MATH553. Topology and Geometry of Surfaces
Problem Sheet 5: Lifts, Homotopy, Coverings

Please hand in your solutions to 1-5 in class on *Thursday 10th November.*

1. For each of the following closed paths in $S^1 = \{z : |z| = 1\}$, find a lift to $\mathbb{R}$ with first endpoint at $0 \in mathbbR$. Hence, determine which (if any) of the paths in S^1 are homotopic.

a) $t \mapsto e^{2\pi i t} : [0,1] \to S^1$, b) $t \mapsto e^{-2\pi i t} : [0,1] \to S^1$,

c) $t \mapsto -e^{\pi i \cos \pi t} : [0,1] \to S^1$

2. Show that your letter is path-connected. In all cases let S_1 be the topological space represented by your letter.

Freddie, Paul and Joel: find a point s_0 which can be removed from your letter so that $S_1 \setminus \{s_0\}$ is not path connected. To show this find a continuous function on $S_1 \setminus \{s_0\}$ which takes exactly two values (for Freddie's letter), and exactly 3 values (for Joel's letter) and exactly 4 values (for Paul's letter).

Cian Be: for your letter, it is not possible to find just a single point s_0 such that $S_1 \setminus \{s_0\}$ is not path-connected, but you do not need to prove this. Instead, find two points s_1, s_2 such that $S_1 \setminus \{s_1, s_2\}$ is path-connected, by producing a continuous function on $S_1 \setminus \{s_1, s_2\}$ which takes exactly 3 values.

Here is how to tackle the problem for the letter X which, as in Problem Sheet 2, is represented by the set

$$S_1 = \{(t,t) : t \in [-1,1]\} \cup (t,-t) : t \in [-1,1]\}.$$

To show this set is path connected: a path from (s,s) to (t,t) is given by a continuous map $f : [0,1] \to S_1$ defined by

$$f(u) = ((1-u)s + ut, (1-u)s + ut),$$

a path from $(s,-s)$ to $(t,-t)$ is given by a continuous map $f : [0,1] \to S_1$ defined by

$$f(u) = ((1-u)s + ut, -(1-u)s - ut),$$

and a path from (s,s) to $(t,-t)$ is given by a continuous map $f : [0,2] \to S_1$ defined by

$$f(u) = \begin{array}{ll} ((1-u)s, (1-u)s) & \text{if } u \in [0,1], \\ ((u-1)t, (1-u)t) & \text{if } u \in [1,2] \end{array}$$

Note that the two definitions of $f(1)$ in this formula coincide, so f is indeed continuous. Also, in the last formula, $u \mapsto f(2-u) : [0,2] \to S_1$ is a path from $(t,-t)$ to (s,s). So any two points in S_1 can be connected by a path.

The set $S_1 \setminus \{(0,0)\}$ is not path-connected. We can find a sontinous surjective function $f : S_1 \setminus \{(0,0)\} \to \{1,2,3,4\}$ defined by

$$\begin{array}{ll} f(t,t) = 1 & \text{if } t < 0 \\ f(t,t) = 2 & \text{if } t > 0 \\ f(t,-t) = 3 & \text{if } t < 0 \\ f(t,-t) = 4 & \text{if } t > 0 \end{array}$$

Each of the sets $f^{-1}(\{j\})$ is open in S_1 so f is continuous. But by the Intermediate Value Theorem the set of values of a continuous real-valued function on a path-connected space must be an interval.
3. Let $\mathbb{Z}$ act on $\mathbb{R} \times (-1,1)$ by

$$n.(x,y) = \varphi_n(x,y) = (x+n, (-1)^n y).$$

a) Check that $\varphi_n \circ \varphi_m = \varphi_{n+m}$ for all n, $m \in \mathbb{Z}$.
b) Let

$$B(\underline{x}_0, r) = \{\underline{x} \in \mathbb{R} \times (-1,1) : \|underline x - \underline{x}_0\| < r\},$$

where $\|.\|$ denotes Euclidean norm. Show that if $r \leq \frac{1}{2}$, the sets $\varphi_n(B(\underline{x}_0, r))$ $(n \in \mathbb{Z})$ are all disjoint, for any $\underline{x}_0 \in \mathbb{R} \times (-1,1)$ and $n \in \mathbb{Z}$.
c) Write $[\underline{x}] = \{n.\underline{x} : n \in \mathbb{Z}\}$. Then by using the sets

$$[B(\underline{x}_0, r)] = \{[\underline{x}] : \underline{x} \in B(\underline{x}_0, r)\},\ r \leq \tfrac{1}{2},$$

or otherwise, show that $\pi : \underline{x} \mapsto [\underline{x}] : \mathbb{R} \times (-1,1) \to (\mathbb{R} \times (-1,1))/\mathbb{Z}$ is a covering. (This then shows that $\mathbb{R} \times (-1,1)/\mathbb{Z}$ is a manifold: it is, in fact, the Möbius strip.)

4a) Show that $\mathbb{R} \times (-1,1)$ is path-connected and simply-connected.
b) Find lifts of the following closed paths in $(\mathbb{R} \times (-1,1))/\mathbb{Z}$ to $\mathbb{R} \times (-1,1)$ with first endpoint at $(0,0)$. Determine which (if any) of the closed paths in $(\mathbb{R} \times (-1,1))/\mathbb{Z}$ are homotopic.
(i) $t \mapsto [(t + \sin \pi t, 0)] : [0,1] \to (\mathbb{R} \times (-1,1))/\mathbb{Z}$,
(ii) $t \mapsto [(-t, 0)] : [0,1] \to (\mathbb{R} \times (-1,1))/\mathbb{Z}$,
(iii) $t \mapsto [(\cos \pi t, 0)] : [0,1] \to (\mathbb{R} \times (-1,1))/\mathbb{Z}$,
(iv) $t \mapsto [(-2t, 0)] : [0,1] \to (\mathbb{R} \times (-1,1))/\mathbb{Z}$.

5. Produce a homeomorphism between $(\mathbb{R} \times (-1,1))/2\mathbb{Z}$ and $S^1 \times (-1,1)$, where $S^1 = \{z : |z| = 1\}$ and $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$. It might be slightly simpler to produce a map which sends $(\mathbb{R} \times [-1+\delta, 1-\delta])/2\mathbb{Z}$ to $S^1 \times [-1+\delta, 1-\delta]$ for each $\delta > 0$, because a continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism.