions on $\mathbb{C}$

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Revision questions solutions

11. To connect $(x_1, 0)$ and $(x_2, 0)$ define $\alpha : [0,1] \to X$ by $\alpha(t) = (x_1(1-t) + tx_2, 0)$. To connect $(0, y_1)$ and $(0, y_2)$, define $\alpha : [0,1] \to X$ by $\alpha(t) = (0, y_1(1-t) + ty_2)$. To connect $(x, 0)$ and $(0, y)$ define $\alpha : [0,1] \to X$ by

$$\alpha(t) = ((1-2t)x, 0) \text{ if } t \in [0, \tfrac{1}{2}],$$

$$\alpha(t) = (0, (2t-1)y) \text{ if } t \in [\tfrac{1}{2}, 1].$$

12a). From the definition of $\mathbb{R}P^2$, we see that $\mathbb{R}P^2 = S^2/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{1, \tau\}$ and $1.\underline{x} = \underline{x}$ for all $\underline{x} \in S^2$, $\tau.x = -\underline{x}$ for all $\underline{x} \in S^2$. This is a free action because if $\tau.\underline{x} = -\underline{x} = \underline{x}$ then $2\underline{x} = \underline{0}$ and $\underline{x} = \underline{0}$, but $\underline{0} \notin S^2$. In fact we claim the action is that of a covering group, and hence $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$ acts on the universal cover S^2 of $\mathbb{R}P^2$ as above. It acts as a covering group because because if $B(\underline{x}, \varepsilon) = \{\underline{y} : \|\underline{y} - \underline{x}\| < \varepsilon\}$ (where $\|.\|$ denotes the Euclidean norm) then $B(\underline{x}, \varepsilon)$ and $\tau.B(\underline{x}, \varepsilon) = B(-\underline{x}, \varepsilon)$ are disjoint for any $\varepsilon < 1$. This is because if there is $\underline{y}$ with $\|\underline{y} - \underline{x}\| < \varepsilon$ and $\|\underline{y} + \underline{x}\| < \varepsilon$ then

$$2\|\underline{x}\| = \|\underline{x} + \underline{y} + (\underline{x} - \underline{y})\| \le \|\underline{x} + \underline{y}\| + \|\underline{x} - \underline{y}\| \le 2\varepsilon < 2,$$

but $\|\underline{x}\| = 1$ for $\underline{x} \in S^1$.
b) Any homomorphism of $\mathbb{Z}_2 = \{1, \tau\}$ maps 1 to 1. The only homomorphisms are $\varphi_1(1) = 1$, $\varphi_1(\tau) = \tau$, and $\varphi_2(1) = \varphi_2(\tau) = 1$.
c) The identity map id of $\mathbb{R}P^2$ has $(\mathrm{id})_* = \varphi_1$ (because for any identity map id, $(\mathrm{Id})_*$ is the identity homomorphism). Any constant map, for example $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ with $f([\underline{x}]) = ([(1,0,0)])$ for all $[\underline{x}] \in \mathbb{R}P^2$, satisfies $f_* = \varphi_2$.
13. We have $f_2([(x,y)]) = f_2([(x',y')]) \Leftrightarrow [(-x,-y)] = [(-x',-y')] \Leftrightarrow (-x',-y') = (-x+n, (-1)^n(-y)$ for some $n \in \mathbb{Z} \Leftrightarrow (x',y') = (x-n, (-1)^{-n}y)$ for some $n \in \mathbb{Z}$ $\Leftrightarrow [(x',y')] = [(x,y)]$. So f_2 is well-defined (and injective, in fact bijective). (It is also continuous because the map without square brackets is continuous.)

14. We have lifts $\tilde{f}_1(x,y) = (x,y)$ and $\tilde{f}_2(x,y) = (-x,-y)$ of f_1, f_2. Hence

$$\tilde{f}_1(x+n, (-1)^n y) = (x+n, (-1)^n y) = (\tilde{f}_1)_*(n).\tilde{f}_1(x,y) = n.(x,y)$$

for all $(x,y) \in \mathbb{R} \times (-1,1)$ and $n \in \mathbb{Z}$, and $(\tilde{f}_1)_*(n) = n$ for all $n \in \mathbb{Z}$. Also

$$\tilde{f}_2(x+n, (-1)^n y) = (-x-n, -(-1)^n y) = (-x-n, -(-1)^{-n} y) = (\tilde{f}_2)_*(n).\tilde{f}_2(x,y) = (-n).\tilde{f}_2(x,y)$$

for all $(x,y) \in \mathbb{R} \times (-1,1)$ and $n \in \mathbb{Z}$. So $(\tilde{f}_2)_*(n) = -n$ for all $n \in \mathbb{Z}$.
15. Define

$$\tilde{F}(\underline{x}, t) = (1-t)\tilde{f}(\underline{x}) + t\tilde{f}_j(\underline{x})$$

if $f_* = f_{j*}$. Then if $\underline{x'} = n.\underline{x}$,

$$\tilde{F}(\underline{x'}, t) = (1-t)\tilde{f}(\underline{x'}) + t\tilde{f}_j(\underline{x'}) = (1-t)\tilde{f}(n.\underline{x}) + t\tilde{f}_j(n.\underline{x})$$

$$= (1-t)(\tilde{f}_j)_*(n).\tilde{f}(\underline{x}) + t(\tilde{f}_j)_*(n).\tilde{f}_j(\underline{x}) = (\tilde{f}_j)_*(n).((1-t)\tilde{f}(x) + t\tilde{f}_j(\underline{x}))$$

for any $\underline{x} \in \mathbb{R} \times (-1,1)$ and $n \in \mathbb{Z}$, because

$$m.(t\underline{u} + (1-t)\underline{v}) = tm.\underline{u} + (1-t)m.\underline{v}$$

for any $\underline{u}$, $\underline{v} \in \mathbb{R} \times (-1,1)$ and $m \in \mathbb{Z}$.

So $F([\underline{x}], t)$ is well-defined and continuous if $F([\underline{x}], t) = \tilde{F}(\underline{x}, t)$ for all $\underline{x} \in (\mathbb{R} \times (-1,1))/\mathbb{Z}$ and $t \in [0,1]$.