MATH348. Harmonic Analysis. Problems 5.

Work is due in on *Friday 5th November.* I shall be away all day on Tuesday 2nd November, so there will be no office hours on that day. I shall be available at the usual times (11-1) on Monday and for part of the afternoon also, but I have to arrange two tutorials to Monday so am not yet sure which times. So I suggest having additional office hours on WEdnesday 3rd November, say 9-10 and 11-12. This is the reason for the later hand-in day, just for this week.

1. Find the Fourier transform $\hat{f}(\xi)$ of f, where a) for some $a > 0$, $f(x) = e^{-ax}$ for $x > 0$ and $= 0$ otherwise,

b) $f(x) = x$ for $0 \leq x \leq 1$ and $= 0$ otherwise,

c) $f(x) = xe^{-|x|}$.

2. Compute $\hat{f}(\xi)$ where

$$f(x) = \frac{1}{2 + 2x + x^2}.$$

In the case $\xi \geq 0$ you might find it helpful to consider the contour integral of $e^{-i\xi z}/(2 + 2z + z^2)$ round a half disc in the lower half plane. To get the formula for all ξ you may find it helpful to show that, as $f(x)$ is real for real x,

$$\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$$

3. Find $\hat{f}(\xi)$ where

$$f(x) = \frac{1}{(1 + x^2)^2}.$$

you can use

$$\overline{\hat{f}(\xi)} = \hat{f}(-\xi).$$

4. Show that the function $1/(1 + ix)$ on $\mathbf{R}$ is not integrable. However, compute

$$I(\xi) = \lim_{\Delta \to \infty} \int_{-\Delta}^{\Delta} \frac{e^{-i\xi x} dx}{1 + ix}$$

by considering separately the cases $\xi = 0$, when you should show that

$$I(\xi) = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi,$$

and $\xi > 0$ and $\xi < 0$, by considering integrals of $e^{-i\xi z}/(1 + iz)$ round half-discs in the lower and upper half-planes respectively. You may assume that the integrals on the curved parts of the contours $\to 0$ as $\Delta \to \infty$. For $\xi > 0$ you should obtain that $I(\xi) = 0$.

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1a) $$\hat{f}(\xi) = \int_0^\infty e^{-ax-i\xi x}dx = \left[\frac{e^{-ax-i\xi x}}{-a-i\xi}\right]_0^\infty = \frac{1}{a+i\xi}.$$

1b) If $\xi = 0$,

$$\hat{f}(0) = \int_0^1 x dx = \frac{1}{2}$$

If $\xi \neq 0$,

$$\begin{aligned}\hat{f}(\xi) = \int_0^1 xe^{-i\xi x}dx &= \left[\frac{xe^{-i\xi x}}{-i\xi}\right]_0^1 + \int_0^1 \frac{e^{-i\xi x}dx}{i\xi} \\ &= \frac{-e^{-i\xi}}{i\xi} + \left[\frac{e^{-i\xi x}}{\xi^2}\right]_0^1 \\ &= \frac{-e^{-i\xi}}{i\xi} + \frac{e^{-i\xi}-1}{\xi^2}.\end{aligned}$$

1c) $$\begin{aligned}\hat{f}(\xi) &= \int_0^\infty xe^{-(1+i\xi)x}dx + \int_{-\infty}^0 xe^{(1-i\xi)x}dx \\ &= \int_0^\infty x(e^{-(1+i\xi)x} - e^{-(1-i\xi)x})dx\end{aligned}$$

$$= \left[x\left(\frac{e^{-(1+i\xi)x}}{-(1+i\xi)} + \frac{e^{-(1-i\xi)x)}}{1-i\xi}\right)\right]_0^\infty + \int_0^\infty \left(\frac{e^{-(1+i\xi)x}}{1+i\xi} - \frac{e^{-(1-i\xi)x}}{1-i\xi}\right)dx.$$

The bracketed term vanishes at $x = 0$ and $\to 0$ as $x \to \infty$. So we are left with the integral. So

$$\begin{aligned}\hat{f}(\xi) &= \left[\frac{e^{-(1+\xi)x}}{-(1+i\xi)^2} + \frac{e^{-(1-i\xi)x}}{(1-i\xi)^2}\right]_0^\infty \\ &= \frac{-1}{(1-i\xi)^2} + \frac{1}{(1+i\xi)^2} \\ &= \frac{-4i\xi}{(1+\xi^2)^2}.\end{aligned}$$

2. Write γ_R for the closed contour in the lower halfplane around the semicircle of radius R, where the circle has centre 0. Let γ_R' be the curved part of the contour.First we consider $\xi \geq 0$ and consider

$$\int_{\gamma_-(R)} \frac{e^{-i\xi z}dz}{2+2z+z^2}$$

We have $2+2z+z^2 = 0$ if and only if $z = -1 \pm i$. So the integrand is holomophic in the complement of the two points $-1 \pm i$. So

$$\int_{\gamma_-(R)} \frac{e^{-i\xi z}}{2+2z+z^2} = 2\pi i \text{Res}\left(\frac{e^{-i\xi z}}{(z+1+i)(z+1-i)}, -1-i\right) = \left(\frac{e^{-\xi z}}{z+1-i}\right)_{z=-1-i} = -\pi e^{(i-1)\xi}.$$

We have

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}dx}{2+2x+x^2} = \lim_{R\to\infty} -\int_{-R}^{R} \frac{e^{-i\xi z}dz}{z^2+2z+2}.$$

We also have

$$\left|\int_{\gamma_R'} \frac{e^{-i\xi z}dz}{z^2+2z+2}\right| \le \frac{\pi R}{R^2-2R-2} \to 0 \text{ as } \Delta \to \infty.$$

This uses that the length of γ_R' is πR, $|e^{-i\xi z}| \le 1$ for $z \in \gamma_R'$ and $|z^2+2z+2| \ge R^2-2R-2$ for $|z| = R$ - in particular for $z \in \gamma_R'$.

So

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-i\xi x}dx}{2+2x+x^2} = \lim_{R\to\infty} -\int_{\gamma_R} \frac{e^{-i\xi z}dz}{2+2z+z^2} = \pi e^{(i-1)\xi}.$$

If $\xi \le 0$ then we have

$$\hat{f}(\xi) = \int f(x)e^{-i\xi x}dx = \overline{\int f(x)e^{i\xi x}dx} = \overline{\hat{f}(-\xi)}.$$

So for all ξ, we have

$$\hat{f}(\xi) = \pi e^{i\xi-|\xi|}.$$

3. Let γ_R and γ_R' be as in question 2. The function $e^{-iz\xi}/(1+z^2)^2$ is holomorphic except where $1+z^2=0$, that is, $z=\pm i$. The only singularity inside γ_R is at $-i$. Then by the Residue Formula,

$$\int_{\gamma_R} \frac{e^{-i\xi z}}{(1+z^2)^2}dz = 2\pi i \operatorname{Res}\left(\frac{e^{-i\xi z}}{(1+z^2)^2}, -i\right) = 2\pi i \operatorname{Res}\left(\frac{e^{-i\xi z}}{(z-i)^2(z+i)^2}, -i\right).$$

Now

$$\begin{aligned}
\operatorname{Res}\left(\frac{e^{-i\xi z}}{(z-i)^2(z+i)^2}, -i\right) &= \frac{d}{dz}\frac{e^{-i\xi z}}{(z+i)^2}\bigg|_{z=-i} \\
&= (-2(z-i)^{-3}e^{-i\xi z} - i\xi(z-i)^{-2}e^{-i\xi z})|_{z=-i} \\
&= \left(\frac{-2}{8i} + \frac{i\xi}{4}\right)e^{-\xi} \\
&= \frac{1+\xi}{4}ie^{-\xi}.
\end{aligned}$$

So

$$\int_{\gamma_R} \frac{e^{-i\xi z}}{(1+z^2)^2}dz = -\frac{1}{2}\pi(1+\xi)e^{-\xi}.$$

Now

$$\int_{\gamma_R} \frac{e^{-i\xi z}}{(1+z^2)^2}dz = -\int_{-R}^{R} \frac{e^{-ix\xi}}{(1+x^2)^2}dx + \int_{\gamma_R'} \frac{e^{-i\xi z}}{(1+z^2)^2}dz.$$

On γ_R' we have $\operatorname{Im} z \le 0$ and $|z| = R$. Writing $z = x+iy$ for x, y real, we have $y \le 0$ and

$$|e^{-i\xi z}| = |e^{-i\xi x+\xi y}| = e^{\xi y} \le 1 \text{ if } y \le 0,\ \xi \ge 0.$$

We also have

$$|(1+z^2)^2| = |1+z^2|^2 \geq (|z|^2-1)^2 \geq (R^2-1)^2.$$

So if $\xi \geq 0$

$$\left|\int_{\gamma_R'} \frac{e^{-i\xi z}}{(1+z^2)^2} dz\right| \leq \text{length}(\gamma_R') \times \frac{1}{(R^2-1)^2} = \frac{\pi R}{(R^2-1)^2} \to 0 \text{ as } R \to \infty.$$

So if $\xi \geq 0$

$$-\frac{1}{2}\pi(1+\xi)e^{-\xi} = \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{-i\xi z}}{(1+z^2)^2} dz = -\int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{(1+x^2)^2} dx$$

This gives, if $\xi \geq 0$

$$\hat{f}(\xi) = \frac{1}{2}\pi(1+\xi)e^{-\xi}$$

which is real. Then using $\overline{\hat{f}(\xi)} = \hat{f}(-\xi)$ we have, for all ξ,

$$\hat{f}(\xi) = \frac{1}{2}\pi(1+|\xi|)e^{-|\xi|}.$$

4. $|(1+ix)^{-1}| \geq 1/2|x|$ if $|x| \geq 1$, and

$$\int_1^\infty \frac{dx}{x} = \infty$$

So $1/(1+ix)$ is not integrable. Taking $\xi = 0$,

$$\lim_{\Delta\to\infty} \int_{-\Delta}^{\Delta} \frac{dx}{1+ix} = \lim_{\Delta\to\infty} \int_{-\Delta}^{\Delta} \frac{(dx(1-ix)}{1+x^2}$$

$$= \lim_{\Delta\to\infty} \int_{-\Delta}^{\Delta} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = [\text{arc tan}x]_{-\infty}^{\infty} = \pi.$$

Now take $\xi > 0$. Let γ_Δ and γ_Δ' be as in question 2. We consider the integral of the function $e^{-i\xi z}/(1+iz)$ around this contour. This function is holomorphic in the complement of the point i, which is ouside the contour. So

$$\int_{\gamma_\Delta} \frac{e^{-\xi z} dz}{1+iz} = 0.$$

We are allowed to assume

$$\lim_{\Delta\to\infty} \int_{\gamma_\Delta'} \frac{e^{-\xi z} dz}{1+iz} = 0.$$

(This is a more sophisticated estimate than some. The integrand is bounded on the contour by $(|z|-1)^{-1} = (\Delta-1)^{-1}$ which is not quite enough, given that the length of the contour γ_Δ' is $\pi\Delta$. But the contour can be written as a union of two bits. on one of which the length is $O(\sqrt{\Delta})$ and on the other of which th integrand is $O(e^{-\sqrt{\Delta}})$.)
So

$$\lim_{\Delta\to\infty} \int_{-\Delta}^{\Delta} \frac{e^{-i\xi x} dx =}{1+ix} = \lim_{\Delta\to\infty} -\int_{\gamma_-(\Delta)} \frac{e^{-i\xi z} dz}{1+iz} = 0.$$

In the case when $\xi < 0$, when we take $\gamma_+(\Delta)$ to be the contour round the half disc in the upper half plane, the main difference is that

$$\int_{\gamma_+(\Delta)} \frac{e^{-\xi iz}dz}{1+iz} = 2\pi i\text{Res}\left(\frac{e^{-i\xi z}}{1+iz}, i\right) = 2\pi e^{\xi}.$$

So altogether we have

$$\hat{f}(\xi) = \begin{matrix} 0 \text{ if } \xi > 0, \\ \pi \text{ if } \xi = 0, \\ 2\pi e^{\xi} \text{ if } \xi < 0. \end{matrix}$$