MATH348. Harmonic Analysis. Problems 4.

Office hours are 11-1 Monday and 3-5 Tuseday. I am also in my office from 9 on Wed

Work due on *Wednesday 27th October.*

1. Work out the Fourier coefficients $\widehat{f}(n)$, $\widehat{g}(n)$, $\widehat{h}(n)$ of the following functions on $[-\pi, \pi]$.

a)
$$f(x) = \begin{matrix} 1 \text{ if } 0 \leq x \leq \pi, \\ 0 \text{ if } -\pi < x < 0, \end{matrix}$$

b) $g(x) = x$,

c) $h(x) = |x|\pi - \frac{1}{2}\pi^2$.

2. Regard the functions f and g of question 1 as 2π-periodic functions on $\mathbf{R}$. Show that $f * g = h$. You may find it helpful to note that if $0 \leq x \leq \pi$ then

$$f * g(x) = \int_{x-\pi}^{x} y dy$$

and if $-\pi \leq x \leq 0$ then

$$f * g(x) = \int_{-\pi}^{x} y dy + \int_{x+\pi}^{\pi} y dy.$$

Verify that $\widehat{h}(n) = \widehat{f}(n)\widehat{g}(n)$.

3. Consider the Laplace equation in $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 > 1\}$ in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

assume that $u(r, \theta)$ is continuous for $1 \leq r < \infty$ and $\theta \in \mathbf{R}$, and write

$$\widehat{u}(r, n) = \int_0^{2\pi} e^{-in\theta} u(r, \theta) d\theta.$$

a) Show that if u is bounded then

$$|\widehat{u}(r, n)| \leq 2\pi \sup\{|u(r, \theta)| : r > 1,\ \theta \in \mathbf{R}\}.$$

b) Assuming that $\widehat{u}(r, n) = A_n r^{-|n|} + B_n r^{|n|}$ if $n \neq 0$ and $\widehat{u}(r, 0) = A_0 + B_0 \log r$, show that $B_n = 0$ for all n.

c) Show that

$$\sum_{n=1}^{\infty} r^{-n}(e^{in\theta} + e^{-in\theta}) + 1 = \frac{1 - r^{-2}}{|1 - r^{-1}e^{i\theta}|^2}.$$

d) Deduce that

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^{-2}}{|1 - r^{-1}e^{i\theta - t}|^2} u(1, t) dt.$$

MATH348. Harmonic Analysis. Solutions 4.

1a) If $n = 0$,

$$\widehat{f}(0) = \int_0^{\pi} 1 dx = \pi.$$

If $n \neq 0$,

$$\widehat{f}(n) = \int_0^{\pi} e^{-inx} dx = \left[\frac{e^{-inx}}{-in} \right]_0^{\pi} = \frac{1 - (-1)^n}{in}.$$

1b) If $n = 0$,

$$\widehat{g}(0) = \int_{-\pi}^{\pi} x dx = \left[\frac{x^2}{2} \right]_{-\pi}^{\pi} = 0.$$

If $n \neq 0$,

$$\widehat{g}(n) = \int_{-\pi}^{\pi} x e^{-inx} dx = \left[\frac{x e^{-inx}}{-in} \right]_{-\pi}^{\pi} + \frac{1}{in} \int_{-\pi}^{\pi} e^{-inx} dx$$

$$= \frac{(-1)^{n+1} 2\pi}{in} + \frac{1}{n^2} \left[e^{-inx} \right]_{-\pi}^{\pi} = \frac{2\pi (-1)^{n+1}}{in}.$$

c) If $n = 0$

$$\widehat{h}(0) = \int_{-\pi}^{\pi} (|x|\pi - \tfrac{1}{2}\pi^2) dx = \int_0^{\pi} 2\pi x dx - \tfrac{1}{2}\pi^2 \int_{-\pi}^{\pi} dx$$

$$= \pi^3 - \pi^3 = 0.$$

If $n \neq 0$,

$$\widehat{h}(n) = \int_{-\pi}^{\pi} (|x|\pi - \tfrac{1}{2}\pi^2) e^{-inx} dx$$

$$= \left(\int_0^{\pi} x\pi e^{-inx} dx + \int_{-\pi}^{0} (-x)\pi e^{-inx} dx \right) - \frac{\pi^2}{2} \int_{-\pi}^{\pi} e^{-inx} dx$$

$$= \pi \int_0^{\pi} x(e^{-inx} + e^{inx}) dx - \frac{\pi^2}{2} \left[\frac{e^{-inx}}{-in} \right]_{-\pi}^{\pi}$$

$$= \left[\frac{\pi x e^{-inx}}{-in} + \frac{\pi x e^{inx}}{in} \right]_0^{\pi} + \int_0^{\pi} \left(\frac{\pi e^{-inx}}{in} - \frac{\pi e^{inx}}{in} \right) dx + 0$$

$$= \frac{\pi^2 (-1)^n}{-in} + \frac{\pi^2 (-1)^n}{in} + \left[\frac{\pi e^{-inx}}{n^2} \right]_0^{\pi} + \left[\frac{\pi e^{inx}}{n^2} \right]_0^{\pi}$$

$$= \frac{2\pi((-1)^n - 1)}{n^2}.$$

2a). Identify f with its 2π-periodic extension, so that $f(x) = 1 \Leftrightarrow 2n\pi \leq x \leq (2n+1)\pi$ $(n \in \mathbf{Z})$. So for fixed x, $f(x - y) = 1 \Leftrightarrow 2n\pi \leq x - y \leq (2n+1)\pi$ $(n \in \mathbf{Z}) \Leftrightarrow x - (2n+1)\pi \leq y \leq x - 2n\pi$ $(n \in \mathbf{Z})$. So for $0 < x \leq \pi$ and

$-\pi \le y \le \pi$, $f(x-y)=1 \Leftrightarrow x-\pi \le y \le x$. For $-\pi \le x \le 0$ and $-\pi \le y \le \pi$, $f(x-y)=1 \Leftrightarrow -\pi \le y \le x$ or $x+\pi \le y \le \pi$. So

$$f*g(x) = \begin{cases} \displaystyle\int_{x-\pi}^{x} y\,dy = \left[\tfrac{1}{2}y^2\right]_{x-\pi}^{x} = \tfrac{1}{2}(x^2-(x-\pi)^2) = \pi x - \tfrac{1}{2}\pi^2 \text{ if } o < x \le \pi \\ \displaystyle\int_{-\pi}^{x} y\,dy + \int_{\pi+x}^{\pi} y\,dy = \left[\tfrac{1}{2}y^2\right]_{-\pi}^{x} + \left[\tfrac{1}{2}y^2\right]_{x+\pi}^{\pi} \text{ if } -\pi \le x \le o. \end{cases}$$

Now

$$\left[\tfrac{1}{2}y^2\right]_{-\pi}^{x} + \left[\tfrac{1}{2}y^2\right]_{x+\pi}^{\pi} = \tfrac{1}{2}(x^2-\pi^2+\pi^2-(x+\pi)^2) = -x\pi - \tfrac{1}{2}\pi^2.$$

So for $\pi \le x \le \pi$ we have

$$f*g(x) = |x|\pi - \frac{\pi^2}{2} = h(x)$$

as required.

2b) If $n=0$ Then $\widehat{h}(0)=0=\widehat{g}(0)$. So $\widehat{h}(0)=2\pi\widehat{f}(0)\widehat{g}(0)$. If $n \ne 0$,

$$\widehat{f}(n)\widehat{g}(n) = \frac{1-(-1)^n}{in}\frac{2\pi(-1)^{n+1}}{in} = \frac{2\pi(-(-1)^n+(-1)^{2n+2})}{-n^2} = \frac{2\pi((-1)^n-1)}{n^2} = \widehat{h}(n),$$

as required.

3a)

$$|\widehat{u}(r,n)| = \left|\ \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}u(r,\theta)d\theta\ \right| \le \frac{1}{2\pi}\int_0^{2\pi}|e^{-in\theta}u(r,\theta)|d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}|u(r,\theta)|d\theta \le \frac{2\pi}{2\pi}\sup\{|u(r,\theta)| : r \ge 1,\ \theta \in \mathbf{R}\}.$$

b) If $n=0$ and $\widehat{u}(r,0)=A_0+B_0\log r$ then the only way $\widehat{u}(r,0)$ can be bounded for $r>1$ is if $B_0=0$. Similarly if $n \ne 0$ and $\widehat{u}(r,n)=A_n r^{-|n|}+B_n r^{|n|}$ then the only way $\widehat{u}(r,n)$ can be bounded for $r>1$ is if $B_n=0$.

c) If $r>1$, by the formula for the sum of a geometric series,

$$\sum_{n=1}^{\infty} r^{-n}(e^{in\theta}+e^{-in\theta})+1 = \sum_{n=0}^{\infty}(r^{-1}e^{i\theta})^n + \sum_{n=0}^{\infty}(r^{-1}e^{-i\theta})^n - 1$$

$$= \frac{1}{1-r^{-1}e^{i\theta}} + \frac{1}{1-r^{-1}e^{-i\theta}} - 1 = 2\pi P(r,\theta).$$

(Take this to be a definition of $P(r,\theta)$.) Since $1-r^{-1}e^{-i\theta} = \overline{1-r^{-1}e^{i\theta}}$, putting everything over a common denominator,

$$2\pi P(r,\theta) = \frac{1-r^{-1}e^{-i\theta}+1-r^{-1}e^{i\theta}-(1-r^{-1}e^{i\theta})(1-r^{-1}e^{i\theta})}{|1-r^{-1}e^{i\theta}|^2}$$

$$= \frac{2-r^{-1}e^{-i\theta}-r^{-1}e^{i\theta}-1+r^{-1}e^{i\theta}+r^{-1}e^{-i\theta}-r^{-2}e^{i\theta-i\theta}}{|1-r^{-1}e^{i\theta}|^2}$$

$$= \frac{1-r^{-2}}{|1-r^{-1}e^{i\theta}|^2}$$

d) We have $\widehat{u}(r,n) = A_n r^{-|n|}$. Putting $n=1$ then gives $A_n = \widehat{u}(1,n)$. Also, from

$$P(r,\theta) = P_r(\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{-|n|}e^{in\theta}$$

$$\widehat{P}_r(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} r^{-|n|}e^{in\theta}e^{-im\theta}d\theta$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} r^{-|n|}e^{in\theta}e^{-im\theta}d\theta = r^{-|m|}.$$

So we see that $\widehat{u}(r,n) = \widehat{P}(r,n)\widehat{u}(1,n)$. So since Fourier functions determine a continuous - or even integrable - function uniquely, $u_r(\theta) = P_r * u_1(\theta)$, where $u_r(\theta) = u(r,\theta)$, $P_r(\theta) = P(r,\theta)$, that is,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^{-2}}{|1-r^{-1}e^{i\theta-t}|^2}u(1,t)dt.$$