MATH553. Topology and Geometry of Surfaces
Problem Sheet 7: Covering Groups of Holomorphic Bijections

Work is due in on *Thursday 24th November*

1. As usual, identify $\mathbb{R}$ with the subgroup $\{x + i0 : x \in \mathbb{R}\}$ of $\mathbb{C}$. Remember that $\mathbb{C}$ is a group under addition. Let Γ be a discrete subgroup of $\mathbb{C}$, that is, there exists $\varepsilon > 0$ such that

$$\Gamma \cap \{z : |z| < \varepsilon\} = \{0\}$$

Suppose also that $\Gamma \cap \mathbb{R} = \mathbb{Z}$.
a) Let $\alpha \in \Gamma$ satisfy

$$|\mathrm{Im}(\alpha)| = \mathrm{Min}\{|\mathrm{Im}(z)| : z \in \Gamma \setminus \mathbb{R}\}.$$

Show that if $z \in \Gamma$ then $\mathrm{Im}(z) = n\mathrm{Im}(\alpha)$ for some $n \in \mathbb{Z}$, and that

$$\Gamma = \mathbb{Z} + \alpha\mathbb{Z}.$$

b) Show that α as in a) does exist, possibly by showing that, for any $M > 0$,

$$\{\mathrm{Im}(z) : z \in \Gamma\, |\mathrm{Im}(z) < M\} = \{\mathrm{Im}(z) : z \in \Gamma\, |\mathrm{Im}(z)| < M\, |\mathrm{Re}(z)| < 1\},$$

and by showing that the righthand set is finite.
Hint: a bounded sequence in $\mathbb{R}^n$ (for any n, including $n = 2$) has a convergent subsequence.

2a). Show that, for $R \in (0, \infty)$,

$$\Phi_R : S_R = \{z : 0 < \mathrm{Im}(z) < \pi R\} \to H = \{z : \mathrm{Im}(z) > 0\} : \Phi_R(z) = e^{z/R}$$

is a holomorphic bijection. Also, find $\lambda > 0$ such that

$$\Phi_R(z + 1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} . \Phi_R(z) (= \lambda^2 \Phi_R(z)).$$

b) Let

$$p_R(z) = e^{2\pi i z}.$$

Show that the image of S_R under p_R is

$$A_R = \{z : e^{-2\pi^2 R} < |z| < 1\}.$$

Assuming that p_R is a covering [which is not hard to prove], both $p_R : S_R \to A_R$ and $p_R \circ \Phi_R^{-1} : H \to A_R$ are coverings. Find the covering group, regarded as a group of holomorphic bijections of
(i) S_R (ii) H.

3. Let $U \subset \mathbb{C}$ be open. A map $\varphi : U \to \mathbb{C}$ is said to be *conformal* if either φ or the composition with conjugation, $z \mapsto \overline{\varphi(z)}$, is holomorphic, A group G of

homeomorphisms of $\mathbb{C}$ is *conformal* if φ is conformal for all $\varphi \in G$. Now let g_1 and g_2 be the homeomorphisms of $\mathbb{C}$ given by

$$g_1(x+iy) = x + \frac{1}{2} - iy \ (x,\ y \in \mathbb{R})$$

$$g_2(x+iy) = x + i(y+1) \ (x,\ y \in \mathbb{R}).$$

a) Check that g_1 and g_2 are conformal, and that g_1^2 and g_2 are holomorphic, where g_1^2 denotes $g_1 \circ g_1$. b) Show also that

$$g_1 \circ g_2 = g_2^{-1} \circ g_1,$$

and if g_1^m denotes the m-fold composition of g_1 and similarly for g_2^n,

$$g_1^m \circ g_2^n(x,y) = (x + \frac{m}{2}, (-1)^m(y+n)(\ x,\ y \in \mathbb{R}).$$

c) Now let G be the group of homeomorphisms generated by g_1, and g_2 - which, by part b), can all be written in the form $g_1^m \circ g_2^n$ for m, $n \in \mathbb{Z}$. Show that the action of G on $\mathbb{C}$ is free, discrete and conformal. [The quotient space $\mathbb{C}/G$ is then a complex manifold - but nonorientable. It is known as the *Klein bottle*.] d) Let H be the subgroup of G generated by g_1^2 and g_2. Show that H is abelian. Let $\mathbb{C}/H = \{[z]_H : z \in \mathbb{C}$, where $[z]_H = \{h(z) : h \in H$ and let $\mathbb{C}/G$ be similarly defined. (This is just the Klein bottle.) Show that $\mathbb{C}/H$ is simply $\mathbb{C}/\sim_i$ where $\sim_i$ is the equivalence relation $z \sim_i z' \Leftrightarrow z' = z + m + ni$ for m, $n \in \mathbb{Z}$. Show also that the map $[z]_H \mapsto [z]_G : \mathbb{C}/H \to \mathbb{C}/G$ is a two-to-one covering map.