MATH348. Harmonic Analysis. Problems 8

Work is due in on *Wednesday 24th November.*

1.

Verify that the function

$$u(x,t) = \frac{e^{t-(x^2/4t)}}{\sqrt{t}}$$

satisfies

$$\frac{\partial u}{\partial t} = u + \frac{\partial^2 u}{\partial x^2}.$$

2. Let $u(x,t)$ be continuous and bounded on $\{(x,t) : x \in \mathbf{R}, \mathbf{t \geq 0}\}$. For all $t > 0$ let $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$ be defined and integrable in x over $\mathbf{R}$, and let

$$\lim_{x\to\infty} u(x,t) = 0, \quad \lim_{x\to\infty} u_x(x,t) = 0.$$

Let $\hat{u}(\xi,t)$ be the Fourier transform of $u(x,t)$ with respect to x, and let $\hat{u}_x(\xi,t)$ and $\hat{u}_{xx}(\xi,t)$ be similarly defined. Using integration by parts, show that

$$\hat{u}_x(\xi,t) = i\xi\hat{u}(\xi,t), \; \hat{u}_{xx}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$

3. Now let $u(x,t)$ be as in question 2. In addition, for all x, and $t > 0$, let u_t be continuous and *locally uniformly integrable in* x, that is for all $a > 0$, let

$$\sup_{0<t<a} \int_{-\infty}^{\infty} |u_t(x,t)|dx < +\infty$$

and

$$\lim_{\Delta\to\infty} \int_{|x|\geq\Delta} |u_t(x,t)|dx = 0$$

uniformly for $t \in (0,a]$. Let

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u. \tag{2}$$

a) Using question 2, show that

$$\frac{\partial \hat{u}}{\partial t}(\xi,t) = -\xi^2\hat{u}(\xi,t) + \hat{u}(\xi,t).$$

You may assumethat $(\partial\hat{u}/\partial t)(\xi,t)$ is the Fourier transform of $(\partial u/\partial t)(x,t)$ with respect to x. (You will be asked for a step towards this in part b). THe conditions above are needded for the full result.)) Now solve this differential equation and show that

$$\hat{u}(\xi,t) = \hat{u}(\xi,0)e^{(1-\xi^2)t}.$$

Look in your notes to find a function that $e^{-\xi^2 t}$ is the Fourier transform of (treating t as a constant). Hence show that

$$u(x,t) = \int_{-\infty}^{\infty} u(y,0)\frac{e^{t-(x-y)^2/4t}dy}{2\sqrt{t\pi}}.$$

b) Show that if $h \neq 0,\ t > 0,\ t+h > 0$,

$$\frac{\hat{u}(\xi, t+h) - \hat{u}(\xi, t)}{h} - \hat{u}_t(\xi, t)$$

$$= \frac{1}{h}\int_0^h \int_{-\infty}^{\infty} e^{-i\xi x}(u_t(x, t+y) - u_t(x, t))dxdy.$$

You should indicate where Tonelli's Theorem is applied. It will help if you can show that

$$\frac{1}{|h|}\int_{[0,h]} {}_{-\infty}^{\infty}|e^{-i\xi x}(u_t(x, t+y) - u_t(x, t))|dxdy \leq 2\sup_{t'>0}\int_{-\infty}^{\infty}|u_t(x, t')dx.$$