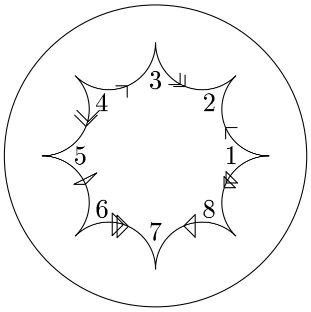
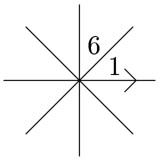
MATH553. Topology and Geometry of Surfaces
Problem Sheet 8: Hyperbolic Surfaces

Work is due in on *Thursday 1st December.*

Let $X \subset \{z : |z| < 1\}$ be an octagon with geodesic sides all of ℓ_D length equal to the same $s \in (0, \infty)$, and with all vertex angles equal to $\pi/4$. (Suppose that X does exist.) Let $X/\sim$ be the hyperbolic manifold formed by identifying sides as shown.



1. Show the order in which the angles labelled 1 to 8 occur round the point $[v]$ in $X/\sim$, where v is any vertex of X. (They are all in the same equivalence class.) The first two have been written in for you.



2. Describe an atlas for the hyperbolic manifold $X/\sim$, considering separately points $x_1 \in X$ where
(i) x_1 is an interior point of X,
(ii) x_1 is an interior point of a side of X, $[x_1] = \{x_1, x_2\}$, in which case there is $\varepsilon > 0$ such that $\{x' \in X : d_P(x_j, x') < \varepsilon\}$ does not meet any vertex of X, and only meets the side of X containing x, and you might consider your chart to have domain

$$\{[x'] : d_P(x', x_j) < \varepsilon\, x' \in X\, j = 1 \text{ or } 2\},$$

and find a chart map from this to (say) $\{z \in H : d_P(z, i) < \varepsilon\}$,
(iii) x_1 is a vertex of X, in which case there is $\varepsilon > 0$ such that $\{x' \in X : d_P(x_1, x') < \varepsilon\}$ does not meet any vertex of X apart from x_1, the set $[x_1]$ is the set of all 8 vertices of X, say x_j, $1 \le j \le 8$, and you might consider your chart to have domain

$$\{[x'] : d_P(x', x_j) < \varepsilon,\ x' \in X,\ 1 \le j \le 8\},$$

and find a chart map from this to (say) $\{z \in H : d_P(z, i) < \varepsilon\}$ (you would need to divide this disc up into 8 equal parts).

3. Now show that this octagon exists in $\{z : |z| < 1\}$, possibly as follows.

A regular octagon centred on 0 is invariant under the rotation $z \mapsto e^{\pi i/4}.z$. All vertices are the same Euclidean distance r from 0, and all vertex angles have the same value $\alpha(r)$, which is continuous in r for $r \in (0,1)$. Show that

$$\lim_{r \to 0} \alpha(r) = 7\pi/8, \ \lim_{r \to 1} \alpha(r) = 0.$$

Hint : For r near 0, the geodesics in which the sides lie are approximately diameters of the circle. For r near 1, the vertices are near the unit circle, and the geodesics in which the sides lie cut the unit circle at rightangles.
Deduce that there is $r \in (0,1)$ with $\alpha(r) = \pi/4$
4. Now show that $r \mapsto \alpha(r)$ is strictly decreasing. You could proceed as follows. Take two radii of the circle making angles $\pm\pi/8$ with the positive real axis. The geodesic segment joining points which are Euclidean distance r from 0 on these radii lies on a circle of radius R with centre on the positive real axis, and cutting the unit circle at rightangles.

Show that

$$r = \sqrt{R^2+1}\cos(\pi/8) - \sqrt{(R^2+1)\cos^2(\pi/8) - 1} = X - \sqrt{X^2-1},$$

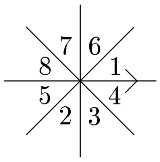
where $X = \sqrt{R^2+1}\cos(\pi/8)$, and, by differentiating or otherwise, that r is therefore a decreasing function of X, and thus also of R (where the above equation gives a real r - which is actually for $X \geq 1$, that is, $R^2 \geq (\sqrt{2} - 1)/(\sqrt{2}+1) = R_0^2$). Show, however, using the sine rule or otherwise, that

$$\frac{\sin((\alpha+\pi)/2)}{\sqrt{R^2+1}} = \frac{\sin(\pi/8)}{R}.$$

Then show that $\alpha \mapsto \sin(\alpha+\pi)/2$ is strictly decreasing for $\alpha \in (0, 3\pi/4)$, and $R \mapsto \sqrt{1+R^2}/R$ is strictly decreasing in $R \in (R_0, \infty)$ and thus that α is a strictly increasing function of R, and a strictly decreasing function of r.
Note that this is enough to show that the r with $\alpha(r) = \pi/4$ is unique.

The Topology and Geometry of Surfaces
Problem Sheet 8 Solutions

1. The order of the angles is as shown.



2.(i) Let τ be a Möbius transformation mapping the unit disc to the upper half plane H. Then (X, τ) is a chart at x_1, since $X \subset H$ and X is an open set containing x_1.
(ii) As suggested, we take the domain of our chart to be

$$V = \{[x'] : x' \in U_1\} \cup \{[x'] : x' \in U_2\},$$

where

$$U_j = \{x' \in X : d(x', x_j) < \varepsilon\}$$

and $U_1 \cap U_2 = \phi$. Then there is a Möbius transformation τ_j from the disc to the upper half plane H which maps U_j to

$$\{z \in H : d_P(z,i) < \varepsilon\} \cap \{z : \mathrm{Re}(z) \geq 0\} \text{ or } \{z \in H : d_P(z,i) < \varepsilon\} \cap \{z : \mathrm{Re}(z) \leq 0\},$$

depending on whether $j = 1$ or 2. Then $\varphi : V\ to \{z \in H : d_P(z,i) < \varepsilon\}$ is well-defined by

$$\varphi([z']) = \tau_j(z') \text{ if } z' \in U_j.$$

To see this, we only need to check that if $z' \in U_1$, $z'' \in U_2$ with $z' \sim z''$ then $\tau_1(z') = \tau_2(z'')$. But if this happens then z', $z'' \in \partial X$, $d_P(z', x_1) = d_P(z'', x_2)$ and the arrows from x_1 to z' and from x_2 to z'' point in the same direction.
(iii) We take chart

$$V = \cup_{j=1}^{8} \{[z'] : z' \in U_j\}\, U_j = \{z' \in X : d_P(z', x_j) < \varepsilon\},$$

at $[x_1]$, where the sets U_j are all disjoint (by choosing $\varepsilon > 0$ small enough). Then we choose a Möbius transformation τ_j from the disc to H which maps x_j to i and U_j to one of the sectors

$$\{z : d_P(z,i) < \varepsilon,\ 2r\pi/8 \leq \mathrm{Arg}(z) \leq 2(r+1)\pi/8\},$$

where $0 \leq r \leq 7$ and the sector is chosen so that the $\tau_j(U_j)$ are in the same order round i as the sets $[U_j]$ are round $[x_1]$. Then we define$\varphi : V \to \{z : d_P(z,i) < \varepsilon\}$ by

$$\varphi([z']) = \tau_j(z') \text{ if } z' \in U_j.$$

Then φ is well-defined, by the same sort of argument as in (ii).

Because the maps identity and τ_j are all Möbius transformations from the disc to H, the transition functions for these charts will all be restrictions of Möbius transformations of H, as required for a hyperbolic manifold.

3. When r is near 0, the geodesic segments making up the sides of the octagon are almost diameters of the unit circle, hence almost straight lines. So the angles between thm are almost what they would be for a Euclidean octagon, that is, $3\pi/4$. For r near 1, the sides of the octagon are almost complete circle arcs meeting the unit circle at rightangles, and thus sides meet each other at an angle close to 0. So

$$\lim_{r\to 0}\alpha(r)=3\pi/4\ \lim_{r\to 1}\alpha(r)=0.$$

Also, the function $r\mapsto\alpha(r)$ is continuous. So by the Intermediate Value Theorem there is $r_0\in(0,1)$ such that $\alpha(r_0)=\pi/8$.

4. Now we show that r_0 is unique. We only have to show that α is a strictly decreasing function of r. Let R, r be as in the diagram in the question. Then by the cosine rule we have

$$R^2=R^2+1+r^2-2r\sqrt{R^2+1}\cos(\pi/8),$$

giving

$$r^2-2r\sqrt{R^2+1}\cos(\pi/8)+1=0,$$

and

$$r=\sqrt{R^2+1}\cos(\pi/8)-\sqrt{(R^2+1)\{rmcos^2(\pi/8)-1}=X-\sqrt{X^2-1}. \quad (1)$$

We take the negative root because $r\in(0,1)$. We can show that

$$\cos^2(\pi/8)=(\sqrt{2}+1)/2\sqrt{2},$$

so we must have

$$R^2>2\sqrt{2}/(\sqrt{2}+1)-1=(\sqrt{2}-1)/(\sqrt{2}+1).$$

Differentiating (1), we obtain

$$\frac{dr}{dX}=1-\frac{X}{\sqrt{X^2-1}}=1-\frac{1}{\sqrt{1-X^{-2}}}<0,$$

$$\frac{dr}{dR}=\frac{dr}{dX}\frac{dX}{dR}<0.$$

So r is a decreasing function of R.

Also, if $\alpha\in(0,3\pi/4)$ then $(\alpha+\pi)/2\in(\pi/2,7\pi/8)$. So $\alpha\mapsto\sin((\alpha+\pi)/2)=y$ is a strictly decreasing function of α. Then by the Sine Rule,

$$\frac{\sin((\alpha+\pi)/2)}{\sqrt{R^2+1}}=\frac{\sin(\pi/8)}{R}.$$

So

$$y=\sin((\alpha+\pi)/2)=(\sqrt{R^2+1}/R)\sin(\pi/8)=\sqrt{1+1/R^2}\sin(\pi/8).$$

So y is a strictly decreasing fuction of R and α is a strictly increasing function of R. So α is a strictly decreasing function of r, as required.